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Multiple Attribute Group Decision-Making Method Based on Generalized Interval-Valued Hesitant Uncertain Linguistic Power Aggregation Operators and Linguistic-Scale Functions

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Abstract As an extension of hesitant linguistic term set, interval-valued hesitant uncertain linguistic set can simultaneously express qualitative information and quantitative information, and reflect the uncertainty and hesitancy of assessment experts. The purpose of this paper is to present a new interval-valued hesitant uncertain linguistic MAGDM method, which can take into account changes in the semantic environment and negative effects caused by experts' extreme assessment values. First, by combining with different linguistic-scale functions, some new operational laws and a new comparison method for IVHULNs are developed to accommodate different semantic environments. Then, several new generalized interval-valued hesitant uncertain linguistic power aggregation operators are proposed, including generalized interval-valued hesitant uncertain linguistic power average operator and its weight form, generalized interval-valued hesitant uncertain linguistic power geometric operator and its weighted form. Some desirable properties and some special cases of these operators are investigated and analyzed. Furthermore, based on the proposed operators, an approach to multiple attribute group decision-making with interval-valued hesitant uncertain linguistic information is developed. Finally, an illustrative example is provided to demonstrate the applicability and feasibility of the proposed approach. A comparative analysis with other existing methods is also conducted to illustrate the effectiveness of the proposed approach.

Keywords Multiple attribute group decision-making · Interval-valued hesitant uncertain linguistic set · Linguistic-scale function · Generalized power aggregation operator

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1 Introduction

As an important extension of the traditional fuzzy set, hesitant fuzzy sets (HFSs) originally proposed by Torra and Narukawa [1, 2] allow the membership degrees of an element to have multiple different values between 0 and 1. HFSs are highly useful to express uncertain information than the traditional fuzzy sets or their extensions, especially when experts are difficult to provide specific assessment values in a decision-making process. HFSs have been investigated in depth and applied in many practical decision problems [3–7]. Similarly to the ordinary fuzzy sets theory, several extensions of HFSs have been extended to accommodate to different decision contexts, including interval-valued hesitant fuzzy sets (IVHFSs) [8], dual hesitant fuzzy sets (DHFSs) [9]. These extensions of HFSs have enriched the applications of HFSs and become a hot topic for many scholars.

However, in many real decision situations, compared with numerical values, experts may prefer to utilize linguistic information to express their options. Therefore, linguistic term sets [10] are widely investigated and applied in the decision-making process to express experts' preference options [11–18]. Motivated by HFSs and linguistic term sets, some useful extensions of HFSs were further proposed, including hesitant fuzzy linguistic term sets (HFLTSs) [19], hesitant fuzzy linguistic sets (HFLSs) [20], interval-valued hesitant fuzzy linguistic sets (IVHFLSs) [21]. For example, when an expert assesses the comfort level of a car, he may choose the linguistic term "good" (s_4) to express his opinion. However, he may be hesitate about several possible membership degrees associated with the linguistic term "good," such as 0.7, 0.8 or 0.9. Under these circumstances, using the hesitate fuzzy linguistic set proposed by Lin et al. [20] , the expert's opinion can be expressed as $\{\langle s_4, (0.7, 0.8, 0.9)\rangle\}$. However, as the complexity of the practical decision-making environment increases, it might not be adequate or sufficient for experts to express their linguistic evaluation and the associated membership degrees by linguistic terms and crisp values. In fact, taking into account the uncertainty and ambiguity of the subjective thinking of experts, interval linguistic terms, i.e., uncertain linguistic variables, and interval values may be more appropriate or convenient for expressing the true opinions of experts. Therefore, by combining with IVHFSs and uncertain linguistic variables, Liu et al. [22] proposed a new extension of the linguistic terms set, called interval-valued hesitant uncertain linguistic set (IVHULS). The desirable characteristic of IVHULSs is that it can simultaneously describe the fuzzy values of two aspects of an evaluation object: One denotes an expert's linguistic evaluation information by an uncertain linguistic variable, and the other uses several possible interval value membership grades to describe the hesitancy of experts. Based on the concept of IVHULSs, the comfort level of a car given by an expert can be expressed as $\{\langle [s_4, s_5], \{[0.7, 0.9]\}\rangle\}$. Comparing with HFLTSs, HFLSs, and IVHFLSs, IVHULSs can more comprehensively and effectively reflect the true preferences of experts. Thus, the research of MADM method based on IVHULSs has important theoretical and applied value.

As we know, aggregation operators are highly useful tools for aggregating experts' preferences to derive the comprehensive value of each alternative. Power average (PA) operator, originally defined by Yager [23], can reduce the negative impact of extreme assessment values provided by experts on the final decision results. Therefore, it has attracted many researchers' attention in recent years. Motivated by the PA operator, Xu and Yager [24] further developed power geometric(PG) average operator and its weighted form. By combining the PA operator and the generalized aggregation operator, Zhou and Chen [25] proposed a new generalized power average(GPA) operator. However, on the one hand, the existing various power averaging operators are based on traditional operational laws and cannot meet the diverse semantic requirements of different experts. On the other hand, they cannot be used to aggregate IVHULSs. Therefore, in this paper, our aim is to present several new power average aggregation operators to integrate interval-valued hesitant uncertain linguistic information.

In the study of linguistic decision-making methods, the processing of language information is an important issue that needs attention. By now, several linguistic information processing methods have been proposed, including the transformation method based on membership function [26, 27], the symbolic calculation method based on the subscripts of linguistic terms [28–30], the transformation method based on cloud model [31], and the 2-tuple linguistic representation model [32–34]. Nevertheless, as mentioned by Martinez and Herrera [35], these above-mentioned existing linguistic modeling methods have certain advantages; however, they cannot be used to handle all types of decision problems. For example, when decision makers assess an object, they may think that the semantic deviation between "good" and "slightly good" is greater or smaller than that between "good" and "very good." That is, as the linguistic term subscript i increases, the semantic deviation between adjacent linguistic terms is not always equal [21]. In many practical decision-making situations, decision makers may have different semantic requirements for predefined linguistic terms. Obviously, these existing linguistic methods cannot successfully solve similar decision-making problems. Thus, in this paper, we introduce the concept of linguistic-scale function (LSF) [21] to redefine the operational laws for IVHULNs to accommodate to different semantic scenarios and improve the flexibility of linguistic information processing, then we further propose four new generalized interval-valued hesitant uncertain linguistic power average operators to solve such decision-making problems.

The main purpose of this paper is to present a MAGDM method under interval-valued hesitant uncertain linguistic environment which cannot only reduce the negative impact of extreme evaluation values on the decision-making results, but also adapt to different semantic environments and satisfy different experts' semantic requirements. To do this, this paper is organized as follows: In Sect. 2, some basic concepts are briefly reviewed. In Sect. 3, by combing with linguistic-scale functions, some new operational laws and a new comparison method for IVHULNs are defined. In Sect. 4, several new generalized power average aggregation operators are developed, some desirable properties and special cases are investigated and analyzed. In Sect. 5, based on the proposed operators, an approach to interval-valued hesitant uncertain fuzzy linguistic MAGDM problems is developed, which considers the relationships among the preference values given by experts and the semantic preferences of experts under different semantic situations. In Sect. 6, a numerical example is provided to illustrate the effectiveness of the proposed approach, subsequently, comparison analyses with other existing MAGDM methods are given. Finally, some conclusions are given in Sect. 7.

2 Preliminaries

2.1 Uncertain Linguistic Variables

Let $S = \{s_0, s_1, s_2, \ldots, s_{2t}\}$ denote a discrete linguistic term set with odd cardinality, where any label s_i represents a possible linguistic variable value. For example, when $t = 3$, S could be represented as $S = \{s_0 = \text{none}, s_1 = \text{very low}, s_2 = \text{low}, s_3 = \text{medium}, s_4 = \text{high}, s_5 = \text{very high}, s_6 = \text{perfect}\}$.

However, in the decision-making process, the linguistic aggregation value is usually not equal to any linguistic term belonging to S. To avoid the computational loss of linguistic terms during the process of linguistic information aggregation, Xu [36] proposed a new continuous linguistic term set $\tilde{S} = \{\tilde{s}_i | i \in [0, q]\}$ to replace the existing discrete linguistic term set S.

Moreover, in many practical decision-making problems, linguistic evaluation values given by experts may be located between any two of the linguistic terms belonging to S. To deal with such situations, Xu [37, 38] further developed the concept of uncertain linguistic variable.

Definition 1 Suppose $\tilde{s} = [s_{a_i}, s_{b_i}], s_{a_i}, s_{b_i} \in S$ and $0 \le a_i \le b_i$, s_{a_i} and s_{b_i} are the lower limit and upper limit of $\tilde{s}$, respectively, then $\tilde{s}$ is called an uncertain linguistic variable.

2.2 Interval Numbers

Definition 2 [39] Let $\gamma = [\gamma^l, \gamma^u] = \{x | 0 \le \gamma^l \le x \le \gamma^u\}$, and then γ is called a positive interval number.

In order to rank the magnitude of any two interval numbers, Ishibuchi and Tanaka [40], Kundu [41], Sengupta and Pal [42], Xu and Da [39] have proposed several ranking methods. In this paper, we utilize the possibility degree formulas mentioned in [39] to rank interval numbers.

Definition 3 Let $\gamma_1 = [\gamma_1^l, \gamma_1^u]$ and $\gamma_2 = [\gamma_2^l, \gamma_2^U]$ be any two interval numbers, and $l_{\gamma_1} = \gamma_1^u - \gamma_1^l, l_{\gamma_2} = \gamma_2^u - \gamma_2^l$, then the possibility degree of $\gamma_1 \ge \gamma_2$ is defined as:

$$P(\gamma_1 \ge \gamma_2) = \max\left\{1 - \max\left\{\frac{\gamma_2^u - \gamma_1^l}{l_{\gamma_1} + l_{\gamma_2}}, 0\right\}, 0\right\}. \quad (1)$$

Similarly, the degree of possibility of $\gamma_2 \ge \gamma_1$ is defined as

$$P(\gamma_2 \ge \gamma_1) = \max\left\{1 - \max\left\{\frac{\gamma_1^u - \gamma_2^l}{l_{\gamma_1} + l_{\gamma_2}}, 0\right\}, 0\right\}. \quad (2)$$

Obviously, the following results can be derived from Eqs. (1) and (2):

(1) $0 \le p(\gamma_1 \ge \gamma_2) \le 1$,
(2) $p(\gamma_1 \ge \gamma_2) + p(\gamma_2 \ge \gamma_1) = 1$,
(3) $p(\gamma_1 \ge \gamma_1) = p(\gamma_2 \ge \gamma_2) = \frac{1}{2}$.

To rank the input arguments $\gamma_i = [\gamma_i^l, \gamma_i^u] (i = 1, 2, \ldots, n)$, using Eq. (1), we can compare each γ_i with all $\gamma_j (j = 1, 2, \ldots, n)$ and then form a complementary matrix $P = [p_{ij}]_{m \times n}$ (for simplicity, suppose $p_{ij} = p(\gamma_i \ge \gamma_j)$), where $p_{ij} \ge 0, p_{ii} = 0.5, p_{ij} + p_{ji} = 1 (i, j = 1, 2, \ldots, n)$. Computing the sum of all of the elements of each line in matrix P, we have

$$p_i = \sum_{j=1}^{n} p_{ij} (i = 1, 2, \ldots, n). \quad (3)$$

According the value of p_i, we can derive the ranking order of γ_i, i.e., the smaller the value of p_i, the greater the value of γ_i $(i = 1, 2, \ldots, n)$.

2.3 IVHFSs

Definition 4 [1, 2] Let X be a reference set, a HFS on X is defined in terms of a function that when applied to X returns a subset of [0, 1].

To be easily understood, a simple mathematical symbol can be utilized to describe the HFS [3]:

$$E = \{(\langle x, h_E(x)\rangle | x \in X)\},$$

where $h_E(x)$ is a set of values in [0, 1], denoting the possible membership degrees of the element $x \in X$ to E.

However, experts may find it difficult to character all possible membership degrees of an assessment object with exact values. Motivated by the concept of HFSs and interval numbers, Chen et al. [8] introduced the interval-valued hesitant fuzzy sets (IVHFSs), which utilize interval numbers instead of exact values to represent the possible membership degrees of an object to a set.

Definition 5 [8] Let X be a fixed set, and M[0, 1] be the set of all closed subintervals of [0, 1]. Then an IVHFS on X is denoted as

$$\tilde{E} = \{\langle x_i, \tilde{h}_{\tilde{E}}(x_i)\rangle | x_i \in X, \quad i = 1, 2, \ldots, n\}, \quad (4)$$

where $\tilde{h}_{\tilde{E}}(x_i) : X \to M[0, 1]$ denotes all possible interval-valued membership degrees of the element $x_i \in X$. For convenience, we call $\tilde{h}_{\tilde{E}}(x_i)$ an interval-valued hesitant fuzzy element (IVHFE), which is expressed by:

$\tilde{h}_{\tilde{E}}(x_i) = \{\tilde{\gamma} | \tilde{\gamma} \in \tilde{h}_{\tilde{E}}(x_i)\}$,

here, $\tilde{\gamma} = [\tilde{\gamma}^-, \tilde{\gamma}^+]$ is an interval value. $\tilde{\gamma}^- = \inf\tilde{\gamma}$ and $\tilde{\gamma}^+ = \sup\tilde{\gamma}$ represent the lower and upper limits of $\tilde{\gamma}$, respectively. It is easy to observe that IVHFS is an extension of HFS. If $\tilde{\gamma}^+ = \tilde{\gamma}^-$, then IVHFS degenerates into HFS.

2.4 IVHULSs

Definition 6 [22] Let X be a reference set, $s_{\theta(x)}, s_{\eta(x)} \in S$, and $M[0,\ 1]$ be the set of all closed subintervals of [0,1]. An interval-valued hesitant fuzzy linguistic set (IVHULS) on X is:

$$A = \{\langle x, [s_{\theta(x)}, s_{\eta(x)}], \delta(x)\rangle | x \in X\},$$

where $\delta(x) = \bigcup_{\gamma(x) \in \delta(x)} \{\gamma_x^l, \gamma_x^u\}$ denotes several possible interval-valued membership degrees of $x \in X$ belongs to $[s_{\theta(x)}, s_{\eta(x)}]$.

For convenience, $\tilde{\alpha} = \langle [s_{\theta(\tilde{\alpha})}, s_{\eta(\tilde{\alpha})}], \delta(\tilde{\alpha})\rangle$ is called an interval-valued hesitant uncertain linguistic number(IVHULN). If $\delta(\tilde{\alpha}) = \{[\gamma^l, \gamma^u]\}$ has only one interval value, then the membership degree of $\tilde{\alpha}$ belongs to $[s_{\theta(\tilde{\alpha})}, s_{\eta(\tilde{\alpha})}]$ is $[\gamma^l, \gamma^u]$. For example, $\tilde{\alpha} = \langle [s_3, s_5], [0.3, 0.7]\rangle$ is called an interval-valued uncertain linguistic number. Moreover, it should be noted that HFLSs and IVHFLSs are special cases of IVHULSs.

Example 1 Here, let us describe the application of IVHULNs by a simple example. Suppose experts need to evaluate the operability of three different production facilities, denoted as f_1, f_2 and f_3. Considering that this attribute is qualitative, it is more appropriate to use linguistic variables for evaluation. For example, the expert may feel that the linguistic evaluation value for f_1 is higher than "good" but lower than "very good"; meanwhile, the expert may also be uncertain and hesitant about such an uncertain linguistic variable, so the expert can express his opinion by providing several possible interval-valued membership. In this case, an IVHULN is more suitable to express such evaluation, which can be expressed as $\langle [s_5, s_6], [0.5, 0.6], [0.7, 0.8]\rangle$. [0.5, 0.6] and [0.7, 0.8] denote the possible interval-valued membership degrees where f_1 belongs to the uncertain linguistic variable $[s_5, s_6]$.

For any two IVHULNs $\tilde{\alpha}_1 = \langle [s_{\theta(\tilde{\alpha}_1)}, s_{\eta(\tilde{\alpha}_1)}], \delta(\tilde{\alpha}_1)\rangle$ and $\tilde{\alpha}_2 = \langle [s_{\theta(\tilde{\alpha}_2)}, s_{\eta(\tilde{\alpha}_2)}], \delta(\tilde{\alpha}_2)\rangle$, $\lambda \in [0, 1]$, Liu and Ju [22] defined the operational rules as follows.

(1)
$$\tilde{\alpha}_1 \oplus \tilde{\alpha}_2 = \left\langle \left[s_{\theta(\tilde{\alpha}_1)+\theta(\tilde{\alpha}_2)}, s_{\eta(\tilde{\alpha}_1)+\eta(\tilde{\alpha}_2)}\right], \bigcup_{\substack{\gamma(\tilde{\alpha}_1) \in \delta(\tilde{\alpha}_1)\\ \gamma(\tilde{\alpha}_2) \in \delta(\tilde{\alpha}_2)}} \left\{\left[\gamma_{\tilde{\alpha}_1}^l + \gamma_{\tilde{\alpha}_2}^l - \gamma_{\tilde{\alpha}_1}^l\gamma_{\tilde{\alpha}_2}^l, \gamma_{\tilde{\alpha}_1}^u + \gamma_{\tilde{\alpha}_2}^u - \gamma_{\tilde{\alpha}_1}^u\gamma_{\tilde{\alpha}_2}^u\right]\right\}\right\rangle,$$

(2)
$$\tilde{\alpha}_1 \otimes \tilde{\alpha}_2 = \left\langle \left[s_{\theta(\tilde{\alpha}_1)\times\theta(\tilde{\alpha}_2)}, s_{\eta(\tilde{\alpha}_1)\times\eta(\tilde{\alpha}_2)}\right], \bigcup_{\substack{\gamma(\tilde{\alpha}_1) \in \delta(\tilde{\alpha}_1)\\ \gamma(\tilde{\alpha}_2) \in \delta(\tilde{\alpha}_2)}} \left\{\left[\gamma_{\tilde{\alpha}_1}^l\gamma_{\tilde{\alpha}_2}^l, \gamma_{\tilde{\alpha}_1}^u\gamma_{\tilde{\alpha}_2}^u\right]\right\}\right\rangle, \tag{6}$$

(3)
$$\lambda\tilde{\alpha}_1 = \left\langle \left[s_{\lambda\times\theta(\tilde{\alpha}_1)}, s_{\lambda\times\theta(\tilde{\alpha}_2)}\right] \bigcup_{\substack{\gamma(\tilde{\alpha}_1) \in \delta(\tilde{\alpha}_1)\\ \gamma(\tilde{\alpha}_2) \in \delta(\tilde{\alpha}_2)}} \left\{\left[1 - (1-\gamma_{\tilde{\alpha}_1}^l)^\lambda, 1 - (1-\gamma_{\tilde{\alpha}_1}^u)^\lambda\right]\right\}\right\rangle, \tag{7}$$

(4)
$$\tilde{\alpha}_1^\lambda = \left\langle \left[s_{(\theta(\tilde{\alpha}_1))^\lambda}, s_{(\eta(\tilde{\alpha}_1))^\lambda}\right], \bigcup_{\substack{\gamma(\tilde{\alpha}_1) \in \delta(\tilde{\alpha}_1)\\ \gamma(\tilde{\alpha}_2) \in \delta(\tilde{\alpha}_2)}} \left\{\left[(\gamma(\tilde{\alpha}_1))^\lambda, (\gamma(\tilde{\alpha}_u))^\lambda\right]\right\}\right\rangle. \tag{8}$$

It is easy to notice that the above operational rules for IVHULNs are based on subscripts of linguistic terms and assume that the absolute deviations of adjacent linguistic terms are always equal. However, in many practical decision situations, experts may have different semantic preferences for linguistic terms and the absolute deviations between adjacent linguistic terms are not always equal. In order to solve such problems, by combining with linguistic-scale functions, we proposed new operational rules, distance measurement and a comparison method for any two IVHULNs in the next section.

3 New Operations for IVHULSs

3.1 Linguistic-Scale Functions

In general, a simple way to deal with linguistic terms is to directly manipulate the subscripts of linguistic terms. However, in many practical situations, we can find that this way cannot flexibly and precisely express the experts' semantic intention under different semantic environments. To solve such situations, Wang et al. [21] summarized three different linguistic-scale functions (LSFs) that are flexible to convert linguistic terms into different semantic values according to different semantic contexts. These linguistic-scale functions are simply introduced as follows:

Definition 7 [21, 43] A linguistic-scale function (LSF) ρ is a mapping from s_i to $\phi_i (i = 0, 1, 2, \ldots, 2t)$ that is denoted by the following formula:

$$\rho : s_i \rightarrow \phi_i (i = 0, 1, 2, \ldots, 2t), \tag{9}$$

where $s_i \in S$, $\phi_i \in [0, 1]$ is an exact value and represent the semantics value of the linguistic term s_i. In addition, ϕ_i need to satisfy the condition that $0 \le \phi_0 \le \phi_1 \le \phi_2, \ldots, \le \phi_{2t}$. Therefore, ρ is a strictly monotonically increasing function with regard to linguistic subscript i.

(1) The most simple LSF is a simple average calculation of the subscripts of linguistic terms, that is:

$$\rho(s_i) = \phi_i = \frac{i}{2t}(i = 0, 1, 2, \ldots, 2t). \quad (10)$$

(2) The second LSF is a composite scale function that is formed by combining the exponential scale and the $-n \sim n$ scale. Its main characteristic is that the closer to both ends of the linguistic term set, the greater the semantic deviation between the adjacent linguistic terms. For example, the semantic deviation between linguistic terms "very high" and "perfect" is greater than that between "high" and "very high."

$$\rho(s_i) = \phi_i = \begin{cases} \dfrac{\varphi^t - \varphi^{t-i}}{2(\varphi^t - 1)} & (i = 0, 1, 2, \ldots, t), \\ \dfrac{\varphi^t + \varphi^{i-t} - 2}{2(\varphi^t - 1)} & (i = t+1, t+2, \ldots, 2t). \end{cases} \quad (11)$$

The parameter value of φ in Eq. (11) is introduced in the Ref. [44].

(3) The third LSF is based on the value function in prospect theory. Contrary to Eq. (11), this function can characterize the phenomenon that the closer to both ends of the linguistic term set, the smaller the absolute deviation between adjacent linguistic terms.

$$\rho(s_i) = \phi_i = \begin{cases} \dfrac{t^{\psi} - (t-i)^{\psi}}{2t^{\psi}} & (i = 0, 1, 2, \ldots, t), \\ \dfrac{t^{\varphi} + (i-t)^{\varphi}}{2t^{\varphi}} & (i = t+1, t+2, \ldots, 2t). \end{cases} \quad (12)$$

where $\psi, \varphi \in (0, 1]$, and if $\psi = \varphi = 1$, then $\rho(s_i) = \phi_i = \frac{i}{2t}$.

Further, in order to avoid the computational loss, Wang et al. [21] further expanded Eq. (9) to the continuous linguistic term set $\tilde{S}$: $\tilde{\rho} : \tilde{S} \to \tilde{R}(\tilde{R} = \{r | r \geq 0, r \in R\})$, which satisfies $\tilde{\rho}(s_i) = \phi_i$. The inverse function of $\tilde{\rho}$ is represented by $\tilde{\rho}^{-1}$.

From the above definitions, we can observe that compared with other existing linguistic information processing methods, the most important advantage of introducing these three types of LSFs is that they can be used in different semantic environments to satisfy different semantic requirements of experts. For example, for s_4 = "high", s_5 = "very high" and s_6 = "perfect" in the linguistic term set $S(t = 3)$ defined in the Sect. 2.1, experts may have different semantic preferences in a real decision-making environment. If the expert feels that the semantic deviations d_{ij} between adjacent linguistic terms s_i and s_j are always equal, then by using Eq. (10), the semantic deviations $d_{45} = d_{56} = 1/7 \approx 0.14$. If the expert feels the absolute deviation between s_6 and s_5 is greater than that of between s_5 and s_4, then by using Eq. (11) (let $\varphi = 1.4$), we can get the linguistic deviations: $d_{54} = \rho(s_5) - \rho(s_4) = 0.1605$, $d_{65} = \rho(s_6) - \rho(s_5) = 0.2248$. If the expert thinks the deviation between s_6 and s_5 is smaller than that of between s_5 and s_4, then by using Eq. (12) (let $\psi = \varphi = 0.8$), we can get $d_{54} = \rho(s_5) - \rho(s_4) = 0.1539$, $d_{65} = \rho(s_6) - \rho(s_5) = 0.1385$. Obviously, existing other linguistic models cannot be effectively used to deal with the latter two semantic situations. Thus, in the next subsection, by combining LSFs, we will propose the operational laws for IVHULNs, which have better flexibility and adaptability for processing linguistic terms.

3.2 New Operations and Distance Measure

In this subsection, based on the operational rules of IVHULNs defined by Liu and Ju [22], by combining with linguistic-scale functions, we redefined the operational laws for IVHULNs to deal with different semantic situations.

Definition 8 Let $h_\alpha = \langle [s_{\theta(\alpha)}, s_{\eta(\alpha)}], \delta(\alpha) \rangle = \langle [s_{\theta(\alpha)}, s_{\eta(\alpha)}], \bigcup_{\gamma_\alpha = [\gamma_\alpha^l, \gamma_\alpha^u] \in \delta_\alpha} \{[\gamma_\alpha^l, \gamma_\alpha^u]\} \rangle$ and $h_\beta = \langle [s_{\theta(\beta)}, s_{\eta(\beta)}], \delta(\beta) \rangle = \langle [s_{\theta(\beta)}, s_{\eta(\beta)}], \bigcup_{\gamma_\beta = [\gamma_\beta^l, \gamma_\beta^u] \in \delta_\beta} \{[\gamma_\beta^l, \gamma_\beta^u]\} \rangle$ be any two IVHULNs, $\lambda \geq 0$, then new operation laws for IVHULNs are defined as follows:

(1)
$$h_\alpha \oplus h_\beta = \left\langle \left[\tilde{\rho}^{-1}\left(\tilde{\rho}(s_{\theta(\alpha)}) + (\tilde{\rho}(s_{\theta(\beta)})\right), \quad \tilde{\rho}^{-1}\left(\tilde{\rho}(s_{\eta(\alpha)}) + (\tilde{\rho}(s_{\eta(\beta)})\right)\right], \bigcup_{\gamma_\alpha \in \delta(\alpha); \gamma_\beta \in \delta(\beta)} \left\{\left[\gamma_\alpha^l + \gamma_\beta^l - \gamma_\alpha^l \gamma_\beta^l], [\gamma_\alpha^u + \gamma_\beta^u - \gamma_\alpha^u \gamma_\beta^u\right]\right\} \right\rangle, \quad (13)$$

(2)
$$h_\alpha \otimes h_\beta = \left\langle \left[\tilde{\rho}^{-1}\left(\tilde{\rho}(s_{\theta(\alpha)})\tilde{\rho}(s_{\theta(\beta)})\right), \quad \tilde{\rho}^{-1}\left(\tilde{\rho}(s_{\eta(\alpha)})\tilde{\rho}(s_{\eta(\beta)})\right)\right], \bigcup_{\substack{\gamma_\alpha \in \delta(\alpha) \\ \gamma_\beta \in \delta(\beta)}} \left\{\left[\gamma_\alpha^l \gamma_\beta^l, \gamma_\alpha^u \gamma_\beta^u\right]\right\} \right\rangle, \quad (14)$$

(3)
$$\lambda h_\alpha = \left\langle \left[\tilde{\rho}^{-1}\left(\lambda \tilde{\rho}(s_{\theta(\alpha)})\right), \quad \tilde{\rho}^{-1}\left(\lambda \tilde{\rho}(s_{\eta(\alpha)})\right)\right], \bigcup_{\gamma_\alpha \in \delta_\alpha} \left\{\left[1 - (1 - \gamma_\alpha^l)^\lambda, 1 - (1 - \gamma_\alpha^u)^\lambda\right]\right\} \right\rangle, \quad (15)$$

(4)
$$h_\alpha^\lambda = \left\langle \left[\tilde{\rho}^{-1}\left((\tilde{\rho}(s_{\theta(\alpha)}))^\lambda\right), \quad \tilde{\rho}^{-1}\left((\tilde{\rho}(s_{\eta(\alpha)}))^\lambda\right)\right], \bigcup_{\gamma_\alpha \in \delta_\alpha} \left\{\left[(\gamma_\alpha^l)^\lambda, (\gamma_\alpha^u)^\lambda\right]\right\} \right\rangle. \quad (16)$$

Example 2 Let $S = \{s_0$ = none, s_1 = very low, s_2 = low, s_3 = medium, s_4 = high, s_5 = very high, s_6 =

perfect} be a linguistic term set. Assume that we choose Eq. (12) as the LSF to manipulate the following two IVHULNs: $h_\alpha = \langle [s_1, s_2]\{[0.1, 0.3][0.2, 0.4]\}\rangle$ and $h_\beta = \langle [s_2, s_3], \{[0.2, 0.3], [0.3, 0.5]\}\rangle$, let $\lambda = 2, \psi = \frac{1}{2}, \varphi = \frac{1}{2}$, then we can get the following results by using (13)–(16):

(1) $h_\alpha \oplus h_\beta = \langle [s_{2.535}, s_{3.536}], \{[0.28, 0.51], [0.37, 0.65], [0.36, 0.58], [0.44, 0.7]\}\rangle$
(2) $h_\alpha \otimes h_\beta = \langle [s_{0.228}, s_{1.134}], \{[0.02, 0.09], [0.03, 0.15], [0.04, 0.12], [0.06, 0.2]\}\rangle$
(3) $\lambda h_\alpha = 2h_\alpha = \langle [s_{1.798}, s_{2.928}, \{[0.19, 0.51], [0.36, 0.64]\}\rangle$
(4) $h_\alpha^\lambda = h_\alpha^2 = \langle [s_{0.1}, s_{0.512}], \{[0.01, 0.09], [0.04, 0.16]\}\rangle$

Further, based on Example 1, if an expert chooses a semantic scenario in which the closer to both ends of the set of linguistic terms, the greater the deviation between adjacent linguistic terms, then Eq. (11) (let $\varphi = 1.4$) is used to participate in the calculation, and we can get the following result:

(1) $h_\alpha \oplus h_\beta = \langle [s_{3.9659}, s_{5.5316}], \{[0.28, 0.51], [0.37, 0.65], [0.36, 0.58], [0.44, 0.7]\}\rangle$
(2) $h_\alpha \otimes h_\beta = \langle [s_{0.3467}, s_{0.8348}], \{[0.02, 0.09], [0.03, 0.15], [0.04, 0.12], [0.06, 0.2]\}\rangle$
(3) $\lambda h_\alpha = 2h_\alpha == \langle [s_{2.5187}, s_{4.9754}, \{[0.19, 0.51], [0.36, 0.64]\}\rangle$
(4) $h_\alpha^\lambda = h_\alpha^2 = \langle [s_{0.1973}, s_{0.6215}], \{[0.01, 0.09], [0.04, 0.16]\}\rangle$

Thus, from Example 1, we can see that if different linguistic-scale functions are used in the calculation process, the results are obviously different. In actual decision-making, experts can flexibly determine the LSF according to individual semantic preferences. Compared with the operational rules proposed Liu and Ju [22], the new operational laws defined above have better flexibility and adaptability to linguistic decision scenarios.

Theorem 1 *Let $h_{\alpha_i} = \langle [s_{\theta(\alpha_i)}, s_{\eta(\alpha_i)}], \delta(\alpha_i)\rangle (i = 1, 2, 3)$ be any three IVHULNs and $\lambda \geq 0$, the following properties hold:*

(1) $h_{\alpha_1} \oplus h_{\alpha_2} = h_{\alpha_2} \oplus h_{\alpha_1}$,
(2) $h_{\alpha_1} \otimes h_{\alpha_2} = h_{\alpha_2} \otimes h_{\alpha_1}$,
(3) $\lambda(\alpha_1 \oplus \alpha_2) = \lambda h_{\alpha_1} \oplus \lambda h_{\alpha_2}$,
(4) $(h_{\alpha_1} \otimes h_{\alpha_2})^\lambda = h_{\alpha_1}^\lambda \otimes h_{\alpha_2}^\lambda$,
(5) $h_{\alpha_1} \oplus (h_{\alpha_2} \oplus h_{\alpha_3}) = (h_{\alpha_1} \oplus h_{\alpha_2}) \oplus h_{\alpha_3}$,
(6) $h_{\alpha_1} \otimes (h_{\alpha_2} \otimes h_{\alpha_3}) = (h_{\alpha_1} \otimes h_{\alpha_2}) \otimes h_{\alpha_3}$.

Based on these abovementioned operational rules of IVHULNs in Definition 8, Theorem 1 can be easily proven, the proof process is omitted here.

Based on Definition 6 and Hamming distance, we propose the distance measures for IVHULNs as follows:

Definition 9 Let $h_\alpha = \left\langle [s_{\theta(\alpha)}, s_{\eta(\alpha)}], \delta(\alpha)\right\rangle = \left\langle [s_{\theta(\alpha)}, s_{\eta(\alpha)}], \bigcup_{\gamma_\alpha = [\gamma_\alpha^l, \gamma_\alpha^u] \in \delta(\alpha)} [\gamma_\alpha^l, \gamma_\alpha^u]\right\rangle$ and $h_\beta = \left\langle [s_{\theta(\beta)}, s_\eta \quad (\beta)], \delta(\beta)\right\rangle = \left\langle [s_{\theta(\beta)}, s_{\eta(\beta)}], \bigcup_{\gamma_\beta = [\gamma_\beta^l, \gamma_\beta^u] \in \delta(\beta)} [\gamma_\beta^l, \gamma_\beta^u]\right\rangle$ be any two IVHULNs, combining the LSF, the normalized Hamming distance measure between h_α and h_β is defined as follows:

$$\begin{aligned} d_H(h_\alpha, h_\beta) = \frac{1}{4\#\text{len}} \sum_{i=1}^{l} \Big(& \left|\tilde{\rho}(s_{\theta(\alpha)})(\gamma_\alpha^l)^{\sigma(j)} - \tilde{\rho}(s_{\theta(\beta)})(\gamma_\beta^l)^{\sigma(j)}\right| \\ & + \left|\tilde{\rho}(s_{\theta(\alpha)})(\gamma_\alpha^u)^{\sigma(j)} - \tilde{\rho}(s_{\theta(\beta)})(\gamma_\beta^u)^{\sigma(j)}\right| \\ & + \left|\tilde{\rho}(s_{\eta(\alpha)}(\gamma_\alpha^l)^{\sigma(j)} - \tilde{\rho}(s_{\eta(\beta)})(\gamma_\beta^u)^{\sigma(j)}\right| \\ & + \left|\tilde{\rho}(s_{\eta(\alpha)}(\gamma_\alpha^u)^{\sigma(j)} - \tilde{\rho}(s_{\eta(\beta)})(\gamma_\beta^u)^{\sigma(j)}\right| \Big), \end{aligned} \quad (17)$$

where $\gamma_\alpha^{\sigma(j)}, \gamma_\beta^{\sigma(j)}$ are the jth largest interval values in $\delta(\alpha)$ and $\delta(\beta)$, respectively. $\#\text{len}(\delta(\alpha))$, $\#\text{len}(\delta(\beta))$ denote the number of interval values in $\delta(\alpha)$ and $\delta(\beta)$, respectively. #len is the maximum of $\#\text{len}(\delta(\alpha))$ and $\#\text{len}(\delta(\beta))$. It is easy to prove that Eq. (17) satisfies the following conditions: $0 \leq d_H(h_\alpha, h_\beta) \leq 1, d_H(h_\alpha, h_\beta) = 0$ if and only if $h_\alpha = h_\beta, d_H(h_\alpha, h_\beta) = d_H(h_\beta, h_\alpha)$. The proof is omitted here.

Note that different IVHULNs may have different interval numbers in most practical cases, that is $\#\text{len}(\delta_\alpha) \neq \#\text{len}(\delta_\beta)$. In order to calculate correctly, we should supplement the shorter IVHULN by adding some interval numbers to it until both of them have the same length. Let h_α be the shorter IVHULN, and γ^l, γ^u be the maximum and minimum interval numbers in $\delta(\alpha)$ respectively, and similarly to the method proposed by Zhu and Xu [45], we can add the interval values to $\delta(\alpha)$ repeatedly by the following formula:

$$\tilde{\gamma} = \lambda\gamma^u \oplus (1 - \lambda)\gamma^l. \quad (18)$$

It is obvious that the parameter λ is employed to represent experts' risk preferences. In general, λ can be set to 1, 0, or 1/2, corresponding to the max, min and the average interval value, respectively. Optimists may expect good results, and thus add a maximum interval value. On the contrary, pessimists will complement a minimum interval value [4]. Here, experts are considered pessimists(that is $\lambda = 0$). For example, let $h_\alpha = \langle [s_2, s_4], \{[0.1, 0.3], [0.4, 0.5]\}\rangle$, $h_\beta = \langle [s_3, s_4], \{[0.1, 0.3], [0.2, 0.4], [0.6, 0.8]\}\rangle$, Obviously, $\#\text{len}(\delta(\alpha)) = 2 < \#\text{len}(\delta(\beta)) = 3$. To operate correctly, we should extend $\delta(\alpha)$ to have the same length with $\delta(\beta)$. By using Eqs. (1) and (2), we can get the minimum interval values in $\delta(\alpha)$, then we can extend h_α as $\langle [s_2, s_4], \{[0.1, 0.3], [0.4, 0.5], [0.1, 0.3]\}\rangle$ (suppose $\lambda = 0$).

3.3 Comparison Method for IVHULNs

In order to compare any two IVHULNs, Liu et al. [22] defined the score function of an IVHULN. However, the accuracy function of IVHULN has not been provided. In the following, the score function and accuracy function of an IVHULN are proposed, a comparison method for two IVHULNs is also defined.

Definition 10 Let $h_\alpha = \langle [s_{\theta(\alpha)}, s_{\eta(\alpha)}], \delta_\alpha \rangle = \langle [s_{\theta(\alpha)}, s_{\eta(\alpha)}], \bigcup_{\gamma_\alpha=[\gamma_\alpha^l,\gamma_\alpha^u]\in\delta(\alpha)} \{[\gamma_\alpha^l, \gamma_\alpha^u]\}] \rangle$ be an IVHULN, then a score function of h_α can be denoted as follows:

$$S(h_\alpha) = \frac{(\tilde{\rho}(s_{\theta(\alpha)}) + \tilde{\rho}(s_{\eta(\alpha)})) \sum_{\gamma_\alpha\in\delta_\alpha} (\gamma_\alpha^l + \gamma_\alpha^u)}{4\#\text{len}},$$

where #len denotes the number of interval values in $\delta(\alpha)$.

Definition 11 Let $h_\alpha = \langle [s_{\theta(\alpha)}, s_{\eta(\alpha)}], \delta_\alpha \rangle = \langle [s_{\theta(\alpha)}, s_{\eta(\alpha)}], \bigcup_{\gamma_\alpha=[\gamma_\alpha^l,\gamma_\alpha^u]\in\delta_\alpha} \{[\gamma_\alpha^l, \gamma_\alpha^u]\}] \rangle$ be an IVHULN, the expectation function $E(\delta_\alpha)$ of h_α can be defined as $E(\delta_\alpha) = \frac{\sum_{\gamma_\alpha\in\delta_\alpha}(\gamma_\alpha^l+\gamma_\alpha^u)}{2\#l}$, then the variance function $V(\delta_\alpha)$ of h_α can be denoted by $V(\delta_\alpha) = \frac{1}{\#l}\sum_{\gamma_\alpha\in\delta_\alpha}\left[\frac{\gamma_\alpha^l+\gamma_\alpha^u}{2} - E(\delta_\alpha)\right]^2$. Therefore, the accuracy function of $D(h_\alpha)$ can be denoted as follows:

$$D(h_\alpha) = \frac{\tilde{\rho}(s_{\theta(\alpha)}) + \tilde{\rho}(s_{\eta(\alpha)})}{2}(1 - V(\delta_\alpha)), \quad (19)$$

where $\#l$ is number of interval values in δ_α

Definition 12 Let $h_\alpha = \langle [s_{\theta(\alpha)}, s_{\eta(\alpha)}], \delta_\alpha \rangle, h_\beta = \langle [s_{\theta(\beta)}, s_{\eta(\beta)}], \delta_\beta \rangle$ be any two IVHULNs, then

(1) *if* $S(h_\alpha) > S(h_\beta)$, *then* $h_\alpha > h_\beta$.
(2) *if* $S(h_\alpha) = S(h_\beta)$, *then*
if $D(h_\alpha) > D(h_\beta)$, *then* $h_\alpha > h_\beta$,
if $D(h_\alpha) = D(h_\beta)$, *then* $h_\alpha = h_\beta$.

Example 3 Let $h_\alpha = \langle [s_1, s_2], \{[0.2, 0.4], [0.3, 0.7]\} \rangle$ and $h_\beta = \langle [s_2, s_4], \{[0.2, 0.3], [0.1, 0.2]\} \rangle$ are two IVHULNs. If $\tilde{\rho}(s_i) = \frac{i}{2t}(i = 0, 1, 2, \ldots, 2t)$ and $t = 3$, then their comparative order can be calculated as follows:

$$S(h_\alpha) = \frac{\left(\frac{1}{6} + \frac{2}{6}\right)(0.2 + 0.4 + 0.3 + 0.7)}{4 * 2} = 0.1,$$
$$E(\delta_\alpha) = \frac{(0.2 + 0.4) + (0.3 + 0.7)}{2 * 2} = 0.4,$$
$$D(\delta_\alpha) = \frac{1}{2} \times \left[\left(\frac{0.2 + 0.4}{2} - 0.4\right)^2 + \left(\frac{0.3 + 0.7}{2} - 0.4\right)^2\right] = 0.01,$$
$$D(h_\alpha) = \frac{\frac{1}{6} + \frac{2}{6}}{2}(1 - 0.01) = 0.2475.$$

Similarly, $S(b_\beta) = 0.1$ and $D(h_\beta) = 0.4938$. Thus, according to Definition 12, we can derive $h_\alpha < h_\beta$.

4 Some Generalized Interval-Valued Hesitant Uncertain Linguistic Power Aggregation Operators

In this section, the PA and PG operators are extended to accommodate interval-valued hesitant uncertain linguistic context and different semantic situations. In addition, some desirable properties and special cases of these newly proposed operators are investigated and discussed.

Power average (PA) operator was originally introduced by Yager [23] to aggregate a collection of real numbers. It is defined as follows:

$$\text{PA}(\ddot{a}_1, \ddot{a}_2, \ldots, \ddot{a}_n) = \frac{\sum_{i=1}^n (1 + \mathbb{T}(\ddot{a}_i))\ddot{a}_i}{\sum_{i=1}^n (1 + \mathbb{T}(\ddot{a}_i))}, \quad (20)$$

where

$$\mathbb{T}(\ddot{a}_i) = \sum_{\substack{i=1 \\ i\neq j}}^{n} \text{Supp}(\ddot{a}_i, \ddot{a}_j), \quad (21)$$

and $\text{Supp}(\ddot{a}_t, \ddot{a}_k)$ denotes the support measure for $\ddot{a}_t$ from $\ddot{a}_k$, which holds the following basic properties: $\text{Supp}(\ddot{a}_i, \ddot{a}_j) \in [0, 1]$, $\text{Supp}(\ddot{a}_i, \ddot{a}_j) = \text{Supp}(\ddot{a}_j, \ddot{a}_i)$, $\text{Supp}(\ddot{a}_i, \ddot{a}_j) \geq \text{Supp}(\ddot{a}_m, \ddot{a}_n)$ if $|(\alpha_i, \alpha_j)| < |\alpha_m, \alpha_n|$.

Based on Eq. (20) and geometric mean, Xu and Yager [24] further presented a power geometric (PG) operator to aggregate $\ddot{a}_i (i = 1, 2, \ldots, n)$:

$$\text{PG}(\ddot{a}_1, \ddot{a}_2, \ldots, \ddot{a}_n) = \prod_{i=1}^n \ddot{a}_i^{\frac{1+\mathbb{T}(\ddot{a}_i)}{\sum_{i=1}^n (1+\mathbb{T}(\ddot{a}_i))}}. \quad (22)$$

Note that the PA and the PG operator are nonlinear aggregation tools. A typical characteristic of these two operators is that the weights of $\ddot{a}_i (i = 1, 2, \ldots, n)$ depends on the support measure of all other inputs for $\ddot{a}_i$, that is, allowing the inputs to support each other.

4.1 GIVHULPA Operator and GIVHULPGA Operator

Definition 13 Let $\hat{h}_i (i = 1, 2, \ldots, n)$ be a collection of IVHULNs, then a generalized interval-valued hesitant uncertain linguistic power average (GIVHULPA) operator is a mapping $\hat{H}^n \to \hat{H}$ and can be defined as follows:

$$\text{GIVHULPA}(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n) = \left(\frac{\bigoplus_{i=1}^n (1 + \mathbb{T}(\hat{h}_i))\hat{h}_i^\lambda}{\sum_{i=1}^n (1 + \mathbb{T}(\hat{h}_i))}\right)^{\frac{1}{\lambda}}, \quad (23)$$

where $\hat{H}$ is the set of all IVHULNs, and

$$\mathbb{T}(\hat{h}_i) = \sum_{\substack{j=1 \\ j \neq i}}^{n} \mathrm{Supp}(\hat{h}_i, \hat{h}_j), \tag{24}$$

in which $\mathrm{Supp}(\hat{h}_i, \hat{h}_j)$ is the support measure for $\hat{h}_i$ from $\hat{h}_j$, the properties listed below hold:

(1) $\mathrm{Supp}(\hat{h}_i, \hat{h}_j) \in [0, 1]$,
(2) $\mathrm{Supp}(\hat{h}_i, \hat{h}_j) = \mathrm{Supp}(\hat{h}_j, \hat{h}_i)$,
(3) $\mathrm{Supp}(\hat{h}_i, \hat{h}_j) \geq \mathrm{Supp}(\hat{h}_m, \hat{h}_n)$ if $d(\hat{h}_i, \hat{h}_j) < d(\hat{h}_m, \hat{h}_n)$, where d denotes a distance measure between any two IVHUFLNs.

It is worth noting that if we assume that $\xi_i = \frac{1+\mathbb{T}(\hat{h}_i)}{\sum_{i=1}^{n}(1+\mathbb{T}(\hat{h}_i))}$ for all i, then $\sum_{i=1}^{n} \xi_i = 1$, and Eq. (23) can be simplified as

$$\mathrm{GIVHULPA}(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n) = \left(\bigoplus_{i=1}^{n} \xi_i \hat{h}_i^{\lambda}\right)^{\frac{1}{\lambda}}. \tag{25}$$

Based on the operational rules of IVHULNs defined in Definition 8, and for computational convenience, in the following, we suppose $\xi_i = \frac{1+\mathbb{T}(\hat{h}_i)}{\sum_{i=1}^{n}(1+\mathbb{T}(\hat{h}_i))}$, then we have the following theorems.

Theorem 2 *Let $h_{\alpha_i} = \langle [s_{\theta(\alpha_i)}, s_{\eta(\alpha_i)}], \delta(\alpha_i)\rangle (i = 1, 2, \ldots, n)$ be a collection of IVHULNs, the result calculated by utilizing the GIVHULPA operator is also an IVHULN, and*

$$\begin{aligned}&\mathrm{GIVHULPA}(h_1, h_2, \ldots, h_n)\\ &= \left\langle \left[\tilde{\rho}^{-1}\left(\sum_{i=1}^{n} \xi_i \left(\tilde{\rho}\left(s_{\theta(\alpha_i)}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}, \tilde{\rho}^{-1}\left(\sum_{i=1}^{n} \xi_i \left(\tilde{\rho}\left(s_{\eta(\alpha_i)}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right], \right.\\ &\bigcup_{\gamma_{\alpha_i}=[\gamma_{\alpha_i}^l, \gamma_{\alpha_i}^u] \in \delta(\alpha_i)} \left\{\left[\left(1 - \prod_{i=1}^{n}\left(1 - (\gamma_{\alpha_i}^l)^{\lambda}\right)^{\xi_i}\right)^{\frac{1}{\lambda}}, \right.\right.\\ &\left.\left.\left.\left(1 - \prod_{i=1}^{n}\left(1 - (\gamma_{\alpha_i}^u)^{\lambda}\right)^{\xi_i}\right)^{\frac{1}{\lambda}}\right]\right\}\right\rangle.\end{aligned} \tag{26}$$

Proof By using the operation of IVHULNs described in Definition 8 and mathematical induction, Eq. (26) can be proven as follows.

(1) For $n = 2$, since

$$\begin{aligned}\xi_1 h_1^{\lambda} &= \left\langle \left[\tilde{\rho}^{-1}\left(\xi_1\left(\tilde{\rho}\left(s_{\theta(\alpha_1)}\right)\right)^{\lambda}\right), \tilde{\rho}^{-1}\left(\xi_1\left(\tilde{\rho}\left(s_{\eta(\alpha_1)}\right)\right)^{\lambda}\right)\right], \right.\\ &\left.\left\{\left[1 - \left(1 - (\gamma_{\alpha_1}^l)^{\lambda}\right)^{\xi_1}, 1 - \left(1 - (\gamma_{\alpha_1}^u)^{\lambda}\right)^{\xi_1}\right]\right\}\right\rangle,\end{aligned}$$

and

$$\begin{aligned}\xi_2 h_2^{\lambda} &= \left\langle \left[\tilde{\rho}^{-1}\left(\xi_2\left(\tilde{\rho}\left(s_{\theta(\alpha_2)}\right)\right)^{\lambda}\right), \tilde{\rho}^{-1}\left(\xi_2\left(\tilde{\rho}\left(s_{\eta(\alpha_2)}\right)\right)^{\lambda}\right)\right], \right.\\ &\left.\left\{\left[1 - \left(1 - (\gamma_{\alpha_2}^l)^{\lambda}\right)^{\xi_2}, 1 - \left(1 - (\gamma_{\alpha_2}^u)^{\lambda}\right)^{\xi_1}\right]\right\}\right\rangle.\end{aligned}$$

then

$$\begin{aligned}&\mathrm{GIVHULPA}(h_1, h_2) = \left(\bigoplus_{i=1}^{2} \xi_i h_i^{\lambda}\right)^{\frac{1}{\lambda}}\\ &= \left(\xi_1 h_1^{\lambda} + \xi_2 h_2^{\lambda}\right)^{\frac{1}{\lambda}}\\ &= \left\langle \left[\tilde{\rho}^{-1}\left(\left(\sum_{i=1}^{2} \xi_i \left(\tilde{\rho}\left(s_{\theta(\alpha_i)}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right), \tilde{\rho}^{-1}\left(\left(\sum_{i=1}^{2} \xi_i \left(\tilde{\rho}\left(s_{\eta(\alpha_i)}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right], \right.\\ &\left.\bigcup_{\substack{\gamma_{\alpha_i} = [\gamma_i^l, \gamma_i^u] \in \delta(\alpha_i) \\ (i=1,2)}} \left\{\left[\left(1 - \prod_{i=1}^{2}\left(1 - (\gamma_{\alpha_i}^l)^{\lambda}\right)^{\xi_i}\right)^{\frac{1}{\lambda}}, \left(1 - \prod_{i=1}^{2}\left(1 - (\gamma_{\alpha_i}^u)^{\lambda}\right)^{\xi_i}\right)^{\frac{1}{\lambda}}\right]\right\}\right\rangle.\end{aligned}$$

so, when $n = 2$, Eq. (26) is right.

(2) Suppose that $n = k$, Eq. (26) is right, i.e.,

$$\begin{aligned}&\mathrm{GIVHULPA}(h_1, h_2, \ldots, h_k) = \left(\oplus_{i=1}^{k} \xi_i h_i^{\lambda}\right)^{\frac{1}{\lambda}}\\ &= \left\langle \left[\tilde{\rho}^{-1}\left(\left(\sum_{i=1}^{k} \xi_i \left(\tilde{\rho}\left(s_{\theta(\alpha_i)}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right), \tilde{\rho}^{-1}\left(\left(\left(\sum_{i=1}^{k} \xi_i \left(\tilde{\rho}\left(s_{\eta(\alpha_i)}\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right], \right.\\ &\left.\bigcup_{\substack{\gamma_{\alpha_i} = [\gamma_i^l, \gamma_i^u] \in \delta(\alpha_i) \\ (i=1,2,\ldots,k)}} \left\{\left[\left(1 - \prod_{i=1}^{k}\left(1 - (\gamma_{\alpha_i}^l)^{\lambda}\right)^{\xi_i}\right)^{\frac{1}{\lambda}}, \left(1 - \prod_{i=1}^{k}\left(1 - (\gamma_{\alpha_i}^u)^{\lambda}\right)^{\xi_i}\right)^{\frac{1}{\lambda}}\right]\right\}\right\rangle,\end{aligned}$$

then when $n = k + 1$, we can obtain:

$$\begin{aligned}&\mathrm{GIVHULPA}(h_1, h_2, \ldots, h_{k+1}) = \left(\bigoplus_{i=1}^{k+1} \xi_i h_i^{\lambda}\right)^{\frac{1}{\lambda}} = \left(\bigoplus_{i=1}^{k} \xi_i h_i^{\lambda} \oplus \xi_{k+1} h_{k+1}^{\lambda}\right)^{\frac{1}{\lambda}}\\ &= \left(\left\langle \left[\tilde{\rho}^{-1}\left(\left(\sum_{i=1}^{k} \xi_i \left(\tilde{\rho}\left(s_{\theta(\alpha_i)}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right), \tilde{\rho}^{-1}\left(\left(\sum_{i=1}^{k} \xi_i \left(\tilde{\rho}\left(s_{\eta(\alpha_i)}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right], \right.\right.\\ &\left.\bigcup_{\substack{\gamma_{\alpha_i} = [\gamma_{\alpha_i}^l, \gamma_{\alpha_i}^u] \in \delta(\alpha_i) \\ (i=1,2,\ldots,k)}} \left\{\left[\left(1 - \prod_{i=1}^{k}\left(1 - (\gamma_{\alpha_i}^l)^{\lambda}\right)^{\xi_i}\right)^{\frac{1}{\lambda}}, \left(1 - \prod_{i=1}^{k}\left(1 - (\gamma_{\alpha_i}^u)^{\lambda}\right)^{\xi_i}\right)^{\frac{1}{\lambda}}\right]\right\}\right\rangle\\ &\bigoplus \left\langle \left[\tilde{\rho}^{-1}\left(\xi_{k+1}\left(\tilde{\rho}\left(s_{\theta(\alpha_{k+1})}\right)\right)^{\lambda}\right), \tilde{\rho}^{-1}\left(\xi_{k+1}\left(\tilde{\rho}\left(s_{\eta(\alpha_{k+1})}\right)\right)^{\lambda}\right)\right], \right.\\ &\left.\left.\left\{\left[1 - \left(1 - (\gamma_{\alpha_{k+1}}^l)^{\lambda}\right)^{\xi_{k+1}}, 1 - \left(1 - (\gamma_{\alpha_{k+1}}^u)^{\lambda}\right)^{\xi_{k+1}}\right]\right\}\right\rangle\right)^{\frac{1}{\lambda}}\\ &= \left\langle \left[\tilde{\rho}^{-1}\left(\left(\sum_{i=1}^{k+1} \xi_i \left(\tilde{\rho}\left(s_{\theta(\alpha_i)}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right), \tilde{\rho}^{-1}\left(\left(\sum_{i=1}^{k+1} \xi_i \left(\tilde{\rho}\left(s_{\eta(\alpha_i)}\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right], \right.\\ &\left.\bigcup_{\substack{\gamma_{\alpha_i} = [\gamma_{\alpha_i}^l, \gamma_{\alpha_i}^u] \in \delta(\alpha_i) \\ (i=1,2,\ldots,k+1)}} \left\{\left[\left(1 - \prod_{i=1}^{k+1}\left(1 - (\gamma_{\alpha_i}^l)^{\lambda}\right)^{\xi_i}\right)^{\frac{1}{\lambda}}, \left(1 - \prod_{i=1}^{k+1}\left(1 - (\gamma_{\alpha_i}^u)^{\lambda}\right)^{\xi_i}\right)^{\frac{1}{\lambda}}\right]\right\}\right\rangle.\end{aligned}$$

Thus, Eq. (26) holds for all $n = k + 1$, which completes the proof. □

Theorem 3 *Let $\mathrm{Supp}(h_{\alpha_i}, h_{\alpha_j}) = \varepsilon$ for all $i \neq j$, then*

$$\mathrm{GIVHULPA}(h_{\alpha_1}, h_{\alpha_2}, \ldots, h_{\alpha_n}) = \left(\bigoplus_{i=1}^{n} \frac{1}{n} h_{\alpha_i}^{\lambda}\right)^{\frac{1}{\lambda}}, \tag{27}$$

which denotes that if all the support measures are equal, Eq. (23) reduces to the generalized interval-valued hesitant uncertain linguistic weighted average (IVHULWA) operator.

Proof Since $\text{Supp}(h_{\alpha_i}, h_{\alpha_j}) = \varepsilon$ for all $i \neq j$, then

$$\mathbb{T}(h_{\alpha_i}) = \sum_{\substack{j=1 \\ i \neq j}}^{n} \text{Supp}(h_{\alpha_i}, h_{\alpha_j}) = (n-1)k.$$

Thus, we have

$$\begin{aligned} \text{GIVHULPA}(h_{\alpha_1}, h_{\alpha_2}, \ldots, h_{\alpha_n}) &= \left(\frac{\bigoplus_{i=1}^{n}(1+\mathbb{T}(h_{\alpha_i})) \otimes h_i^{\lambda}}{\sum_{i=1}^{n}(1+\mathbb{T}(h_{\alpha_i}))}\right)^{\frac{1}{\lambda}} \\ &= \left(\bigoplus_{i=1}^{n}(\frac{1}{n}h_{\alpha_i}^{\lambda})\right)^{\frac{1}{\lambda}}. \end{aligned}$$

□

Theorem 4 (Commutativity) Let $h_{\alpha_i}(i=1,2,\ldots,n)$ *be a collection of IVHUFLNs and* $\tilde{h}_{\alpha_i}(i=1,2,\ldots,n)$ *be any permutation of* $h_{\alpha_i}(i=1,2,\ldots,n)$ *then*

$$\text{GIVHULPA}(h_{\alpha_1}, h_{\alpha_2}, \ldots, h_{\alpha_n}) = \text{GIVHULPA}(\tilde{h}_{\alpha_1}, \tilde{h}_{\alpha_2}, \ldots, \tilde{h}_{\alpha_n}).$$

Proof Since $\tilde{h}_{\alpha_i}$ is any permutation of $h_{\alpha_i}(i=1,2,\ldots,n)$, then for each h_{α_i}, there must be exist one and only one $\tilde{h}_{\alpha_i}$ such that $h_{\alpha_i} = \tilde{h}_{\alpha_i}, \mathbb{T}(h_{\alpha_i}) = \mathbb{T}(\tilde{h}_{\alpha_i})$, and vice verse. Thus, we have:

$$\sum_{i=1}^{n}(1+\mathbb{T}(h_{\alpha_i})) \otimes h_{\alpha_i}^{\lambda} = \sum_{i=1}^{n}(1+\mathbb{T}(\tilde{h}_{\alpha_i})) \otimes h_{\alpha_i}^{\lambda}.$$

According to Eq. (23), we have

$$\text{GIVHULPA}(h_{\alpha_1}, h_{\alpha_2}, \ldots, h_{\alpha_n}) = \left(\frac{\bigoplus_{i=1}^{n}(1+\mathbb{T}(h_{\alpha_i})) \otimes h_{\alpha_i}^{\lambda}}{\sum_{i=1}^{n}(1+\mathbb{T}(h_{\alpha_i}))}\right)^{\frac{1}{\lambda}},$$

$$\text{GIVHULPA}(\tilde{h}_{\alpha_1}, \tilde{h}_{\alpha_2}, \ldots, \tilde{h}_{\alpha_n}) = \left(\frac{\bigoplus_{i=1}^{n}(1+\mathbb{T}(\tilde{h}_{\alpha_i})) \otimes \tilde{h}_{\alpha_i}^{\lambda}}{\sum_{i=1}^{n}(1+\mathbb{T}(\tilde{h}_{\alpha_i}))}\right)^{\frac{1}{\lambda}}.$$

Therefore

$$\text{GIVHULPA}(h_{\alpha_1}, h_{\alpha_2}, \ldots, h_{\alpha_n}) = \text{GIVHULPA}(\tilde{h}_{\alpha_1}, \tilde{h}_{\alpha_2}, \ldots, \tilde{h}_{\alpha_n}).$$

□

Note that the GIVHULPA operator is neither monotonic nor idempotent, which can be illustrated by the following example.

Example 4 Let $h_1 = \langle[s_1, s_3], \{[0.4, 0.6], [0.5, 0.7]\}\rangle, h_2 = \langle[s_2, s_4], \{[0.2, 0.4], [0.3, 0.5]\}\rangle, h_3 = \langle[s_1, s_2], \{[0.2, 0.3], [0.4, 0.6]\}\rangle$, $h_4 = \rangle[s_3, s_5], \{[0.1, 0.3], [0.2, 0.4]\}\rangle$ be four IVHUFLNs. Suppose $\text{Supp}(h_i, h_j) = 1 - d_H(h_i, h_j)$, $\lambda = 1$ and $\tilde{\rho}(s_{\theta(i)}) = \frac{i}{2t}$ (for computational convenience). Then, by Eq. (30), we have

$$\begin{aligned} \text{GIVHULPA}(h_1, h_1, h_1) = \langle[s_1, s_3], \{&[0.4, 0.6][0.4354, 0.6362], \\ &[0.4354, 0.6368], [0.4687, 0.6697], \\ &[0.4354, 0.6367], [0.4687, 0.6696], \\ &[0.4687, 0.6702], [0.5, 0.7]\}\rangle, \\ (h_1, h_2, h_3) = \langle[s_{1.3358}, s_{3.0065}], \{&[0.2735, 0.4489], \\ &[0.3391, 0.5416], [0.3053, 0.4816], \\ &[0.3681, 0.5689], [0.3165, 0.4995], \\ &[0.3783, 0.5837], [0.3465, 0.5292], \\ &[0.4055, 0.6085]\}\rangle, \\ \text{GIVHULPA}(h_1, h_2, h_4) = \langle[s_{2.0014}, s_{4.0014}], \{&[0.2437, 0.4489], \\ &[0.2728, 0.4762], [0.2768, 0.4816], \\ &[0.3046, 0.5073], [0.2881, 0.4995], \\ &[0.3155, 0.5243], [0.3192, 0.5292], \\ &[0.3454, 0.5525]\}\rangle. \end{aligned}$$

According to Definition 10, we have $S(h_1) = 0.1833$, $S(h_2) = 0.175$, $S(h_3) = 0.0938$, $S(h_4) = 0.1667$, $S(\text{GIVHULPA}(h_1, h_1, h_1)) = 0.184$, $S(\text{GIVHULPA}(h_1, h_2, h_3)) = 0.1582$ and $S(\text{GIVHULPA}(h_1, h_2, h_4)) = 0.1997$.

Obviously, $S(\text{GIVHULPA}(h_1, h_2, h_3)) \neq S(h_1)$. Therefore, $\text{GIVHULPA}(h_1, h_2, h_3) \neq h_1$, which explains that the GIVHULPA operator is not idempotent. Moreover, because $S(\text{GIVHULPA}\,(h_1, h_2, h_4)) = 0.1997 > \max s(h_1), (h_2), (h_4)\} = S(h_1) = 0.1833$, the inequation $\min_{\{i=1,2,4\}} \leq \text{GIVHULPA}(h_1, h_2, h_4) \leq \max_{\{i=1,2,4\}}$ does not hold. Thus, the GIVHULPA operator is not bounded.

Lemma 1 *Let* $\tau_i > 0, \gamma_i > 0, i = 1, 2, \ldots, t$, and $\sum_{i=1}^{t} \gamma_i = 1$, then

$$\prod_{i=1}^{t} \tau_j^{\gamma_j} \leq \sum_{i=1}^{t} \gamma_j \tau_j,$$

with equality if and only if each $\tau_i(i = 1, 2, \ldots, t)$ *is equal.*

Theorem 5 $h_{\alpha_i} = \langle[s_{\theta(\alpha_i)}, s_{\eta(\alpha_i)}], \delta(\alpha_i)\rangle = \langle\ [s_{\theta(\alpha_i)}, s_{\eta(\alpha_i)}], \bigcup_{\gamma_{\alpha_i}=[\gamma_{\alpha_i}^l, \gamma_{\alpha_i}^u] \in \delta(\alpha_i)} \{[\gamma_{\alpha_i}^l, \gamma_{\alpha_i}^u]\}]\rangle(i = 1, 2, \ldots, n)$ be a *collection of IVHULNs and* $\lambda > 0$. *Then,*

$$\text{GIVHULPG}(h_{\alpha_1}, h_{\alpha_2}, \ldots, h_{\alpha_n}) \leq \text{GIVHULPA}(h_{\alpha_1}, h_{\alpha_2}, \ldots, h_{\alpha_n}).$$

Proof For any uncertain fuzzy linguistic information part of the aggregated result by using the GIVHULPG operator, by Lemma 1, we have

$$\begin{aligned} \tilde{\rho}^{-1}\left(\prod_{i=1}^{n}\left(\tilde{\rho}(s_{\theta(\alpha_i)})\right)^{\xi_i}\right) &= \tilde{\rho}^{-1}\left(\left(\prod_{i=1}^{n}\left(\left(\tilde{\rho}(s_{\theta(\alpha_i)})\right)^{\lambda}\right)^{\xi_i}\right)^{\frac{1}{\lambda}}\right) \\ &\leq \tilde{\rho}^{-1}\left(\left(\sum_{i=1}^{n}\xi_i\left(\tilde{\rho}(s_{\theta(\alpha_i)})\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right), \end{aligned}$$

similarly,

$$\tilde{\rho}^{-1}\left(\prod_{i=1}^{n}\left(\tilde{\rho}(s_{\eta(\alpha_i)})\right)^{\xi_i}\right) \leq \tilde{\rho}^{-1}\left(\left(\sum_{i=1}^{n}\xi_i\left(\tilde{\rho}(s_{\eta(\alpha_i)})\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right),$$

and for the membership of the aggregated result by using the GIVHULPG operator, we have

$$\prod_{i=1}^{n}(\gamma_{\alpha_i}^{l})^{\xi_i} = \left(\prod_{i=1}^{n}\left((\gamma_{\alpha_i}^{l})^{\lambda}\right)^{\xi_i}\right)^{\frac{1}{\lambda}} \leq \left(\sum_{i=1}^{n}\xi_i(\gamma_{\alpha_i}^{l})^{\lambda}\right)^{\frac{1}{\lambda}} = \left(1-\sum_{i=1}^{n}\xi_i\left(1-(\gamma_{\alpha_i}^{l})^{\lambda}\right)\right)^{\frac{1}{\lambda}} \leq \left(1-\prod_{i=1}^{n}\left(1-(\gamma_{\alpha_i}^{l})^{\lambda}\right)^{\xi_i}\right)^{\frac{1}{\lambda}}.$$

Similarly, we can derive

$$\prod_{i=1}^{n}(\gamma_{\alpha_i}^{u})^{\xi_i} = \left(\prod_{i=1}^{n}\left((\gamma_{\alpha_i}^{u})^{\lambda}\right)^{\xi_i}\right)^{\frac{1}{\lambda}} \leq \left(\sum_{i=1}^{n}\xi_i(\gamma_{\alpha_i}^{u})^{\lambda}\right)^{\frac{1}{\lambda}} = \left(1-\sum_{i=1}^{n}\xi_i\left(1-(\gamma_{\alpha_i}^{u})^{\lambda}\right)\right)^{\frac{1}{\lambda}} \leq \left(1-\prod_{i=1}^{n}\left(1-(\gamma_{\alpha_i}^{u})^{\lambda}\right)^{\xi_i}\right)^{\frac{1}{\lambda}}.$$

By Definition 12, we can conclude that

$$S(\mathrm{GIVHULPG}(h_{\alpha_1}, h_{\alpha_2}, \ldots, h_{\alpha_n})) \leq S(\mathrm{GIVHULPA}(h_{\alpha_1}, h_{\alpha_2}, \ldots, h_{\alpha_n})),$$

which implies that

$$\mathrm{GIVHULPG}(h_{\alpha_1}, h_{\alpha_2}, \ldots, h_{\alpha_n}) \leq \mathrm{GIVHULPA}(h_{\alpha_1}, h_{\alpha_2}, \ldots, h_{\alpha_n}).$$

□

Now, we consider some special cases of GIVHULPA operator. For computational convenience, we suppose $\xi_i = (1+\mathbb{T}(h_{\alpha_i}))/\sum_{i=1}^{n}(1+\mathbb{T}(h_{\alpha_i}))$.

(1) If $\lambda \to 0$, then the GIVHULPA operator achieves the following limit:

$$\lim_{\lambda\to 0}\mathrm{GIVHULPA}(h_{\alpha_1}, h_{\alpha_2}, \ldots, h_{\alpha_n}) = \left\langle \left[\tilde{\rho}^{-1}\left(\prod_{i=1}^{n}\left(\tilde{\rho}(s_{\theta(\alpha_i)})\right)^{\xi_i}\right), \tilde{\rho}^{-1}\left(\prod_{i=1}^{n}\left(\tilde{\rho}(s_{\eta(\alpha_i)})\right)^{\xi_i}\right)\right], \bigcup_{\gamma_{\alpha_i}=[\gamma_{\alpha_i}^{l},\gamma_{\alpha_i}^{u}]\in\delta(\alpha_i)}\left\{\left[e^{\prod_{i=1}^{n}\left(\ln(\gamma_{\alpha_i}^{l})\right)^{\xi_i}}, e^{\prod_{i=1}^{n}\left(\ln(\gamma_{\alpha_i}^{u})\right)^{\xi_i}}\right]\right\}\right\rangle.$$

(2) If $\lambda = 1$, the GIVHULPA operator reduces to the IVHULPA operator [46]:

$$\mathrm{GIVHULPA}(h_{\alpha_1}, h_{\alpha_2}, \ldots, h_{\alpha_n}) = \bigoplus_{i=1}^{n}\frac{(1+\mathbb{T}(h_{\alpha_i}))\otimes h_{\alpha_i}}{\sum_{i=1}^{n}(1+\mathbb{T}(h_{\alpha_i}))}.$$

Furthermore, if $\mathrm{Supp}(h_{\alpha_i}, h_{\alpha_j}) = k$ for all $i \leq j$ and $\lambda = 1$, then the GIVHULPA operator reduces to the IVHUFLA operator.

(3) If $\lambda = -1$, the GIVHULPA operator becomes the interval-valued hesitant uncertain fuzzy linguistic power harmonic average(IVHULPHA) operator:

$$\mathrm{GIVHULPA}(h_{\alpha_1}, h_{\alpha_2}, \ldots, h_{\alpha_n}) = \left(\bigoplus_{i=1}^{n}\frac{(1+\mathbb{T}(h_{\alpha_i}))\otimes h_{\alpha_i}^{-1}}{\sum_{i=1}^{n}(1+\mathbb{T}(h_{\alpha_i}))}\right)^{-1}.$$

(4) If $\lambda = 2$, the GIVHULPA operator becomes the interval-valued hesitant uncertain fuzzy linguistic power quadratic average(IVHULPQA) operator:

$$\mathrm{GIVHULPA}(h_{\alpha_1}, h_{\alpha_2}, \ldots, h_{\alpha_n}) = \left(\bigoplus_{i=1}^{n}\frac{(1+\mathbb{T}(h_{\alpha_i}))\otimes h_{\alpha_i}^{2}}{\sum_{i=1}^{n}(1+\mathbb{T}(h_{\alpha_i}))}\right)^{\frac{1}{2}}.$$

In the following, by combining Eq. (20) and the geometric mean, we further present a generalized interval-valued hesitant uncertain linguistic power geometric average (GIVHULPGA) operator.

Definition 14 Let $h_{\alpha_i}(i = 1, 2, \ldots, n)$ be a collection of IVHULNs, then a generalized interval-valued hesitant uncertain linguistic power (GIVHULPG) geometric operator is a mapping $H^n \to H$ and can be defined as follows:

$$\mathrm{GIVHULPG}(h_{\alpha_1}, h_{\alpha_2}, \ldots, h_{\alpha_n}) = \frac{1}{\lambda}\left(\bigotimes_{i=1}^{n}(\lambda\otimes h_{\alpha_i})^{\frac{1+\mathbb{T}(h_{\alpha_i})}{\sum_{i=1}^{n}(1+\mathbb{T}(h_{\alpha_i}))}}\right), \quad (28)$$

where H is the set of all IVHULNs, λ is a parameter such that $\lambda \in (-\infty, +\infty)$ and $\lambda \neq 0$, $\mathbb{T}(h_{\alpha_i})$ satisfies Eq. (24).

Similarly to Definition 13, if we suppose that $\xi_i = \frac{1+\mathbb{T}(h_i)}{\sum_{i=1}^{n}(1+\mathbb{T}(h_i))}$ for all i, then $\sum_{i=1}^{n}\xi_i = 1$, and Eq. (28) can be transformed into the following Eq. (29):

$$\mathrm{GIVHUFLPG}(h_1, h_2, \ldots, h_n) = \frac{1}{\lambda}\left(\bigotimes_{i=1}^{n}(\lambda\otimes h_i)^{\xi_i}\right). \quad (29)$$

According to the operations of IVHULNs and mathematical induction on n, we can obtain Theorem 6.

Theorem 6 *Let $h_{\alpha_i} = \langle[s_{\theta(\alpha_i)}, s_{\eta(\alpha_i)}], \delta(\alpha_i)\rangle(i = 1, 2, \ldots, n)$ be a collection of IVHULNs, the calculated result by utilizing Eq. (29) is also a IVHULN, and*

$$\mathrm{GIVHULPG}(h_{\alpha_1},h_{\alpha_2},\ldots,h_{\alpha_n}) = \left\langle \left[\tilde{\rho}^{-1}\left(\frac{1}{\lambda}\prod_{i=1}^{n}\left(\lambda\tilde{\rho}(s_{\theta(\alpha_i)})\right)^{\xi_i}\right), \tilde{\rho}^{-1}\left(\frac{1}{\lambda}\prod_{i=1}^{n}\left(\lambda\tilde{\rho}(s_{\eta(\alpha_i)})\right)^{\xi_i}\right)\right], \bigcup_{\gamma_{\alpha_i}=[\gamma^l_{\alpha_i},\gamma^u_{\alpha_i}]\in\delta(\alpha_i)}\left\{\left[1-\left(1-\prod_{i=1}^{n}\left(1-(1-\gamma^l_{\alpha_i})^{\lambda}\right)^{\xi_i}\right)^{\frac{1}{\lambda}}, 1-\left(1-\prod_{i=1}^{n}\left(1-(1-\gamma^u_{\alpha_i})^{\lambda}\right)^{\xi_i}\right)^{\frac{1}{\lambda}}\right]\right\}\right\rangle, \tag{30}$$

where $\xi_i = \frac{1+\mathbb{T}(h_{\alpha_i})}{\sum_{i=1}^{n}(1+\mathbb{T}(h_{\alpha_i}))}$.

Theorem 7 *Let* $h_{\alpha_i} = \langle [s_{\theta(\alpha_i)}, s_{\eta(\alpha_i)}], \delta(\alpha_i)\rangle = \langle [s_{\theta(\alpha_i)}, s_{\eta(\alpha_i)}], \bigcup_{\gamma_{\alpha_i}=[\gamma^l_{\alpha_i},\gamma^u_{\alpha_i}]\in\delta(\alpha_i)}\{[\gamma^l_{\alpha_i},\gamma^u_{\alpha_i}]\}]\rangle (i=1,2,\ldots,n)$ *be a collection of IVHULNs and* $\lambda > 0$, *then,*

$$GVHULPG(h_{\alpha_1},h_{\alpha_2},\ldots,h_{\alpha_n}) \le IVHULPA(h_{\alpha_1},h_{\alpha_2},\ldots,h_{\alpha_n}).$$

Proof For the uncertain linguistic information part of the aggregated result by using the GIVHULPG operator, by Lemma 1, we have:

$$\tilde{\rho}^{-1}\left(\frac{1}{\lambda}\prod_{i=1}^{n}\left(\lambda\tilde{\rho}(s_{\theta(\alpha_i)})\right)^{\xi_i}\right) \le \tilde{\rho}^{-1}\left(\frac{1}{\lambda}\sum_{i=1}^{n}\xi_i\lambda\tilde{\rho}(s_{\theta(\alpha_i)})\right) = \tilde{\rho}^{-1}\left(\sum_{i=1}^{n}\xi_i\tilde{\rho}(s_{\theta(\alpha_i)})\right),$$

$$\tilde{\rho}^{-1}\left(\frac{1}{\lambda}\prod_{i=1}^{n}\left(\lambda\tilde{\rho}(s_{\eta(\alpha_i)})\right)^{\xi_i}\right) \le \tilde{\rho}^{-1}\left(\frac{1}{\lambda}\sum_{i=1}^{n}\xi_i\lambda\tilde{\rho}(s_{\eta(\alpha_i)})\right) = \tilde{\rho}^{-1}\left(\sum_{i=1}^{n}\xi_i\tilde{\rho}(s_{\eta(\alpha_i)})\right),$$

and for the interval number part of the aggregated result by using the GIVHULPG operator, we have:

$$1-\left(1-\prod_{i=1}^{n}\left(1-(1-\gamma^l_{\alpha_i})^{\lambda}\right)^{\xi_i}\right)^{\frac{1}{\lambda}} \le 1-\left(1-\sum_{i=1}^{n}\xi_i\left(1-(1-\gamma^l_{\alpha_i})^{\lambda}\right)\right)^{\frac{1}{\lambda}} = 1-\left(\sum_{i=1}^{n}\xi_i(1-\gamma^l_{\alpha_i})^{\lambda}\right)^{\frac{1}{\lambda}} 1-\left(\prod_{i=1}^{n}(1-\gamma^l_{\alpha_i})^{\xi_i\lambda}\right)^{\frac{1}{\lambda}} = 1-\prod_{i=1}^{n}(1-\gamma^l_{\alpha_i})^{\xi_i}.$$

Similarly, we have

$$1-\left(1-\prod_{i=1}^{n}\left(1-(1-\gamma^u_{\alpha_i})^{\lambda}\right)^{\xi_i}\right)^{\frac{1}{\lambda}} \le 1-\prod_{i=1}^{n}(1-\gamma^u_{\alpha_i})^{\xi_i}.$$

By Definition 12, we can conclude that

$$S(\mathrm{GIVHULPG}(h_{\alpha_1},h_{\alpha_2},\ldots,h_{\alpha_n})) \le S(\mathrm{IVHULPA}(h_{\alpha_1},h_{\alpha_2},\ldots,h_{\alpha_n})).$$

Thus, we can derive the result

$$\mathrm{GIVHULPG}(h_{\alpha_1},h_{\alpha_2},\ldots,h_{\alpha_n}) \le \mathrm{IVHULPA}(h_{\alpha_1},h_{\alpha_2},\ldots,h_{\alpha_n}).$$

□

Next, several special cases of the GIVHULPG operator will be discussed. For computational convenience, we suppose $\xi_i = (1+\mathbb{T}(h_{\alpha_i}))/\sum_{i=1}^{n}(1+\mathbb{T}(h_{\alpha_i}))$.

(1) If $\lambda \to 0$, then

$$\lim_{\lambda\to 0}\mathrm{GIVHULPG}(h_{\alpha_1},h_{\alpha_2},\ldots,h_{\alpha_n}) = \left\langle\left[\tilde{\rho}^{-1}\left(\prod_{i=1}^{n}(s_{\theta(\alpha_i)})^{\xi_i}\right), \tilde{\rho}^{-1}\left(\prod_{i=1}^{n}(s_{\eta(\alpha_i)})^{\xi_i}\right)\right], \bigcup_{\gamma_{\alpha_i}=[\gamma^l_{\alpha_i},\gamma^u_{\alpha_i}]\in\delta(\alpha_i)}\left\{1-\mathrm{e}^{\prod_{i=1}^{n}\left(\ln(1-\gamma^l_{\alpha_i})\right)^{\xi_i}}, \mathrm{e}^{\prod_{i=1}^{n}\left(\ln|\gamma^u_{\alpha_i}|\right)^{\xi_i}}\right\}\right\rangle.$$

(2) If $\lambda = 1$, then the GIVHULPG operator reduces to the IVHULPG operator:

$$\mathrm{GIVHULPG}(h_{\alpha_1},h_{\alpha_2},\ldots,h_{\alpha_n}) = \bigotimes_{i=1}^{n}h_{\alpha_i}^{\xi_i} = \left\langle\left[\tilde{\rho}^{-1}\left(\prod_{i=1}^{n}\left(\tilde{\rho}(s_{\theta(\alpha_i)})\right)^{\xi_i}\right), \tilde{\rho}^{-1}\left(\prod_{i=1}^{n}\left(\tilde{\rho}(s_{\eta(\alpha_i)})\right)^{\xi_i}\right)\right], \bigcup_{\gamma_{\alpha_i}=[\gamma^l_{\alpha},\gamma^u_{\alpha}]\in\delta(\alpha_i)}\left\{\left[\prod_{i=1}^{n}(\gamma^l_{\alpha_i})^{\xi_i}, \prod_{i=1}^{n}(\gamma^u_{\alpha_i})^{\xi_i}\right]\right\}\right\rangle.$$

(3) If $\mathrm{Supp}(h_{\alpha_i},h_{\alpha_j}) = \varepsilon$ for all $i \ne j$, then

$$\mathrm{GIVHULPG}(h_{\alpha_1},h_{\alpha_2},\ldots,h_{\alpha_n}) = \frac{1}{\lambda}\left(\bigotimes_{i=1}^{n}(\lambda h_i)^{\frac{1}{n}}\right) = \left\langle\left[\tilde{\rho}^{-1}\left(\frac{1}{\lambda}\prod_{i=1}^{n}\left(\lambda\tilde{\rho}(s_{\theta(\alpha_i)})\right)^{\frac{1}{n}}\right), \tilde{\rho}^{-1}\left(\frac{1}{\lambda}\prod_{i=1}^{n}\left(\lambda\tilde{\rho}(s_{\eta(\alpha_i)})\right)^{\frac{1}{n}}\right)\right], \bigcup_{\gamma_{\alpha_i}=[\gamma^l_{\alpha_i},\gamma^u_{\alpha_i}]\in\delta(\alpha_i)}\left\{\left[1-\left(1-\prod_{i=1}^{n}\left(1-(1-\gamma^l_{\alpha_i})^{\lambda}\right)^{\frac{1}{n}}\right)^{\frac{1}{\lambda}}, 1-\left(1-\prod_{i=1}^{n}\left(1-(1-\gamma^u_{\alpha_i})^{\lambda}\right)^{\frac{1}{n}}\right)^{\frac{1}{\lambda}}\right]\right\}\right\rangle.$$

which shows that if all the support measure are equal, Eq. (28) reduces to the generalized interval-valued hesitant uncertain linguistic geometric(GIVHULG) operator. Further, if $\mathrm{Supp}(h_{\alpha_i},h_{\alpha_j}) = \varepsilon$ for all $i \ne j$ and $\lambda = 1$, then Eq. (28) reduces to the interval-valued hesitant uncertain linguistic geometric(IVHULG) operator.

In the GIVHULPA operator and the GIVHULPG operator, all of the arguments $(h_{\alpha_1},h_{\alpha_2},\ldots,h_{\alpha_n})$ are of equal importance. However, this may not be consistent with the practical decision-making situations. In most practical decision-making problems, different evaluation attributes have different degrees of importance. Therefore, by considering the importance of different attributes, we further present their weighted forms, which are defined in next section.

4.2 Weighted Forms of GIVHULPA and GIVHULPG

Definition 15 Let $h_{\alpha_i}(i=1,2,\ldots,n)$ be a collection of IVHULNs, $\sigma=(\sigma_1,\sigma_2,\ldots,\sigma_n)^T$ be the weight vector with $\sigma_i\in[0,1]$ and $\sum_{i=1}^n\sigma_i=1$. Then, the weighted form of GIVHULPA operator is defined as

$$\mathrm{WGIVHULPA}_{\sigma,\lambda}(h_{\alpha_1},h_{\alpha_2},\ldots,h_{\alpha_n}) = \left(\frac{\bigoplus_{i=1}^n\sigma_i(1+\mathbb{T}(h_{\alpha_i}))\otimes h_{\alpha_i}^{\lambda}}{\sum_{i=1}^n\sigma_i(1+\mathbb{T}(h_{\alpha_i}))}\right)^{\frac{1}{\lambda}}, \quad (31)$$

where

$$\mathbb{T}(h_{\alpha_i})=\sum_{\substack{j=1\\ j\neq i}}^{n}\sigma_j\mathrm{Supp}(h_{\alpha_i},h_{\alpha_j}), \quad (32)$$

Theorem 8 *Let $h_{\alpha_i}=\langle[s_{\theta(\alpha_i)},s_{\eta(\alpha_i)},\delta(\alpha_i)]\rangle(i=1,2,\ldots,n)$ be a collection of IVHULNs. The result using Eq. (31) is also an IVHULN, and*

$$\mathrm{WGIVHULPA}(h_{\alpha_1},h_{\alpha_2},\ldots,h_{\alpha_n}) = \left\langle\left[\tilde{\rho}^{-1}\left(\left(\bigoplus_{i=1}^n v_i(\tilde{\rho}(s_{\theta(\alpha_i)}))^{\lambda}\right)^{\frac{1}{\lambda}}\right),\tilde{\rho}^{-1}\left(\left(\bigoplus_{i=1}^n v_i(\tilde{\rho}(s_{\eta(\alpha_i)}))^{\lambda}\right)^{\frac{1}{\lambda}}\right)\right],\right.$$
$$\left.\bigcup_{\gamma_{\alpha_i}=[\gamma^l_{\alpha_i},\gamma^u_{\alpha_i}]\in\delta(\alpha_i)}\left\{\left[\left(1-\prod_{i=1}^n(1-(\gamma^l_{\alpha_i})^{\lambda})^{v_i}\right)^{\frac{1}{\lambda}},\left(1-\prod_{i=1}^n(1-(\gamma^u_{\alpha_i})^{\lambda})^{v_i}\right)^{\frac{1}{\lambda}}\right]\right\}\right\rangle, \quad (33)$$

where $v_i=\frac{\sigma_i(1+\mathbb{T}(h_{\alpha_i}))}{\sum_{i=1}^n\sigma_i(1+\mathbb{T}(h_{\alpha_i}))}$.

Especially, if $\sigma=(\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n})^T$, then Eq. (31) reduces to the GIVHULPA operator. In addition, if $\mathrm{Supp}(h_{\alpha_i},h_{\alpha_j})=0$ for $i\neq j$, then $\mathbb{T}(h_{\alpha_i})=0$, thus we can derive $\mathrm{WGIVHULPA}(h_{\alpha_1},h_{\alpha_2},\ldots,h_{\alpha_n})=\left(\bigoplus_{i=1}^n\sigma_i\otimes h_{\alpha_i}^{\lambda}\right)^{\frac{1}{\lambda}}$. That is to say, Eq. (8) reduces to the generalized interval-valued hesitant uncertain linguistic mean.

Definition 16 Let $h_{\alpha_i}(i=1,2,\ldots,n)$ be a collection of IVHULNs, $\sigma=(\sigma_1,\sigma_2,\ldots,\sigma_n)^T$ be the weight vector with $\sigma_i\in[0,1]$ and $\sum_{i=1}^n\sigma_i=1$. Then, the weighted form of GIVHULPG is defined as

$$\mathrm{WGIVHULPG}_{\sigma,\lambda}(h_{\alpha_1},h_{\alpha_2},\ldots,h_{\alpha_n}) = \frac{1}{\lambda}\left(\bigotimes_{i=1}^n(\lambda\otimes h_{\alpha_i})^{\frac{\sigma_i(1+\mathbb{T}(h_{\alpha_i}))}{\sum_{i=1}^n\sigma_i(1+\mathbb{T}(h_{\alpha_i}))}}\right), \quad (34)$$

where $\mathbb{T}(h_{\alpha_i})$ satisfies Eq. (32).

Analogously, we can have the following Theorem 9.

Theorem 9 *Let $h_{\alpha_i}=\langle[s_{\theta(\alpha_i)},s_{\eta(\alpha_i)}],\delta(\alpha_i)\rangle(i=1,2,\ldots,n)$ be a collection of IVHULNs. The result using Eq. (34) is also an IVHULN, and*

$$\mathrm{WGIVHULPG}(h_{\alpha_1},h_{\alpha_2},\ldots,h_{\alpha_n}) = \left\langle\left[\tilde{\rho}^{-1}\left(\frac{1}{\lambda}\bigotimes_{i=1}^n(\lambda\tilde{\rho}(s_{\theta(\alpha_i)}))^{v_i}\right),\tilde{\rho}^{-1}\left(\frac{1}{\lambda}\bigotimes_{i=1}^n(\lambda\tilde{\rho}(s_{\eta(\alpha_i)}))^{v_i}\right)\right],\right.$$
$$\left.\bigcup_{\gamma_{\alpha_i}=[\gamma^l_{\alpha_i},\gamma^u_{\alpha_i}]\in\delta(\alpha_i)}\left\{\left[1-\left(1-\prod_{i=1}^n(1-(1-\gamma^l_{\alpha_i})^{\lambda})^{v_i}\right)^{\frac{1}{\lambda}},1-\left(1-\prod_{i=1}^n(1-(1-\gamma^u_{\alpha_i})^{\lambda})^{v_i}\right)^{\frac{1}{\lambda}}\right]\right\}\right\rangle, \quad (35)$$

where $v_i=\frac{\sigma_i(1+T(h_i))}{\sum_{i=1}^n(\sigma_i(1+T(h_i)))}$.

Note that the WGIVHULPA and WGIVHULPG operators, similar to the GIVHULPA and GIVHULPG operators, are neither idempotent nor bounded. Moreover, the WIVGHULPA and WIVGHULPG operators are not commutative. In fact, if $(\tilde{h}_{\alpha_1},\tilde{h}_{\alpha_2},\ldots,\tilde{h}_{\alpha_n})$ is any permutation of $(h_{\alpha_1},h_{\alpha_2},\ldots,h_{\alpha_n})$, $\mathbb{T}(\tilde{h}_{\alpha_i})=\sum_{j=1,j\neq i}^n\sigma_j\mathrm{Supp}(\tilde{h}_{\alpha_i},\tilde{h}_{\alpha_j})$. Since σ_i usually are not equal, then $\mathbb{T}(\tilde{h}_{\alpha_1},\tilde{h}_{\alpha_2},\ldots,\tilde{h}_{\alpha_n})$ may not be the permutation of $\mathbb{T}(h_{\alpha_1},h_{\alpha_2},\ldots,h_{\alpha_n})$. As a result, equations $\mathrm{WGIVHULPA}(\tilde{h}_{\alpha_1},\tilde{h}_{\alpha_2},\ldots,\tilde{h}_{\alpha_n})=\mathrm{WGIVHULPA}(h_{\alpha_1} h_{\alpha_2},\ldots,h_{\alpha_n})$ and $\mathrm{WGIVHULPG}(\tilde{h}_{\alpha_1},\tilde{h}_{\alpha_2},\ldots,\tilde{h}_{\alpha_n})=\mathrm{WGIVHULPG}(h_{\alpha_1},h_{\alpha_2},\ldots,h_{\alpha_n})$ generally do not hold, as illustrated in the following example.

Example 5 Let $h_{\alpha_1}=\langle[s_1,s_3],\{[0.4,0.6],[0.5,0.7]\}\rangle$, $h_{\alpha_2}=\langle[s_2,s_4],\{[0.2,0.4],[0.3,0.5]\}\rangle$, $h_3=\langle[s_1,s_2],\{[0.2,0.3],[0.4,0.6]\}\rangle$ be three IVHULNs and $w=(0.2,0.5,0.3)^T$. Suppose $\mathrm{Supp}(h_{\alpha_i},h_{\alpha_j})=1-d(h_{\alpha_i},h_{\alpha_j})(i,j=1,2,3)$, $\lambda=2$, and $\tilde{\rho}(s_{\theta(\alpha_i)})=\frac{i}{2t}$. Then, by Eqs. (33) and (35), we have:

$$\mathrm{WGIVHULPA}(h_{\alpha_1},h_{\alpha_2},h_{\alpha_3})=\langle[s_{1.5665},s_{3.3129}],\{[0.2632,0.4396],[0.3222,0.5195],[0.3051,0.4861],[0.3561,0.5558],[0.3056,0.4843],[0.3565,0.5543],,[0.34150.5249],[0.3865,0.5868]\}\rangle,$$

$$\mathrm{WGIVHULPA}(h_{\alpha_2},h_{\alpha_3},h_{\alpha_1})=\langle[s_{1.2913},s_{2.8708}],\{[0.2819,0.4469],[0.3342,0.5048],[0.3670,,0.5658],[0.4062,0.6058],[0.3006,0.4687],[0.3496,0.5229],[0.3807,0.5805],[0.4181,0.6186]\}\rangle,$$

$$\mathrm{WGIVHULPG}(h_{\alpha_1},h_{\alpha_2},h_{\alpha_3})=\langle[s_{1.3993},s_{3.0743}],\{[0.2328,0.3993],[0.2818,0.4865],[0.2838,,0.4434],[0.3468,0.5473],[0.2429,0.4093],[0.2946,0.5001],,[0.2966,0.4552],[0.3634,0.5641]\}\rangle,$$

$$\text{WGIVHULPG}(h_{\alpha_2}, h_{\alpha_3}, h_{\alpha_1}) = \langle[s_{1.1667}, s_{2.6507}], \{[0.2459, 0.3897], [0.2607, 0.403], [0.339, , 0.5425], [0.3612, 0.5651], [0.2693, 0.4085], [0.2858, 0.4226], [0.3743, 0.5747], [0.3997, 0.6]\}\rangle.$$

According to Definition 10, we have

$S(\text{WGIVHULPA}(h_{\alpha_1}, h_{\alpha_2}, h_{\alpha_3})) = 0.1730$, $S(\text{WGIVHULPA}(h_2, h_3, h_1)) = 0.155$,
$S(\text{WGIVHULPG}(h_{\alpha_1}, h_{\alpha_2}, h_{\alpha_3})) = 0.1432$, $S(\text{WGIVHULPG}(h_2, h_3, h_1)) = 0.130$.

Obviously, $S(\text{WGIVHULPA}(h_{\alpha_1}, h_{\alpha_2}, h_{\alpha_3})) \neq S(\text{WGIVHULPA}(h_{\alpha_2}, h_{\alpha_3}, h_{\alpha_1}))$, and $S(\text{WGIVHULPG}(h_{\alpha_1}, h_{\alpha_2}, h_{\alpha_3})) \neq S(\text{WGIVHULPG}(h_2, h_3, h_1))$. Therefore, the WGIVHULPA and WGIVHULPG operators are not commutative.

5 An Approach to Multiple Attribute Group Decision-Making with Interval-Valued Hesitant Uncertain Linguistic Information

Let $\widetilde{AX} = \{\widetilde{ax}_1, \widetilde{ax}_2, \ldots, \widetilde{ax}_m\}$ be a set of alternatives, $\widetilde{AC} = \{\widetilde{ac}_1, \widetilde{ac}_2, \ldots, \widetilde{ac}_n\}$ be the collection of attributes, and $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)^T$ be the weight vector, where $\sigma_j \geq 0$ and $\sum_{i=1}^{n} \sigma_i = 1$. Let $AE = \{ae_1, ae_2, \ldots, ae_p\}$ be the set of assessment experts, and $\omega = \{\omega_1, \omega_2, \ldots, \omega_p\}$ is the experts' weight vector, with $\omega_i \geq 0 (i = 1, 2, \ldots, p), \sum_{i=1}^{p} \omega_i = 1$. Suppose that $\text{DM}^k = [a_{ij}^k]_{m \times n}$ is an interval-valued hesitant uncertain linguistic decision matrix given by expert $ae_k (k = 1, 2, \ldots, p)$, where $a_{ij}^k = \langle[s_{\theta(a_{ij}^k)}, s_{\eta(a_{ij}^k)}], \delta(a_{ij}^k)]\rangle$ takes the form of IVHULNs, given by the assessment expert D_k, for alternative ax_i under attribute ac_j, where $\theta(a_{ij}^k) \geq \eta(a_{ij}^k), s_{\theta(a_{ij}^k)}, s_{\eta(a_{ij}^k)} \in S = \{s_0, s_1, \ldots, s_{2t}\}$ and $\delta(a_{ij}^k) = \bigcup_{\gamma_{a_{ij}^k} \in \delta(a_{ij}^k)} \{[\gamma_{a_{ij}^k}^l, \gamma_{a_{ij}^k}^u]\}, 0 \leq \gamma_{a_{ij}^k}^l, \gamma_{a_{ij}^k}^u \leq 1$.

Next, the GIVHULPA (or GIVHULPG) operator is employed to develop an approach to MAGDM with IVHULNs, the detailed steps are illustrated as following:

Step 1 Calculate the supports.

$$\text{Supp}\left(a_{ij}^k, a_{ij}^t\right) = 1 - d\left(a_{ij}^k, a_{ij}^t\right), k,\ t = 1, 2, \ldots, p,\ i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n. \tag{36}$$

which satisfy the conditions defined in Definition 23. Here, $d(a_{ij}^k, a_{ij}^t)$ is the distance measure given by Eq. (17).

Step 2 Utilize expert weights $\omega_k (k = 1, 2, \ldots, p)$ to calculate the weighted support $\mathbb{T}(a_{ij}^k)$ of a_{ij}^k from $a_{ij}^t (t = 1, 2, \ldots, p$, and $t \neq k)$.

$$\mathbb{T}(a_{ij}^k) = \sum_{\substack{t=1 \\ t \neq k}}^{p} \omega_t \text{Supp}(a_{ij}^k, a_{ij}^t), \tag{37}$$

and calculate the weights ξ_{ij}^k with respect to $a_{ij}^k (k = 1, 2, \ldots, p)$:

$$\xi_{ij}^k = \frac{\omega_k (1 + T(a_{ij}^k))}{\sum_{k=1}^{p} \omega_k (1 + T(a_{ij}^k))}, k = 1, 2, \ldots, p. \tag{38}$$

where $\xi_{ij}^k \geq 0$ and $\sum_{k=1}^{p} \xi_{ij}^k = 1$.

Step 3 Utilize the WGIVHULPA operator Eq. (33):

$$\begin{aligned} a_{ij} &= \text{WGIVHULPA}(a_{ij}^1, a_{ij}^2, \ldots, a_{ij}^p) \\ &= \left(\bigoplus_{k=1}^{p} \xi_{ij}^k (a_{ij}^k)^{\lambda}\right)^{\frac{1}{\lambda}} \\ &= \left\langle \left[\tilde{\rho}^{-1}\left(\sum_{k=1}^{p} \xi_{ij}^k (\tilde{\rho}(s_{\theta(a_{ij}^k)}))^{\lambda}\right)^{\frac{1}{\lambda}},\right.\right. \\ &\quad \left. \tilde{\rho}^{-1}\left(\sum_{k=1}^{p} \xi_i (\tilde{\rho}(s_{\eta(a_{ij}^k)}))^{\lambda}\right)^{\frac{1}{\lambda}}\right], \\ &\quad \bigcup_{\gamma_{ij}^k = [\gamma_{ij}^{k-}, \gamma_{ij}^{k+}] \in \delta_{ij}^k} \left\{\left[\left(1 - \prod_{k=1}^{p} (1 - (\gamma_{ij}^{k-})^{\lambda})^{\xi_{ij}^k}\right)^{\frac{1}{\lambda}},\right.\right. \\ &\quad \left.\left.\left.\left(1 - \prod_{k=1}^{p} (1 - (\gamma_{ij}^{k+})^{\lambda})^{\xi_{ij}^k}\right)^{\frac{1}{\lambda}}\right]\right\}\right\rangle, \end{aligned} \tag{39}$$

or the WGIVHULPG operator Eq. (35):

$$\begin{aligned} a_{ij} &= \text{WGIVHULPG}(a_{ij}^1, a_{ij}^2, \ldots, a_{ij}^p) = \frac{1}{\lambda}\left(\bigotimes_{k=1}^{p} (\lambda a_{ij}^k)^{\xi_{ij}^k}\right) \\ &= \left\langle \left[\tilde{\rho}^{-1}\left(\frac{1}{\lambda}\prod_{k=1}^{p} (\lambda\tilde{\rho}(s_{\theta(a_{ij}^k)}))^{\xi_{ij}^k}\right),\right.\right. \\ &\quad \left. \tilde{\rho}^{-1}\left(\frac{1}{\lambda}\prod_{k=1}^{p} (\lambda\tilde{\rho}(s_{\eta(a_{ij}^k)}))^{\xi_{ij}^k}\right)\right], \\ &\quad \bigcup_{\gamma_{ij}^k = [\gamma_{ij}^{k-}, \gamma_{ij}^{k+}] \in \delta_{ij}^k} \left\{\left[1 - \left(1 - \prod_{k=1}^{p} (1 - (1 - \gamma_{ij}^{k-})^{\lambda})^{\xi_{ij}^k}\right)^{\frac{1}{\lambda}}, 1\right.\right. \\ &\quad \left.\left.\left. - \left(1 - \prod_{k=1}^{p} (1 - (1 - \gamma_{ij}^{k+})^{\lambda})^{\xi_{ij}^k}\right)^{\frac{1}{\lambda}}\right]\right\}\right\rangle \end{aligned} \tag{40}$$

to integrate decision matrices $\text{DM}^k = (a_{ij}^k)_{m \times n} (k = 1, 2, \ldots, p)$ given by all experts into the collective decision matrix $\text{DM} = (a_{ij})_{m \times n}$.

Step 4 Calculate the support degrees:

$$\text{Supp}(a_{ij}, a_{it}) = 1 - d(a_{ij}, a_{it}), i = 1, 2, \ldots, m; j, t = 1, 2, \ldots, n. \tag{41}$$

which satisfy the conditions defined in Definition 23. Here, $d(a_{ij}^k, a_{ij}^t)$ is the distance measure given by Eq. (17).

Step 5 Utilize the attribute weights $\sigma_j (j = 1, 2, \ldots, n)$ to calculate the weighted support $\mathbb{T}(a_{ij})$ of a_{ij} from $a_{it}(t = 1, 2, \ldots, n, and\ t \neq j)$:

$$\mathbb{T}(a_{ij}) = \sum_{\substack{t=1 \\ t \neq j}}^{n} \sigma_t \mathrm{Supp}(a_{ij}, a_{it}), \tag{42}$$

and calculate ξ_{ij} associated with a_{ij}:

$$\xi_{ij} = \frac{\sigma_j (1 + \mathbb{T}(a_{ij}))}{\sum_{j=1}^{n} \sigma_j (1 + \mathbb{T}(a_{ij}))}, j = 1, 2, \ldots, n, \tag{43}$$

where $\xi_{ij} \geq 0$, and $\sum_{j=1}^{n} \xi_{ij} = 1$.

Step 6 Utilize the WGIVHULPA operator(Eq. (33))

$$\zeta_i = \mathrm{WGIVHULPA}(a_{i1}, a_{i2}, \ldots, a_{in}) = \left(\bigoplus_{j=1}^{n} \xi_{ij} (a_{ij})^{\lambda}\right)^{\frac{1}{\lambda}},$$

or the WGIVHULPG operator(Eq. (35))

$$\zeta_i = \mathrm{WGIVHULPG}(a_{i1}, a_{i2}, \ldots, a_{in}) = \frac{1}{\lambda}\left(\bigotimes_{j=1}^{n} (\lambda a_{ij})^{\xi_{ij}}\right),$$

to aggregate all attributes values in each row of decision matrix DM and derive the comprehensive assessment value ζ_i corresponding to the alternative $\widetilde{ax}_i$.

Step 7 Rank $\zeta_i (i = 1, 2, \ldots, m)$ in descending order by using the comparison method of IVHULNs proposed in Definition 12.

Step 8 Rank the alternatives and select the best alternative(s) according to the ranking of $\zeta_i (i = 1, 2, \ldots, m)$.

Step 9 End.

6 Illustrative Examples

Example 6 In the following, an illustrative example adapted from [21] is cited to demonstrate the application of the proposed approach. Let us reconsider its background. The main business of a large state-owned enterprise in China is the production and sale of non-ferrous metals. In order to further expand its business, the company needs to choose a partner from serval alternative countries to cooperate. After the previous investigation and research, four alternatives $(\{\widetilde{ax}_1, \widetilde{ax}_2, \widetilde{ax}_3, \widetilde{ax}_4\})$ are considered. Four attributes are under consideration (suppose the weight vector is $\sigma = (0.25, 0.2, 0.3, 0.25)^T$, including $\widetilde{ac}_1$: available mineral resources; $\widetilde{ac}_2$: political environment; $\widetilde{ac}_3$: economic conditions; $\widetilde{ac}_4$: domestic infrastructure

Three assessment experts $\{DM_1, DM_2, DM_3\}$, with the weight vector $\omega = (0.35, 0.4, 0.25)^T$, form an evaluation committee to evaluate alternatives $\widetilde{ax}_i (i = 1, 2, 3, 4)$ under attributes $\widetilde{ac}_j (j = 1, 2, 3, 4)$. IVHULNs are selected by experts to express evaluation information and linguistic term set $S = \{s_0 = \text{very poor},\ s_1 = \text{poor},\ s_2 = \text{slightly poor}, s_3 = \text{fair},\ s_4 = \text{slightly good},\ s_5 = \text{good},\ s_6 = \text{very good}\}$. Three evaluation decision matrices $\left(A^k = (a_{ij}^k)_{m \times n}, k = 1, 2, 3\right)$ are shown in Tables 1, 2 and 3.

6.1 An Illustration of the Developed Approach

In the following, the above approach and the WGIV-HULPA operator are used to rank the alternatives. In order to facilitate the calculation, the first type of LSF, i.e., Eq. (10), is selected for handling linguistic terms. The detailed calculation steps are shown as follows:

Step 1 Calculate the support $\mathrm{Supp}(a_{ij}^m, a_{ij}^n)(i = 1, 2, 3, 4; j = 1, 2, 3, 4, 5; m, n = 1, 2, 3, m \neq n)$ based on Eq. (36) (for simplicity, we denote $\mathrm{Supp}(a_{ij}^m, a_{ij}^n)$ by Supp_{ij}^{mn}), which means the supports between A^m and A^n, and they are shown as follows:

$$\mathrm{Supp}^{12} = \mathrm{Supp}^{21} = \begin{bmatrix} 0.8917 & 0.8854 & 0.9542 & 0.9548 \\ 0.8750 & 0.9000 & 0.7667 & 0.9083 \\ 0.8833 & 0.8208 & 0.8875 & 0.9375 \\ 0.9792 & 0.9333 & 0.9208 & 0.8833 \end{bmatrix}$$

$$\mathrm{Supp}^{13} = \mathrm{Supp}^{31} = \begin{bmatrix} 0.8021 & 0.9125 & 0.8667 & 0.8792 \\ 0.8667 & 0.8500 & 0.8917 & 0.8417 \\ 0.8833 & 0.9125 & 0.9125 & 0.8542 \\ 0.9000 & 0.7625 & 0.9583 & 0.9542 \end{bmatrix}$$

$$\mathrm{Supp}^{23} = \mathrm{Supp}^{32} = \begin{bmatrix} 0.9104 & 0.7979 & 0.8208 & 0.8667 \\ 0.9583 & 0.9167 & 0.7500 & 0.9333 \\ 0.9500 & 0.9083 & 0.8917 & 0.7917 \\ 0.9208 & 0.7458 & 0.9375 & 0.8417 \end{bmatrix}$$

Step 2 Use the weights ω_p of experts $DM_p (p = 1, 2, 3)$ and Eq. (37) to obtain the weighted support $\mathbb{T}(r_{ij}^k)(k = 1, 2, 3)$ of r_{ij}^k from other IVHULNs $r_{ij}^t (t = 1, 2, 3 \text{ and } t \neq k)$:

$$\mathbb{T}_1 = \begin{bmatrix} 0.5572 & 0.5823 & 0.5983 & 0.5981 \\ 0.5667 & 0.5725 & 0.5296 & 0.5738 \\ 0.5742 & 0.5565 & 0.5831 & 0.5885 \\ 0.6167 & 0.5640 & 0.6079 & 0.5919 \end{bmatrix}$$

$$\mathbb{T}_2 = \begin{bmatrix} 0.5397 & 0.5094 & 0.5392 & 0.5477 \\ 0.5458 & 0.5442 & 0.4558 & 0.5513 \\ 0.5467 & 0.5144 & 0.5335 & 0.5260 \\ 0.5729 & 0.5131 & 0.5567 & 0.5169 \end{bmatrix}$$

$$\mathbb{T}_3 = \begin{bmatrix} 0.6449 & 0.6385 & 0.6317 & 0.6544 \\ 0.6867 & 0.6642 & 0.6121 & 0.6679 \\ 0.6892 & 0.6827 & 0.6760 & 0.6156 \\ 0.6833 & 0.5652 & 0.7104 & 0.6706 \end{bmatrix}$$

Then, use Eq. (38) to calculate ξ_{ij}^k of

Table 1 Decision matrix given by DM_1

	$\widetilde{ac}_1$	$\widetilde{ac}_2$	$\widetilde{ac}_3$	$\widetilde{ac}_4$
$\widetilde{ax}_1$	$\langle [s_1,s_2],\{[0.4,0.6]\}\rangle$	$\langle [s_3,s_5],\{[0.6,0.7],[0.6,0.9]\}\rangle$	$\langle [s_2,s_4],\{[0.5,0.6]\}\rangle$	$\langle [s_3,s_4],\{[0.5,0.8]\}\rangle$
$\widetilde{ax}_2$	$\langle [s_2,s_3],\{[0.7,0.9]\}\rangle$	$\langle [s_2,s_4],\{[0.5,0.7]\}\rangle$	$\langle [s_2,s_2],\{[0.6,0.8]\}\rangle$	$\langle [s_2,s_4],\{[0.6,0.8],[0.7,0.9]\}\rangle$
$\widetilde{ax}_3$	$\langle [s_3,s_5],\{[0.6,0.8]\}\rangle$	$\langle [s_3,s_4],\{[0.4,0.6],[0.5,0.7]\}\rangle$	$\langle [s_3,s_4],\{[0.6,0.9]\}\rangle$	$\langle [s_2,s_3],\{[0.7,0.7]\}\rangle$
$\widetilde{ax}_4$	$\langle [s_2,s_3],\{[0.5,0.7]\}\rangle$	$\langle [s_4,s_4],\{[0.6,0.7],[0.6,0.8]\}\rangle$	$\langle [s_2,s_3],\{[0.6,0.7]\}\rangle$	$\langle [s_4,s_5],\{[0.5,0.7]\}\rangle$

Table 2 Decision matrix given by DM_2

	$\widetilde{ac}_1$	$\widetilde{ac}_2$	$\widetilde{ac}_3$	$\widetilde{ac}_4$
$\widetilde{ax}_1$	$\langle [s_1,s_3],\{[0.6,0.8]\}\rangle$	$\langle [s_4,s_5],\{[0.6,0.8],[0.8,0.9]\}\rangle$	$\langle [s_2,s_3],\{[0.5,0.6]\}\rangle$	$\langle [s_3,s_5],\{[0.5,0.6]\}\rangle$
$\widetilde{ax}_2$	$\langle [s_2,s_3],\{[0.4,0.6]\}\rangle$	$\langle [s_4,s_4],\{[0.5,0.6]\}\rangle$	$\langle [s_3,s_4],\{[0.7,0.9]\}\rangle$	$\langle [s_3,s_4],\{[0.7,0.9]\}\rangle$
$\widetilde{ax}_3$	$\langle [s_3,s_4],\{[0.5,0.7]\}\rangle$	$\langle [s_3,s_5],\{[0.7,0.8]\}\rangle$	$\langle [s_2,s_4],\{[0.6,0.7]\}\rangle$	$\langle [s_2,s_3],\{[0.5,0.6]\}\rangle$
$\widetilde{ax}_4$	$\langle [s_2,s_3],\{[0.6,0.7]\}\rangle$	$\langle [s_3,s_4],\{[0.7,0.9]\}\rangle$	$\langle [s_3,s_3],\{[0.5,0.6],[0.7,0.8]\}\rangle$	$\langle [s_4,s_4],\{[0.8,0.9]\}\rangle$

Table 3 Decision matrix given by DM_3

	$\widetilde{ac}_1$	$\widetilde{ac}_2$	$\widetilde{ac}_3$	$\widetilde{ac}_4$
$\widetilde{ax}_1$	$\langle [s_2,s_3],\{[0.7,0.8],[0.7,0.9]\}\rangle$	$\langle [s_3,s_4],\{[0.6,0.7]\}\rangle$	$\langle [s_3,s_4],\{[0.6,0.8]\}\rangle$	$\langle [s_4,s_4],\{[0.7,0.8]\}\rangle$
$\widetilde{ax}_2$	$\langle [s_2,s_2],\{[0.5,0.7]\}\rangle$	$\langle [s_4,s_5],\{[0.5,0.7]\}\rangle$	$\langle [s_1,s_3],\{[0.5,0.7],[0.6,0.8]\}\rangle$	$\langle [s_3,s_5],\{[0.7,0.9]\}\rangle$
$\widetilde{ax}_3$	$\langle [s_3,s_3],\{[0.6,0.8],[0.6,0.9]\}\rangle$	$\langle [s_3,s_4],\{[0.6,0.8]\}\rangle$	$\langle [s_3,s_4],\{[0.6,0.8]\}\rangle$	$\langle [s_3,s_4],\{[0.6,0.9]\}\rangle$
$\widetilde{ax}_4$	$\langle [s_2,s_4],\{[0.6,0.8]\}\rangle$	$\langle [s_1,s_2],\{[0.8,0.9]\}\rangle$	$\langle [s_2,s_3],\{[0.7,0.8]\}\rangle$	$\langle [s_3,s_4],\{[0.6,0.7],[0.6,0.9]\}\rangle$

$r_{ij}^k\,(k=1,2,3;i,j=1,2,3,4)$. We denote $(\xi_{ij}^k)_{4\times4}$ by V_k as shown in the following, respectively.

$$V_1=\begin{bmatrix}0.3467 & 0.3534 & 0.3534 & 0.3513\\ 0.3452 & 0.3474 & 0.3520 & 0.3468\\ 0.3461 & 0.3467 & 0.3493 & 0.3541\\ 0.3502 & 0.3545 & 0.3489 & 0.3520\end{bmatrix}$$

$$V_2=\begin{bmatrix}0.3917 & 0.3852 & 0.3889 & 0.3889\\ 0.3893 & 0.3899 & 0.3829 & 0.3907\\ 0.3886 & 0.3855 & 0.3866 & 0.3887\\ 0.3894 & 0.3920 & 0.3860 & 0.3841\end{bmatrix}$$

$$V_3=\begin{bmatrix}0.2616 & 0.2614 & 0.2577 & 0.2598\\ 0.2655 & 0.2626 & 0.2650 & 0.2625\\ 0.2653 & 0.2677 & 0.2641 & 0.2572\\ 0.2604 & 0.2534 & 0.2651 & 0.2639\end{bmatrix}$$

Step 3 Utilize Eq. (33) to integrate the three decision matrices $A^k=(a_{ij}^k)_{4\times4}(k=1,2,3)$ into the collective decision matrix $A=(a_{ij})_{4\times4}$(see Table 4).

Step 4 Calculate the supports $\mathrm{Supp}(r_{ij},r_{it})(i=1,2,3,4,j,t=1,2,3,4,j\neq t)$ by utilizing Eq. (41). For simplicity, we denote $(\mathrm{Supp}(r_{ij},r_{ip}))_{4\times1}$ by Supp_{jp}, which means the supports between the jth and the pth columns of A.

$$\mathrm{Supp}_{12}=Sup_{21}=\begin{bmatrix}0.7237\\0.8933\\0.9463\\0.7432\end{bmatrix},\mathrm{Supp}_{13}=Sup_{31}=\begin{bmatrix}0.9267\\0.9452\\0.9532\\0.9604\end{bmatrix},$$

$$\mathrm{Supp}_{14}=Sup_{41}=\begin{bmatrix}0.8042\\0.8079\\0.9128\\0.7801\end{bmatrix},\mathrm{Supp}_{23}=Sup_{32}=\begin{bmatrix}0.7987\\0.9481\\0.9567\\0.8938\end{bmatrix},$$

$$\mathrm{Supp}_{24}=Sup_{42}=\begin{bmatrix}0.9213\\0.9146\\0.8929\\0.9085\end{bmatrix},\mathrm{Supp}_{34}=Sup_{43}=\begin{bmatrix}0.8774\\0.8626\\0.9362\\0.8023\end{bmatrix}.$$

Step 5 Calculate the weighted support $\mathbb{T}(a_{ij})$ of a_{ij} from other IVHULNs $a_{it}(t=1,2,3,4,t\neq j)$. We denote $(T(a_{ij}))_{4\times4}$ by T:

$$T = \begin{bmatrix} 0.6238 & 0.6509 & 0.6108 & 0.6485 \\ 0.6642 & 0.7364 & 0.6416 & 0.6437 \\ 0.7034 & 0.7468 & 0.6637 & 0.6876 \\ 0.6318 & 0.6811 & 0.6194 & 0.6174 \end{bmatrix}.$$

Further, utilize Eq. (43) to obtain the weights ε_{ij} of $a_{ij}(i = j = 1,2,3,4.)$ which is shown as follows:

$$V = \begin{bmatrix} 0.2488 & 0.2024 & 0.2962 & 0.2526 \\ 0.2496 & 0.2084 & 0.2955 & 0.2465 \\ 0.2511 & 0.2060 & 0.2942 & 0.2487 \\ 0.2496 & 0.2057 & 0.2973 & 0.2474 \end{bmatrix}.$$

Step 6. Utilize Eq. (33) to integrate all the assessment values $a_{ij}(i,j = 1,2,3,4)$ in the ith row of A and obtain the comprehensive assessment value ζ_i of the alternative $\widetilde{ax}_i$:

$\zeta_1 = \langle [s_{2.4912}, s_{3.7974}], \{[0.5631, 0.7216], [0.5861, 0.7362], [0.5631, 0.7426], [0.5861, 0.7562], [0.5631, 0.7339], [0.5861, 0.7479], [0.5631, 0.754], [0.5861, 0.7669] \}\rangle$,

$\zeta_2 = \langle [s_{2.4678}, s_{3.5172}], \{[0.5942, 0.8032], [0.6040, 0.8145], [0.6197, 0.8096], [0.6111, 0.8205]\}\rangle$,

$\zeta_3 = \langle [s_{2.7015}, s_{3.8372}], \{[0.5902, 0.774], [0.5955\ 0.7786], [0.5902, 0.7842], [0.5955, 0.7647]\}\rangle$,

$\zeta_4 = \langle [s_{2.7187}, s_{3.5010}], [0.6320, 0.7723], [0.6320, 0.7880], [0.6530, 0.8132], [0.632, 0.7789][0.632, 0.7942], [0.6529, 0.8099], [0.6529, 0.8099], [0.653, 0.8187]\rangle$.

Step 7 Using Definition 12, we calculate the scores of $\zeta_i (i = 1,2,3,4)$:

$S(\zeta_1) = 0.3458$, $S(\zeta_2) = 0.3539$, $S(\zeta_3) = 0.3728$, $S(\zeta_4) = 0.3732$.

Then, the descending order of ζ_i can be derived based on the value of $S(\zeta_i)$:

$\zeta_4 > \zeta_3 > \zeta_2 > \zeta_1$.

Step 8 The ranking of $\widetilde{ax}_i (i = 1,2,3,4)$ is the same as the ranking of ζ_i. Thus, we can derive: $\widetilde{ax}_4 > \widetilde{ax}_3 > \widetilde{ax}_2 > \widetilde{ax}_1$.

Thus, the best alternative is $\widetilde{ax}_4$.

6.2 The Influence of the LSFs on Ranking Results

Further, to illustrate the influence of the LSF on the ranking results , other two types of LSFs are also applied to the above decision-making process to derive the ranking results. The results are represented in Table 5.

From Table 5, we can observe that when the second LSF is used to process linguistic terms, we can obtain the same ranking result $\widetilde{ax}_4 > \widetilde{ax}_3 > \widetilde{ax}_2 > \widetilde{ax}_1$ as the above calculation. However, when the third type of LSF, i.e., Eq. (12), is employed in the decision-making process, $\widetilde{ax}_3$ and $\widetilde{ax}_4$ are identified as the first and second best solutions, respectively, which are different from those obtained by other two types of LSFs. The main reason for this difference is that three different types of LSFs depict different semantic situations respectively, and may produce different semantic preferences and semantic deviations, which eventually result in different ranking results. For example, expert DM_2 provides $\langle [s_3, s_4], [0.5, 0.7]$ as his preference for alternative $\widetilde{ax}_3$ with respect to attribute $\widetilde{ac}_1$. If we use these three different types of LSFs to deal with this attribute value, absolute semantic deviations between s_3 and s_4 are 0.17, 0.11 and 0.21, respectively. Obviously, different semantic deviations will have a certain impact on the final decision-making results. Thus, one of the advantages of our proposed method is that it can accommodate to different semantic decision-making environments and satisfy the semantic requirements of different experts. In the actual decision-making, experts can flexibly select the appropriate LSF according to their real decision-making linguistic preferences.

6.3 Comparison Analyses and Discussions

In this subsection, to illustrate the effectiveness and feasibility of the proposed method, we compare the proposed method with other existing methods based on different interval-valued hesitant uncertain linguistic aggregation operators: generalized interval-valued hesitant uncertain linguistic weighted average (GIVHULWA) operator [22], generalized interval-valued hesitant uncertain linguistic weighted geometric (GIVHULWG) operator [22] and interval-valued hesitant uncertain linguistic power weighted average (IVHFULPWA) operator [46].

6.3.1 Compared with the Method Based on the IVHULPWA Operator

In Ref. [46], Wei proposed an interval-valued hesitant uncertain linguistic power weighted average (IVHULPWA) operator to derive the comprehensive values of alternatives. Now, we utilize the MAGDM method based on the IVHULPWA operator to solve Example 6. The ranking results are presented in Table 6.

From Table 6, it can be observed that the ranking result obtained by Wei's method [46] is exactly the same as the result obtained by our proposed method with the first and second type of LSFs. This shows the effectiveness and feasibility of the proposed method. However, when the third type of LSF is applied to the calculation process, the ranking results obtained by Wei's method [46] are slightly different from that obtained by the proposed method. By using Wei' method [46], $\widetilde{ax}_4$ and $\widetilde{ax}_3$ are evaluated as the best and second best alternatives, respectively, whereas the best alternative identified by our method is $\widetilde{ax}_3$ and the second best alternative is $\widetilde{ax}_4$. Next, it is necessary to look

Table 4 The collective interval-valued hesitant uncertain linguistic decision matrix A

$\widetilde{ac}_1$	$\widetilde{ac}_2$	$\widetilde{ac}_3$	$\widetilde{ac}_4$
$\widetilde{ax}_1$			
$\langle[s_{1.2616}, s_{2.6533}]$, {[0.5730, 0.7457], [0.5730, 0.7878]}$\rangle$	$\langle[s_{3.3852}, s_{4.7386}]$, {[0.6000, 0.7434], [0.6937, 0.8035], [0.6, 0.8259], [0.6937, 0.8667]}$\rangle$	$\langle[s_{2.2577}, s_{3.6111}]$, {[0.5297, 0.6654]}$\rangle$	$\langle[s_{3.2598}, s_{4.3889}]$, {[0.5621, 0.7381]}$\rangle$
$\widetilde{ax}_2$			
$\langle[s_{2.0}, s_{2.7345}]$, { [0.55, 0.7704]}$\rangle$	$\langle[s_{3.3051}, s_{4.2626}]$, {[0.5, 0.6644]}$\rangle$	$\langle[s_{2.1179}, s_{3.0309}]$, {[0.6199, 0.8292], [0.6423, 0.8473]}$\rangle$	$\langle[s_{2.6532}, s_{4.2625}]$, {[0.6685, 0.8728], [0.7, 0.9]}$\rangle$
$\widetilde{ax}_3$			
$\langle[s_{3.0}, S_{4.0808}]$, {[0.5638, 0.7659, [0.5638, 0.8052]}$\rangle$	$\langle[s_{3.0}, s_{4.3855}]$, {[0.5880, 0.7457], [0.6132, 0.7698]}$\rangle$	$\langle[s_{2.6134}, s_{3.7359}]$, {[0.6, 0.8164]}$\rangle$	$\langle[s_{2.2572}, s_{3.2572}]$, {[0.6060, 0.7471]}$\rangle$
$\widetilde{ax}_4$			
$\langle[s_{2.0}, s_{3.2604}]$, {[0.5675, 0.7301]}$\rangle$	$\langle[s_{2.8477}, s_{3.4931}]$, {[0.7002, 0.8524], [0.7002, 0.8721]}$\rangle$	$\langle[s_{2.3860}, s_{3.0}]$, {[0.5960, 0.6989], [066836, 0.7697]}$\rangle$	$\langle[s_{3.7361}, s_{4.3520}]$, {[0.6684, 0.8033], [0.6684, 0.8528]}$\rangle$

Table 5 Ranking results obtained by different LSFs

LSF	Score values	Ranking
$\tilde{\rho}(s_i) = \frac{i}{2t}(0 \le i \le 2t)$	$S(\zeta_1) = 0.3458, S(\zeta_2) = 0.3539$ $S(\zeta_3) = 0.3728, S(\zeta_4) = 0.3731.$	$\widetilde{ax}_4 > \widetilde{ax}_3 > \widetilde{ax}_2 > \widetilde{ax}_1$
$\tilde{\rho}(s_i) = \begin{cases} \dfrac{\tilde{\rho}^t - \varphi^{t-i}}{2\tilde{\rho}^t - 2} & (0 \le i \le t), \\ \dfrac{\tilde{\rho}^t + \tilde{\rho}^{i-t} - 2}{2\tilde{\rho}^t - 2} & (t < i \le 2t). \end{cases}$	$S(\zeta_1) = 0.3417, S(\zeta_2) = 0.3552$ $S(\zeta_3) = 0.3661, S(\zeta_4) = 0.3697.$	$\widetilde{ax}_4 > \widetilde{ax}_3 > \widetilde{ax}_2 > \widetilde{ax}_1$
$\rho(s_i) = \begin{cases} \dfrac{t^\mu - (t-i)^\mu}{2t^\mu} & (0 \le i \le t), \\ \dfrac{t^\nu + (i-t)^\nu}{2t^\nu} & (t < i \le 2t). \end{cases}$	$S(\zeta_1) = 0.3486, S(\zeta_2) = 0.3535$ $S(\zeta_3) = 0.3776, S(\zeta_4) = 0.3753.$	$\widetilde{ax}_3 > \widetilde{ax}_4 > \widetilde{ax}_2 > \widetilde{ax}_1$

at Wei' method and our proposed method in depth and identify the reasons for such ranking results.

(1) In Ref. [46], the IVHULPWA operator is proposed based on the traditional operations of IVHULNs, i.e., Eqs. (5)–(8). In essence, our new operational laws that combines with the first type of LSF are equivalent to Eqs. (5)–(8). They are all based on a basic assumption that the semantic deviation between any two adjacent linguistic terms is always equal. But, in practice, decision makers may not think so. For example, they may think that the semantic deviation between linguistic terms "very high" and "perfect" is smaller than that between high and very high. Obviously, the traditional operation laws for IVHULNs are not appropriate for handling such decision situations. Our proposed operations with the third type of LSF can be used to properly and effectively solve such semantic situations. In addition, we can observe that when $\lambda = 1$, the proposed GIVHULPWA operator reduces to the IVHULPWA operator proposed by Wei [46]. That is, the IVHULPWA operator proposed by Wei [46] is a special case of our proposed operator. Thus, from Table 6, it can be seen that when we use the first type of LSF, i.e., $\tilde{\rho}(s_i) = \frac{i}{2t}$, although the score values derived by these two methods are different, the ranking results are exactly the same.

(2) From Table 6, we can also observed that when the third type of LSF is used in the calculation process, the ranking result is different from that obtained by Wei's method [46]. The main reason for this difference is that the third LSF is based on the value function of prospect theory, which describes such a semantic situation that the closer to the ends of a linguistic terms set, the smaller the absolute deviation between adjacent linguistic terms. Its

Table 6 Ranking results by different methods

Methods	Score values	Ranking
Our proposed method with the first type of LSF	$S(\widetilde{ax}_1) = 0.3458, S(\widetilde{ax}_2) = 0.3539,$ $S(\widetilde{ax}_3) = 0.3728, S(\widetilde{ax}_4) = 0.3731.$	$\widetilde{ax}_4 \succ \widetilde{ax}_3 \succ \widetilde{ax}_2 \succ \widetilde{ax}_1$
Our proposed method with the second type of LSF	$S(\widetilde{ax}_1) = 0.3417, S(\widetilde{ax}_2) = 0.3552,$ $S(\widetilde{ax}_3) = 0.3661, S(\widetilde{ax}_4) = 0.3697.$	$\widetilde{ax}_4 \succ \widetilde{ax}_3 \succ \widetilde{ax}_2 \succ \widetilde{ax}_1$
Our proposed method with the third type of LSF	$S(\widetilde{ax}_1) = 0.3486, S(\widetilde{ax}_2) = 0.3535,$ $S(\widetilde{ax}_3) = 0.3776, S(\widetilde{ax}_4) = 0.3753.$	$\widetilde{ax}_3 \succ \widetilde{ax}_4 \succ \widetilde{ax}_2 \succ \widetilde{ax}_1$
Wei's method based on IVHULPWA operator [46]	$S(\widetilde{ax}_1) = 1.9290, S(\widetilde{ax}_2) = 1.9756.$ $S(\widetilde{ax}_3) = 2.1135, S(\widetilde{ax}_4) = 2.1301.$	$\widetilde{ax}_4 \succ \widetilde{ax}_3 \succ \widetilde{ax}_2 \succ \widetilde{ax}_1$

application provides us different semantic values and semantic deviations between adjacent linguistic terms. Correspondingly, the support measure for each attribute value is different from that obtained by the traditional operations. Finally, the ranking results obtained by these two methods are different. From this point of view, compared with Wei's method [46], our proposed method can adapt to different semantic environment requirements and therefore has better adaptability and flexibility.

6.3.2 Compared with the Method Based on the GIVHULWA and GIVHULWG Operators

In the above subsection, we have illustrated and analyzed the effectiveness and flexibility of the proposed method by comparing with the method based on IVHULPWA operator proposed by Wei [46]. In the following, we further present another example to show the advantages of our proposed method by comparing with the two existing methods proposed by Liu et al. [22].

Example 7 A traditional manufacturing enterprise plans to implement the construction of ERP system. After the pre-investigation and screening, four suitable ERP systems r_1, r_2, r_3, r_4 are chosen as alternatives. Three experts t_1, t_2, t_3 with the weight vector $\varpi = \{0.4, 0.3, 0.3\}$ form an expert panel to evaluate these four alternatives under the following four attributes: (1) system functions and technical level c_1, (2) software developer's reputation c_2, (3) software developer's technical capabilities c_3, (4) software developer service capabilities c_4. The attribute weight vector is $w = (0.32, 0.26, 0.22, 0.2)$. Experts $t_k (k = 1, 2, 3)$ provide their interval-valued hesitant uncertain linguistic decision matrices $D^{(k)} = (\tau_{ij}^{(k)}) (i = j = 1, 2, 3, 4, k = 1, 2, 3)$ are presented in Tables 7, 8 and 9.

In Ref. [22], Liu et al. presented two generalized interval-valued hesitant uncertain linguistic aggregation operators, including the generalized interval-valued hesitant uncertain linguistic weighted average(GIVHULWA) operator and the generalized interval-valued hesitant uncertain linguistic weighted geometric (GIVHULWG) operator. In the following, we utilize Liu et al.'s [22] method and our proposed method to solve Example 7, respectively. The ranking results are shown in Table 10.

From Table 10, the ranking results obtained by our method with different types of LSFs are different from that obtained by Liu et al.'s method [22]. Based on the GIVHULWA and GIVHULWG operators, Liu et al. [22] identifies r_2 and r_3 as the best alternatives, respectively. Our proposed method with the first and second types of LSFs choose r_1 as the best alternative, whereas r_3 is considered as the optimal ERP system under the third type of semantic environment. One reason for this difference is that, as mentioned in Example 5, these two operators proposed by Liu et al. [22] are based on traditional operations. Therefore, they cannot be employed to deal with different semantic situations. Another important reason is that Liu et al.' method [22] uses the basic weighted averaging and weighted geometric operators to aggregate decision information, which do not consider the influence of extreme evaluation attribute values on the ranking results. However, our proposed method can assign lower weights to irrational evaluation values by introducing the concept of support measures, and then reduce the impact of them on final decision results.

Based on the above analysis and discussion, we can summarize the main advantages of the proposed method:

(1) IVHULNs can describe experts' preferences more flexibly and adequately. As an extension and generalization of uncertain linguistic variable and interval-valued hesitant fuzzy number, an IVHULN can effectively and objectively describe the experts' point of view and characterize their uncertainty, hesitancy and inadequacy, which is the prerequisite

Table 7 Decision matrix given by t_1

	c_1	c_2	c_3	c_4
r_1	$\langle[s_2,s_3],\{[0.6,0.7]\}\rangle$	$\langle[s_3,s_4],\{[0.7,0.8],[0.8,0.9]\}\rangle$	$\langle[s_1,s_2],\{[0.5,0.6]\}\rangle$	$\langle[s_5,s_5],\{[0.8,0.9]\}\rangle$
r_2	$\langle[s_4,s_5],\{[0.8,0.9]\}\rangle$	$\langle[s_4,s_5],\{[0.7,0.8]\}\rangle$	$\langle[s_2,s_3],\{[0.8,0.9]\}\rangle$	$\langle[s_4,s_5],\{[0.6,0.7],[0.6,0.8]\}\rangle$
r_3	$\langle[s_3,s_4],\{(0.5,0.7)\}\rangle$	$\langle[s_4,s_5],\{[(0.6,0.8),[0.7,0.9]\}\rangle$	$\langle[s_1,s_2],\{[(0.5,0.7)\}\rangle$	$\langle[s_3,s_4],\{[0.6,0.9]\}\rangle$
r_4	$\langle[s_3,s_3],\{(0.5,0.6)\}\rangle$	$\langle[s_2,s_3],\{[0.5,0.6],[0.6,0.8]\}\rangle$	$\langle[s_2,s_2],\{[0.6,0.7]\}\rangle$	$\langle[s_3,s_5],\{[0.7,0.8]\}\rangle$

Table 8 Decision matrix given by T_2

	c_1	c_2	c_3	c_4
r_1	$\langle[s_4,s_4],\{[0.5,0.6]\}\rangle$	$\langle[s_3,s_5],\{[0.5,0.6],[0.7,0.8]\}\rangle$	$\langle[s_2,s_3],\{[0.6,0.8]\}\rangle$	$\langle[s_4,s_5],\{[0.6,0.8]\}\rangle$
r_2	$\langle[s_4,s_4],\{[0.6,0.6]\}\rangle$	$\langle[s_3,s_4],\{[0.6,0.8]\}\rangle$	$\langle[s_3,s_3],\{[0.6,0.7]\}\rangle$	$\langle[s_2,s_3],\{[0.5,0.8]\}\rangle$
r_3	$\langle[s_4,s_4],\{[0.8,0.9]\}\rangle$	$\langle[s_3,s_4],\{[0.6,0.8]\}\rangle$	$\langle[s_2,s_4],\{[0.5,0.8]\}\rangle$	$\langle[s_3,s_4],\{[0.6,0.7]\}\rangle$
r_4	$\langle[s_2,s_3],\{(0.7,0.8)\}\rangle$	$\langle[s_2,s_3],\{[0.5,0.8]\}\rangle$	$\langle[s_1,s_3],\{[0.6,0.7],[0.6,0.8]\}\rangle$	$\langle[s_5,s_5],\{[0.8,0.9]\}\rangle$

Table 9 Decision matrix given by T_3

	c_1	c_2	c_3	c_4
r_1	$\langle[s_3,s_3],\{[0.5,0.6],[0.7,0.8]\}\rangle$	$\langle[s_4,s_5],\{[0.6,0.8]\}\rangle$	$\langle[s_2,s_4],\{[0.7,0.8]\}\rangle$	$\langle[s_3,s_5],\{[0.6,0.9]\}\rangle$
r_2	$\langle[s_2,s_2],\{[0.2,0.3]\}\rangle$	$\langle[s_3,s_4],\{[0.5,0.9]\}\rangle$	$\langle[s_1,s_2],\{[0.5,0.6],[0.8,0.9]\}\rangle$	$\langle[s_3,s_3],\{[0.6,0.9]\}\rangle$
r_3	$\langle[s_2,s_4],\{[0.5,0.6],[0.7,0.9]\}\rangle$	$\langle[s_2,s_4],\{[0.6,0.8]\}\rangle$	$\langle[s_2,s_4],\{[0.7,0.9]\}\rangle$	$\langle[s_4,s_5],\{[0.6,0.7]\}\rangle$
r_4	$\langle[s_1,s_3],\{(0.5,0.7)\}\rangle$	$\langle[s_2,s_2],\{[0.7,0.9]\}\rangle$	$\langle[s_2,s_4],\{[0.6,0.9]\}\rangle$	$\langle[s_4,s_5],\{[0.5,0.6],[0.8,0.9]\}\rangle$

Table 10 Ranking results by different methods

Methods	Score values	Ranking
Liu et al.'s method [22](based on the GIVHULWA operator)	$S(r_1)=2.4162, S(r_2)=2.4285.$ $S(r_3)=2.4160, S(r_4)=1.9819.$	$r_2 \succ r_1 \succ r_3 \succ r_4$
Liu et al.'s method [22](based on the GIVHULWG operator)	$S(r_1)=2.1975, S(r_2)=2.0677.$ $S(r_3)=2.2523, S(r_4)=1.7905.$	$r_3 \succ r_1 \succ r_2 \succ r_4$
Our proposed method with the first type of LSF	$S(r_1)=0.4037, S(r_3)=0.3996,$ $S(r_3)=0.4009, S(r_4)=0.3311.$	$r_1 \succ r_3 \succ r_2 \succ r_4$
Our proposed method with the second type of LSF	$S(r_1)=0.3935, S(r_2)=0.3896,$ $S(r_3)=0.3874, S(r_4)=0.3407.$	$r_1 \succ r_2 \succ r_3 \succ r_4$
Our proposed method with the third type of LSF	$S(r_1)=0.4096, S(r_2)=0.4062,$ $S(r_3)=0.4110, S(r_4)=0.3231.$	$r_3 \succ r_1 \succ r_2 \succ r_4$

for ensuring the accuracy of the result. Moreover, although the GIVHULPA and the GIVHULPG operators, from the computational point of view, are more complicated than the GIVHULWA, GIVHULWG [22] and IVHULPA[46] operators, the results can be quickly derived by using application software.

(2) LSFs are utilized to defined the operations of IVHULNs and aggregation operators presented in Sect. 4. As a result, different ranking results can be derived when different LSFs $\tilde{\rho}$ are employed in the aggregation process. Experts can autonomously choose different LSFs $\tilde{\rho}$ according to actual semantic contexts, this provides better flexibility for experts to evaluate alternatives. Hence, the developed approach is more flexible and practical than Wei's method [46].

(3) the proposed approach based on the GIVHULPA or GIVHULPG operator takes into account the impact of extreme evaluation values on the final decision results. By introducing the concept of support measures, our approach can assign lower weights to irrational evaluation values, and then reduce the impact of them on final decision results. Therefore, our approach is more reliable than the approach proposed in Ref. [22].

7 Conclusions

In this paper, combining linguistic-scale functions (LSFs) and generalized power average operator, an approach is proposed to solve interval-valued hesitant uncertain linguistic MAGDM problems and accommodate to different semantic situations. Firstly, new operational laws, Hamming distance and comparison method for IVHULNs are defined by combining LSFs. Then, aiming at the traditional PA operators cannot accommodate to situations in which evaluation values given by experts are IVHULNs, some new generalized power aggregation operators are presented to aggregate IVHULNs. The most important feature of these operators is that they cannot only accommodate to different semantic scenes but also reduce the negative impact of unreasonable evaluation values. Meanwhile, we have investigated some desired properties and analyzed some special cases of these operators. Furthermore, using the newly proposed aggregation operators, a new MAGDM approach is proposed. Finally, an illustrative example is provided to demonstrate the effectiveness of the developed approach. In addition, detailed comparison analyses are also made with the existing methods. In the further research, we will further investigate LSFs and their application in other linguistic sets, and continue working on the extension and application of the developed operators to other domains.

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Compliance with Ethical Standards

Conflicts of interest The authors declare that they have no conflict of interest.

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References

1. Torra, V.: Hesitant fuzzy sets. Int. J. Intell. Syst. **25**(6), 529–539 (2010)
2. Torra, V., Narukawa, Y.: On hesitant fuzzy sets and decision. In: IEEE International Conference on Fuzzy Systems, 2009. FUZZ-IEEE 2009. IEEE, pp. 1378–1382 (2009)
3. Xia, M., Xu, Z.: Hesitant fuzzy information aggregation in decision making. Int. J. Approx. Reason. **52**(3), 395–407 (2011)
4. Xu, Z., Xia, M.: Distance and similarity measures for hesitant fuzzy sets. Inf. Sci. **181**(11), 2128–2138 (2011)
5. Zhang, Z.: Hesitant fuzzy power aggregation operators and their application to multiple attribute group decision making. Inf. Sci. **234**, 150–181 (2013)
6. Xu, Z., Zhang, X.: Hesitant fuzzy multi-attribute decision making based on topsis with incomplete weight information. Knowl. Based Syst. **52**, 53–64 (2013)
7. Liu, P., Shi, L.: The generalized hybrid weighted average operator based on interval neutrosophic hesitant set and its application to multiple attribute decision making. Neural Comput. Appl. **26**(2), 457–471 (2015)
8. Chen, N., Xu, Z., Xia, M.: Interval-valued hesitant preference relations and their applications to group decision making. Knowl. Based Syst. **37**, 528–540 (2013)
9. Zhu, B., Xu, Z., Xia, M.: Dual hesitant fuzzy sets. J. Appl. Math. **2012**, 1–13 (2012)
10. Zadeh, L.A.: The concept of a linguistic variable and its application to approximate reasoning—I. Inf. Sci. **8**(3), 199–249 (1975)
11. Pedrycz, W., Song, M.: A granulation of linguistic information in AHP decision-making problems. Inf. Fusion **17**, 93–101 (2014)
12. Herrera, F., Herrera-Viedma, E., Martınez, L.: A fusion approach for managing multi-granularity linguistic term sets in decision making. Fuzzy Sets Syst. **114**(1), 43–58 (2000)
13. Liu, Z., Liu, P.: Normal intuitionistic fuzzy Bonferroni mean operators and their applications to multiple attribute group decision making. J. Intell. Fuzzy Syst. **29**(5), 2205–2216 (2015)
14. Liu, P., Jin, F.: Methods for aggregating intuitionistic uncertain linguistic variables and their application to group decision making. Inf. Sci. **205**, 58–71 (2012)
15. Wang, J.Q., Wang, D.D., Zhang, H.Y., Chen, X.-H.: Multi-criteria group decision making method based on interval 2-tuple linguistic information and Choquet integral aggregation operators. Soft. Comput. **19**(2), 389–405 (2015)
16. Meng, F., Chen, X., Zhang, Q.: Some interval-valued intuitionistic uncertain linguistic choquet operators and their application to multi-attribute group decision making. Appl. Math. Model. **38**(9), 2543–2557 (2014)
17. Liu, Z., Liu, P.: Intuitionistic uncertain linguistic partitioned Bonferroni means and their application to multiple attribute decision-making. Int. J. Syst. Sci. **48**(5), 1092–1105 (2017)
18. Liu, Z., Liu, P., Liu, W., Pang, J.: Pythagorean uncertain linguistic partitioned Bonferroni mean operators and their application in multi-attribute decision making. J. Intell. Fuzzy Syst. **32**(3), 2779–2790 (2017)
19. Rodriguez, R.M., Martinez, L., Herrera, F.: Hesitant fuzzy linguistic term sets for decision making. IEEE Trans. Fuzzy Syst. **20**(1), 109–119 (2012)
20. Lin, R., Zhao, X., Wei, G.: Models for selecting an ERP system with hesitant fuzzy linguistic information. J. Intell. Fuzzy Syst. **26**(5), 2155–2165 (2014)
21. Wang, J.Q., Wu, J.T., Wang, J., Zhang, H.Y., Chen, X.H.: Interval-valued hesitant fuzzy linguistic sets and their applications in multi-criteria decision-making problems. Inf. Sci. **288**, 55–72 (2014)

22. Liu, X., Ju, Y., Yang, S.: Some generalized interval-valued hesitant uncertain linguistic aggregation operators and their applications to multiple attribute group decision making. Soft. Comput. **20**(2), 495–510 (2016)
23. Yager, R.R.: The power average operator. IEEE Trans. Syst. Man Cybern. Part A Syst. Hum. **31**(6), 724–731 (2001)
24. Xu, Z., Yager, R.R.: Power-geometric operators and their use in group decision making. IEEE Trans. Fuzzy Syst. **18**(1), 94–105 (2010)
25. Zhou, L., Chen, H., Liu, J.: Generalized power aggregation operators and their applications in group decision making. Comput. Ind. Eng. **62**(4), 989–999 (2012)
26. Jiang, Y.P., Fan, Z.P., Ma, J.: A method for group decision making with multi-granularity linguistic assessment information. Inf. Sci. **178**(4), 1098–1109 (2008)
27. Delgado, M., Verdegay, J., Vila, M.: Linguistic decision-making models. Int. J. Intell. Syst. **7**(5), 479–492 (1992)
28. Herrera, F., Herrera-Viedma, E.: Linguistic decision analysis: steps for solving decision problems under linguistic information. Fuzzy Sets Syst. **115**(1), 67–82 (2000)
29. Delgado, M., Verdegay, J.L., Vila, M.: On aggregation operations of linguistic labels. Int. J. Intell. Syst. **8**(3), 351–370 (1993)
30. Merigó, J.M., Casanovas, M.: Decision making with distance measures and linguistic aggregation operators. Int. J. Fuzzy Syst. **12**(3), 190–198 (2010)
31. Wang, J.Q., Peng, L., Zhang, H.Y., Chen, X.H.: Method of multi-criteria group decision-making based on cloud aggregation operators with linguistic information. Inf. Sci. **274**, 177–191 (2014)
32. Herrera, F., Martínez, L.: A 2-tuple fuzzy linguistic representation model for computing with words. IEEE Trans. Fuzzy Syst. **8**(6), 746–752 (2000)
33. Herrera, F., Martínez, L.: A model based on linguistic 2-tuples for dealing with multigranular hierarchical linguistic contexts in multi-expert decision-making. IEEE Trans. Syst. Man Cybern. Part B Cybern. **31**(2), 227–234 (2001)
34. Merigó, J.M., Gil-Lafuente, A.M.: Induced 2-tuple linguistic generalized aggregation operators and their application in decision-making. Inf. Sci. **236**, 1–16 (2013)
35. Martı, L., Herrera, F., et al.: An overview on the 2-tuple linguistic model for computing with words in decision making: extensions, applications and challenges. Inf. Sci. **207**, 1–18 (2012)
36. Xu, Z.: Deviation measures of linguistic preference relations in group decision making. Omega **33**(3), 249–254 (2005)
37. Xu, Z.: Uncertain linguistic aggregation operators based approach to multiple attribute group decision making under uncertain linguistic environment. Inf. Sci. **168**(1), 171–184 (2004)
38. Xu, Z.: Induced uncertain linguistic OWA operators applied to group decision making. Inf. Fusion **7**(2), 231–238 (2006)
39. Xu, Z.S., Da, Q.L.: The uncertain OWA operator. Int. J. Intell. Syst. **17**(6), 569–575 (2002)
40. Ishibuchi, H., Tanaka, H.: Multiobjective programming in optimization of the interval objective function. Eur. J. Oper. Res. **48**(2), 219–225 (1990)
41. Kundu, S.: Min-transitivity of fuzzy leftness relationship and its application to decision making. Fuzzy Sets Syst. **86**(3), 357–367 (1997)
42. Sengupta, A., Pal, T.K.: On comparing interval numbers. Eur. J. Oper. Res. **127**(1), 28–43 (2000)
43. Dong, Y., Xu, Y., Li, H., Dai, M.: A comparative study of the numerical scales and the prioritization methods in ahp. Eur. J. Oper. Res. **186**(1), 229–242 (2008)
44. Bao, G.Y., Lian, X.L., He, M., Wang, L.L.: Improved two-tuple linguistic representation model based on new linguistic evaluation scale. Control Decis. **25**(5), 780–784 (2010)
45. Zhu, B., Xu, Z.: Consistency measures for hesitant fuzzy linguistic preference relations. IEEE Trans. Fuzzy Syst. **22**(1), 35–45 (2014)
46. Wei, G.: Interval valued hesitant fuzzy uncertain linguistic aggregation operators in multiple attribute decision making. Int. J. Mach. Learn. Cybern. **7**(6), 1093–1114 (2016)

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