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Some Generalized Intuitionistic Fuzzy Einstein Hybrid Aggregation Operators and Their Application to Multiple Attribute Group Decision Making

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Abstract The objective of the present work is divided into threefold. Firstly, we developed intuitionistic fuzzy Einstein hybrid averaging (IFEHA) aggregation operator and intuitionistic fuzzy Einstein hybrid geometric (IFEHG) aggregation operator along with their desirable properties. Secondly, we introduced two generalized aggregation operators along with their desirable properties, namely generalized intuitionistic fuzzy Einstein hybrid averaging (GIFEHA) aggregation operator and generalized intuitionistic fuzzy Einstein hybrid geometric (GIFEHG) aggregation operator. The main advantage of using the proposed methods is that these operators and methods give a more complete view of the problem to the decision makers. These methods provide more general, more accurate and precise results as compared to the existing methods. Therefore, these methods play a vital role in real-world problems. Finally the proposed operators have been applied to decision-making problems to show the validity, practicality and effectiveness of the new approach.

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1 Introduction

In 1965, Zadeh introduced the notion of fuzzy set [1], which has a basic component called membership function. After its positive applications, Atanassov [2] developed the notion of intuitionistic fuzzy set to generalize the concept of FS. In IFS, each element is expressed by an ordered pair, and each pair is characterized by a membership degree and a nonmembership degree, and the sum of the membership degree and the nonmembership degree of each ordered pair is less than or equal to one. Since the appearance of IFS, several researchers [3–19] have made important contributions to the development of intuitionistic fuzzy set theory and its applications, resulting in the great success from the theoretical and technological points of view. Xu and Yager [20–22] introduced the notion of some aggregation operators, such as IFWA operator, IFOWA operator, IFHA operator, IFWG operator, IFOWG operator, IFHG operator, and applied them to multiple attribute group decision making. Wang and Liu [23, 24] introduced the concept of some Einstein aggregation operators, namely intuitionistic fuzzy Einstein weighted geometric operator, intuitionistic fuzzy Einstein ordered weighted geometric operator, intuitionistic fuzzy Einstein weighted averaging operator, intuitionistic fuzzy Einstein ordered weighted averaging operator, and applied them to group decision making. Zhao and Wei [25] introduced the notion of some Einstein hybrid aggregation operators, namely intuitionistic fuzzy Einstein hybrid averaging operator, intuitionistic fuzzy Einstein hybrid geometric operator, and applied them to group decision making. Zhao et al. [26] introduced the notion of GIFWA operator, GIFOWA operator, GIFHA operator, GIIFWA operator, GIIFOWA operator, GIIFHA operator and applied them to group decision making.

Thus keeping the advantages of the above-mentioned aggregation operators, in this paper we introduce the notion of two new generalized Einstein hybrid aggregation operators based on IFNs such as generalized intuitionistic fuzzy Einstein hybrid averaging operator and generalized intuitionistic fuzzy Einstein hybrid geometric operator. Comparing with Einstein hybrid aggregation operators proposed by Zhao and Wei [25], they are only the special cases of the proposed operators in this paper. Obviously, the operators proposed in this paper are more general. Of course, superficially, it is more complicated in calculation. However, in real applications, we need assign the specific parameter λ, firstly.

The remainder of this paper is structured as follows. In Sect. 2, we give some basic definitions and results which will be used in our later sections. In Sect. 3, we discuss IFEHA operator and IFEHA operator. In Sect. 4, we introduce the notion of generalized intuitionistic fuzzy Einstein hybrid averaging operator and generalized intuitionistic fuzzy Einstein hybrid geometric operator. In Sect. 5, we apply these propose operators to group decision making. In Sect. 6, we develop a numerical example. In Sect. 7, we have conclusion.

2 Preliminaries

This section reviews the basic concepts of fuzzy set, intuitionistic fuzzy set, score function, accuracy function, some Einstein operations and intuitionistic fuzzy hybrid aggregation operators.

Definition 1 [1] Let Z be a fixed set, then a fuzzy set can be defined as follows:

$$F = \{\langle z, \mu_F(z)\rangle | z \in Z\}, \tag{1}$$

where $\mu_F(z)$ is a mapping from Z to $[0,1]$, and denotes the degree of membership of the element z to the set F.

Definition 2 [2] Let Z be a fixed set, then an intuitionistic fuzzy set can be defined as follows:

$$I = \{\langle z, \mu_I(z), \eta_I(z)\rangle | z \in Z\}, \tag{2}$$

where $\mu_I(z)$ and $\eta_I(z)$ are mappings from Z to $[0,1]$, such that $0 \le \mu_I(z) + \eta_I(z) \le 1, \forall z \in Z$. Let $\pi_I(z) = 1 - \mu_I(z) - \eta_I(z), \forall z \in Z$, then $\pi_I(z)$ is called the degree of indeterminacy of the element z to $I, \forall z \in Z$.

Definition 3 [23] Let $\alpha = (\mu_\alpha, \eta_\alpha)$ be an intuitionistic fuzzy value, then score of α can be defined as follows:

$$S(\alpha) = \mu_\alpha - \eta_\alpha, \tag{3}$$

where $S(\alpha) \in [-1, 1]$.

Definition 4 [23] Let $\alpha = (\mu_\alpha, \eta_\alpha)$ be an intuitionistic fuzzy value, then the accuracy degree of α can be defined as follows:

$$H(\alpha) = \mu_\alpha + \eta_\alpha, \tag{4}$$

where $H(\alpha) \in [0, 1]$.

Definition 5 [23] Let $\alpha_1 = \left(\mu_{\alpha_1}, \eta_{\alpha_1}\right)$ and $\alpha_2 = \left(\mu_{\alpha_2}, \eta_{\alpha_2}\right)$ are two intuitionistic fuzzy values then the following conditions hold:

(i) If $S(\alpha_2) \succ S(\alpha_1)$, then $\alpha_1 \prec \alpha_2$
(ii) If $S(\alpha_1) = S(\alpha_2)$, then
(a) If $H(\alpha_1) = H(\alpha_2)$, then $\alpha_1 = \alpha_2$
(b) If $H(\alpha_1) \prec H(\alpha_2)$, then $\alpha_1 \prec \alpha_2$

Definition 6 [23] Let $\alpha = (\mu_\alpha, \eta_\alpha)$, $\alpha_1 = \left(\mu_{\alpha_1}, \eta_{\alpha_1}\right)$ and $\alpha_2 = \left(\mu_{\alpha_2}, \eta_{\alpha_2}\right)$ be the three intuitionistic fuzzy values and $\lambda \succ 0$ be any real number, then

$$\alpha_1 \oplus_\varepsilon \alpha_2 = \left(\frac{\mu_{\alpha_1} + \mu_{\alpha_2}}{1 + \mu_{\alpha_1}\mu_{\alpha_2}}, \frac{\eta_{\alpha_1}\eta_{\alpha_2}}{1 + \left(1 - \eta_{\alpha_1}\right)\left(1 - \eta_{\alpha_2}\right)}\right), \tag{5}$$

$$\alpha_1 \otimes_\varepsilon \alpha_2 = \left(\frac{\mu_{\alpha_1}\mu_{\alpha_2}}{1 + \left(1 - \mu_{\alpha_1}\right)\left(1 - \mu_{\alpha_2}\right)}, \frac{\eta_{\alpha_1} + \eta_{\alpha_2}}{1 + \eta_{\alpha_1}\eta_{\alpha_2}}\right), \tag{6}$$

$$(\alpha)^\lambda = \left(\frac{2(\mu_\alpha)^\lambda}{(2 - \mu_\alpha)^\lambda + (\mu_\alpha)^\lambda}, \frac{(1 + \eta_\alpha)^\lambda - (1 - \eta_\alpha)^\lambda}{(1 + \eta_\alpha)^\lambda + (1 - \eta_\alpha)^\lambda}\right), \tag{7}$$

$$\lambda(\alpha) = \left(\frac{(1 + \mu_\alpha)^\lambda - \left(1 - \mu_{\alpha_1}\right)^\lambda}{(1 + \mu_\alpha)^\lambda + (1 - \mu_\alpha)^\lambda}, \frac{2(\eta_\alpha)^\lambda}{(2 - \eta_\alpha)^\lambda + (\eta_\alpha)^\lambda}\right). \tag{8}$$

Definition 7 [20] The intuitionistic fuzzy hybrid averaging operator can be defined as follows:

$$\mathrm{IFHA}_{w,w}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left(1 - \prod_{j=1}^{n}\left(1 - \mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}, \prod_{j=1}^{n}\left(\eta_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}\right), \tag{9}$$

where $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ is the weighted vector of $\alpha_j (j = 1, 2, \ldots, n)$ such that such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$.

Definition 8 [21] The intuitionistic fuzzy hybrid geometric aggregation operator can be defined as follows:

$$\mathrm{IFHG}_{w,w}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left(\prod_{j=1}^{n}\left(\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}, 1 - \prod_{j=1}^{n}\left(1 - \eta_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}\right), \tag{10}$$

where $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ is the weighted vector of $\alpha_j (j = 1, 2, \ldots, n)$ such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$.

3 Intuitionistic Fuzzy Einstein Hybrid Aggregation Operators

In this section, we develop some properties of intuitionistic fuzzy Einstein hybrid averaging aggregation operator and intuitionistic fuzzy Einstein hybrid geometric aggregation operator.

3.1 Intuitionistic Fuzzy Einstein Hybrid Averaging Aggregation Operator

Definition 9 [25] The intuitionistic fuzzy Einstein hybrid averaging operator can be defined as:

$$\mathrm{IFEHA}_{w,w}(\alpha_1,\alpha_2,\ldots,\alpha_n)=\left(\frac{\prod_{j=1}^{n}\left(1+\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}},\frac{2\prod_{j=1}^{n}\left(\eta_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-\eta_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}+\prod_{j=1}^{n}\left(\eta_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}}\right), \tag{11}$$

where $\dot{\alpha}_{\delta(j)}$ is the jth largest of the weighted intuitionistic fuzzy values $\dot{\alpha}_j(\dot{\alpha}_j = nw_j\alpha_j, j = 1, 2, \ldots, n))$, $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ is the weighted vector of $\alpha_j (j = 1,\ 2, \ldots, j)$, such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, where n is the balancing coefficient, if the vector $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ goes to $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^{\mathrm{T}}$, then the vector $(nw_1\alpha_1, nw_2\alpha_2, \ldots, nw_n\alpha_n)^{\mathrm{T}}$ goes to $(\alpha_1, \alpha_2, \ldots, \alpha_n)^{\mathrm{T}}$.

Theorem 1 *Let $\alpha_j = (\mu_{\alpha_j}, \eta_{\alpha_j})(j = 1, 2, \ldots, n)$ be a collection of intuitionistic fuzzy values , then their aggregated value by using the intuitionistic fuzzy Einstein hybrid averaging operator is also a intuitionistic fuzzy value and*

$$\mathrm{IFEHA}_{w,w}(\alpha_1,\alpha_2,\ldots,\alpha_n)=\left(\frac{\prod_{j=1}^{n}\left(1+\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}},\frac{2\prod_{j=1}^{n}\left(\eta_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-\eta_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}+\prod_{j=1}^{n}\left(\eta_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}}\right), \tag{12}$$

where $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ is the weighted vector of $\alpha_j (j = 1, 2, \ldots, n)$ such that $w_j \in [0, 1], \sum_{j=1}^{n} w_j = 1$.

Proof This theorem can be proved by mathematical induction. For $n = 2$,

$$w_1\alpha_1 = \left(\frac{(1+\mu_{\alpha_1})^{w_1}-(1-\mu_{\alpha_1})^{w_1}}{(1+\mu_{\alpha_1})^{w_1}+(1-\mu_{\alpha_1})^{w_1}}, \frac{2(\eta_{\alpha_1})^{w_1}}{(2-\eta_{\alpha_1})^{w_1}+(\eta_{\alpha_1})^{w_1}}\right)$$

$$w_2\alpha_2 = \left(\frac{(1+\mu_{\alpha_2})^{w_2}-(1-\mu_{\alpha_2})^{w_2}}{(1+\mu_{\alpha_2})^{w_2}+(1-\mu_{\alpha_2})^{w_2}}, \frac{2(\eta_{\alpha_2})^{w_2}}{(2-\eta_{\alpha_2})^{w_2}+(\eta_{\alpha_2})^{w_2}}\right)$$

Then

$$\mathrm{IFEHA}_{w,w}(\alpha_1,\alpha_2)=\left(\frac{\prod_{j=1}^{2}\left(1+\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}-\prod_{j=1}^{2}\left(1-\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}}{\prod_{j=1}^{2}\left(1+\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}+\prod_{j=1}^{2}\left(1-\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}},\frac{2\prod_{j=1}^{2}\left(\eta_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}}{\prod_{j=1}^{2}\left(2-\eta_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}+\prod_{j=1}^{2}\left(\eta_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}}\right).$$

Thus the result is true for $n = 2$; now, we assume that Eq. (12) holds for $n = k$; then,

$$\mathrm{IFEHA}_{w,w}(\alpha_1,\alpha_2,\ldots,\alpha_k)=\left(\frac{\prod_{j=1}^{k}\left(1+\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}-\prod_{j=1}^{k}\left(1-\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}}{\prod_{j=1}^{k}\left(1+\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}+\prod_{j=1}^{k}\left(1-\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}},\frac{2\prod_{j=1}^{k}\left(\eta_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}}{\prod_{j=1}^{k}\left(2-\eta_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}+\prod_{j=1}^{k}\left(\eta_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}}\right)$$

If Eq. (12) is true for $n = k$,, then we show that (12) is true for $n = k + 1$,; thus,

$$\begin{aligned}\mathrm{IFEHA}_{w,w}(\alpha_1,\alpha_2,\ldots\alpha_{k+1})=&\left(\frac{\prod_{j=1}^{k}\left(1+\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}-\prod_{j=1}^{k}\left(1-\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}}{\prod_{j=1}^{k}\left(1+\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}+\prod_{j=1}^{k}\left(1-\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}},\frac{2\prod_{j=1}^{k}\left(\eta_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}}{\prod_{j=1}^{k}\left(2-\eta_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}+\prod_{j=1}^{k}\left(\eta_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}}\right)\\ &\oplus_{\varepsilon}\left(\frac{(1+\mu_{\dot{\alpha}_{k+1}})^{w_{k+1}}-(1-\mu_{\dot{\alpha}_{k+1}})^{w_{k+1}}}{(1+\mu_{\dot{\alpha}_{k+1}})^{w_{k+1}}+(1-\mu_{\dot{\alpha}_{k+1}})^{w_{k+1}}},\frac{2(\eta_{\dot{\alpha}_{k+1}})^{w_{k+1}}}{(2-\eta_{\dot{\alpha}_{k+1}})^{w_{k+1}}+(\eta_{\dot{\alpha}_{k+1}})^{w_{k+1}}}\right).\end{aligned} \tag{13}$$

Let

$$p_1=\prod_{j=1}^{k}\left(1+\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}-\prod_{j=1}^{k}\left(1-\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}, r_1=2\prod_{j=1}^{k}\left(\eta_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}, r_2=2(\eta_{\dot{\alpha}_{k+1}})^{w_{k+1}}$$

$$q_1=\prod_{j=1}^{k}\left(1+\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}+\prod_{j=1}^{k}\left(1-\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}, p_2=(1+\mu_{\dot{\alpha}_{k+1}})^{w_{k+1}}-(1-\mu_{\dot{\alpha}_{k+1}})^{w_{k+1}}$$

$$q_2=(1+\mu_{\dot{\alpha}_{k+1}})^{w_{k+1}}+(1-\mu_{\dot{\alpha}_{k+1}})^{w_{k+1}}, s_1=\prod_{j=1}^{k}\left(2-\eta_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}+\prod_{j=1}^{k}\left(\eta_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}$$

$$s_2=(2-\eta_{\dot{\alpha}_{k+1}})^{w_{k+1}}+(\eta_{\dot{\alpha}_{k+1}})^{w_{k+1}}.$$

Now putting these values in Eq. (13), we have

$$\begin{aligned}\mathrm{IFEHA}_{w,w}(\alpha_1,\alpha_2,\ldots,\alpha_{k+1}) &= \left(\frac{p_1}{q_1},\frac{r_1}{s_1}\right)\oplus_{\varepsilon}\left(\frac{p_2}{q_2},\frac{r_2}{s_2}\right)\\ &=\left(\frac{\left(\frac{p_1}{q_1}\right)+\left(\frac{p_2}{q_2}\right)}{1+\left(\frac{p_1}{q_1}\right)\left(\frac{p_2}{q_2}\right)},\frac{\frac{r_1r_2}{s_1s_2}}{1+\left(1-\left(\frac{r_1}{s_1}\right)\right)\left(1-\left(\frac{r_2}{s_2}\right)\right)}\right)\\ &=\left(\frac{p_1q_2+p_2q_1}{q_1q_2+p_1p_2},\frac{r_1r_2}{2s_1s_2-s_1r_2-r_1s_2+r_1r_2}\right).\end{aligned} \tag{14}$$

Now putting the values of $p_1q_2 + p_2q_1, q_1q_2 + p_1p_2, r_1r_2, 2s_1s_2 - s_1r_2 - r_1s_2 + r_1r_2$ in Eq. (14), we have

$$\mathrm{IFEHA}_{w,w}(\alpha_1,\alpha_2,\ldots,\alpha_{k+1})=\left(\frac{\prod_{j=1}^{k+1}\left(1+\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}-\prod_{j=1}^{k+1}\left(1-\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}}{\prod_{j=1}^{k+1}\left(1+\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}+\prod_{j=1}^{k+1}\left(1-\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}},\frac{2\prod_{j=1}^{k+1}\left(\eta_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}}{\prod_{j=1}^{k+1}\left(2-\eta_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}+\prod_{j=1}^{k+1}\left(\eta_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}}\right).$$

Thus Eq. (12) is true for $n = k + 1$. Hence Eq. (12) is true for all n. □

Lemma 1 [23] *Let $\alpha_j \succ 0, w_j \succ 0 (j = 1, 2, \ldots n)$ and $w_j \in [0, 1], \sum_{j=1}^{n} w_j = 1$. then*

$$\prod_{j=1}^{n}(\alpha_j)^{w_j} \le \sum_{j=1}^{n} w_j\alpha_j, \tag{15}$$

where the equality holds if and only if $\alpha_j (j = 1, 2, \ldots, n) = \alpha$.

Theorem 2 *Let* $\alpha_j = \left(\mu_{\alpha_j}, \eta_{\alpha_j}\right) (j = 1,2,\ldots,n)$ *be a collection of intuitionistic fuzzy values, then*

$$\mathrm{IFEHA}_{w,w}(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq \mathrm{IFHA}_{w,w}(\alpha_1, \alpha_2, \ldots, \alpha_n), \tag{16}$$

where $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ *is the weighted vector of* $\alpha_j (j = 1,2,\ldots,n)$ *such that such that* $w_j \in [0,1]$ *and* $\sum_{j=1}^{n} w_j = 1$.

Proof Since

$$\prod_{j=1}^{n}\left(1+\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j} + \prod_{j=1}^{n}\left(1-\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j} \leq \sum_{j=1}^{n} w_j\left(1+\mu_{\dot{\alpha}_{\delta(j)}}\right) + \sum_{j=1}^{n} w_j\left(1-\mu_{\dot{\alpha}_{\delta(j)}}\right)$$

As

$$\sum_{j=1}^{n} w_j\left(1+\mu_{\dot{\alpha}_{\delta(j)}}\right) + \sum_{j=1}^{n} w_j\left(1-\mu_{\dot{\alpha}_{\delta(j)}}\right) = 2.$$

Thus

$$\prod_{j=1}^{n}\left(1+\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j} + \prod_{j=1}^{n}\left(1-\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j} \leq 2.$$

Also

$$\frac{\prod_{j=1}^{n}\left(1+\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j} - \prod_{j=1}^{n}\left(1-\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j} + \prod_{j=1}^{n}\left(1-\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}} = 1 - \frac{2\prod_{j=1}^{n}\left(1-\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j} + \prod_{j=1}^{n}\left(1-\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}} \leq 1 - \prod_{j=1}^{n}\left(1-\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}, \tag{17}$$

where the equality holds if and only if $\mu_{\dot{\alpha}_{\delta(j)}} (j = 1,2,\ldots n)$ are equal.

Again

$$\prod_{j=1}^{n}\left(2-\eta_{\dot{\alpha}_{\delta(j)}}\right)^{w_j} + \prod_{j=1}^{n}\left(\eta_{\dot{\alpha}_{\delta(j)}}\right)^{w_j} \leq \sum_{j=1}^{n} w_j\left(2-\eta_{\dot{\alpha}_{\delta(j)}}\right) + \sum_{j=1}^{n} w_j\left(\eta_{\dot{\alpha}_{\delta(j)}}\right).$$

As

$$\sum_{j=1}^{n} w_j\left(2-\eta_{\dot{\alpha}_{\delta(j)}}\right) + \sum_{j=1}^{n} w_j\left(\eta_{\dot{\alpha}_{\delta(j)}}\right) = 2,$$

then

$$\prod_{j=1}^{n}\left(2-\eta_{\dot{\alpha}_{\delta(j)}}\right)^{w_j} + \prod_{j=1}^{n}\left(\eta_{\dot{\alpha}_{\delta(j)}}\right)^{w_j} \leq 2.$$

Thus

$$\frac{2\prod_{j=1}^{n}\left(\eta_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-\eta_{\dot{\alpha}_{\delta(j)}}\right)^{w_j} + \prod_{j=1}^{n}\left(\eta_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}} \geq \prod_{j=1}^{n}\left(\eta_{\dot{\tilde{a}}_{\delta(j)}}\right)^{w_j}, \tag{18}$$

where the quality holds if and only if $\eta_{\dot{\alpha}_{\delta(j)}} (j = 1,2,\ldots n)$ are equal.

Let

$$\mathrm{IFHA}_{w,w}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha = (\mu_\alpha, \eta_\alpha) \tag{19}$$

and

$$\mathrm{IFEHA}_{w,w}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha^\varepsilon = (\mu_{\alpha^\varepsilon}, \eta_{\alpha^\varepsilon}). \tag{20}$$

Then Eqs. (17) and (18) can be transformed into the following forms: $\mu_\alpha \geq \mu_{\alpha^\varepsilon}$ and $\eta_\alpha \leq \eta_{\alpha^\varepsilon}$, respectively . Hence

$$S(\alpha) = \mu_\alpha - \eta_\alpha \geq \mu_{\alpha^\varepsilon} - \eta_{\alpha^\varepsilon} = S(\alpha^\varepsilon).$$

Thus

$$S(\alpha) \geq S(\alpha^\varepsilon).$$

If

$$S(\alpha) \succ S(\alpha^\varepsilon),$$

then

$$\mathrm{IFEHA}_{w,w}(\alpha_1, \alpha_2, \ldots, \alpha_n) \prec \mathrm{IFHA}_{w,w}(\alpha_1, \alpha_2, \ldots, \alpha_n). \tag{21}$$

If

$$S(\alpha) = S(\alpha^\varepsilon),$$

then

$$H(\alpha) = \mu_\alpha + \eta_\alpha = \mu_{\alpha^\varepsilon} + \eta_{\alpha^\varepsilon} = H(\alpha^\varepsilon).$$

Thus

$$\mathrm{IFEHA}_{w,w}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \mathrm{IFHA}_{w,w}(\alpha_1, \alpha_2, \ldots, \alpha_n). \tag{22}$$

Hence from Eqs. (21) and (22), we have

$$\mathrm{IFEHA}_{w,w}(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq \mathrm{IFHA}_{w,w}(\alpha_1, \alpha_2, \ldots, \alpha_n).$$

□

3.2 Intuitionistic Fuzzy Einstein Hybrid Geometric Aggregation Operator

Definition 10 [25] The intuitionistic fuzzy Einstein hybrid geometric operator can be defined as:

$$\mathrm{IFEHG}_{w,w}(\alpha_1,\alpha_2,\ldots,\alpha_n)=\left(\frac{2\prod_{j=1}^{n}\left(\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}+\prod_{j=1}^{n}\left(\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}},\frac{\prod_{j=1}^{n}\left(1+\eta_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\eta_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\eta_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\eta_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}}\right), \tag{23}$$

where $\dot{\alpha}_{\delta(j)}$ is the j^{th} largest of the weighted intuitionistic fuzzy values $\dot{\alpha}_j\left(\dot{\alpha}_j=\left(\alpha_j\right)^{nw_j}, j=1,2,\ldots,n\right)$, $w=(w_1, w_2,\ldots,w_n)^{\mathrm{T}}$ is the weighted vector of $\alpha_j(j=1,2,\ldots,j)$, such that $w_j\in[0,1]$ and $\sum_{j=1}^{n} w_j=1$, where n is the balancing coefficient, if the vector $w=(w_1,w_2,\ldots,w_n)^{\mathrm{T}}$ goes to $(\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n})^{\mathrm{T}}$, then the vector $\left(\alpha_1^{nw_1},\alpha_2^{nw_2},\ldots,\alpha_n^{nw_n}\right)^{\mathrm{T}}$ goes to $(\alpha_1,\alpha_2,\ldots,\alpha_n)^{\mathrm{T}}$.

Theorem 3 *Let $\alpha_j=\left(\mu_{\alpha_j},\eta_{\alpha_j}\right)(j=1,2,\ldots,n)$ be a collection of intuitionistic fuzzy values, then their aggregated value by using the intuitionistic fuzzy Einstein hybrid geometric operator is also a intuitionistic fuzzy value, and*

$$\mathrm{IFEHG}_{w,w}(\alpha_1,\alpha_2,\ldots,\alpha_n)=\left(\frac{2\prod_{j=1}^{n}\left(\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}+\prod_{j=1}^{n}\left(\mu_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}},\frac{\prod_{j=1}^{n}\left(1+\eta_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\eta_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\eta_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\eta_{\dot{\alpha}_{\delta(j)}}\right)^{w_j}}\right), \tag{24}$$

where $w=(w_1,w_2,\ldots,w_n)^{\mathrm{T}}$ is the weighted vector of $\alpha_j(j=1,2,\ldots,n)$ such that $w_j\in[0,1]$ and $\sum_{j=1}^{n} w_j=1$.

Proof For proof follows the proof of Theorem 1. □

Theorem 4 *Let $\alpha_j=\left(\mu_{\alpha_j},\eta_{\alpha_j}\right)(j=1,2,\ldots,n)$ be a collection of intuitionistic fuzzy values, then*

$$\mathrm{IFHG}_{w,w}(\alpha_1,\alpha_2,\ldots,\alpha_n)\le \mathrm{IFEHG}_{w,w}(\alpha_1,\alpha_2,\ldots,\alpha_n), \tag{25}$$

where $w=(w_1,w_2,\ldots,w_n)^{\mathrm{T}}$ is the weighted vector of $\alpha_j(j=1,\ 2,\ldots,\ n)$ such that $w_j\in[0,1]$ and $\sum_{j=1}^{n} w_j=1$.

Proof For proof follows the proof of Theorem 2. □

4 Generalized Intuitionistic Fuzzy Einstein Hybrid Aggregation Operators

In this section, we introduce generalized intuitionistic fuzzy Einstein hybrid geometric aggregation operator and generalized intuitionistic fuzzy Einstein hybrid averaging aggregation operator.

4.1 Generalized Intuitionistic Fuzzy Einstein Hybrid Geometric Aggregation Operator

Definition 11 Let $\alpha_j=(\mu_{\alpha_j},\eta_{\alpha_j})\,(j=1,2,\ldots,n)$ be a collection of IFVs, then the generalized intuitionistic fuzzy Einstein hybrid geometric operator can be define as:

$$\mathrm{GIFEHG}_{w,w}(\alpha_1,\alpha_2,\ldots,\alpha_n)=\frac{1}{\lambda}\cdot_{\varepsilon}\left(\underset{j=1}{\overset{n}{\otimes}}\left(\lambda\cdot_{\varepsilon}\dot{\alpha}_{\delta(j)}\right)^{w_j}\right), \tag{26}$$

where $\dot{\alpha}_{\delta(j)}$ is the jth largest of the weighted intuitionistic fuzzy values $\dot{\alpha}_j\left(\dot{\alpha}_j=\left(\alpha_j\right)^{nw_j}, j=1,2,\ldots,n\right)$, $w=(w_1,w_2,\ldots,w_n)^{\mathrm{T}}$ is the weighted vector of $\alpha_j(j=1,2,\ldots,j)$, such that $w_j\in[0,1]$ and $\sum_{j=1}^{n} w_j=1$, where n is the balancing coefficient, if the vector $w=(w_1,w_2,\ldots,w_n)^{\mathrm{T}}$ goes to $(\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n})^{\mathrm{T}}$, then the vector $\left(\alpha_1^{nw_1},\alpha_2^{nw_2},\ldots,\alpha_n^{nw_n}\right)^{\mathrm{T}}$ goes to $(\alpha_1,\alpha_2,\ldots,\alpha_n)^{\mathrm{T}}$, and λ is a real number greater than zero.

Theorem 5 *Let $\alpha_j=(\mu_{\alpha_j},\eta_{\alpha_j})(j=1,2,\ldots,n)$ be a collection of intuitionistic fuzzy values and $w=(w_1,w_2,\ldots,w_n)^{\mathrm{T}}$ be the associated weighted vector of $\alpha_j(j=1,2,\ldots,n)$ such that $w_j\in[0,1]$ and $\sum_{j=1}^{n} w_j=1$, then their aggregated value by using the GIFEHG operator can be expressed as:*

$$\mathrm{GIFEHG}_{w,w}(\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_n)$$

$$=\left(\begin{array}{c}\dfrac{\left(\prod_{j=1}^{n}\left\{\left(1+\mu_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}+3\left(1-\mu_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}\right\}^{w_j}+3\prod_{j=1}^{n}\left\{\left(1+\mu_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}-\left(1-\mu_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}\right\}^{w_j}\right)^{\frac{1}{\lambda}}-\left(\prod_{j=1}^{n}\left\{\left(1+\mu_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}+3\left(1-\mu_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}\right\}^{w_j}-\prod_{j=1}^{n}\left\{\left(1+\mu_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}-\left(1-\mu_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}\right\}^{w_j}\right)^{\frac{1}{\lambda}}}{\left(\prod_{j=1}^{n}\left\{\left(1+\mu_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}+3\left(1-\mu_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}\right\}^{w_j}+3\prod_{j=1}^{n}\left\{\left(1+\mu_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}-\left(1-\mu_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}\right\}^{w_j}\right)^{\frac{1}{\lambda}}+\left(\prod_{j=1}^{n}\left\{\left(1+\mu_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}+3\left(1-\mu_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}\right\}^{w_j}-\prod_{j=1}^{n}\left\{\left(1+\mu_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}-\left(1-\mu_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}\right\}^{w_j}\right)^{\frac{1}{\lambda}}},\\ \dfrac{2\left(\prod_{j=1}^{n}\left\{\left(2-\eta_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}+3\left(\eta_{\dot{\alpha}_{\sigma(j)}}\right)^{\lambda}\right\}^{w_j}-\prod_{j=1}^{n}\left\{\left(2-\eta_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}-\left(\eta_{\dot{\alpha}_{\sigma(j)}}\right)^{\lambda}\right\}^{w_j}\right)^{\frac{1}{\lambda}}}{\left(\prod_{j=1}^{n}\left\{\left(2-\eta_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}+3\left(\eta_{\dot{\alpha}_{\sigma(j)}}\right)^{\lambda}\right\}^{w_j}+3\prod_{j=1}^{n}\left\{\left(2-\eta_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}-\left(\eta_{\dot{\alpha}_{\sigma(j)}}\right)^{\lambda}\right\}^{w_j}\right)^{\frac{1}{\lambda}}+\left(\prod_{j=1}^{n}\left\{\left(2-\eta_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}+3\left(\eta_{\dot{\alpha}_{\sigma(j)}}\right)^{\lambda}\right\}^{w_j}-\prod_{j=1}^{n}\left\{\left(2-\eta_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}-\left(\eta_{\dot{\alpha}_{\sigma(j)}}\right)^{\lambda}\right\}^{w_j}\right)^{\frac{1}{\lambda}}}\end{array}\right)$$

Proof Proof is easy so it is omitted here. □

Theorem 6 *Let $\alpha_j=(\mu_{\alpha_j},\eta_{\alpha_j})\,(j=1,2,\ldots,n)$ be a collection of intuitionistic fuzzy values, then*

$$\mathrm{GIFHG}_{w,w}(\alpha_1,\alpha_2,\ldots,\alpha_n)\le \mathrm{GIFEHG}_{w,w}(\alpha_1,\alpha_2,\ldots,\alpha_n), \tag{27}$$

$w=(w_1,w_2,\ldots,w_n)^{\mathrm{T}}$ be the associated weighted vector of $\alpha_j(j=1,2,\ldots,j)$, such that $w_j\in[0,1]$ and $\sum_{j=1}^{n} w_j=1$.

Proof Proof is easy so it is omitted here. □

4.2 Generalized Intuitionistic Fuzzy Einstein Hybrid Averaging Aggregation Operator

Definition 12 Let $\alpha_j=(\mu_{\alpha_j},\eta_{\alpha_j})\,(j=1,2,\ldots,n)$ be a collection of IFVs, then the generalized intuitionistic fuzzy Einstein hybrid averaging operator can be define as:

$$\mathrm{GIFEHA}_{w,w}(\alpha_1,\alpha_2,\ldots,\alpha_n)=\left(\underset{j=1}{\overset{n}{\oplus}}\left(w\cdot_{\varepsilon}\dot{\alpha}_{\delta(j)}\right)^{\frac{1}{\lambda}}\right), \tag{28}$$

where $\dot{\alpha}_{\delta(j)}$ is the jth largest of the weighted intuitionistic fuzzy values $\dot{\alpha}_j\left(\dot{\alpha}_j = nw_j\alpha_j, j = 1, 2, \ldots, n\right)$, $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ is the weighted vector of $\alpha_j (j = 1, 2, \ldots, j)$, such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, where n is the balancing coefficient, if the vector $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ goes to $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^{\mathrm{T}}$, then the vector $(nw_1\alpha_1, nw_2\alpha_2, \ldots, nw_n\alpha_n)^{\mathrm{T}}$ goes to $(\alpha_1, \alpha_2, \ldots, \alpha_n)^{\mathrm{T}}$, and λ is a real number greater than zero.

Theorem 7 *Let $\alpha_j = (\mu_{\alpha_j}, \eta_{\alpha_j})\,(j = 1, 2, \ldots, n)$ be a collection of intuitionistic fuzzy values and $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ is the weighted vector of $\alpha_j (j = 1, 2, \ldots, j)$, such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, using the generalized intuitionistic fuzzy Einstein hybrid averaging aggregation operator, then the aggregated intuitionistic fuzzy value can be expressed as:*

$$\mathrm{GIFEHA}_{w,w}(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)$$
$$= \left(\begin{array}{c} \dfrac{2\left(\prod_{j=1}^{n}\left\{\left(2-\mu_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}+3\left(\mu_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}\right\}^{w_j}-\prod_{j=1}^{n}\left\{\left(2-\mu_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}-\left(\mu_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}\right\}^{w_j}\right)^{\frac{1}{\lambda}}}{\begin{array}{c}\left(\prod_{j=1}^{n}\left\{\left(2-\mu_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}+3\left(\mu_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}\right\}^{w_j}+3\prod_{j=1}^{n}\left\{\left(2-\mu_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}-\left(\mu_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}\right\}^{w_j}\right)^{\frac{1}{\lambda}} \\ +\left(\prod_{j=1}^{n}\left\{\left(2-\mu_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}+3\left(\mu_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}\right\}^{w_j}-\prod_{j=1}^{n}\left\{\left(2-\mu_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}+3\left(\mu_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}\right\}^{w_j}\right)^{\frac{1}{\lambda}}\end{array}}, \\ \dfrac{\begin{array}{c}\left(\prod_{j=1}^{n}\left\{\left(1+\eta_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}+3\left(1-\eta_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}\right\}^{w_j}+3\prod_{j=1}^{n}\left\{\left(1+\eta_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}-\left(1-\eta_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}\right\}^{w_j}\right)^{\frac{1}{\lambda}} \\ -\left(\prod_{j=1}^{n}\left\{\left(1+\mu_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}+3\left(1-\mu_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}\right\}^{w_j}-\prod_{j=1}^{n}\left\{\left(1+\eta_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}-\left(1-\eta_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}\right\}^{w_j}\right)^{\frac{1}{\lambda}}\end{array}}{\begin{array}{c}\left(\prod_{j=1}^{n}\left\{\left(1+\mu_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}+3\left(1-\mu_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}\right\}^{w_j}+3\prod_{j=1}^{n}\left\{\left(1+\mu_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}-\left(1-\mu_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}\right\}^{w_j}\right)^{\frac{1}{\lambda}} \\ +\left(\prod_{j=1}^{n}\left\{\left(1+\mu_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}+3\left(1-\mu_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}\right\}^{w_j}-\prod_{j=1}^{n}\left\{\left(1+\mu_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}-\left(1-\mu_{\dot{\alpha}_{\delta(j)}}\right)^{\lambda}\right\}^{w_j}\right)^{\frac{1}{\lambda}}\end{array}} \end{array} \right)$$

Proof Proof is easy so it is omitted here. □

Theorem 8 *Let $\alpha_j = (\mu_{\alpha_j}, \eta_{\alpha_j})\,(j = 1, 2, \ldots, n)$ be a collection of intuitionistic fuzzy values, then*

$$\mathrm{GIFEHA}_{w,w}(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq \mathrm{GIFHA}_{w,w}(\alpha_1, \alpha_2, \ldots, \alpha_n), \tag{29}$$

where $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ is the weighted vector of $\alpha_j (j = 1, 2, \ldots, j)$, such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$.

Proof Proof is easy so it is omitted here. □

5 An Application of the Generalized Intuitionistic Fuzzy Einstein Hybrid Aggregation Operators to Group Decision Making

In this section, we will use these generalized intuitionistic fuzzy Einstein hybrid aggregation operators to the multiple attribute group decision-making problems.

Algorithm Let $G = \{G_1, G_2, \ldots, G_m\}$ be the set of a finite m alternatives, and $A = \{A_1, A_2, \ldots, A_n\}$ be the set of a finite n attributes, and $D = \{D_1, D_2, \ldots, D_k\}$ be the set of k decision makers. Let $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ is the weighted vector of $A_j (j = 1, 2, \ldots, j)$, such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. Let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^{\mathrm{T}}$ be the weighted vector of the decision makers $D^s (s = 1, 2, \ldots, k)$, such that $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$.

Step 1 Construct $D^s = \left[\alpha_{ij}^{(s)}\right]_{m\times n} (s = 1, 2, \ldots, k)$ for decision. If the criteria have two types, such as benefit criteria and cost criteria, then $D^s = \left[\alpha_{ij}^{s}\right]_{m\times n}$ can be converted into $R^s = \left[r_{ij}^{s}\right]_{m\times n}$, where $r_{ij}^{s} = \begin{cases} \alpha_{ij}^{s}, & \text{for benefit criteria } A_j \\ \bar{\alpha}_{ij}^{s}, & \text{for cost criteria } A_j, \end{cases}$ $j = 1, 2, \ldots, n$, and $\bar{\alpha}_{ij}^{s}$ is the complement of α_{ij}^{s}.

Step 2 Utilize (IFEWA, IFEWG) operator to aggregate $R^s = \left[r_{ij}^{s}\right]_{m\times n}$ into a single $R = \left[r_{ij}\right]_{m\times n}$

Step 3 Utilize $\dot{\alpha}_{ij} = nw_j\alpha_{ij}$ and $\dot{\alpha}_{ij} = \left(\alpha_{ij}\right)^{nw_j}$ to derive the overall preference values, and then calculate the scores of $r_i (i = 1, 2, 3, \ldots, m)$.

Step 4 Utilize (GIFEWA, GIFEWA) operator to derive the overall preference values, and then calculate the scores of $r_i (i = 1, 2, 3, \ldots, m)$.

Step 5 Arrange the scores of the all alternatives in the form of descending order and select that alternative which has the highest score function.

6 Illustrative Example

Suppose a man wants to invest has money in the following best option G_1: Medicine Company, G_2: Army Company, G_3: Computer Company and G_4: Mobile Company. The man must take a decision according to the following four attributes, whose weighted vector is $w = (0.1, 0.2, 0.3, 0.4)^{\mathrm{T}}$. A_1: risk analysis, A_2: growth analysis, A_3: social political impact analysis and A_4: environmental analysis, where A_1, A_3 are cost-type criteria and A_2, A_4 are benefit-type criteria. There are three experts $D^s (s = 1, 2, 3)$, from a group to act as decision makers, whose weight vector is $\omega = (0.3, 0.3, 0.4)^{\mathrm{T}}$.

Step 1 Construct the decision matrices (Tables 1, 2, 3)

Step 2 Construct the normalized decision matrices (Tables 4, 5, 6)

Step 3 Utilize IFEWA operator, where $\omega = (0.3, 0.3, 0.4)^{\mathrm{T}}$ (Table 7)

Table 1 Intuitionistic fuzzy decision matrix D_1

	A_1	A_2	A_3	A_4
G_1	(0.5, 0.4)	(0.5, 0.4)	(0.6, 0.4)	(0.6, 0.4)
G_2	(0.6, 0.3)	(0.4, 0.5)	(0.4, 0.5)	(0.6, 0.3)
G_3	(0.5, 0.4)	(0.5, 0.3)	(0.5, 0.3)	(0.4, 0.3)
G_4	(0.6, 0.3)	(0.5, 0.4)	(0.5, 0.4)	(0.5, 0.4)

Table 2 Intuitionistic fuzzy decision matrix D_2

	A_1	A_2	A_3	A_4
G_1	(0.6, 0.3)	(0.5, 0.4)	(0.6, 0.3)	(0.6, 0.4)
G_2	(0.5, 0.3)	(0.6, 0.4)	(0.4, 0.5)	(0.5, 0.3)
G_3	(0.6, 0.3)	(0.6, 0.3)	(0.6, 0.4)	(0.5, 0.3)
G_4	(0.4, 0.5)	(0.6, 0.2)	(0.5, 0.4)	(0.5, 0.4)

Table 3 Intuitionistic fuzzy decision matrix D_3

	A_1	A_2	A_3	A_4
G_1	(0.5, 0.4)	(0.5, 0.4)	(0.5, 0.4)	(0.4, 0.4)
G_2	(0.5, 0.3)	(0.4, 0.3)	(0.5, 0.3)	(0.6, 0.3)
G_3	(0.6, 0.3)	(0.6, 0.3)	(0.5, 0.4)	(0.6, 0.3)
G_4	(0.6, 0.2)	(0.5, 0.4)	(0.7, 0.1)	(0.6, 0.4)

Table 4 Normalized decision matrix R_1

	A_1	A_2	A_3	A_4
G_1	(0.4, 0.5)	(0.5, 0.4)	(0.4, 0.6)	(0.6, 0.4)
G_2	(0.3, 0.6)	(0.4, 0.5)	(0.5, 0.4)	(0.6, 0.3)
G_3	(0.4, 0.5)	(0.5, 0.3)	(0.3, 0.5)	(0.4, 0.3)
G_4	(0.3, 0.6)	(0.5, 0.4)	(0.4, 0.5)	(0.5, 0.4)

Table 5 Normalized decision matrix R_2

	A_1	A_2	A_3	A_4
G_1	(0.3, 0.6)	(0.5, 0.4)	(0.3, 0.6)	(0.6, 0.4)
G_3	(0.3, 0.5)	(0.6, 0.4)	(0.5, 0.4)	(0.5, 0.3)
G_3	(0.3, 0.6)	(0.6, 0.3)	(0.4, 0.6)	(0.5, 0.3)
G_4	(0.5, 0.4)	(0.6, 0.2)	(0.4, 0.5)	(0.5, 0.4)

Table 6 Normalized decision matrix R_3

	A_1	A_2	A_3	A_4
G_1	(0.4, 0.5)	(0.5, 0.4)	(0.4, 0.5)	(0.4, 0.4)
G_2	(0.3, 0.5)	(0.4, 0.3)	(0.3, 0.5)	(0.6, 0.3)
G_3	(0.3, 0.6)	(0.6, 0.3)	(0.4, 0.5)	(0.6, 0.3)
G_4	(0.2, 0.6)	(0.5, 0.4)	(0.1, 0.7)	(0.6, 0.4)

Table 7 Collective intuitionistic fuzzy decision matrix R

		A_2	A_3	A_4
G_1	(0.37, 0.52)	(0.50, 0.40)	(0.37, 0.55)	(0.52, 0.40)
G_2	(0.30, 0.52)	(0.46, 0.38)	(0.42, 0.43)	(0.54, 0.30)
G_3	(0.33, 0.56)	(0.57, 0.30)	(0.39, 0.52)	(0.55, 0.30)
G_4	(0.32, 0.53)	(0.53, 0.32)	(0.28, 0.57)	(0.57, 0.40)

Table 8 Intuitionistic fuzzy hybrid decision matrix

	A_1	A_2	A_3	A_4
G_1	(0.73, 0.19)	(0.41, 0.49)	(0.43, 0.46)	(0.15, 0.79)
G_2	(0.74, 0.11)	(0.45, 0.35)	(0.38, 0.48)	(0.12, 0.79)
G_3	(0.75, 0.11)	(0.47, 0.39)	(0.45, 0.45)	(0.13, 0.81)
G_4	(0.77, 0.19)	(0.44, 0.42)	(0.33, 0.50)	(0.13, 0.80)

Step 4 Utilize $\dot{\alpha}_{ij} = nw_j\alpha_{ij}$ we have

$\dot{\alpha}_{11} = (0.15, 0.79),\ \dot{\alpha}_{12} = (0.41, 0.49),\ \dot{\alpha}_{13} = (0.43, 0.46),$
$\dot{\alpha}_{14} = (0.73, 0.19)\ \dot{\alpha}_{21} = (0.12, 0.79),\ \dot{\alpha}_{22} = (0.38, 0.48),$
$\dot{\alpha}_{23} = (0.45, 0.35),\ \dot{\alpha}_{24} = (0.74, 0.11)\ \dot{\alpha}_{31} = (0.13, 0.81),$
$\dot{\alpha}_{32} = (0.47, 0.39),\ \dot{\alpha}_{33} = (0.45, 0.45),\ \dot{\alpha}_{34} = (0.75, 0.11)$
$\dot{\alpha}_{41} = (0.13, 0.80),\ \dot{\alpha}_{42} = (0.44, 0.42),\ \dot{\alpha}_{43} = (0.33, 0.50),$
$\dot{\alpha}_{44} = (0.77, 0.19)$

Now we calculate the score functions:

$S(\dot{\alpha}_{11}) = -0.64,\ S(\dot{\alpha}_{12}) = -0.03,\ S(\dot{\alpha}_{13}) = -0.33,$
$S(\dot{\alpha}_{14}) = 0.53,\ S(\dot{\alpha}_{21}) - 0.67$
$S(\dot{\alpha}_{22}) = -0.09,\ S(\dot{\alpha}_{23}) = 0.09,\ S(\dot{\alpha}_{24}) = 0.63,$
$S(\dot{\alpha}_{31}) = -0.68,\ S(\dot{\alpha}_{32}) = 0.07$
$S(\dot{\alpha}_{33}) = 0.00,\ S(\dot{\alpha}_{34}) = 0.64,\ S(\dot{\alpha}_{41}) = -0.66,\ S(\dot{\alpha}_{42})$
$S(\dot{\alpha}_{44}) = 0.58$

Thus we obtain Table 8

Step 4 Utilize IFEWA operator, where $w = (0.1, 0.2, 0.3, 0.4)^{\mathrm{T}}$, we have

$r_1 = (0.36, 0.55), r_2 = (0.35, 0.50), r_3 = (0.38, 0.55),$
$r_4 = (0.30, 0.54)$

By calculating the scores, we have

$S(r_1) = -0.18, S(r_2) = -0.15, S(r_3) = -0.17,$
$S(r_4) = -0.24$

Step 5 Thus the best option is G_2 (Table 9).

Table 9 Ranking of the alternative at different values of λ

λ	Operators	Ranking
1	IFEHA aggregation operator	$G_2 \succ G_3 \succ G_1 \succ G_4$
	IFEHG aggregation operator	$G_2 \succ G_3 \succ G_1 \succ G_4$
	GIFEHA aggregation operator	$G_2 \succ G_3 \succ G_1 \succ G_4$
	GIFEHG aggregation operator	$G_2 \succ G_3 \succ G_1 \succ G_4$
2	IFEHA aggregation operator	$G_2 \succ G_3 \succ G_1 \succ G_4$
	IFEHG aggregation operator	$G_2 \succ G_3 \succ G_1 \succ G_4$
	GIFEHA aggregation operator	$G_2 \succ G_1 \succ G_3 \succ G_4$
	GIFEHG aggregation operator	$G_2 \succ G_1 \succ G_3 \succ G_4$

6.1 Compare with the Other Methods

First we compare the propose methods with methods proposed by Zhao et al. [26], the aggregation operators proposed by H. Zhao et al. [26] are based on algebraic operations, and those in this paper are based on the generalized Einstein operations. Obviously the methods proposed in this paper are more general, more accurate and more flexible. Comparing with Einstein operators proposed by Zhao and Wei [25], they are only the special cases of the proposed operators in this paper. When $\lambda = 1$, then the GIFEHA aggregation operator and GIFEHG aggregation operator proposed in this paper can be reduce to IFEHA aggregation operator and IFEHG aggregation operator, respectively. If we change the value of parameter λ, we get different ranking results.

7 Conclusion

The objective of this paper is to present some generalized Einstein hybrid aggregation operators based on intuitionistic fuzzy numbers and applied them to the multi-attribute group decision-making problems where attribute values are the intuitionistic fuzzy numbers. Firstly, we have developed two Einstein hybrid aggregation operators along with their properties, namely intuitionistic fuzzy Einstein hybrid averaging aggregation operator and intuitionistic fuzzy Einstein hybrid geometric aggregation operator. Furthermore, these operators have been extended to its generalized operators by incorporating the attitude parameter of the decision making λ during the aggregation process. It has been obtained from these operator that when $\lambda = 1$, then the generalized operators reduce to intuitionistic fuzzy Einstein hybrid averaging aggregation operator and intuitionistic fuzzy Einstein hybrid geometric aggregation operator. Furthermore, have developed a method for multi-criteria group decision making based on these operators, and the operational processes have illustrated in detail. An illustrative example of selecting the best company to invest money has been considered for demonstrating the approach. The results corresponding to the proposed approach have been compared with the existing operator results, and it has been found that the decision-making method proposed in this paper is more stable and practical.

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