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Consensus Building for Probabilistic Hesitant Fuzzy Preference Relations with Expected Additive Consistency

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Abstract As an extension of hesitant fuzzy element, probabilistic hesitant fuzzy element (PHFE) has received increasing attention. However, some important issues in PHFE utilization remain to be addressed. This study aims to build a consensus among the decision makers for probabilistic hesitant fuzzy preference relations (PHFPRs) with expected additive consistency. First, several generalized operations that are suitable for PHFPRs are defined. Then, expected additive consistent PHFPRs are introduced, and a theorem is proposed to obtain them. Second, the consistency index is defined based on the Hausdorff distance of PHFEs to measure whether individual PHFPRs exhibit acceptable expected additive consistency. If not, then an automatic iterative algorithm is designed to obtain acceptable ones. Third, group PHFPR is obtained based on the proposed generalized operations, and then the consensus index is determined according to the Hausdorff distance. If at least one of the consensus levels of the decision makers is lower than a given threshold, then an automatic iterative algorithm is used to update the PHFPRs to reach a predefined consensus level. Finally, a numerical example is provided, and comparative analyses with existing methods are performed to demonstrate the validity of the proposed method in addressing group decision-making problems.

Keywords Probabilistic hesitant fuzzy element · Group decision making · Probabilistic hesitant fuzzy preference relations · Expected additive consistency · Consensus

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1 Introduction

Group decision making (GDM) is a common activity among humans and is widely applied to many fields [1–3]. For typical multi-criteria group decision-making (MCGDM) problems, a group of experts is invited to participate in finding the best among several alternatives. In decision-making processes, experts provide their assessments for alternatives with respect to the predetermined criteria. Afterward, a ranking of alternatives can be obtained based on the evaluation information of experts. In the MCGDM process, a group of experts reaching a high degree of consensus among their opinions is preferable [4]. For this reason, the consensus reaching algorithms for handling MCGDM problems has been studied deeply [5–9]. For example, Wu et al. [5] built a consensus model under social network with distributed linguistic trust. Zhang et al. [6] constructed a minimum-cost consensus model based on random opinions. Quesada et al. [7] presented an expert weighting methodology for consensus reaching. Zhang et al. [8] established a consensus model with heterogeneous information according to individual concerns and satisfactions. Moreover, an overview of consensus models under fuzzy environments was investigated by Herrera-Viedma et al. [10].

With the contribution of fuzzy set and hesitant fuzzy linguistic term set, which were investigated by Liu and

Table 1 Consistency thresholds of PHFPRs

$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$
0.9118	0.8789	0.8604	0.8490	0.8414	0.8357

Liao [11], and Liao et al. [12], some scholars focused on studying the properties of hesitant fuzzy set (HFS) [13]. The primary advantage of HFS is that they can represent the hesitant information of experts. During assessments, experts are occasionally hesitant about certain values, that is, they cannot determine which value can best express their assessments of alternatives. In such case, hesitant values are used to express their assessments. Thus, the information represented by HFS makes describing uncertain evaluation more convenient than using fuzzy set. Fuzzy preference relations can accomplish diverse properties, and some of the properties have been introduced into consensus models for addressing MCGDM problems [7]. Consequently, several types of preference relations have been developed, including hesitant fuzzy preference relations (HFPRs) [14, 15], hesitant multiplicative preference relations [16], hesitant fuzzy linguistic preference relations [17], and hesitant intuitionistic fuzzy preference relations [18]. Although the aforementioned preference relations have been extensively studied, controversies remain. For example, Xu et al. [19] pointed out that a shortcoming in the normalization process was introduced in [17]; they then proposed a method for normalizing HFPRs. Meng and An [20] identified shortcomings in [17, 21] and then constructed 0–1 mixed programming models to derive priority weight vectors. The shortcoming of HFPRs was also presented in [22].

To overcome the shortcoming of HFSs in representing evaluation information, extended fuzzy sets are developed. Zhu [23] introduced the concept of probabilistic hesitant fuzzy set (PHFS). The primary advantage of PHFS is that it not only represents the hesitant information of experts, but also the corresponding probabilities of hesitant information. Consequently, PHFS has gained the attention of scholars. Xu and Zhou [24] established the score, deviation function, comparison laws, and basic operations of PHFS. Li and Wang [25] presented an information integration method based on statistical knowledge. Zhang et al. [26] reported that the probabilistic hesitant fuzzy element (PHFE) might have missing values and then proposed an improved PHFS. Wang and Li [27] presented the correlation coefficient of PHFE. Similar studies were also conducted in [28–30].

To provide more accurate assessment information than HFPRs, the concept of probabilistic hesitant fuzzy preference relations (PHFPRs) have been recently proposed by scholars [31, 32], and MCGDM methods based on consensus with PHFPRs have been developed in [23, 31, 33–35]. Zhou and Xu [31] discussed consensus among decision makers under uncertain PHFPRs based on additive consistency. Wu et al. [26] developed consensus measures based on the distance between individuals PHFPRs. A consensus model was constructed for large-scale group decision making with probabilistic hesitant fuzzy information and changeable clusters by Wu and Xu [33]. Zhou and Xu [34] presented the probability calculation and element optimization of PHFPRs based on expected consistency. Zhu [23] discussed individual consistency and consensus level based on multiplicative consistency. Zhu et al. [35] developed probabilistic hesitant multiplicative preference relations that used the 1/9–9 scale instead of the 0.1–0.9 scale to express the membership degree in PHFEs.

Although the concept of PHFPRs has been introduced and decision-making methods have been proposed by some scholars, some important issues remain to be solved. (1) Existing works have mainly utilized expected consistency to develop PHFPRs [23, 31, 32, 34]. PHFPRs have been translated into fuzzy preference relations with expected consistency. This method is simple and easy to implement, but it cannot reflect the hesitancy of decision makers. (2) When the expected consistency of PHFPR was unacceptable, iterative optimization algorithms were developed in [31, 34], which provided formulas to calculate the corrected expected values in the proposed algorithms. Since every element in PHFPR has two parts: γ_i and p_i, they should be considered simultaneously. However, the corrected expected value developed in [31, 34] considered only the membership degree part γ_i, whereas the probability part p_i received insufficient attention. (3) The Euclidean distance was defined to obtain the consistency index [28]. This distance required all PHFEs offered by decision makers to have the same number of elements. Therefore, a normalization process is necessary. However, the normalization process will derive different priority weight vectors with respect to different PHFEs because the normalization process will obtain various normalization results. (4) Consensus is an essential aspect of group decision-making problems [36]. However, some studies [28, 35] considered only the consistency of preference relations and provided less attention to consensus in the decision-making process.

To overcome the aforementioned limitations, the current study focuses on building consensus among PHFPRs with expected additive consistency and developing an effective method to address probabilistic hesitant MCGDM problems. The main objectives of this study are summarized as follows.

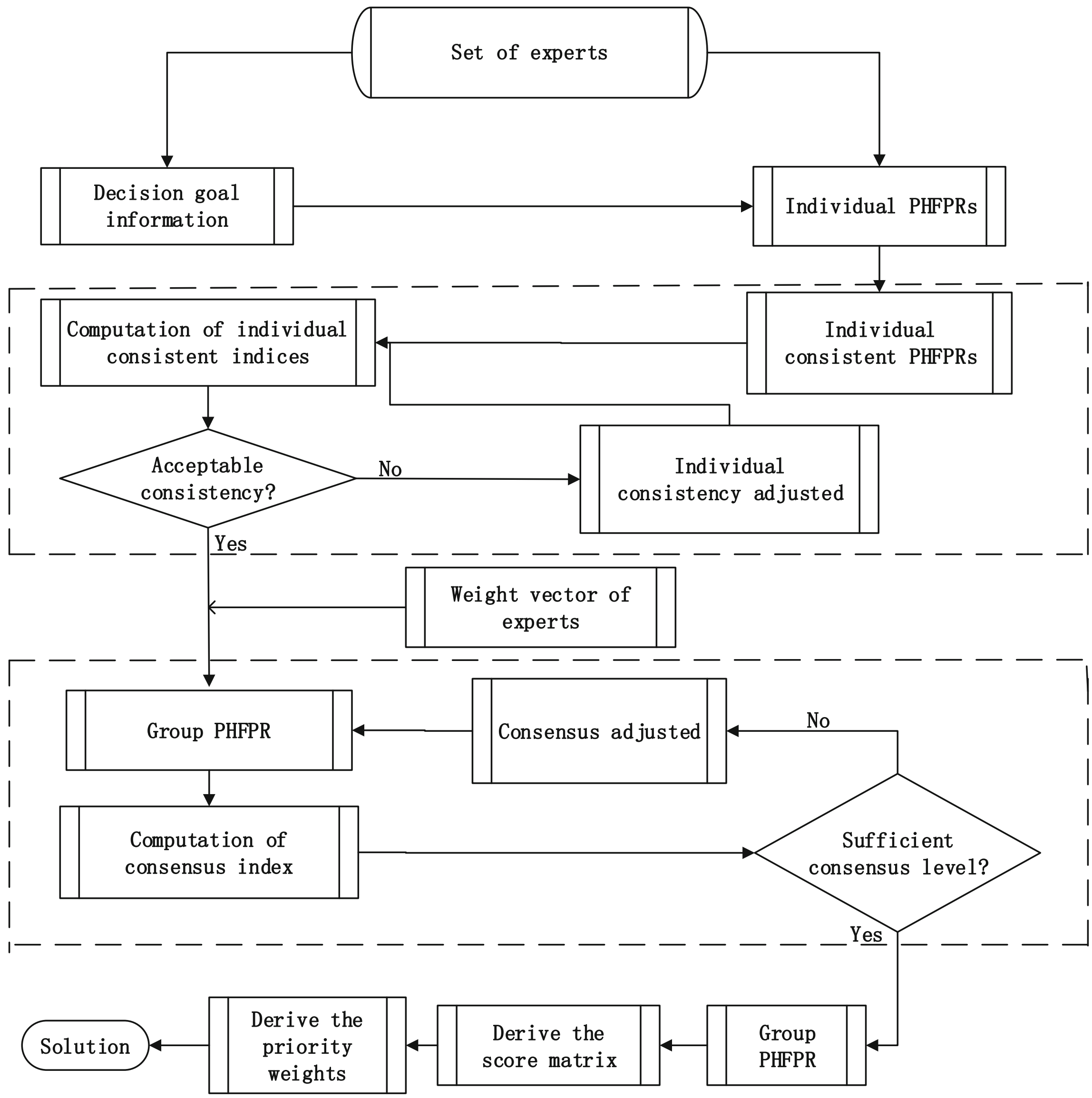

Fig. 1 Group decision-making process framework with PHFPRs

(1) To better reflect the hesitancy of decision makers in utilizing expected consistency to develop PHFPRs. Expected additive consistent PHFPRs are proposed in this study. Moreover, a theorem is introduced to obtain expected additive consistent PHFPRs.

(2) To overcome the shortcoming of iterative optimization algorithms that merely used the membership degree part γ_i. In the current study, two automatic iterative algorithms are designed to improve individual consistency and consensus among decision makers. The adjusted formulas developed in the algorithms not only consider the membership degree part γ_i, but also the probability part p_i.

(3) The automatic iterative Algorithm 2 developed in this study not only considers the acceptable consistency of PHFPRs, but also consensus in the group decision-making process, which can overcome the shortcoming of iterative optimization algorithms that considered only consistency.

(4) The Hausdorff distance is defined to obtain the consistency index. The distance developed in this study does not require a normalization process. It can effectively calculate the distance when two PHFEs have different number of elements.

The remainder of the paper is organized as follows. In Sect. 2, basic concepts related to HFS, HFPRs, and PHFS are reviewed. In Sect. 3, expected additive consistent PHFPRs are introduced and a theorem is provided to obtain expected additive consistent PHFPRs. In Sect. 4, two automatic iterative algorithms are designed to improve individual consistency and consensus among the decision makers. In Sect. 5, the proposed algorithms are applied to investment evaluation problems, a comparative study is conducted, and discussions are presented. Finally, conclusions are provided in Sect. 6.

2 Preliminaries

In this section, basic definitions related to HFS, HFPRs, and PHFS are reviewed.

Orlovshy [37] introduced the concept of fuzzy preference relations (FPRs), which can be described as follows.

Definition 1 [37] Let $R = (\gamma_{ij})_{n\times n}$ be a preference relation. Then, R is regarded as an FPR if

$$\gamma_{ij} + \gamma_{ji} = 1,\ \gamma_{ii} = 0.5,\ \gamma_{ij} \in [0,1],\ \ i,j = 1,2,\ldots,n. \quad (1)$$

Tanino [38] defined R as additive consistent if it satisfies the following additive transitivity:

$$\gamma_{ij} + \gamma_{jk} = \gamma_{ik} + 0.5,\ \forall\, i,j,k \in n. \quad (2)$$

Theorem 1 [38] *Let $R = (\gamma_{ij})_{n\times n}$ be an additive consistent FPR. Then, the following formulas are equivalent:*

1. $\gamma_{ij} + \gamma_{jk} = \gamma_{ik} + 0.5,\ \forall\, i,j,k \in n;$
2. $\gamma_{ij} + \gamma_{jk} + \gamma_{ki} = \gamma_{ji} + \gamma_{kj} + \gamma_{ik},\ \forall\, i,j,k \in n;$
3. $\gamma_{ij} + \gamma_{jk} + \gamma_{ki} = \frac{3}{2},\ \forall\, i,j,k \in n;$
4. $\gamma_{ij} = \frac{1}{n}\sum_{k=1}^{n}(\gamma_{ik} + \gamma_{kj}) - 0.5,\ \forall\, i,j,k \in n.$

To represent hesitant information, Torra [13] *developed the concept of HFS.*

Definition 2 [13] Let X be a reference set. An HFS A on X is defined in terms of function $h_A(x)$. When applied to X, the function returns a finite subset of [0, 1].

For easy understanding, Xia and Xu [39] expressed HFS using a mathematical symbol, as follows:

$$A = \{ < x, h_A(x) > | x \in X\},$$

where function $h_A(x)$ is a set of different values in [0, 1], which represent the possible membership degrees of element x in X to A. For convenience, $h_A(x)$ is called hesitant fuzzy element (HFE).

Xia and Xu [40] first introduced the definition of HFPRs. Afterward, a revised definition was presented by Xu et al. in [19], which can be restated as follows.

Definition 3 [19] Let $X = \{x_1, x_2, \ldots, x_n\}$ be a fixed set. Then, the HFPRs on X is presented by a matrix $H = (h_{ij})_{n\times n} \subset X \times X$, where $h_{ij} = \left\{ \gamma_{ij}^{l} \middle| l = 1,2,\ldots,\#h_{ij} \right\}$ is an HFE that indicates all possible preference degrees of the objective x_i over x_j. Moreover, h_{ij} should satisfy the following conditions:

$$\gamma_{ij}^{\sigma(l)} + \gamma_{ji}^{\sigma(l)} = 1,\ \gamma_{ii} = 0.5,\ \#h_{ij} = \#h_{ji},\ i,j = 1,2,\ldots,n, \quad (3)$$

where $\#h_{ij}$ is the number of values in h_{ij}, and $\gamma_{ij}^{\sigma(l)}$ is the l-th element in h_{ij}.

Remark 1 The difference of the definitions related to HFPRs between Xia and Xu [40], and Xu et al. [19] is that the elements in HFEs should be arranged in ascending order in [40], whereas such arrangement is not required in [19].

To collect the preferences of decision makers, Zhu [23] introduced the concept of PHFS. Afterward, Zhang et al. [26] noted several missing values in PHFE and proposed the improved PHFS, which can be restated as follows.

Definition 4 [26] Let R be a fixed set. Then, PHFS on R is expressed by a mathematical symbol:

$$A = \{ < x, h_x(p_x) > | x \in X\},$$

where function h_x is a set of different values in [0, 1], which is described by the probability distribution p_x that denotes possible membership degrees of element x in X to A. For convenience, $h_x(p_x)$ is called PHFE and indicated by

$$h_x(p_x) = \{\gamma_i(p_i) | i = 1,2,\ldots,\#h\},$$

where p_i satisfies $\sum_{i=1}^{\#h} p_i \le 1$, and it is the probability of the possible value γ_i, and $\#h$ is the number of all $\gamma_i(p_i)$ in $h_x(p_x)$. If $\sum_{i=1}^{\#h} p_i < 1$, then several values are missing in PHFE. If $\sum_{i=1}^{\#h} p_i = 1$, then no value is missing in PHFE. In such case, PHFS has transformed into the concept introduced in [23, 24]. No special explanation is provided in this paper because we discuss only condition $\sum_{i=1}^{\#h} p_i = 1$.

Example 1 Let $X = \{x_1, x_2\}$ be a reference set, and $h_{x_1}(p_{x_1}) = \{0.2(0.3), 0.4(0.2), 0.5(0.1), 0.7(0.4)\}$ and $h_{x_2}(p_{x_2}) = \{0.3(0.1), 0.4(0.9)\}$ be two PHFEs of $x_i (i = 1, 2)$ to a set A. Then, A can be considered a set of PHFS,

$$A = \{\langle x_1, \{0.2(0.3),\ 0.4(0.2),\ 0.5(0.1), 0.7(0.4)\}\rangle,\ \langle x_2, \{0.3(0.1),\ 0.4(0.9)\}\rangle\}.$$

If we disregard the probabilities of the possible values in a PHFE, then the values have the same probability. In this case, PHFEs are transformed into HFEs.

Xu and Zhou [24] introduced the score function for ranking PHFEs.

Definition 5 [24] Let $h_x(p_x) = \{\gamma_i(p_i) | i=1,2,\ldots,\#h\}$ be a PHFE. Then, its score function is defined as

$$E(h_x(p_x)) = \sum_{i=1}^{\#h} \gamma_i p_i. \quad (4)$$

Li and Wang [41] introduced the Hausdorff distance between two PHFEs as follows.

Definition 6 [41] Let $h_x(p_x) = \{\gamma_i(p_i)|i=1,2,\ldots,\#h_x\}$ and $h_y(p_y) = \{\gamma_j(p_j)|j=1,2,\ldots,\#h_y\}$ be two arbitrary PHFEs. The Hausdorff distance between them is defined as

$$D_1(h_x(p_x), h_y(p_y)) = \frac{1}{2}\left\{\frac{1}{\#h_x}\sum_{i=1}^{\#h_x}\min_{\gamma_j(p_j)\in h_y(p_y)}|\gamma_i p_i - \gamma_j p_j| + \frac{1}{\#h_y}\sum_{j=1}^{\#h_y}\min_{\gamma_i(p_i)\in h_x(p_x)}|\gamma_j p_j - \gamma_i p_i|\right\}. \quad (5)$$

Example 2 On the basis of the data provided in Example 1 and according to Definition 6, the Hausdorff distance between the two PHFEs is $D_1(h_{x_1}(p_{x_1}), h_{x_2}(p_{x_2})) = 0.0475$.

3 Expected Additive Consistency of PHFPRs

In this section, several operations related to PHFEs are proposed, and some concepts, including PHFPRs and expected additive consistency of PHFPRs, are introduced.

3.1 PHFPRs

The widely applied operational laws for PHFEs were proposed by Zhang et al. [26] and Xu and Zhou [24]. These operational laws may be unsuitable for PHFPRs. Motivated by the work of Zhu and Xu [42], several operations are proposed as follows.

Definition 7 Let $h(p)$, $h_1(p_1)$, and $h_2(p_2)$ be three PHFEs, and a $(a > 0)$ be a real number. Then, the generalized operations "$\oplus$," "$\ominus$," and "$\otimes$" are defined as follows:

$$h_1(p_1) \oplus h_2(p_2) = \cup_{\gamma_1(p_1)\in h_1(p_1),\gamma_2(p_2)\in h_2(p_2)}\{(\gamma_1+\gamma_2)(p_1 p_2)\}, \quad (6)$$

$$h(p) \oplus a = \cup_{\gamma(p)\in h(p)}\{(\gamma + a)(p)\}, \quad (7)$$

$$h(p) \ominus a = \cup_{\gamma(p)\in h(p)}\{(\gamma - a)(p)\}, \quad (8)$$

$$a \otimes h(p) = \cup_{\gamma(p)\in h(p)}\{(a\gamma)(p)\}. \quad (9)$$

Remark 2 The corresponding probability of real number a is 1, that is, a is represented in a PHFE as $a(1)$. Therefore, Eq. (7) is a special case of Eq. (6).

Remark 3 Every element in a PHFE has two parts: the membership part γ_i and the corresponding probability part p_i. From the operations related to HFEs presented in Definition 4 [42], the membership part $\gamma_1 + \gamma_2$ obtained from Eq. (6) is occasionally higher than 1, which is meaningless. However, considering that the operations are involved only in the calculation process, the final result is only slightly influenced.

On the basis of the definition of HFPRs, PHFPRs can be obtained in a similar manner as follows.

Definition 8 Let $X = \{x_1, x_2, \ldots, x_n\}$ be a fixed set, and PHFPRs on X is presented by a matrix $H = (h_{ij}(p_{ij}))_{n\times n} \subset X \times X$, where $h_{ij}(p_{ij}) = \left\{\gamma_{ij}^l\left(p_{ij}^l\right)\middle| l = 1,2,\ldots,\#h_{ij}\right\}$ is a PHFE that indicates all possible probabilistic preference degrees of objective x_i over x_j. Moreover, $h_{ij}(p_{ij})$ should satisfy the following conditions:

$$\gamma_{ij}^{\sigma(l)} + \gamma_{ji}^{\sigma(l)} = 1,\ p_{ij}^{\sigma(l)} = p_{ji}^{\sigma(l)},\ \gamma_{ii}(p) = \gamma_{ii}(1) = 0.5,\ \#h_{ij} = \#h_{ji},\ i,j = 1,2,\ldots,n, \quad (10)$$

where $\#h_{ij}$ is the number of the values in $h_{ij}(p_{ij})$, and $\gamma_{ij}^{\sigma(l)}$ is the l-th element in $h_{ij}(p_{ij})$.

Remark 4 The PHFPRs introduced in Definition 8 are similar to those in the definition presented by Zhou and Xu[31] and Wu et al. [26]. However, a slight difference exists. PHFEs do not need to be normalized in this study; thus, the condition $\gamma_{ij}^{\sigma(l)} < \gamma_{ij}^{\sigma(l+1)} = 1,\ i,j = 1,2,\ldots,n$ has been omitted.

On the basis of the operations introduced in Definition 7 and additive transitivity, the expected additive consistent PHFPR is defined as follows.

Definition 9 The PHFPR $H = (h_{ij}(p_{ij}))_{n\times n}$ is called an expected additive consistent PHFPR if it satisfies the following formula:

$$E(h_{ij}(p_{ij})) = E(h_{ik}(p_{ik}) \oplus h_{kj}(p_{kj}) \ominus 0.5),\ \forall i,j,k \in n, \quad (11)$$

where $E(h_{ij}(p_{ij}))$ and $E(h_{ik}(p_{ik}) \oplus h_{kj}(p_{kj}) \ominus 0.5)$ denote the score functions of $h_{ij}(p_{ij})$ and $h_{ik}(p_{ik}) \oplus h_{kj}(p_{kj}) \ominus 0.5$, respectively. Equation (11) can be interpreted as follows. For an expected additive consistent PHFPR, the averaging preference degree of objective V_i preferred to objective V_j is equal to the averaging preference degree of objective V_i preferred to objective V_k. When probabilistic hesitation emerges in judgments, add ($\oplus$) that of objective V_k preferred to objective V_j plus ($\ominus$) 0.5.

Example 3 When a PHFPR is given,

$$H = \begin{bmatrix} \{0.5\} & \{0.6(0.2), 0.7(0.8)\} & \{0.4(0.4), 0.5(0.6)\} \\ \{0.4(0.2), 0.3(0.8)\} & \{0.5\} & \{0.7(0.3), 0.8(0.7)\} \\ \{0.6(0.4), 0.5(0.6)\} & \{0.3(0.3), 0.2(0.7)\} & \{0.5\} \end{bmatrix}.$$

$h_{12}(p_{12}) = \{0.6(0.2), 0.7(0.8)\}$, $h_{13}(p_{13}) \oplus h_{32}(p_{32}) \ominus 0.5 = \{0.1(0.28), 0.2(0.54), 0.3(0.18)\}$ and $E(h_{12}(p_{12})) = 0.68$, $E(h_{13}(p_{13}) \oplus h_{32}(p_{32}) \ominus 0.5) = 0.19$, and $E(h_{12}(p_{12})) \neq E(h_{13}(p_{13}) \oplus h_{32}(p_{32}) \ominus 0.5)$. From

Definition 9, H is not an expected additive consistent PHFPR.

Motivated by the idea of Zhang et al. [28], the following theorem can be developed to obtain expected additive consistent PHFPRs.

Theorem 2 *Given a PHFPR* $H=\left(h_{ij}(p_{ij})\right)_{n\times n}$. *Then,* $H'=\left(h'_{ij}\left(p'_{ij}\right)\right)_{n\times n}$ *is an expected additive consistent PHFPR if*

$$h'_{ij}\left(p'_{ij}\right)=\begin{cases}\left\{\frac{1}{n}\left(\oplus_{k=1}^{n}\left(h_{ik}(p_{ik})\oplus h_{kj}(p_{kj})\right)\right)\ominus 0.5\right\}, & i,j=1,2,\ldots,n,\ i\neq j\\ \{0.5\}, & \text{otherwise}\end{cases}. \tag{12}$$

Proof For any $i=j=1,2,\ldots,n,\ i\neq j$, we have

$$E\left(h'_{ij}\left(p'_{ij}\right)\right)=E\left(\frac{1}{n}\left(\oplus_{k=1}^{n}\left(h_{ik}(p_{ik})\oplus h_{kj}(p_{kj})\right)\right)\ominus 0.5\right)$$
$$=\frac{1}{n}E\left(\oplus_{k=1}^{n}\left(h_{ik}(p_{ik})\oplus h_{kj}(p_{kj})\right)\right)\ominus 0.5.$$

Furthermore,

$$\begin{aligned}E\left(h'_{ik}\left(p'_{ik}\right)\oplus h'_{kj}\left(p'_{kj}\right)\right)&=E\left(\frac{1}{n}\left(\oplus_{q=1}^{n}\left(h_{iq}(p_{iq})\oplus h_{qk}(p_{qk})\right)\right)\ominus 0.5\right)\\&\quad\oplus\frac{1}{n}\left(\oplus_{q=1}^{n}\left(h_{kq}(p_{kq})\oplus h_{qj}(p_{qj})\right)\ominus 0.5\right)\\&=\frac{1}{n}E\left(\oplus_{q=1}^{n}\left(h_{iq}(p_{iq})\oplus h_{qk}(p_{qk})\right.\right.\\&\quad\left.\left.\oplus h_{kq}(p_{kq})\oplus h_{qj}(p_{qj})\right)\right)\ominus 1\\&=\frac{1}{n}E\left(\oplus_{q=1}^{n}\left(h_{kq}(p_{kq})\oplus h_{qk}(p_{qk})\right)\right)\\&\quad\oplus\left(\oplus_{q=1}^{n}\left(h_{iq}(p_{iq})\oplus h_{qj}(p_{qj})\right)\right)\ominus 1\\&=\frac{1}{n}\left(E\left(\oplus_{q=1}^{n}\left(h_{kq}(p_{kq})\oplus h_{qk}(p_{qk})\right)\right)\ominus 0.5\right)\\&\quad\oplus E\left(\oplus_{q=1}^{n}\left(h_{iq}(p_{iq})\oplus h_{qj}(p_{qj})\right)\ominus 0.5\right)\\&=E\left(\frac{1}{n}\left(\oplus_{q=1}^{n}\left(h_{kq}(p_{kq})\oplus h_{qk}(p_{qk})\right)\right)\ominus 0.5\right)\\&\quad\oplus E\left(\frac{1}{n}\left(\oplus_{q=1}^{n}\left(h_{iq}(p_{iq})\oplus h_{qj}(p_{qj})\right)\right)\ominus 0.5\right)\\&=0.5\oplus E\left(\frac{1}{n}\left(\oplus_{q=1}^{n}\left(h_{iq}(p_{iq})\oplus h_{qj}(p_{qj})\right)\right)\ominus 0.5\right)\\&=E\left(h'_{ij}\left(p'_{ij}\right)\right)\oplus 0.5.\end{aligned}$$

Therefore, $E\left(h'_{ij}\left(p'_{ij}\right)\right)=E\left(h'_{iq}\left(p'_{iq}\right)\oplus h'_{qj}\left(p'_{qj}\right)\ominus 0.5\right)$, which indicates that $H'=\left(h'_{ij}\left(p'_{ij}\right)\right)_{n\times n}$ is an expected additive consistent PHFPR.

Hence, the proof is completed.

Example 4 Let H be the PHFPR shown in Example 3. Then, its corresponding expected additive consistent PHFPR H' can be obtained from Eq. (12) as follows:

$$H'=\begin{bmatrix}\{0.5\} & \left\{\begin{matrix}0.43(0.01),0.47(0.11),0.5(0.36)\\0.53(0.4),0.57(0.12)\end{matrix}\right\} & \left\{\begin{matrix}0.53(0.01),0.57(0.09),0.6(0.29)\\0.63(0.41),0.67(0.2)\end{matrix}\right\}\\ \left\{\begin{matrix}0.57(0.01),0.53(0.11),0.5(0.36)\\0.47(0.4),0.43(0.12)\end{matrix}\right\} & \{0.5\} & \left\{\begin{matrix}0.53(0.03),0.57(0.19),0.6(0.4)\\0.63(0.32),0.67(0.06)\end{matrix}\right\}\\ \left\{\begin{matrix}0.47(0.01),0.43(0.09),0.4(0.29)\\0.37(0.41),0.33(0.2)\end{matrix}\right\} & \left\{\begin{matrix}0.47(0.03),0.43(0.19),0.4(0.4)\\0.37(0.32),0.33(0.06)\end{matrix}\right\} & \{0.5\}\end{bmatrix}.$$

$h'_{12}\left(p'_{12}\right)$ is used as an example. Then, from Eq. (12), we derive

$$\begin{aligned}h'_{12}\left(p'_{12}\right)&=\left\{\frac{1}{3}\left(\oplus_{k=1}^{3}(h_{1k}\oplus h_{k2})\right)\ominus 0.5\right\}\\&=\left\{\frac{1}{3}\left((h_{11}\oplus h_{12})\oplus(h_{12}\oplus h_{22})\oplus(h_{13}\oplus h_{32})\right)\ominus 0.5\right\}\\&=\left\{\frac{1}{3}\left((1.1(0.2),1.2(0.8))\oplus(1.1(0.2),1.2(0.8))\right.\right.\\&\quad\left.\left.\oplus(0.6(0.28),0.7(0.54),0.8(0.18))\right)\ominus 0.5\right\}\\&=\left\{\frac{1}{3}\left((2.2(0.04),2.3(0.32),2.4(0.64))\right.\right.\\&\quad\left.\left.\oplus(0.6(0.28),0.7(0.54),0.8(0.18))\right)\ominus 0.5\right\}\\&=\left\{\frac{1}{3}(2.8(0.01),2.9(0.11),3(0.36),3.1(0.4),3.2(0.12))\ominus 0.5\right\}\\&=\left\{(0.93(0.01),0.97(0.11),1(0.36),1.03(0.4),1.07(0.12))\ominus 0.5\right\}\\&=\left\{0.43(0.01),0.47(0.11),0.5(0.36),0.53(0.4),0.57(0.12)\right\}.\end{aligned}$$

Remark 5 Occasionally, the membership values in PHFEs obtained from the formula proposed in Theorem 2 will lie outside the range of [0, 1], but in an interval of $[-a,1+a]$, $(a>0)$. In this case, a transformation function $f(x)=\frac{x+a}{1+2a}$ developed by Herrera-Viedma et al. [43] is applied to normalize the obtained values.

Example 5 Let H be a PHFPR, which is shown as follows:

$$H=\begin{bmatrix}\{0.5\} & \{0.1(0.2),0.2(0.8)\} & \{0.1(0.4),0.2(0.6)\}\\ \{0.9(0.2),0.8(0.8)\} & \{0.5\} & \{0.8(0.3),0.9(0.7)\}\\ \{0.9(0.4),0.8(0.6)\} & \{0.2(0.3),0.2(0.7)\} & \{0.5\}\end{bmatrix}.$$

Then, its corresponding expected additive consistent PHFPR H' can be obtained according to Eq. (12) as follows:

$$H'=\begin{bmatrix}\{0.5\} & \left\{\begin{matrix}-0.03(0.01),0(0.11),0.03(0.36)\\0.07(0.4),0.1(0.12)\end{matrix}\right\} & \left\{\begin{matrix}0.2(0.01),0.23(0.09),0.27(0.29)\\0.3(0.41),0.33(0.2)\end{matrix}\right\}\\ \left\{\begin{matrix}1.03(0.01),1(0.11),0.97(0.36)\\0.93(0.4),0.9(0.12)\end{matrix}\right\} & \{0.5\} & \left\{\begin{matrix}0.5(0.03),0.53(0.19),0.57(0.4)\\0.6(0.32),0.63(0.06)\end{matrix}\right\}\\ \left\{\begin{matrix}0.8(0.01),0.77(0.09),0.73(0.29)\\0.7(0.41),0.67(0.2)\end{matrix}\right\} & \left\{\begin{matrix}0.5(0.03),0.47(0.19),0.43(0.4)\\0.4(0.32),0.37(0.06)\end{matrix}\right\} & \{0.5\}\end{bmatrix}.$$

Evidently, the membership values in H' lie in the interval of $[-0.03,1.03]$, where $a=0.03$. When the transformation function $f(x)=\frac{x+a}{1+2a}$ is utilized, the normalized expected additive consistent PHFPR H^N can be obtained as follows:

$$H^N = \begin{bmatrix} \{0.5\} & \left\{ \begin{array}{r} 0(0.01), 0.03(0.11), 0.06(0.36) \\ 0.09(0.4), 0.12(0.12) \end{array} \right\} & \left\{ \begin{array}{r} 0.22(0.01), 0.25(0.09), 0.28(0.29) \\ 0.31(0.41), 0.34(0.2) \end{array} \right\} \\ \left\{ \begin{array}{r} 1(0.01), 0.97(0.11), 0.94(0.36) \\ 0.91(0.4), 0.88(0.12) \end{array} \right\} & \{0.5\} & \left\{ \begin{array}{r} 0.5(0.03), 0.53(0.19), 0.57(0.4) \\ 0.59(0.32), 0.62(0.06) \end{array} \right\} \\ \left\{ \begin{array}{r} 0.78(0.01), 0.75(0.09), 0.72(0.29) \\ 0.69(0.41), 0.66(0.2) \end{array} \right\} & \left\{ \begin{array}{r} 0.5(0.03), 0.47(0.19), 0.43(0.4) \\ 0.41(0.32), 0.38(0.06) \end{array} \right\} & \{0.5\} \end{bmatrix}.$$

Remark 6 Remark 5 indicates that the membership part γ_i obtained from Theorem 2 occasionally lies outside the range of [0, 1], but not the PHFE $\gamma_i(p_i)$.

4 Consistency Improvement for PHFPRs with Acceptable Consistency

In a decision-making process, obtaining consistent preference relation is difficult. In such case, decision makers strive for acceptable consistent preference relation. In this section, two automatic iterative algorithms are developed to obtain individual acceptable consistent PHFPRs and consensus PHFPRs.

4.1 Automatic Iterative Algorithm for Individual Acceptable Consistent PHFPRs

In this subsection, we first introduce the Hausdorff distance between two PHFPRs, and then the consistency index of the PHFPR is developed based on the Hausdorff distance. Finally, an automatic iterative algorithm is designed to obtain individual acceptable consistent PHFPRs.

4.1.1 Consistency Index of PHFPRs

The Hausdorff distance is a formula for calculating the distance between two objects. The characteristic of the Hausdorff distance introduced in Definition 6 is that it does not require two PHFEs to have the same number of elements. However, Eq. (5) may be unsuitable for PHFPRs. In the succeeding section, a new Hausdorff distance formula is proposed as follows.

Definition 10 Let $h_x(p_x) = \{\gamma_i(p_i) | i = 1, 2, \ldots, \#h_x\}$ and $h_y(p_y) = \{\gamma_j(p_j) | j = 1, 2, \ldots, \#h_y\}$ be two arbitrary PHFEs. Then, the Hausdorff distance between them is defined as

$$\begin{aligned} &D_2(h_x(p_x), h_y(p_y)) \\ &= \max\left\{ \max_{\gamma_i(p_i)\in h_x(p_x)} \min_{\gamma_j(p_j)\in h_y(p_y)} |\gamma_i - \gamma_j| p_i p_j, \max_{\gamma_j(p_j)\in h_y(p_y)} \min_{\gamma_i(p_i)\in h_x(p_x)} |\gamma_j - \gamma_i| p_j p_i \right\}. \end{aligned} \tag{13}$$

The Hausdorff distance introduced in Eq. (13) is similar to that in Eq. (5), and the basic properties can be easily obtained in the same manner, and thus, the procedures are no longer discussed in this study.

Definition 11 Let $H_1 = \left(h_{ij}(p_{ij})_1\right)_{n\times n}$ and $H_2 = \left(h_{ij}(p_{ij})_2\right)_{n\times n}$ be two PHFPRs. Then, the Hausdorff distance between them is calculated as

$$D(H_1, H_2) = \frac{2}{n(n-1)} \sum_{j=i+1}^{n} \sum_{i=1}^{n-1} D_2\left(h_{ij}(p_{ij})_1, h_{ij}(p_{ij})_2\right), \tag{14}$$

where $D_2\left(h_{ij}(p_{ij})_1, h_{ij}(p_{ij})_2\right)$ is the Hausdorff distance between $h_{ij}(p_{ij})_1$ and $h_{ij}(p_{ij})_2$, which is introduced in Definition 10.

From Eq. (14), the consistency index of the PHFPRs is defined as follows.

Definition 12 Let $H = \left(h_{ij}(p_{ij})\right)_{n\times n}$ be a PHFPR and $H' = \left(h'_{ij}\left(p'_{ij}\right)\right)_{n\times n}$ be its corresponding consistent PHFPR, which is obtained from Theorem 2. Then, the consistency index of H can be measured by the Hausdorff distance between H and H', which is denoted as

$$\mathrm{CI}(H) = 1 - D\left(H, H'\right). \tag{15}$$

When the value $\mathrm{CI}(H)$ is large, the consistency level of H is good. If $\mathrm{CI}(H) = 1$, then H is consistent. The consistency threshold $\overline{\mathrm{CI}}(H)$ is discussed in the succeeding section. First, $\mathrm{CI}(H)$ can be concretely expressed as

$$\begin{aligned} \mathrm{CI}(H) &= 1 - D\left(H, H'\right) \\ &= 1 - \frac{2}{n(n-1)} \sum_{j=i+1}^{n} \sum_{i=1}^{n-1} D_2\left(h_{ij}(p_{ij}), h'_{ij}\left(p'_{ij}\right)\right) \\ &= 1 - \frac{2}{n(n-1)} \sum_{j=i+1}^{n} \sum_{i=1}^{n-1} \max\left\{ \max_{\gamma_{ij,l}(p_{ij,l})\in h_{ij}(p_{ij})} \right. \\ &\quad \left. \min_{\gamma_{ij,m}(p_{ij,m})\in h'_{ij}(p'_{ij})} |\gamma_{ij,l} - \gamma_{ij,m}| p_{ij,l} p_{ij,m}, \max_{\gamma_{ij,m}(p_{ij,m})\in h'_{ij}(p'_{ij})} \min_{\gamma_{ij,l}(p_{ij,l})\in h_{ij}(p_{ij})} |\gamma_{ij,m} - \gamma_{ij,l}| p_{ij,m} p_{ij,l} \right\}. \end{aligned}$$

Let $\varepsilon_{ij} = \max\left\{ \max_{\gamma_{ij,l}(p_{ij,l})\in h_{ij}(p_{ij})} \min_{\gamma_{ij,m}(p_{ij,m})\in h'_{ij}(p'_{ij})} |\gamma_{ij,l} - \gamma_{ij,m}| p_{ij,l} p_{ij,m}, \max_{\gamma_{ij,m}(p_{ij,m})\in h'_{ij}(p'_{ij})} \min_{\gamma_{ij,l}(p_{ij,l})\in h_{ij}(p_{ij})} |\gamma_{ij,m} - \gamma_{ij,l}| p_{ij,m} p_{ij,l} \right\}$.

Then, we have

$$\mathrm{CI}(H) = 1 - \frac{2}{n(n-1)} \sum_{j=i+1}^{n} \sum_{i=1}^{n-1} \varepsilon_{ij}.$$

Decision makers frequently exhibit a tendency to achieve certain consistency in making pairwise comparisons [28, 44]. PHFS is extensions of HFS; thus, decision makers also exhibit such a tendency when making pairwise comparisons. Motivated by this idea, the values of the Hausdorff distance ε_{ij} should be relatively centralized in the domain close to zero. ε_{ij} $(i < j)$ can be assumed as independently and normally distributed, with a mean of zero and a standard deviation of σ, that is, $\varepsilon_{ij} \sim N(0, \sigma^2)$. On the basis of this condition, $\frac{n(n-1)}{2}\left(\frac{1}{\sigma} \times (1 - \mathrm{CI}(H))\right)^2$ is a Chi-square distribution with $\frac{n(n-1)}{2}$ degrees of freedom, that is, $\frac{n(n-1)}{2}\left(\frac{1}{\sigma}\times(1-\mathrm{CI}(H))\right)^2 \sim \chi^2\left(\frac{n(n-1)}{2}\right)$, which is a one-sided right-tailed test [28, 44]. The critical value of χ^2 is λ_α at a significance level α. Then, we obtain

$$\overline{\mathrm{CI}}(H) = 1-\sigma\sqrt{\frac{2}{n(n-1)}\lambda_\alpha}.$$

If $\mathrm{CI}(H) \geq \overline{\mathrm{CI}}(H)$, then the obtained consistency level of H is acceptable. From the given values σ and α of Dong [44], i.e., $\sigma = 2$ and $\alpha = 0.1$, Zhu [23] provided the values of the consistency thresholds $\overline{\mathrm{CI}}(H)$ for different n values, as shown in Table 1.

Example 6 Let H be the PHFPR shown in Example 3 and H' be its corresponding expected additive consistent PHFPR shown in Example 4. Then, the consistency index $\mathrm{CI}(H)$ of H can be obtained as follows:

$$\begin{aligned}\mathrm{CI}(H) &= 1-\frac{2}{3\times 2}\sum_{j=i+1}^{3}\sum_{i=1}^{2} D\left(h_{ij}(p_{ij}), h'_{ij}\left(p'_{ij}\right)\right)\\ &= 0.9488 > \overline{\mathrm{CI}}(H) = 0.9118,\ (n=3).\end{aligned}$$

Evidently, H is acceptable consistent PHFPR.

4.1.2 Automatic Iterative Algorithm for Individual Acceptable Consistent PHFPRs

If PHFPRs provided by decision makers exhibit unacceptable consistency, then their consistency should be improved to meet the established consistency level. The weights of the decision makers can be derived with the individual acceptable consistent PHFPRs. Motivated by the automatic iterative algorithm proposed by Wu and Xu [45], a similar automatic iterative algorithm is developed to obtain individual acceptable consistent PHFPRs as follows.

Algorithm 1

Input: PHFPRs $H = \left(h_{ij}(p_{ij})\right)_{n\times n}$

Output: Acceptable consistent PHFPRs $H^* = \left(h^*_{ij}\left(p^*_{ij}\right)\right)_{n\times n}$, iteration times k, and consistency index $CI(H)$

Step 1: Let $H^{(k)} = \left(h_{ij}(p_{ij})^{(k)}\right)_{n\times n}$ and $k=0$;

Step 2: Use Eq. (12) to calculate its corresponding consistent PHFPRs $H' = \left(h'_{ij}\left(p'_{ij}\right)\right)_{n\times n}$;

Step 3: Use Eq. (15) to calculate the consistency index $CI(H)$ of H;

Step 4: If the consistency index does not meet the consistency level listed in Table 1, then continue to the next step; otherwise, proceed to the Step 7;

Step 5: Assume an adjusted parameter $\theta(0<\theta<1)$, and let

$$h_{ij}(p_{ij})^{(k+1)} = (1-\theta)h_{ij}(p_{ij})^{(k)} \oplus \theta h'_{ij}\left(p'_{ij}\right);$$

Step 6: Let $k = k+1$, and return to Step 3;

Step 7: Let $H^* = H^k$, then output the acceptable consistent PHFPRs H^*, the iteration times k, and the consistency index $\mathrm{CI}(H)$;

Step 8: End

Theorem 3 *Let $H = \left(h_{ij}(p_{ij})\right)_{n\times n}$ be a PHFPR. Then, $\left\{H^{(k)} = \left(h_{ij}(p_{ij})^{(k)}\right)_{n\times n}\right\}$ is the matrix sequence generated in Algorithm* 1. *It follows that* $\mathrm{CI}\left(H^{(k+1)}\right) > \mathrm{CI}\left(H^{(k)}\right)$ *for each* k, *and* $\lim_{k\to+\infty} \mathrm{CI}\left(H^{(k)}\right) = 1$.

Proof Let $h_{ij}(p_{ij})^{(k)} = \left\{\gamma_{ij,l}(p_{ij,l}) \middle| l = 1,2,\ldots,\#h_{ij}\right\}$ and $h'_{ij}\left(p'_{ij}\right) = \left\{\gamma_{ij,m}(p_{ij,m}) \middle| m = 1,2,\ldots,\#h'_{ij}\right\}$. Given that $h_{ij}(p_{ij})^{(k+1)} = (1-\theta)h_{ij}(p_{ij})^{(k)} \oplus \theta h'_{ij}\left(p'_{ij}\right) = \{((1-\theta)\gamma_{ij,l} + \theta\gamma_{ij,m})(p_{ij,l}p_{ij,m}) | l = 1,2,\ldots,\#h_{ij}; m = 1,2,\ldots,\#h'_{ij}\}$, we derive

$$\begin{aligned}
\mathrm{CI}\left(H^{(k+1)}\right) &= 1-\frac{2}{n(n-1)}\sum_{j=i+1}^{n}\sum_{i=1}^{n-1} D_2\left(h_{ij}(p_{ij})^{(k+1)}, h'_{ij}\left(p'_{ij}\right)\right)\\
&= 1-\frac{2}{n(n-1)}\sum_{j=i+1}^{n}\sum_{i=1}^{n-1}\max\left\{\begin{array}{l}\max\limits_{\substack{\gamma_{ij,l}(p_{ij,l})\in h_{ij}(p_{ij})\\ \gamma_{ij,m}(p_{ij,m})\in h'_{ij}(p'_{ij})}}\min\limits_{\gamma_{ij,m}(p_{ij,m})\in h'_{ij}(p'_{ij})}\left|(1-\theta)\gamma_{ij,l}+\theta\gamma_{ij,m}-\gamma_{ij,m}\right|p_{ij,l}p_{ij,m}p_{ij,m},\\ \max\limits_{\gamma_{ij,m}(p_{ij,m})\in h'_{ij}(p'_{ij})}\min\limits_{\substack{\gamma_{ij,l}(p_{ij,l})\in h_{ij}(p_{ij})\\ \gamma_{ij,m}(p_{ij,m})\in h'_{ij}(p'_{ij})}}\left|\gamma_{ij,m}-((1-\theta)\gamma_{ij,l}+\theta\gamma_{ij,m})\right|p_{ij,m}p_{ij,l}p_{ij,m}\end{array}\right\}\\
&\geq 1-\frac{2}{n(n-1)}\sum_{j=i+1}^{n}\sum_{i=1}^{n-1}\max\left\{\begin{array}{l}\max\limits_{\gamma_{ij,l}(p_{ij,l})\in h_{ij}(p_{ij})}\min\limits_{\gamma_{ij,m}(p_{ij,m})\in h'_{ij}(p'_{ij})}\left|(1-\theta)\gamma_{ij,l}+\theta\gamma_{ij,m}-\gamma_{ij,m}\right|p_{ij,l}p_{ij,m}p_{ij,m},\\ \max\limits_{\gamma_{ij,m}(p_{ij,m})\in h'_{ij}(p'_{ij})}\min\limits_{\gamma_{ij,l}(p_{ij,l})\in h_{ij}(p_{ij})}\left|\gamma_{ij,m}-((1-\theta)\gamma_{ij,l}+\theta\gamma_{ij,m})\right|p_{ij,m}p_{ij,l}p_{ij,m}\end{array}\right\}\\
&\geq 1-\frac{2}{n(n-1)}\sum_{j=i+1}^{n}\sum_{i=1}^{n-1}\max\left\{\begin{array}{l}\max\limits_{\gamma_{ij,l}(p_{ij,l})\in h_{ij}(p_{ij})}\min\limits_{\gamma_{ij,m}(p_{ij,m})\in h'_{ij}(p'_{ij})}\left|(1-\theta)\gamma_{ij,l}+\theta\gamma_{ij,m}-\gamma_{ij,m}\right|p_{ij,l}p_{ij,m},\\ \max\limits_{\gamma_{ij,m}(p_{ij,m})\in h'_{ij}(p'_{ij})}\min\limits_{\gamma_{ij,l}(p_{ij,l})\in h_{ij}(p_{ij})}\left|\gamma_{ij,m}-((1-\theta)\gamma_{ij,l}+\theta\gamma_{ij,m})\right|p_{ij,m}p_{ij,l}\end{array}\right\}\\
&= 1-\frac{2}{n(n-1)}\sum_{j=i+1}^{n}\sum_{i=1}^{n-1}(1-\theta)\max\left\{\begin{array}{l}\max\limits_{\gamma_{ij,l}(p_{ij,l})\in h_{ij}(p_{ij})}\min\limits_{\gamma_{ij,m}(p_{ij,m})\in h'_{ij}(p'_{ij})}\left|\gamma_{ij,l}-\gamma_{ij,m}\right|p_{ij,l}p_{ij,m},\\ \max\limits_{\gamma_{ij,m}(p_{ij,m})\in h'_{ij}(p'_{ij})}\min\limits_{\gamma_{ij,l}(p_{ij,l})\in h_{ij}(p_{ij})}\left|\gamma_{ij,m}-\gamma_{ij,l}\right|p_{ij,m}p_{ij,l}\end{array}\right\}\\
&> 1-\frac{2}{n(n-1)}\sum_{j=i+1}^{n}\sum_{i=1}^{n-1}\max\left\{\begin{array}{l}\max\limits_{\gamma_{ij,l}(p_{ij,l})\in h_{ij}(p_{ij})}\min\limits_{\gamma_{ij,m}(p_{ij,m})\in h'_{ij}(p'_{ij})}\left|\gamma_{ij,l}-\gamma_{ij,m}\right|p_{ij,l}p_{ij,m},\\ \max\limits_{\gamma_{ij,m}(p_{ij,m})\in h'_{ij}(p'_{ij})}\min\limits_{\gamma_{ij,l}(p_{ij,l})\in h_{ij}(p_{ij})}\left|\gamma_{ij,m}-\gamma_{ij,l}\right|p_{ij,m}p_{ij,l}\end{array}\right\}\\
&= \mathrm{CI}\left(H^{(k)}\right).
\end{aligned}$$

Furthermore, we obtain

$$\begin{aligned}
&\lim_{\lim k\to+\infty} \mathrm{CI}\left(H^{(k)}\right)\\
&= \lim_{\lim k\to+\infty}\left\{1-\frac{2}{n(n-1)}\sum_{j=i+1}^{n}\sum_{i=1}^{n-1}(1-\theta)^k\max\left\{\begin{array}{l}\max\limits_{\gamma_{ij,l}(p_{ij,l})\in h_{ij}(p_{ij})}\min\limits_{\gamma_{ij,m}(p_{ij,m})\in h'_{ij}(p'_{ij})}\left|\gamma_{ij,l}-\gamma_{ij,m}\right|p_{ij,l}p_{ij,m},\\ \max\limits_{\gamma_{ij,m}(p_{ij,m})\in h'_{ij}(p'_{ij})}\min\limits_{\gamma_{ij,l}(p_{ij,l})\in h_{ij}(p_{ij})}\left|\gamma_{ij,m}-\gamma_{ij,l}\right|p_{ij,m}p_{ij,l}\end{array}\right\}\right\}\\
&= 1-\lim_{\lim k\to+\infty}(1-\theta)^k\frac{2}{n(n-1)}\sum_{j=i+1}^{n}\sum_{i=1}^{n-1}\max\left\{\begin{array}{l}\max\limits_{\gamma_{ij,l}(p_{ij,l})\in h_{ij}(p_{ij})}\min\limits_{\gamma_{ij,m}(p_{ij,m})\in h'_{ij}(p'_{ij})}\left|\gamma_{ij,l}-\gamma_{ij,m}\right|p_{ij,l}p_{ij,m},\\ \max\limits_{\gamma_{ij,m}(p_{ij,m})\in h'_{ij}(p'_{ij})}\min\limits_{\gamma_{ij,l}(p_{ij,l})\in h_{ij}(p_{ij})}\left|\gamma_{ij,m}-\gamma_{ij,l}\right|p_{ij,m}p_{ij,l}\end{array}\right\}\\
&= 1.
\end{aligned}$$

Hence, the proof is completed.

4.2 Automatic Iterative Algorithm for Consensus PHFPRs

Acceptable individual consistency PHFPRs can be obtained using Algorithm 1. In group context, consensus decision making is frequently considered a desirable outcome. In the succeeding section, we first utilize the generalized add operator "$\oplus$," which is defined in Eq. (6), for collecting individual PHFPRs. Then, a consensus index is introduced to measure the consensus level among group members. When the consensus level is lower than a specified threshold, an automatic iterative algorithm is designed to update the PHFPRs to reach a predefined consensus level.

4.2.1 Consensus Index of PHFPRs

Definition 13 Let $H_l = \left(h_{ij}^l\left(p_{ij}^l\right)\right)_{n\times n}$, $l = 1, 2, \ldots m$ be any m PHFPRs provided from m decision makers. Then, the collective PHFPR $H_c = \left(h_{ij}^c\left(p_{ij}^c\right)\right)_{n\times n}$ can be calculated as follows:

$$h_{ij}^c\left(p_{ij}^c\right) = \varpi_1 h_{ij}^1\left(p_{ij}^1\right) \oplus \varpi_2 h_{ij}^2\left(p_{ij}^2\right) \oplus \cdots \oplus \varpi_m h_{ij}^m\left(p_{ij}^m\right), \tag{16}$$

where $(\varpi_1, \varpi_2, \ldots, \varpi_m)$ is the weight vector of the decision makers, and the symbol "$\oplus$" is the generalized add operation introduced to Eq. (6).

On the basis of Eq. (16) and the Hausdorff distance introduced in Eq. (13), the consensus index related to the l-th decision maker is defined as follows.

Definition 14 Let $H_l = \left(h_{ij}^l\left(p_{ij}^l\right)\right)_{n\times n}$ be any m PHFPRs are provided from m decision makers and $H_c = \left(h_{ij}^c\left(p_{ij}^c\right)\right)_{n\times n}$ is the collective PHFPR obtained from Definition 13. Then, the consensus index related to the l-th decision maker is defined as

$$\mathrm{GCI}(H_l) = 1 - \frac{2}{n(n-1)}\sum_{j=i+1}^{n}\sum_{i=1}^{n-1} D_2\left(h_{ij}^l\left(p_{ij}^l\right), h_{ij}^c\left(p_{ij}^c\right)\right). \tag{17}$$

When the value of $\mathrm{GCI}(H_l)$ is large, the decision makers are close to the group. If $\mathrm{GCI}(H_l) = 1$, then the l-th decision maker achieves full consensus with the group. The threshold of the consensus level $\overline{\mathrm{GCI}}$ can be determined in advance by decision makers in the decision-making process. As suggested in [45, 46], the threshold of the acceptable consensus level can be set at 0.9.

Theorem 4 *Let $H_l = \left(h_{ij}^l\left(p_{ij}^l\right)\right)_{n\times n}$ be any m PHFPRs provided from m decision makers and $H_c = \left(h_{ij}^c\left(p_{ij}^c\right)\right)_{n\times n}$ is the collective PHFPR. Then,*

$$D_2(H_l, H_c) \le \max_{t=1,2,\ldots,m;\ t\ne l}\{D_2(H_l, H_t)\}. \tag{18}$$

Proof From Definition 11, we have

$$\begin{aligned}
D_2(H_l,H_c) &= \frac{2}{n(n-1)}\sum_{j=i+1}^{n}\sum_{i=1}^{n-1} D_2\left(h_{ij}^l\left(p_{ij}^l\right), h_{ij}^c\left(p_{ij}^c\right)\right)\\
&= \frac{2}{n(n-1)}\sum_{j=i+1}^{n}\sum_{i=1}^{n-1}\max\left\{\begin{array}{l}
\max\limits_{\gamma_{ij,l}(p_{ij,l})\in h_{ij}^l(p_{ij}^l)} \min\limits_{\substack{\gamma_{ij,1}(p_{ij,1})\in h_{ij}^1(p_{ij}^1)\\ \gamma_{ij,2}(p_{ij,2})\in h_{ij}^2(p_{ij}^2)\\ \cdots\\ \gamma_{ij,m}(p_{ij,m})\in h_{ij}^m(p_{ij}^m)}} \left|\gamma_{ij,l} - (\varpi_1\gamma_{ij,1} + \varpi_2\gamma_{ij,2} + \cdots + \varpi_m\gamma_{ij,m})\right| p_{ij,l}p_{ij,1}p_{ij,2}\cdots p_{ij,m},\\
\max\limits_{\substack{\gamma_{ij,1}(p_{ij,1})\in h_{ij}^1(p_{ij}^1)\\ \gamma_{ij,2}(p_{ij,2})\in h_{ij}^2(p_{ij}^2)\\ \cdots\\ \gamma_{ij,m}(p_{ij,m})\in h_{ij}^m(p_{ij}^m)}} \min\limits_{\gamma_{ij,l}(p_{ij,l})\in h_{ij}^l(p_{ij}^l)} \left|(\varpi_1\gamma_{ij,1} + \varpi_2\gamma_{ij,2} + \cdots + \varpi_m\gamma_{ij,m}) - \gamma_{ij,l}\right| p_{ij,1}p_{ij,2}\cdots p_{ij,m}p_{ij,l}
\end{array}\right\}\\
&\le \frac{2}{n(n-1)}\sum_{j=i+1}^{n}\sum_{i=1}^{n-1}\max\left\{\begin{array}{l}
\max\limits_{\gamma_{ij,l}(p_{ij,l})\in h_{ij}^l(p_{ij}^l)}\ \min\limits_{\gamma_{ij,t}(p_{ij,t})\in h_{ij}^t(p_{ij}^t)} \left|\gamma_{ij,l} - (\varpi_1\gamma_{ij,t} + \varpi_2\gamma_{ij,t} + \cdots + \varpi_t\gamma_{ij,t})\right| p_{ij,l}p_{ij,t}p_{ij,t}\cdots p_{ij,t},\\
\max\limits_{\gamma_{ij,t}(p_{ij,t})\in h_{ij}^t(p_{ij}^t)}\ \min\limits_{\gamma_{ij,l}(p_{ij,l})\in h_{ij}^l(p_{ij}^l)} \left|(\varpi_1\gamma_{ij,t} + \varpi_2\gamma_{ij,t} + \cdots + \varpi_t\gamma_{ij,t}) - \gamma_{ij,l}\right| p_{ij,t}p_{ij,t}\cdots p_{ij,t}p_{ij,l}
\end{array}\right\}\\
&\le \frac{2}{n(n-1)}\sum_{j=i+1}^{n}\sum_{i=1}^{n-1}\max\left\{\begin{array}{l}
\max\limits_{\gamma_{ij,l}(p_{ij,l})\in h_{ij}^l(p_{ij}^l)}\ \min\limits_{\gamma_{ij,t}(p_{ij,t})\in h_{ij}^t(p_{ij}^t)} \left|\gamma_{ij,l} - (\varpi_1\gamma_{ij,t} + \varpi_2\gamma_{ij,t} + \cdots + \varpi_t\gamma_{ij,t})\right| p_{ij,l}p_{ij,t},\\
\max\limits_{\gamma_{ij,t}(p_{ij,t})\in h_{ij}^t(p_{ij}^t)}\ \min\limits_{\gamma_{ij,l}(p_{ij,l})\in h_{ij}^l(p_{ij}^l)} \left|(\varpi_1\gamma_{ij,t} + \varpi_2\gamma_{ij,t} + \cdots + \varpi_t\gamma_{ij,t}) - \gamma_{ij,l}\right| p_{ij,t}p_{ij,l}
\end{array}\right\}\\
&= \frac{2}{n(n-1)}\sum_{j=i+1}^{n}\sum_{i=1}^{n-1}\max\left\{\begin{array}{l}
\max\limits_{\gamma_{ij,l}(p_{ij,l})\in h_{ij}^l(p_{ij}^l)}\ \min\limits_{\gamma_{ij,t}(p_{ij,t})\in h_{ij}^t(p_{ij}^t)} \left|\gamma_{ij,l} - \gamma_{ij,t}\right| p_{ij,l}p_{ij,t},\\
\max\limits_{\gamma_{ij,t}(p_{ij,t})\in h_{ij}^t(p_{ij}^t)}\ \min\limits_{\gamma_{ij,l}(p_{ij,l})\in h_{ij}^l(p_{ij}^l)} \left|\gamma_{ij,t} - \gamma_{ij,l}\right| p_{ij,t}p_{ij,l}
\end{array}\right\}\\
&= D_2(H_l,H_t).
\end{aligned}$$

Therefore, the proof is completed.

Theorem 4 shows that the consensus index related to the l-th decision maker is higher than one minus the Hausdorff distance between any two individual PHFPRs.

Theorem 5 *Let $H_l = \left(h_{ij}^l\left(p_{ij}^l\right)\right)_{n\times n}$ be any m PHFPRs provided from m decision makers and $H_c = \left(h_{ij}^c\left(p_{ij}^c\right)\right)_{n\times n}$ is the collective PHFPR. Then,*

$$\mathrm{CI}(H_c) \ge \max_{l=1,2,\ldots,m}\{\mathrm{CI}(H_l)\}. \tag{19}$$

Proof Let

$$\begin{aligned}
h_{ij}^c\left(p_{ij}^c\right) = \{&(\gamma_{ij,1} + \gamma_{ij,2} + \cdots + \gamma_{ij,m})(p_{ij,1}p_{ij,2}\cdots p_{ij,m})\\
&|\gamma_{ij,1}(p_{ij,1}) \in h_{ij}^1\left(p_{ij}^1\right), \gamma_{ij,2}(p_{ij,2}) \in h_{ij}^2\left(p_{ij}^2\right), \ldots, \gamma_{ij,m}(p_{ij,m}) \in h_{ij}^m\left(p_{ij}^m\right).\}
\end{aligned}$$

From Definition 11, we derive

$$
\begin{aligned}
&\mathrm{CI}(H_c)\\
&= 1 - D_2\left(H_c, H_c^{'}\right)\\
&= 1 - \frac{2}{n(n-1)} \sum_{j=i+1}^{n} \sum_{i=1}^{n-1} \max \left\{ \begin{array}{l}
\max\limits_{\gamma_{ij,cl}(p_{ij,cl}) \in h_{ij}^{c}(p_{ij}^{c})} \min\limits_{\substack{\gamma_{ik,cl}(p_{ik,cl}) \in h_{ij}^{c}(p_{ij}^{c}) \\ \gamma_{kj,cl}(p_{kj,cl}) \in h_{ij}^{c}(p_{ij}^{c})}} \left| \gamma_{ij,cl} - \left\{ \frac{1}{n} \left(\oplus_{k=1}^{n} \left(\gamma_{ik,cl} + \gamma_{kj,cl} \right) \right) \ominus 0.5 \right\} \right| p_{ij,cl} p_{ik,cl} p_{kj,cl}, \\
\max\limits_{\substack{\gamma_{ik,cl}(p_{ik,cl}) \in h_{ij}^{c}(p_{ij}^{c}) \\ \gamma_{kj,cl}(p_{kj,cl}) \in h_{ij}^{c}(p_{ij}^{c})}} \min\limits_{\gamma_{ij,cl}(p_{ij,cl}) \in h_{ij}^{c}(p_{ij}^{c})} \left| \left\{ \frac{1}{n} \left(\oplus_{k=1}^{n} \left(\gamma_{ik,cl} + \gamma_{kj,cl} \right) \right) \ominus 0.5 - \gamma_{ij,cl} \right\} \right| p_{ik,cl} p_{kj,cl} p_{ij,cl}
\end{array} \right\}\\
&\geq 1 - \frac{2}{n(n-1)} \sum_{j=i+1}^{n} \sum_{i=1}^{n-1} \max \left\{ \begin{array}{l}
\max\limits_{\gamma_{ij,l}(p_{ij,l}) \in h_{ij}^{l}(p_{ij}^{l})} \min\limits_{\substack{\gamma_{ik,l}(p_{ik,l}) \in h_{ij}^{l}(p_{ij}^{l}) \\ \gamma_{kj,l}(p_{kj,l}) \in h_{ij}^{l}(p_{ij}^{l})}} \left| \gamma_{ij,l} - \left\{ \frac{1}{n} \left(\oplus_{k=1}^{n} \left(\gamma_{ik,l} + \gamma_{kj,l} \right) \right) \ominus 0.5 \right\} \right| p_{ij,l} p_{ik,l} p_{kj,l}, \\
\max\limits_{\substack{\gamma_{ik,l}(p_{ik,l}) \in h_{ij}^{l}(p_{ij}^{l}) \\ \gamma_{kj,l}(p_{kj,l}) \in h_{ij}^{l}(p_{ij}^{l})}} \min\limits_{\gamma_{ij,l}(p_{ij,l}) \in h_{ij}^{l}(p_{ij}^{l})} \left| \left\{ \frac{1}{n} \left(\oplus_{k=1}^{n} \left(\gamma_{ik,l} + \gamma_{kj,l} \right) \right) \ominus 0.5 - \gamma_{ij,l} \right\} \right| p_{ik,l} p_{kj,l} p_{ij,l}
\end{array} \right\}\\
&= 1 - D_2\left(H_l, H_l^{'}\right)
\end{aligned}
$$

Hence, the proof is completed.

Corollaries 1 and 2 can be easily obtained with Theorem 5 as follows.

Corollary 1 *Let* $H_l = \left(h_{ij}^{l}\left(p_{ij}^{l}\right)\right)_{n\times n}$ *be any m PHFPRs provided from m decision makers and* $H_c = \left(h_{ij}^{c}\left(p_{ij}^{c}\right)\right)_{n\times n}$ *is the collective PHFPR. Then,* $\mathrm{CI}(H_c) \geq \overline{\mathrm{CI}}$ *under the condition of* $\mathrm{CI}(H_l) \geq \overline{\mathrm{CI}}$, $l = 1, 2, \ldots m$.

Corollary 2 *If* $\mathrm{CI}(H_l) = 1$ *and* $l = 1, 2, \ldots m$, *then* $\mathrm{CI}(H_c) = 1$.

Theorem 5 shows that the consistency of the group is higher than the highest individual consistency. Corollary 1 indicates that if every PHFPR exhibits acceptable consistency, then the group PHFPR also has acceptable consistency. Corollary 2 implies that if all individual PHFPRs are completely consistent, then the group PHFPR is also completely consistent.

4.2.2 *Automatic Iterative Algorithm for Consensus PHFPRs*

If for all PHFPRs H_l, $l = 1, 2, \ldots, m$ provided from m decision makers, then we have $\mathrm{GCI}(H_l) \geq 0.9$. An acceptable consensus level is then concluded to be achieved among decision makers. If at least one consensus index related to the l-th decision maker exists, such that $\mathrm{GCI}(H_l) < 0.9$, then consensus is not achieved. Accordingly, the following algorithm is developed to improve the consensus level and obtain the consensus PHFPRs.

Algorithm 2

Input: Individual PHFPRs $H_l = \left(h_{ij}^{l}\left(p_{ij}^{l}\right)\right)_{n\times n}$, H_l, $l = 1, 2, \ldots m$, and the weight vector of decision makers $\varpi = (\varpi_1, \varpi_2, \ldots, \varpi_m)$

Output: Consensus PHFPRs $H^{*} = \left(h_{ij}^{*}\left(p_{ij}^{*}\right)\right)_{n\times n}$, iteration times k, and the consensus index $\mathrm{GCI}(H_l)$ related to the l-th decision maker

Step 1: Let $H_l^{k} = \left(h_{ij}^{l}\left(p_{ij}^{l}\right)^{k}\right)_{n\times n}$ and $k = 0$;

Step 2: Use Eq. (16) to calculate collective PHFPRs $H_c = \left(h_{ij}^{c}\left(p_{ij}^{c}\right)\right)_{n\times n}$;

Step 3: Use Eq. (17) to calculate the consensus index $\mathrm{GCI}(H_l)$, $l = 1, 2, \ldots m$;

Step 4: If at least one consensus index related to the l-th decision maker exists, such that $\mathrm{GCI}(H_l) < 0.9$, then continue to the next step; otherwise, proceed to Step 7;

Step 5: Assume an adjusted parameter $\theta (0 < \theta < 1)$, and let

$$
h_{ij}^{l}\left(p_{ij}^{l}\right)^{(k+1)} = (1-\theta) h_{ij}^{l}\left(p_{ij}^{l}\right)^{(k)} \oplus \theta h_{ij}^{c}\left(p_{ij}^{c}\right)^{(k)};
$$

Step 6: Let $k = k + 1$, and return to Step 2;

Step 7: Let $H^{*} = H_l^{k}$, then output the consensus PHFPRs H^{*}, iteration times k, and consensus index $\mathrm{GCI}(H_l)$;

Step 8: End

Remark 7 In Step 5 of Algorithm 2, more than one consensus index related to the l-th decision maker may exist, such that $\mathrm{GCI}(H_l) < 0.9$. In such case, the minimum $\mathrm{GCI}(H_l)$ should be determined and its value should be adjusted, thereby reducing the number of iterations.

Theorem 6 *Let* $H_l = \left(h_{ij}^{l}\left(p_{ij}^{l}\right)\right)_{n\times n}$ *be any m PHFPRs provided from m decision makers and* $\mathrm{G\overline{C}I}$ *be the corresponding consensus threshold,* $\left\{H_l^{(k)}\right\}$ *be the PHFPR sequence that is generated in Algorithm 2 for the l-th decision maker, and* $\left\{\mathrm{GCI}\left(H_l^{(k)}\right)\right\}$ *be the consensus index of* $H_l^{(k)}$. *Then, we have* $\mathrm{GCI}\left(H_l^{(k+1)}\right) > \mathrm{GCI}\left(H_l^{(k)}\right)$ *for each k.*

The proof of Theorem 6 is similar to that of Theorem 3 and, thus, is omitted from this paper.

Theorem 7 *Let* $H_l = \left(h_{ij}^l\left(p_{ij}^l\right)\right)_{n\times n}$ *be any m PHFPRs provided from m decision makers and* $\left\{H_l^{(k)}\right\}$ *be the PHFPR sequence generated in Algorithm 2 for the l-th decision maker. If* $\max\left\{\mathrm{CI}\left(H_l^{(k)}\right)\right\} \geq \bar{\mathrm{CI}}$, *then* $l = 1, 2, \ldots m$. *Thus, we have* $\max\left\{\mathrm{CI}\left(H_l^{(k+1)}\right)\right\} \geq \max\left\{\mathrm{CI}\left(H_l^{(k)}\right)\right\} \geq \bar{\mathrm{CI}}$.

The proof of Theorem 7 can be easily obtained according to Theorems 4 and 6 and, thus, is omitted from this paper.

Theorem 7 implies that in Algorithm 2, we start with PHFPRs with acceptable consistency and end with modified PHFPRs that not only achieve the predefined consensus level but also the acceptable individual consistency.

5 Group Decision-Making Based on PHFPRs

In this section, a group decision-making problem under a probabilistic hesitant fuzzy environment is proposed to demonstrate the application of the proposed method.

5.1 Problems Description

For group decision-making problems, let $A = \{A_1, A_2, \ldots, A_n\}$ be a set of alternatives; $D = \{D_1, D_2, \ldots, D_m\}$ be a set of decision makers; $\varpi = (\varpi_1, \varpi_2, \ldots, \varpi_m)$, where $\varpi_i \geq 0$; and $\sum_{i=1}^m \varpi_i = 1$ be weight vector of the decision makers, which are completely unknown. The evaluation of alternative A_i over alternative A_j by the decision makers with PHFPRs is represented by $H_l = \left(h_{ij}^l\left(p_{ij}^l\right)\right)_{n\times n}$, $l = 1, 2, \ldots m$.

The complete support model for group decision making with PHFPRs is depicted in Fig. 1, and the main procedures are described as follows.

Step 1 Construct the individual PHFPRs
The individual PHFPRs can be constructed and denoted as $H_l = \left(h_{ij}^l\left(p_{ij}^l\right)\right)_{n\times n}$, $l = 1, 2, \ldots m$

Step 2 Calculate the individual consistent PHFPRs
The individual consistent PHFPRs can be obtained according to Eq. (12)

Step 3 Calculate the individual consistency index
The individual consistency index can be calculated according to Eq. (15). If the consistency index does not meet the established consistency level, then the acceptable individual consistent PHFPRs $H_l = \left(h_{ij}^l\left(p_{ij}^l\right)\right)_{n\times n}$ and the consistency index $\mathrm{CI}(H_l)$, $l = 1, 2, \ldots m$ can be obtained using Algorithm 1

Step 4 Derive the weights of the decision makers
The weights of the decision makers can be derived using the following formula:

$$\varpi_i = \frac{\mathrm{CI}(H_i)}{\sum_{l=1}^m \mathrm{CI}(H_l)},\ i = 1, 2, \ldots m. \tag{20}$$

Step 5 Calculate the group PHFPR
The group PHFPR can be obtained according to Eq. (16)

Step 6 Calculate the consensus index
The consensus index can be obtained according to Eq. (17). If at least one consensus index related to the l-th decision maker exists, such that $\mathrm{GCI}(H_l) < 0.9$, then the consensus PHFPRs $H^* = \left(h_{ij}^*\left(p_{ij}^*\right)\right)_{n\times n}$ can be derived using Algorithm 2. Otherwise, proceed to next step.

Step 7 Calculate the score matrix
The score matrix $E(H^*)$ related to the consensus PHFPRs can be obtained according to Eq. (4)

Step 8 Calculate the priority weights of alternatives
The priority weights of fuzzy preference relations have been discussed in depth by several scholars [47, 48]. In this study, we utilize the formula proposed in [48] to derive the priority weights of the alternatives:

$$w_i = \frac{\sum_{j=1}^n H_{ij}^*}{\frac{n^2}{2}},\ i = 1, 2, \ldots m. \tag{21}$$

Step 9 Rank all the alternatives

The best alternative can be determined according to the values of w_i, $i = 1, 2, \ldots m$.

5.2 Illustrative Example

For an investment company, the investment project selection problem is significant for its long-term development. This problem not only affects the economic benefits of the company, but also its future development. Accordingly, three investment experts are invited to participate in making the investment plan of the company. In accordance with an earlier survey, three investment projects are determined, namely: (1) education and training for the middle school, (2) civil serve examination training, and (3) pre-job training. The problem is selecting the best among the aforementioned projects for the company to invest in. Let A_1, A_2, and A_3 indicate the three investment projects, respectively. To evaluate which project to choose, five main influence factors are considered, namely: (1) economic returns, (2) market capacity, (3) existing resources, (4) investment cost, and (5) future market. After fully considering these factors, the experts conduct project pairwise comparisons by utilizing the PHFPRs. Let H_1, H_2, and H_3 be three PHFPRs provided by the three experts, which are denoted as follows.

To obtain the best investment plan, the decision-making process can be proposed in the following steps.

Step 1 Construct the individual PHFPRs

The individual PHFPRs are constructed and are shown in H_1, H_2, and H_3:

$$H_1 = \begin{bmatrix} \{0.5\} & \{0.1\} & \{0.8\} \\ \{0.9\} & \{0.5\} & \{0.2(0.8), 0.7(0.2)\} \\ \{0.2\} & \{0.8(0.8), 0.3(0.2)\} & \{0.5\} \end{bmatrix},$$

$$H_2 = \begin{bmatrix} \{0.5\} & \{0.4\} & \{0.3(0.5), 0.4(0.5)\} \\ \{0.6\} & \{0.5\} & \{0.5\} \\ \{0.7(0.5), 0.6(0.5)\} & \{0.5\} & \{0.5\} \end{bmatrix},$$

$$H_3 = \begin{bmatrix} \{0.5\} & \{0.6\} & \{0.7\} \\ \{0.4\} & \{0.5\} & \{0.4\} \\ \{0.3\} & \{0.6\} & \{0.5\} \end{bmatrix}.$$

Step 2 Calculate the individual consistent PHFPRs

The individual consistent PHFPRs, namely $H_1^{'}$, $H_2^{'}$, and $H_3^{'}$, can be derived by using Eq. (12) as follows:

$$H_1^{'} = \begin{bmatrix} \{0.5\} & \{0.27(0.2), 0.43(0.8)\} & \{0.47(0.8), 0.61(0.2)\} \\ \{0.73(0.2), 0.57(0.8)\} & \{0.5\} & \left\{ \begin{array}{r} 0.37(0.64), 0.53(0.32) \\ 0.7(0.04) \end{array} \right\} \\ \{0.53(0.8), 0.39(0.2)\} & \left\{ \begin{array}{r} 0.63(0.64), 0.47(0.32) \\ 0.3(0.04) \end{array} \right\} & \{0.5\} \end{bmatrix},$$

$$H_2^{'} = \begin{bmatrix} \{0.5\} & \{0.37(0.5), 0.4(0.5)\} & \left\{ \begin{array}{r} 0.33(0.25), 0.4(0.25) \\ 0.37(0.5) \end{array} \right\} \\ \{0.63(0.5), 0.6(0.5)\} & \{0.5\} & \{0.47(0.5), 0.5(0.5)\} \\ \left\{ \begin{array}{r} 0.67(0.25), 0.6(0.25) \\ 0.63(0.5) \end{array} \right\} & \{0.53(0.5), 0.5(0.5)\} & \{0.5\} \end{bmatrix},$$

$$H_3^{'} = \begin{bmatrix} \{0.5\} & \{0.6\} & \{0.63\} \\ \{0.4\} & \{0.5\} & \{0.47\} \\ \{0.37\} & \{0.53\} & \{0.5\} \end{bmatrix}.$$

Step 3 Calculate the individual consistency index

The individual consistency index can be calculated using Eq. (15) as follows:

$\mathrm{CI}(H_1) = 0.8099$, $\mathrm{CI}(H_2) = 0.9875$, and $\mathrm{CI}(H_3) = 0.9633$.

For $\mathrm{CI}(H_2) = 0.9875 > 0.9118$ and $\mathrm{CI}(H_3) = 0.9633 > 0.9118$, $(n = 3)$, then H_2 and H_3 satisfy the consistency level and do not need to be adjusted.

In addition, $\mathrm{CI}(H_1) = 0.8099 < 0.9118$, H_1 dissatisfies the consistency level and should be further adjusted according to Algorithm 1 $(\theta = 0.5,\ k = 1)$. The acceptable consistent PHFPRs of H_1^* can be obtained as follows:

$$H_1^* = \begin{bmatrix} \{0.5\} & \{0.19(0.2), 0.27(0.8)\} & \{0.64(0.8), 0.7(0.2)\} \\ \{0.81(0.2), 0.73(0.8)\} & \{0.5\} & \left\{ \begin{array}{l} 0.29(0.51), 0.37(0.26), 0.45(0.03) \\ 0.54(0.13), 0.62(0.06), 0.07(0.01) \end{array} \right\} \\ \{0.36(0.8), 0.3(0.2)\} & \left\{ \begin{array}{l} 0.71(0.51), 0.63(0.26), 0.55(0.03) \\ 0.46(0.13), 0.38(0.06), 0.93(0.01) \end{array} \right\} & \{0.5\} \end{bmatrix}.$$

The consistency index of H_1^* is $\mathrm{CI}\left(H_1^*\right) = 0.9722$.

Step 4 Derive the weights of the experts

The weights of the experts can be derived using Eq. (20) as follows:

$\varpi_1 = 0.3$, $\varpi_2 = 0.4$, and $\varpi_3 = 0.3$.

Step 5 Calculate the group PHFPR

The group PHFPR $H_c = \left(h_{ij}^c\left(p_{ij}^c\right)\right)_{n \times n}$ can be derived using Eq. (16) as follows:

$$H_c = \begin{bmatrix} \{0.5\} & \{0.55\} & \{0.45(0.5), 0.49(0.5)\} \\ \{0.45\} & \{0.5\} & \{0.44(0.8), 0.47(0.2)\} \\ \{0.55(0.5), 0.51(0.5)\} & \{0.56(0.8), 0.53(0.2)\} & \{0.5\} \end{bmatrix}.$$

Step 6 Calculate the consensus index

The consensus index related to the three experts can be obtained using Eq. (17) as follows:

$\mathrm{GCI}\left(H_1^c\right) = 0.9318$, $\mathrm{GCI}\left(H_2^c\right) = 0.9393$, and $\mathrm{GCI}\left(H_3^c\right) = 0.9310$.

All the consensus indexes are greater than 0.9; thus, consensus among the three experts can be obtained. Let $H^* = H_c$ and proceed to the next step.

Step 7 Calculate the score matrix
The score matrix $E(H^*)$ is calculated using Eq. (4) as follows:

$$E(H^*) = \begin{bmatrix} \{0.5\} & \{0.55\} & \{0.47\} \\ \{0.45\} & \{0.5\} & \{0.45\} \\ \{0.53\} & \{0.55\} & \{0.5\} \end{bmatrix}.$$

Step 8 Calculate the priority weights of alternatives
The priority weights of alternatives can be calculated using Eq. (21) as follows:
$w_1 = 0.3378$, $w_2 = 0.3111$, and $w_3 = 0.3511$.

Step 9 Rank all the alternatives

$w_3 > w_1 > w_2$; therefore, $A_3 \succ A_1 \succ A_2$, which implies that pre-job training is the best choice among the three investment projects.

5.3 Comparative Study and Discussion

To validate the feasibility of the proposed method, a comparative study was conducted with other methods based on the same illustrative example. In this section, we apply the goal programming model introduced by Zhou and Xu [34] to the illustrative example. The following steps are involved.

Step 1 Construct the goal programming model to derive the individual priority weight vector

For H_1, use Eq. (11) in Zhou and Xu [34]. The goal programming model can be constructed as follows:

$$\min z = d_{12}^{+} + d_{12}^{-} + d_{13}^{+} + d_{13}^{-} + d_{23}^{+} + d_{23}^{-}$$
$$s.t.\begin{cases} 0.1\left(w_1^1 + w_2^1\right) - w_1^1 + d_{12}^{-} - d_{12}^{+} = 0 \\ 0.8\left(w_1^1 + w_3^1\right) - w_1^1 + d_{13}^{-} - d_{13}^{+} = 0 \\ (0.2 \times 0.8 + 0.7 \times 0.2)\left(w_2^1 + w_3^1\right) - w_2^1 + d_{23}^{-} - d_{23}^{+} = 0 \\ w_1^1 + w_2^1 + w_3^1 = 1 \\ w_1^1,\ w_2^1,\ w_3^1 \geq 0 \\ d_{12}^{-},\ d_{13}^{-},\ d_{23}^{-},\ d_{12}^{+},\ d_{13}^{+},\ d_{23}^{+} \geq 0;\ d_{12}^{-} \cdot d_{12}^{+} \\ \quad = 0,\ d_{13}^{-} \cdot d_{13}^{+} = 0,\ d_{23}^{-} \cdot d_{23}^{+} = 0 \end{cases}.$$

By solving this optimization model, we obtain $w_1^1 = 0.03$, $w_2^1 = 0.29$, $w_3^1 = 0.68$, $d_{12}^{+} = 0$, $d_{12}^{-} = 0$, $d_{13}^{+} = 0.54$, $d_{13}^{-} = 0$, $d_{23}^{+} = 0$, and $d_{23}^{-} = 0$.

Similarly, for H_2, we can obtain $w_1^2 = 0.22$, $w_2^2 = 0.39$, $w_3^2 = 0.39$, $d_{12}^{+} = 0.03$, $d_{12}^{-} = 0$, $d_{13}^{+} = 0$, $d_{13}^{-} = 0$, $d_{23}^{+} = 0$, and $d_{23}^{-} = 0$.

For H_3, we can derive $w_1^3 = 0.48$, $w_2^3 = 0.32$, $w_3^3 = 0.2$, $d_{12}^{+} = 0$, $d_{12}^{-} = 0$, $d_{13}^{+} = 0$, $d_{13}^{-} = 0$, $d_{23}^{+} = 0$, and $d_{23}^{-} = 0.11$.

Step 2 Expected consistency checking and improving

The expected consistency for H_1, H_2, and H_3 can be obtained as follows by utilizing Eq. (13) in Zhou and Xu [34]: $\mathrm{ECI}_1 = 0.18$, $\mathrm{ECI}_2 = 0.01$, and $\mathrm{ECI}_3 = 0.04$. $\mathrm{ECI}_2 = 0.01 < 0.02$, and thus, the expected consistency of H_2 is acceptable. Meanwhile, $\mathrm{ECI}_1 = 0.18 > 0.02$ and $\mathrm{ECI}_3 = 0.04 > 0.02$; hence, H_1 and H_3 are unacceptable. The iterative optimization algorithm [34] is used to improve the expected consistency as follows:

For H_1, we obtain $d^{\max} = d_{13}^{+} = 0.54$ by calculating the maximum deviation, and then the adjustment amount $d^{*} = \frac{d_{13}^{+}}{w_1^1 + w_3^1} = 0.76$. The optimal element is $\gamma_{13}^{*} = 0.04$. $\gamma_{13} = 0.8$ is replaced with $\gamma_{13}^{*} = 0.04$, the goal programming model is constructed, and the adjustment optimum solution is obtained as follows:

$w_1^{1*} = 0.03$, $w_2^{1*} = 0.29$, $w_3^{1*} = 0.68$, $d_{12}^{+*} = 0.003$, $d_{12}^{-*} = 0$, $d_{13}^{+*} = 0$, $d_{13}^{-*} = 0$, $d_{23}^{+*} = 0$, and $d_{23}^{-*} = 0$.

$\mathrm{ECI}_1^{*} = 0.001 < 0.02$, and thus, the expected consistency of H_1^{*} is acceptable.

Similarly, the adjustment optimum solution for H_3 can be obtained as follows:

$w_1^{3*} = 0.48$, $w_2^{3*} = 0.32$, $w_3^{3*} = 0.2$, $d_{12}^{+*} = 0$, $d_{12}^{-*} = 0$, $d_{13}^{+*} = 0$, $d_{13}^{-*} = 0$, $d_{23}^{+*} = 0.0007$, and $d_{23}^{-*} = 0$.

$\mathrm{ECI}_3^{*} = 0.0002 < 0.02$, and thus, the expected consistency of H_3^{*} is acceptable.

Step 3 Derive the collective priority weight

Suppose that the weights of the decision makers are $\varpi_1 = 0.3$, $\varpi_2 = 0.4$, and $\varpi_3 = 0.3$. Then, the ranking matrix is obtained as follows:

$$[0.3,\ 0.4,\ 0.3] \times \begin{bmatrix} 0.03, & 0.29, & 0.68 \\ 0.22, & 0.39, & 0.39 \\ 0.48, & 0.32, & 0.20 \end{bmatrix} = [0.241,\ 0.339,\ 0.420].$$

Step 4 Rank all the alternatives

$w_3 > w_2 > w_1$; therefore, $A_3 \succ A_2 \succ A_1$, which implies that pre-job training is the best choice among the three investment projects.

For a better comparison, the calculation results of Zhou and Xu [34] and the proposed method are listed in Table 2.

As shown in Table 2, the ranking result derived from the method of Zhou and Xu [34] differs from that of the proposed method. The ranking result derived from the method of Zhou and Xu [34] is $A_3 \succ A_2 \succ A_1$, whereas that of the proposed method is $A_3 \succ A_1 \succ A_2$. However, the best choice obtained by the method of Zhou and Xu [34] is the same as that of the proposed method, i.e., A_3. The possible reason for the differences between the two methods is as follows. The method of Zhou and Xu [34] requires higher consistency than the proposed method. In the illustrative example, the consistencies of H_1 and H_3 are unacceptable,

Table 2 Ranking results of different methods

Methods	Results	Ranking	Best alternative
Zhou and Xu's method [34]	(0.241, 0.339, 0.420)	$A_3 \succ A_2 \succ A_1$	A_3
Proposed method	(0.3378, 0.3111, 0.3511)	$A_3 \succ A_1 \succ A_2$	A_3

whereas in the proposed method, only H_1 is unacceptable. Furthermore, we provide a possible reason why the best choice is the same between the two methods, i.e., these methods are both based on consistency checking and improvement.

Moreover, to demonstrate the validity of the proposed method, comparative studies with existing methods are provided as follows:

1. Comparison of consensus models. The studies in [7–9] present some current trends in the field of consensus models, that is, the experts behavior is integrated with uninorm aggregation operators to reach consensus in [7], individual concerns and satisfactions are considered comprehensively in [8], and trust degree of experts in social network is determined in [9]. These consensus models fully take into account the expert behavior, preference and trust, and they can effectively solve many MCGDM problems. However, the probabilistic hesitant information receives insufficient attention in these models, and how to establish models based on the PHFPRs to reach consensus deserves further study. Fortunately, the consensus method presented in this study appropriately deals with probabilistic hesitant fuzzy information.
2. Comparison of expected consistency-based methods. The studies in [23, 31, 32, 34] utilize expected consistency to develop PHFPRs, and then PHFPRs are translated into FPRs. Subsequently, the consensus reaching process is implemented based on FPRs. These methods are simple and easy to use, but they cannot reflect the hesitancy of decision makers. Moreover, these methods do not fully take advantage of probabilistic hesitant fuzzy information, and they may cause the loss of information. However, the proposed method employs the expected additive PHFPRs, and it can then avoid the loss of information and better reflect the hesitancy of decision makers.
3. Comparison of the iterative optimization algorithms proposed in [31, 34]. When the expected consistency of PHFPR is unacceptable, iterative optimization algorithms are developed to improve the consistency [31, 34]. Unfortunately, these algorithms only considered the membership degree part of PHFE, whereas the probability part is ignored. These methods do not fully take advantage of probabilistic hesitant fuzzy information, and they may lead to the loss of information. However, the adjusted formulas presented in the proposed algorithms consider not only the membership degree part but also the probability part, and they can then avoid the loss of information.
4. Comparison of the automatic iterative algorithms proposed in [28, 35]. These studies only consider the consistency of preference relations and pay less attention to consensus in decision-making process. Due to different experiences and knowledge background of the decision makers, the provided evaluation values based on the decision makers' preferences may be divergent. Therefore, the decision results obtained without consensus check may be unreasonable and cannot be accepted by most of decision makers. However, Algorithm 2 developed in this study considers not only the acceptable consistency of PHFPRs, but also the consensus in the group decision-making process. Therefore, the final decision results are easy to be accepted by decision makers in the proposed algorithm.
5. Comparison of the Euclidean distance defined to obtain the consistency index in [28]. This distance defined in [28] requires all PHFEs offered by decision makers to have the same number of elements. Therefore, a normalization process is necessary. Unfortunately, the normalization process will generate different priority weight vectors with respect to different PHFEs because the normalization process will obtain various normalization results. However, the Hausdorff distance proposed in this study does not require the normalization. It can calculate distance when two PHFEs have different number of elements and effectively avoid the influence of different additional values on decision results.

Compared with the above methods, the features and advantages of the proposed method in this study can be described as follows.

1. The expected additive consistent PHFPR presented in this study can better reflect the hesitancy of decision makers. And the adjusted formulas in the proposed automatic iterative algorithms consider not only the membership degree, but also the corresponding probability. Moreover, both the expected additive consistent PHFPRs and automatic iterative algorithms presented in this study can avoid the loss of information.
2. The automatic iterative Algorithm 2 developed in this study considers not only the acceptable consistency of PHFPRs, but also the consensus among decision makers. In this way, the final decision results are reasonable and can be accepted by decision makers.
3. The Hausdorff distance developed in this study does not require the normalization, it can effectively avoid the influences from different additional values on various decision results.

6 Conclusion

As an extension of HFE, PHFE is a useful tool for describing uncertain and fuzzy information. This study aims to introduce a new method to address group decision-making problems based on the consensus among decision makers with PHFPRs. First, a theorem is introduced to determine expected additive consistent PHFPRs. Then, the consistency index of PHFPR is defined based on the Hausdorff distance of PHFEs. Next, two automatic iterative algorithms are constructed to identify the individual acceptable consistent PHFPRs and the consensus of PHFPRs. Finally, an illustrative example in conjunction with comparative analyses is employed to demonstrate that our proposed method is feasible for practical MCGDM problems. In addition, the advantages of the proposed method are summarized by further comparing our method with existing methods.

This study makes several significant contributions to MCGDM fields, which are summarized as follows. (1) The expected additive consistent PHFPRs presented in this study can better reflect the hesitancy of decision makers. (2) The adjusted formulas in the proposed automatic iterative algorithms consider not only the membership degree, but also the corresponding probability. (3) The automatic iterative Algorithm 2 developed in this study considers not only the acceptable consistency of PHFPRs, but also the consensus among decision makers. (4) The Hausdorff distance used in this study can effectively avoid the influences from different additional values on the final decision results. In future, we will investigate the linear programming model for developing group decision-making methods with probabilistic hesitant information.

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