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Intuitionistic Fuzzy Quantifier and Its Application in Feature Selection

Shivani Singh[1] · Shivam Shreevastava[2] · Tanmoy Som[2] · Pankhuri Jain[2]

Abstract Nowadays, databases expand rapidly due to electronically generated information from different fields like bioinformatics, census data, social media, business transactions, etc. Hence, feature selection/attribute reduction in databases is necessary in order to reduce time, cost, storage and noise for better accuracy. For this purpose, the rough set theory has been played a very significant role, but this theory is inefficient in case of real-valued data set due to information loss through discretization process. Hybridization of rough set with intuitionistic fuzzy set successfully dealt with this issue, but it may radically change the outcome of the approximations by adding or ignoring a single element. To handle this situation, we reconsider the hybridization process by introducing intuitionistic fuzzy quantifiers into the idea of upper and lower approximations. Supremacy of intuitionistic fuzzy quantifier over VPRS and VQRS is presented with the help of some examples. A novel process for feature selection is given by using the degree of dependency approach with intuitionistic fuzzy quantifier-based lower approximation. A greedy algorithm along with two supportive examples is presented in order to demonstrate the proposed approach. Finally, proposed algorithm is implemented on some benchmark datasets and classification accuracies for different classifiers are compared.

1 Introduction

Data size has increased explosively not only due to increase in samples but also due to attributes in many real-life applications such as text mining, biomedical and bioinformatics. The high-dimensional dataset not only makes the machine learning algorithm very slow but also can corrupt the actual performance due to the presence of redundant, irrelevant and noisy informations. It is necessary to learn how to handle these large size data for knowledge extraction. Feature selection or attribute reduction is one of the dimensionality reduction techniques, which is capable of selecting a small subset of relevant and non-redundant features from the original information systems [1–3].

Rough set theory, proposed by Pawlak, is a key tool to tackle high-dimensional datasets in a much efficient manner [4–7]. However, this method is applicable to discrete data only. Therefore, discretization is required in order to tackle the real-valued information system before feature selection, but this may lead to information loss. Zadeh's fuzzy set concept is very much useful in handling vagueness and uncertainty available in datasets [8]. Dubois and Prade combined fuzzy set with rough set and proposed fuzzy rough set by approximating a set by its lower and upper approximations [9, 10]. Further many researchers

✉ Shivam Shreevastava
  shivam.rs.apm@itbhu.ac.in

  Shivani Singh
  shivanithakur030@gmail.com

  Tanmoy Som
  tsom.apm@itbhu.ac.in

  Pankhuri Jain
  pankhurijain20@gmail.com

[1] DST-Centre for Interdisciplinary Mathematical Sciences, Institute of Science, BHU, Varanasi 221005, India

[2] Department of Mathematical Sciences, IIT (BHU), Varanasi 221005, India

extended this idea for feature selection and classification problems of high-dimensional datasets [11–14].

In variable precision rough set (VPRS) model, Ziarko presented a generalization of the rough set concept by introducing quantifiers at least 100 * u per cent and more than 100 * l per cent to replace the universal and existential quantifiers, respectively [15]. Further, in vaguely quantified rough set (VQRS) model, Cornelis et al. extended the concept of VPRS model by introducing vague quantifiers most (an object x belongs to the lower approximation of $X \subseteq U$, if most of the object related to X are included in X)) and some (an object x belongs to the upper approximation of $X \subseteq U$, if some of the object related to X are included in X) into the model [16]. They used fuzzy quantifiers based on Zadeh's notion [17]. They also discussed vaguely quantified fuzzy rough set (VQFRS) model by using a fuzzy relation in place of crisp relation.

An intuitionistic fuzzy set is an extension of a fuzzy set as it considers positive as well as negative and hesitancy degree of an object to belong to a set [18]. So, it has a strong ability in better describing the vagueness when compared to the traditional fuzzy set theory. Some researchers proposed different kinds of intuitionistic fuzzy rough sets by combining rough set with an intuitionistic fuzzy set [19–24]. The intuitionistic fuzzy rough set theory is evolving as a successful and powerful tool to deal with uncertainty and implemented for decision-making to take care of numerous genuine issues of the real world. However, not much research works have been published for attribute reduction based on intuitionistic fuzzy rough set [25–34]. Intuitionistic fuzzy rough set-based approaches for attribute reduction are much better than fuzzy rough set-based approaches when hesitancy is involved in an information system, but it may drastically change lower and upper approximations on addition or rejection of a single object.

To overcome the above issue, intuitionistic fuzzy quantifier-based feature selection approach is needed. However, some researchers defined intuitionistic fuzzy quantifier, but they did not apply it on the problem of feature selection of high-dimensional datasets. Atanassov [35] defined intuitionistic fuzzy quantifier. Szmidt and Kacprzyk [36] proposed the concept of intuitionistic fuzzy linguistic quantifiers to find the degree of truth of intuitionistic fuzzy linguistic quantified statements. Cui represented an intuitionistic fuzzy linguistic quantifier by a family of intuitionistic fuzzy-valued fuzzy measures and provided a method for calculating the intuitionistic fuzzy truth value of a quantified proposition [37]. Recently, Atanassov proposed the basic concept of intuitionistic fuzzy quantifier and discussed multi-dimensional intuitionistic fuzzy quantifiers along with level operators [38, 39].

We present an intuitionistic fuzzy quantifier-based rough set method for attribute reduction using the degree of dependency approach to overcome above mentioned issues. In this paper, we first define intuitionistic fuzzy quantifiers most and some and compared their effects with already existed VPRS and VQRS through an example. Lower and upper approximations are constructed by using intuitionistic fuzzy quantifiers. A method for attribute reduction in a decision system is presented via the degree of dependency approach. Algorithm and illustrative examples are given for a better understanding of the proposed method.

By using intuitionistic fuzzy linguistic quantifier in the definition of lower approximation, we get extra information about the degree of non-involvement (non-membership grade) of objects in the lower approximation together with involvement degree (membership grade). Proposed model contributes in finding close-to-minimal reduct set and improves classification accuracy of real benchmark datasets. Our approach also works on the fuzzy and intuitionistic fuzzy decision systems, and hence, it is a generalized approach. Moreover, the proposed approach provides better results in terms of classification accuracy for reduced datasets.

The rest of the paper is structured as follows. In this paper, some preliminaries are discussed in Sect. 2. In Sect. 3, intuitionistic fuzzy quantifier and intuitionistic fuzzy quantified rough set (IFQRS) are defined. An approach for feature selection of a decision system is presented by using IFQRS model in Sect. 4. In Sect. 5, an algorithm for the proposed method is presented. Illustrative examples based on proposed model are given for demonstration in Sect. 6. In Sect. 7, we conclude our work.

2 Preliminaries

In this section, an overview of proposed models like VPRS, VQRS and VQFRS is given.

2.1 Variable Precision Rough Sets (VPRS) [15]

Traditional lower and upper approximations of a set A proposed in Pawlak's rough set can be redefined as follows [4]:

$$y \in R \uparrow A \quad \text{iff} \quad R_A(y) > 0 \tag{1}$$

$$y \in R \downarrow A \quad \text{iff} \quad R_A(y) = 1 \tag{2}$$

where $R_A(y) = \frac{|R_y \cap A|}{|R_y|}$ represents the degree of involvement of R_y into A.

However, a small change in the involvement of R_y into A may discard the entire class from the lower approximation.

Example 2.1 Let us consider a collection of reports $R = \{r_1, r_2, \ldots, r_{16}\}$ in any manufacturing company. The reports are characterized into four categories: $R_1 = \{r_1, r_2, r_3, r_4\}$, $R_2 = \{r_5, r_6, r_7, r_8\}$, $R_3 = \{r_9, r_{10}, r_{11}, r_{12}\}$ and $R_4 = \{r_{13}, r_{14}, r_{15}, r_{16}\}$. These grouping form an equivalence relation R in X. If any query was launched which is relevant to the report set $A = \{r_2, r_3, \ldots, r_{10}\}$, then it is clear that information retrieval system has all documents from R_1 are in A except r_1. Moreover, only r_9 and r_{10} are recaptured from R_3, showing the fact that these reports from R_3 are less relevant to the query, than the reports of R_2, which all comprised in A. Pawlak's notion of rough sets does not pay attention to these nuances, since using Eqs. (1) and (2) we get, $R \downarrow A = R_2$ and $R \uparrow A = R_1 \cup R_2 \cup R_3$, putting R_1 and R_3 at the same level.

In many practical situations, classification analysis in datasets is not easy due to the inadequacy of available information. Ziarko [15] dealt with classification error problem by introducing parameters into Eqs. (1) and (2) to get the definitions:

$$y \in R \uparrow_l A \quad \text{iff} \quad R_A(y) > l \tag{3}$$

$$y \in R \downarrow_u A \quad \text{iff} \quad R_A(y) \geq u \tag{4}$$

Using quantifiers, the above-defined formulas (3) and (4) can be interpreted as:

$$y \in R \uparrow_l A \quad \text{iff} \quad \text{at least } 100 * u\% \text{ elements of } R_y \text{ are in } A \tag{5}$$

$$\begin{aligned} &y \in R \downarrow_u A \\ &\text{iff more than } 100 * l\% \text{ elements of } R_y \text{ are in } A \end{aligned} \tag{6}$$

Both above-defined threshold quantifiers $> 100 * l\%$ and $\geq 100 * u\%$ are crisp. Despite the fact that VPRS model permits a measure of tolerance towards errors, but it obeys old-fashioned manner of a binary system. Due to its dependency on particular values of l and u, any element either fully belongs to lower or upper approximations or does not belong at all.

Example 2.2 In the information retrieval problem from Example 2.1, Ziarko's VPRS model proposes a more flexible way to differentiate between the roles of R_1 and R_3 as described in Eqs. (5) and (6), but the choice of the threshold value is very vital. If we choose $u = 0.7$ and $l = 1 - u = 0.3$, in a symmetric VPRS model [15], this provides $R \downarrow_{.7} = R_1 \cup R_2$ and $R \uparrow_{.3} = R_1 \cup R_2 \cup R_3$. However, for $u = 0.8$, it gives the same outcome as in Example 2.1.

2.2 Vaguely Quantified Rough Sets (VQRS) [16]

The definitions of upper and lower approximations in the VPRS model can be relaxed by presenting vague quantifiers, deliberating the fact that y belongs to the lower and upper approximations of A to the extent that *most* and *some* elements of R_y are in A, respectively. It is assumed that approximations used in the approach are fuzzy sets. A fuzzy quantifier is used to model quantifier Q suitably [17]. Q is an increasing mapping from [0, 1] to [0, 1] which fulfils the boundary conditions $Q(0) = 0$ and $Q(1) = 1$.

With the notion of Zadeh's concept of s-number, a fuzzy quantifier [17] that takes values in-between 0 and 1 is given by the parameterized formula, for $0 \leq \alpha < \beta \leq 1$, and x in [0, 1],

$$Q_{(\alpha,\beta)}(x) = \begin{cases} 0, & x \leq \alpha \\ \dfrac{2(x-\alpha)^2}{(\beta-\alpha)^2}, & \alpha \leq x \leq \dfrac{\alpha+\beta}{2} \\ 1 - \dfrac{2(x-\beta)^2}{(\beta-\alpha)^2}, & \dfrac{\alpha+\beta}{2} \leq x \leq \beta \\ 1, & \beta \leq x \end{cases} \tag{7}$$

In this example, quantifiers $Q_{(0.2,0.6)}$ and $Q_{(0.3,1)}$ could be chosen to reveal vague quantifiers *some* and *most* from natural language, respectively.

If Q_l and Q_u are fixed fuzzy quantifiers, the Q_l-upper and Q_u-lower approximations of any set A can be defined as

$$R \uparrow_{Q_l} A(y) = Q_l\left(\frac{|R_y \cap A|}{|R_y|}\right) = Q_l(R_A(y)) \tag{8}$$

$$R \downarrow_{Q_u} A(y) = Q_u\left(\frac{|R_y \cap A|}{|R_y|}\right) = Q_u(R_A(y)) \tag{9}$$

where $Q_l\left(\frac{|R_y \cap A|}{|R_y|}\right)$ and $Q_u\left(\frac{|R_y \cap A|}{|R_y|}\right)$, respectively, quantify the truth values of the statement "Q_l and $Q_u R_y' s$ are also $A's$".

Example 2.3 Continuing Examples 2.1 and 2.2, on applying VQRS model with fuzzy quantifiers $Q_u = Q_{(0.3,1)}$ and $Q_l = Q_{(0.2,0.6)}$, results the lower approximation $R \downarrow_{Q_u} A = \{(r_5, 1), (r_6, 1), \ldots, (r_8, 1), (r_1, 0.74), (r_2, 0.74), \ldots, (r_4, 0.74), (r_9, 0.16), \ldots, (r_{12}, 0.16)\}$. In this list, a report will have a higher grade if most of its elements relevant to query report set A. With this reason, category R_3 appears at the bottom of the list. Therefore, different roles of the categories in the search process can be determined in a desirable way by assigning membership grades through this model. Similarly, upper approximation can be found as $R \downarrow_{Q_u} A = \{(r_1, 1), (r_2, 1), \ldots, (r_8, 1), (r_9, 0.87), \ldots, (r_{12}, 0.87)\}$.

2.3 Vaguely Quantified Fuzzy Rough Sets (VQFRS) [16]

In VQFRS, the Q_l-upper and Q_u-lower approximations of A are same as defined in Eqs. (8) and (9) by using the following fuzzy convention.

If P and Q are two fuzzy sets in X, then

(1) $(P \cap Q)(x) = \min (P(x), Q(x))$,
(2) Cardinality of P, $|P| = \sum_{x \in X} P(x)$
(3) $R_y(x) = R(x, y)$, where $R(x, y)$ is a fuzzy relation.

3 Intuitionistic Fuzzy Quantifier

Intuitionistic fuzzy sets, a generalization of fuzzy sets, are more effective to handle uncertainty as it contains hesitancy degree together with membership and non-membership degree. Combining VQRS model to intuitionistic fuzzy sets, we can better deal with the complexity of vague (uncertain) problems.

Definition 3.1 [18] Let U be a finite non-empty set. A set A on the universe U of the form $A = \{\langle x, \mu_A(x), \vartheta_A(x) \rangle | x \in U\}$ is said to be an intuitionistic fuzzy set (IFS), where $\mu_A : U \to [0, 1]$ and $\vartheta_A : U \to [0, 1]$ satisfy the condition $0 \le \mu_A(x) + \vartheta_A(x) \le 1$ for all x in U. $\mu_A(x)$ and $\vartheta_A(x)$ are the membership degree and non-membership degree of the element x in A, respectively, and $\pi_A(x) = 1 - \mu_A(x) - \vartheta_A(x)$ is the degree of hesitancy (or non-determinacy) of the element x in IFS A.

Any fuzzy set $A = \{\langle x, \mu_A(x) \rangle | x \in U\}$ can be characterized by an IFS having the form $\{\langle x, \mu_A(x), 1 - \mu_A(x) \rangle \ | x \in U\}$. Thus, every fuzzy set is an IFS.

Definition 3.2 [18] An ordered pair $\langle \mu, \vartheta \rangle$ is called an intuitionistic fuzzy value, where $0 \le \mu, \vartheta \le 1$ and $0 \le \mu + \vartheta \le 1$.

Definition 3.3 [40] Let $A = \{\langle x, \mu_A(x), \vartheta_A(x) \rangle | x \in U\}$ be an IFS on the real line R. It is called an intuitionistic fuzzy number (IFN) if

(a) A is normal,
     i.e. there exist at least two points $x_1, x_2 \in X$ such that $\mu_A(x_1) = 1, \vartheta_A(x_2) = 1$,
(b) A is convex,
     i.e. $\forall x_1, x_2 \in R, \forall \lambda \in [0, 1], \mu_A(\lambda x_1 + (1 - \lambda) x_2) \ge \mu_A(x_1) \wedge \mu_A(x_2)$ and $\vartheta_A(\lambda x_1 + (1 - \lambda) x_2) \le \vartheta_A(x_1) \vee \vartheta_A(x_2)$,
(c) μ_A is upper semi-continuous and ϑ_A is lower semi-continuous,
(d) supp $A = \{x \in U | \vartheta_A(x) < 1\}$ is bounded.

Definition 3.4 [41, 42] Let $A = \{\langle x, \mu_A(x), \vartheta_A(x) \rangle | x \in U\}$ be an IFS, its cardinality is given by $|A| = \sum_{x \in A} \frac{1 + \mu_A(x) - \vartheta_A(x)}{2}$.

We generalize the concept of Zadeh's fuzzy quantifiers, which rely on fuzzy cardinality, into the notion of the intuitionistic fuzzy quantifier. We propose two quantifiers in natural language with imprecise meaning such as *most* and *some* in order to find a degree of truth of intuitionistic fuzzy linguistically quantified statements as follows:

$$\mu_Q(x) = Q_{(a_1, a_2)}(x)$$
$$= \begin{cases} 0, & x \le a_1 \\ \dfrac{2(x - a_1)^2}{(a_2 - a_1)^2}, & a_1 \le x \le \dfrac{a_1 + a_2}{2} \\ 1 - \dfrac{2(x - a_2)^2}{(a_2 - a_1)^2}, & \dfrac{a_1 + a_2}{2} \le x \le a_2 \\ 1, & x \ge a_2 \end{cases} \quad (10)$$

and

$$v_Q(x) = Q_{(b_1, b_2)}(x)$$
$$= \begin{cases} 1, & x \le b_1 \\ 1 - \dfrac{2(x - b_1)^2}{(b_2 - b_1)^2}, & b_1 \le x \le \dfrac{b_1 + b_2}{2} \\ \dfrac{2(x - b_2)^2}{(b_2 - b_1)^2}, & \dfrac{b_1 + b_2}{2} \le x \le b_2 \\ 0, & x \ge b_2 \end{cases} \quad (11)$$

where $a_1, a_2, b_1, b_2 \in [0, 1]$ are parameters such that $0 \le b_1 \le a_1 \le b_2 \le a_2 \le 1$. These parametric values are expert dependent/user oriented. $Q = \{\langle x, \mu_A(x), \vartheta_A(x) \rangle | x \in U\}$ is an intuitionistic fuzzy quantifier and $\langle \mu_Q, \vartheta_Q \rangle$ interprets the quantifiers *most* and *some*. For example, $\langle Q_{(0.3,1)}, Q_{(0.2, 0.8)} \rangle$ and $\langle Q_{(0.2, 0.6)}, Q_{(0.1, 0.55)} \rangle$ could be used to express the intuitionistic quantifiers *most* and *some* from natural language.

These mathematical formulae can be interpreted as follows for the quantifier *most*: if less than $a_1 * 100\%$ of some elements $x \in X$ satisfy a property P and less than $b_1 * 100\%$ do not satisfy it, then we say *most* of them certainly do not satisfy it (satisfy to degree $\langle 0, 1 \rangle$). If at least $a_2 * 100\%$ of them satisfy it and also at least $b_2 * 100\%$ do not satisfy it, then *most* of them certainly satisfy it (to degree $\langle 1, 0 \rangle$). For rest of the cases, we will get some values in-between $\langle 0, 1 \rangle$ and $\langle 1, 0 \rangle$, which means the more of them satisfy it, the higher the degree of satisfaction by *most* of the elements. Similarly, the quantifier *some* can be interpreted. The geometrical interpretation of the intuitionistic fuzzy quantifier is illustrated in Fig. 1.

Lemma 3.1

$$0 \le \mu_Q(x) + v_Q(x) \le 1.$$
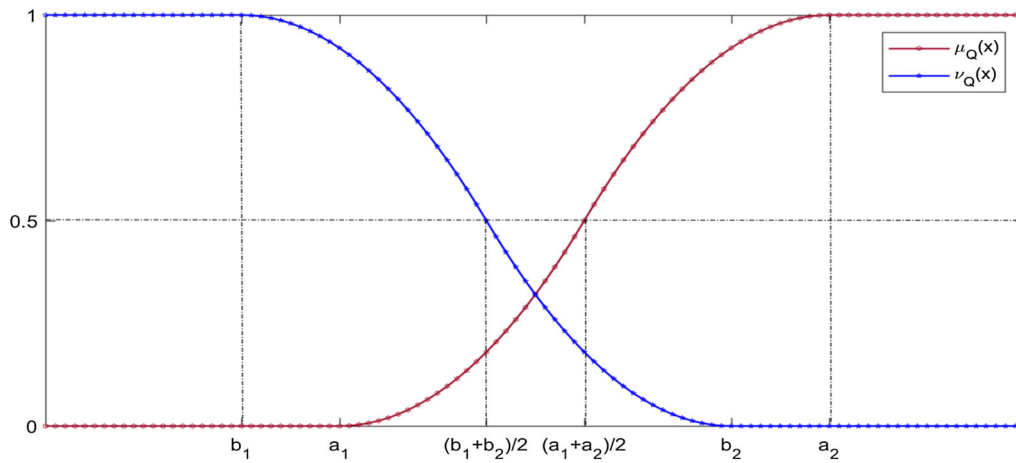
Proof From Eqs. (10) and (11),

Fig. 1 Intuitionistic fuzzy quantifiers

$\mu_Q(x) + v_Q(x)$

$$= \begin{cases} 1, & x \le b_1, x \ge a_2 \\ 1 - \dfrac{2(x-b_1)^2}{(b_2-b_1)^2}, & b_1 \le x \le a_1 \\ 1 + \dfrac{2(x-a_1)^2}{(a_2-a_1)^2} - \dfrac{2(x-b_1)^2}{(b_2-b_1)^2}, & a_1 \le x \le \dfrac{b_1+b_2}{2} \\ \dfrac{2(x-a_1)^2}{(a_2-a_1)^2} + \dfrac{2(x-b_2)^2}{(b_2-b_1)^2}, & \dfrac{b_1+b_2}{2} \le x \le \dfrac{a_1+a_2}{2} \\ 1 + \dfrac{2(x-b_2)^2}{(b_2-b_1)^2} - \dfrac{2(x-a_2)^2}{(a_2-a_1)^2}, & \dfrac{a_1+a_2}{2} \le x \le b_2 \\ 1 - \dfrac{2(x-a_2)^2}{(a_2-a_1)^2}, & b_2 \le x \le a_2 \end{cases}$$

It is clear from the above expression and Fig. 1 that $0 \le \mu_Q(x) + v_Q(x) \le 1$. □

Lemma 3.2 *An Intuitionistic fuzzy quantifier is an intuitionistic fuzzy number.*

Proof For the intuitionistic fuzzy quantifier defined in Eqs. (10) and (11),

(1) *Normality* Since $\mu_Q(x) = 1, \forall x \ge a_2$ and $v_Q(x) = 1, \forall x \le b_1$, Q is normal.

(2) *Convexity* Let $x_1, x_2 \in \mathbb{R}, \lambda \in [0,1]$.
 Since μ_Q is an increasing function. It implies $\mu_Q(x_1) \ge \mu_Q(x_2)$ when $x_1 \ge x_2$, and v_Q is a decreasing function. It implies $v_Q(x_1) \le v_Q(x_2)$ when $x_1 \ge x_2$

 Case 1 If $x_1 \ge x_2 \Rightarrow \lambda x_1 + (1-\lambda)x_2 \ge x_2 \Rightarrow \mu_Q(\lambda x_1 + (1-\lambda)x_2) \ge \mu_Q(x_2)$ and $\mu_Q(x_1) \ge \mu_Q(x_2) \Rightarrow \mu_Q(x_1) \wedge \mu_Q(x_2) = \mu_Q(x_2)$ hence, $\mu_Q(\lambda x_1$

$+(1-\lambda)x_2) \ge \mu_Q(x_1) \wedge \mu_Q(x_2)$. Similarly, $x_1 \ge x_2 \Rightarrow \lambda x_1 + (1-\lambda)x_2 \ge x_2 \Rightarrow v_Q(\lambda x_1 + (1-\lambda)x_2) \le v_Q(x_2)$ and $v_Q(x_1) \ge v_Q(x_2) \Rightarrow v_Q(x_1) \vee v_Q(x_2) = v_Q(x_2)$ hence, $v_Q(\lambda x_1 + (1-\lambda)x_2) \ge v_Q(x_1) \vee v_Q(x_2)$.
 Case 2 If $x_2 \ge x_1 \Rightarrow \lambda x_1 + (1-\lambda)x_2 \ge x_1$. With the similar process, $\mu_Q(\lambda x_1 + (1-\lambda)x_2) \ge \mu_Q(x_1) \wedge \mu_Q(x_2)$ and $v_Q(\lambda x_1 + (1-\lambda)x_2) \ge v_Q(x_1) \vee v_Q(x_2)$.

(3) *Semi-continuity*
 Since μ_Q and v_Q are continuous, μ_Q is upper semi-continuous and v_Q is lower semi-continuous also.

(4) *Boundedness*
 supp $(Q) = \{x \in X | v_Q(x) < 1\} = [b_1, b_2] \subseteq [0,1]$ is bounded. □

The definition of lower and upper approximations in VQFRS model can be more generalized by presenting intuitionistic fuzzy quantifiers, expressing the fact that y belongs to the lower approximations of A to the extent that *most* $= \langle Q_l, Q_{l'} \rangle$ elements of R_y are in A and y belongs to the upper approximations of A to the extent that *some* $=$ elements of R_y are in A.

If we fix couples $\langle Q_l, Q_{l'} \rangle, \langle Q_u, Q_{u'} \rangle$ of intuitionistic fuzzy quantifiers, the $Q_{(l,l')}$-upper and $Q_{(u,u')}$-lower approximations of an approximation set A in X are defined as

$$R \uparrow_{Q_{(l,l')}} A(y) = \left\langle Q_l \left(\frac{|R_y \cap A|}{|R_y|} \right), Q_{l'} \left(\frac{|R_y \cap A|}{|R_y|} \right) \right\rangle$$
$$= \langle Q_l(R_A(y)), Q_{l'}(R_A(y)) \rangle \tag{12}$$

$$R \downarrow_{Q_{(u,u')}} A(y) = \left\langle Q_u \left(\frac{|R_y \cap A|}{|R_y|} \right), Q_{u'} \left(\frac{|R_y \cap A|}{|R_y|} \right) \right\rangle$$
$$= \left\langle Q_u(R_A(y)), Q_{u'}(R_A(y)) \right\rangle$$

$$(13)$$

Here, Q_l, Q_u are determined by Eq. (10) and $Q_{l'}, Q_{u'}$ are determined by Eq. (11). l, u and l', u' are parameters in tuple form (a_i, a_j) and (b_i, b_j), respectively. Pair $\left(R \downarrow_{Q_{(u,u')}} A(y), R \uparrow_{Q_{(l,l')}} A(y) \right)$ is called intuitionistic fuzzy quantified rough set.

Example 3.1 In the continuation of Example 2.1, we have applied the proposed IFQRS model on the above information retrieval problem. We have chosen intuitionistic fuzzy quantifiers $\langle Q_u, Q_{u'} \rangle$ and $\langle Q_l, Q_{l'} \rangle$ with $Q_u = Q_{(0.3,1)}, Q_{u'} = Q_{(0.2,0.8)}, Q_l = Q_{(0.2,0.6)}$ and $Q_{l'} = Q_{(0.1,0.55)}$. We get, the lower approximation $R \downarrow_{Q_u} A = \{(r_5, 1, 0), (r_6, 1, 0), \ldots, (r_8, 1, 0), (r_1, 0.74, 0.01), (r_2, 0.74, 0.01), \ldots, (r_4, 0.74, 0.01), (r_9, 0.16, 0.5), \ldots, (r_{12}, 0.16, 0.5)\}$. Here, we are assigning membership grade together with non-membership grade to reports in each category so that we can get extra information to choose any category relevant to the query by choosing desirable parameters. We can see, the reports which are highly related to the query report set have high membership grade and low non-membership grade. On the similar manner, we get upper approximation $R \downarrow_{Q_u} A = \{(r_1, 1, 0), (r_2, 1, 0), \ldots, (r_8, 1, 0), (r_9, 0.87, 0.05), \ldots, (r_{12}, 0.87, 0.05)\}$.

Now, we extend the results on lower and upper approximations similar to Dubois and Prade [10] for our approach.

Let $\langle Q_l, Q_{l'} \rangle, \langle Q_u, Q_{u'} \rangle$ be intuitionistic fuzzy quantifiers with $Q_l, Q_{l'}, Q_u, Q_{u'} \in [0, 1]$

Theorem 3.1 If $Q_u \subseteq Q_l$ and $Q_{l'} \subseteq Q_{u'}$, i.e. $Q_u(x) \leq Q_l(x)$ and $Q_{l'}(x) \leq Q_{u'}(x), \forall x \in [0, 1]$ $\quad R \downarrow_{Q_{(u,u')}} A(y) \subseteq R \uparrow_{Q_{(l,l')}} A(y)$

Proof

If $Q_u(x) \leq Q_l(x)$ and $Q_{l'}(x) \leq Q_{u'}(x), \forall x \in [0, 1]$

Replacing x by $R_A(y)$,

$$Q_u(R_A(y)) \leq Q_l(R_A(y)) \tag{14}$$

$$Q_{l'}(R_A(y)) \leq Q_{u'}(R_A(y)) \tag{15}$$

Using Eqs. (14) and (15), we get

$$\left\langle Q_u(R_A(y)), Q_{u'}(R_A(y)) \right\rangle \leq \left\langle Q_l(R_A(y)), Q_{l'}(R_A(y)) \right\rangle$$

Hence, $R \downarrow_{Q_{(u,u')}} A(y) \subseteq R \uparrow_{Q_{(l,l')}} A(y)$. □

Theorem 3.2 If $A_1 \subseteq A_2 \subseteq X$,

$$R \downarrow_{Q_{(u,u')}} A_1(y) \subseteq R \downarrow_{Q_{(u,u')}} A_2(y)$$

$$R \uparrow_{Q_{(l,l')}} A_1(y) \supseteq R \uparrow_{Q_{(l,l')}} A_2(y)$$

Proof Since $A_1 \subseteq A_2$

$$\therefore R_y \cap A_1 \subseteq R_y \cap A_2 \Rightarrow |R_y \cap A_1| \leq |R_y \cap A_2|$$
$$\Rightarrow \frac{|R_y \cap A_1|}{|R_y|} \leq \frac{|R_y \cap A_2|}{|R_y|} \Rightarrow R_{A_1}(y) \leq R_{A_2}(y)$$

Since Q_u and Q_l are increasing functions,

$$Q_u(R_{A_1}(y)) \leq Q_u(R_{A_2}(y)) \tag{16}$$

$$Q_l(R_{A_1}(y)) \leq Q_l(R_{A_2}(y)) \tag{17}$$

and $Q_{u'}$ and $Q_{l'}$ are decreasing functions,

$$Q_{u'}(R_{A_1}(y)) \leq Q_{u'}(R_{A_2}(y)) \tag{18}$$

$$Q_{l'}(R_{A_1}(y)) \leq Q_{l'}(R_{A_2}(y)) \tag{19}$$

By Eqs. 16–19, we get the required result. □

4 Feature Selection by IFQRS

Feature selection has become center of the attention of much research in application areas for which datasets with tens or hundreds of thousands of features are available. Feature selection for the information systems with the help of the proposed model is done in this section. Here, the degree of dependency method for the feature selection is used.

Definition 4.1 [43] A quadruple $IS = (U, AT, V, h)$ is called an information system, where $U = \{u_1, u_2, \ldots, u_n\}$ is a non-empty finite set of objects, called the universe of discourse, $AT = \{a_1, a_2, \ldots, a_m\}$ is a non-empty finite set of attributes. $V = \bigcup_{a \in AT} V_a$, where V_a is the set of attribute values associated with each attribute $a \in AT$ and $h : U \times AT \to V$ is an information function that assigns particular values to the objects against attribute set such that $\forall a \in AT, \forall u \in U, h(u, a) \in V_a$.

An information system is called an intuitionistic fuzzy information system if attribute values associated with objects are intuitionistic fuzzy numbers.

Definition 4.2 [43] An information system $IS = (U, AT, V, h)$ is said to be decision system if $AT = C \cup D$ where C is a non-empty finite set of conditional attributes and D is a non-empty collection of decision attributes with $C \cap D = \emptyset$. Here, $V = V_C \cup V_D$ with V_C and V_D as the set of conditional attribute values and decision attribute

Table 1 Real-valued decision system

Objects	Attributes			
	c_1	c_2	c_3	d
u_1	− 0.4	− 0.3	− 0.5	No
u_2	− 0.4	0.2	− 0.1	Yes
u_3	− 0.3	− 0.4	− 0.3	No
u_4	0.3	− 0.3	0	Yes
u_4	0.2	− 0.3	0	Yes
u_6	0.2	0	0	No

values, respectively. h be a mapping from $U \times C \cup D$ to V, such that $h : U \times C \to V_C$ and $h : U \times D \to V_D$. For example, Table 1 represents real-valued decision system, while Table 2 represents intuitionistic fuzzy decision system.

Different similarity relations for different type of information systems are available in the literature. We consider following fuzzy similarity relation between two objects of the information system with respect to a conditional attribute $c \in C$ as given in [12].

$$
\begin{aligned}
& S_c(u_i, u_j) \\
& = \max\left(\min\left(\frac{(\mu(u_j) - (\mu(u_i) - \sigma_c))}{(\mu(u_i) - (\mu(u_i) - \sigma_c))}, \frac{((\mu(u_i) + \sigma_c) - \mu(u_j))}{((\mu(u_i) + \sigma_c) - \mu(u_i))}\right), 0\right)
\end{aligned}
\tag{20}
$$

where σ_c^2 is the variance of feature c and $\mu(u_i)$ is membership grade of object u_i.

And following intuitionistic fuzzy similarity relation as in [41] is used

$$
\begin{aligned}
& S_c(u_i, u_j) \\
& = 1 - \sqrt{(\mu(u_i) - \mu(u_j))^2 + (v(u_i) - v(u_j))^2 + (\pi(u_i) - \pi(u_j))^2}
\end{aligned}
\tag{21}
$$

where u_i and u_j are objects. $\mu(u_i), v(u_i)$ and $\pi(u_i)$ are membership, non-membership and hesitancy degree of u_i, respectively.

For partitioning the universe of discourse, we define R-foreset of the object u_i, i.e. R_{u_i} with respect to attribute $c \in C$ as:

$$
R_{u_i} = [T_c^\alpha](u_i) = \{u_j : s_c(u_i, u_j) \geq \alpha\}
\tag{22}
$$

where $\alpha \in (0, 1)$ is a similarity threshold, which provides a degree of similarity for addition of objects within tolerance classes.

With the help of lower approximation defined in terms of quantifiers, the positive region of decision attribute D over set of conditional attributes C can be calculated as:

$$
\begin{aligned}
\text{POS}_C(D) &= \sup_{A \in U/D} R \downarrow_{Q_{(u,u')}} A(u_i) \\
&= \left\langle \sup_{A \in U/D} Q_u(R_A(u_i)), \inf_{A \in U/D} Q_{u'}(R_A(u_i)) \right\rangle
\end{aligned}
\tag{23}
$$

where $R \downarrow_{Q_{(u,u')}} A(u_i)$ is defined in Eq. (13) and $U \backslash D$ contain sets of objects having same decision values.

After calculating positive region, the degree of dependency of decision attribute D over set of conditional attributes C can be computed as:

$$
\Gamma_C(D) = \frac{|\text{POS}_C(D)|}{|U|}
\tag{24}
$$

where $|.|$ in the numerator is the cardinality of an intuitionistic fuzzy set, while $|.|$ in the denominator is the cardinality of a crisp set and $\Gamma_C(D) \in [0, 1]$.

Definition 4.3 [11] A subset R of the conditional attribute set C is said to be reduct set of a decision system if

$$
\begin{aligned}
& \Gamma_R(D) = \Gamma_C(D) \\
& \Gamma_{R - \{c_i\}}(D) < \Gamma_R(D), \forall c_i \in C
\end{aligned}
\tag{25}
$$

Required reduct set can be obtained by comparing the degree of dependencies of decision attribute over sets of conditional attributes. Attributes are selected one by one until the reduct set gives the same degree of dependency value as the original set.

5 Algorithm for *Intuitionistic Fuzzy Rough Set Approach*

In this section, a quick reduct algorithm for feature selection of a decision system based on intuitionistic fuzzy quantifier approach is presented by using degree of dependency method. Initially, the reduct set is an empty set. We add conditional attributes one by one in reduct set and calculate degree of dependencies of decision attribute over obtained reduct set. The proposed algorithm selects only those conditional attributes, which causes a maximum increase in the degree of dependency of decision attribute. The proposed algorithm is given as follows:

Step 1 Take an intuitionistic fuzzy information system/fuzzy information system.
Step 2 Calculate similarity relations between objects with respect to set of conditional attributes.
Step 3 On the basis of similarity relations, tolerance class of each object with respect to conditional attributes can be determined by introducing a similarity threshold between object.
Step 4 Compute the lower approximations of each object with respect to $U \backslash D$ by using quantifier $Q_{(u,u')}$.

Step 5 Evaluate the degree of dependency of decision attribute over set of conditional attributes by calculating positive region.

Step 6 Select the conditional attribute that has the highest degree of dependency (Select any one if there are more than one conditional attributes having the highest and same degree of dependencies). That will be the first member of reduct set.

Step 7 Add other conditional attributes to obtained conditional attribute and again calculate the degree of dependencies of such coupled conditional attributes.

Step 8 Apply the same process for so obtained attributes as in Step 6 and 7.

Step 9 The process will terminate until the newly obtained subset of conditional attributes causes no increment in the degree of dependency than the previous one.

In the proposed algorithm, for a dataset with dimension n, we require n evaluations in calculating degrees of dependency of decision attribute over n conditional attributes. After selecting the conditional attribute with the highest degree of dependency, we need $(n-1)$ evaluations and repeating this process one more time, we need $(n-2)$ evaluations and so on. Hence, time complexity of proposed approach is $n + (n-1) + (n-2) + \cdots + 3 + 2 + 1$ for the worst-case dataset. Therefore, the proposed intuitionistic fuzzy quantified rough set-based algorithm requires $(n^2 + n)/2$ evaluations of the dependency function.

6 Experimental Analysis

In this section, proposed algorithm is implemented on two different types of example datasets and on eight real datasets for feature selection and classification.

Example 6.1 Consider a decision system as given in Table 1, which contains real-valued conditional attributes $\{c_1, c_2, c_3\}$ and a decision attribute $\{d\}$[11].

The similarity degree between objects with respect to each conditional attributes is calculated by using Eq. (20) and presented in Table 3.

Here, $D = \{d\}$, $U \backslash D = \{A_1, A_2\}$, $A_1 = \{u_1, u_3, u_6\}$, $A_2 = \{u_2, u_4, u_5\}$

Taking $\alpha = 0.7$, R-foreset of each object corresponding to attribute c_1 is calculated by Eq. (22),

$$R_{u_1} = \{u_1, u_2\}, \quad R_{u_2} = \{u_1, u_2\}, \quad R_{u_3} = \{u_3\},$$
$$R_{u_4} = \{u_4\}, \quad R_{u_5} = \{u_5, u_6\}, \quad R_{u_6} = \{u_5, u_6\}.$$

By using Eq. (13), intuitionistic fuzzy lower approximations of objects over $U \backslash D$ are given as:

$$R \downarrow_{Q_{(u,u')}} A_1(u_1) = \langle 0.16, 0.5 \rangle, \quad R \downarrow_{Q_{(u,u')}} A_1(u_2)$$
$$= \langle 0.16, 0.5 \rangle, \quad R \downarrow_{Q_{(u,u')}} A_1(u_3) = \langle 1, 0 \rangle$$
$$R \downarrow_{Q_{(u,u')}} A_1(u_4) = \langle 0, 1 \rangle, \quad R \downarrow_{Q_{(u,u')}} A_1(u_5)$$
$$= \langle 0.16, 0.5 \rangle, \quad R \downarrow_{Q_{(u,u')}} A_1(u_6) = \langle 0.16, 0.5 \rangle$$
$$R \downarrow_{Q_{(u,u')}} A_2(u_1) = \langle 0.16, 0.5 \rangle, \quad R \downarrow_{Q_{(u,u')}} A_2(u_2)$$
$$= \langle 0.16, 0.5 \rangle, \quad R \downarrow_{Q_{(u,u')}} A_2(u_3) = \langle 0, 1 \rangle$$
$$R \downarrow_{Q_{(u,u')}} A_2(u_4) = \langle 1, 0 \rangle, \quad R \downarrow_{Q_{(u,u')}} A_2(u_5)$$
$$= \langle 0.16, 0.5 \rangle, \quad R \downarrow_{Q_{(u,u')}} A_2(u_6) = \langle 0.16, 0.5 \rangle$$

After calculating fuzzy lower approximation, the positive region of decision attribute over conditional attributes computed by Eq. (23) results

$$\text{POS}_{c_1}(D)(u_1) = \langle 0.16, 0.5 \rangle, \quad \text{POS}_{c_1}(D)(u_2) = \langle 0.16, 0.5 \rangle,$$
$$\text{POS}_{c_1}(D)(u_3) = \langle 1, 0 \rangle, \text{POS}_{c_1}(D)(u_4) = \langle 1, 0 \rangle,$$
$$\text{POS}_{c_1}(D)(u_5) = \langle 0.16, 0.5 \rangle, \quad \text{POS}_{c_1}(D)(u_6) = \langle 0.16, 0.5 \rangle.$$

Now, the degree of dependency of decision attribute D over conditional attribute c_1 can be calculated by Eq. (24) as follows:

$$\Gamma_{c_1}(D) = 0.554$$

In a similar manner, $\Gamma_{c_2}(D) = 0.854, \Gamma_{c_3}(D) = 0.854$.

The degrees of dependency of c_2 and c_3 are equal and highest. Any one of them can be taken as the first candidate of reduct set of given information system, say c_2. Now, combining c_1 and c_3 to c_2 one by one and calculating the corresponding degrees of dependency, we get

$$\Gamma_{\{c_1, c_2\}}(d) = 1, \ \Gamma_{\{c_2, c_3\}}(d) = 1$$

Since degree of dependency cannot exceed 1, the reduct set of this dataset is either $\{c_1, c_2\}$ or $\{c_2, c_3\}$.

Example 6.2 An intuitionistic fuzzy decision system is chosen which is a judgment problem of CISA's information systems security audit risk [44]. In this dataset, the object set $U = \{u_1, u_2, \ldots, u_{10}\}$ comprises 10 audited objects. Five conditional attributes $C = \{c_1, c_2, c_3, c_4, c_5\}$ are there in conditional attribute set, where c_1 = "Better Systems Total Security", c_2 = "Better Systems Operation Security", c_3 = "Safer Data Centre", c_4 = "Credible Hardware Device", and c_5 = "Credible Network Security". Every value which condition attribute is taken on has special actual meaning. For example,$f(x_1, c_1) = \langle 0.2, 0.4 \rangle$ means that the membership degree of systems total security is 0.2, and non-membership degree of systems total security is 0.4. The decision attribute set, d = "Risk Judgement Order of Information Systems Security Audit". The

Table 2 An information systems security risk judgment decision table

Objects	Attributes					
	C_1	C_2	C_3	C_4	C_5	d
U_1	$\langle 0.2, 0.4 \rangle$	$\langle 0.1, 0.7 \rangle$	$\langle 0.2, 0.6 \rangle$	$\langle 0.6, 0.4 \rangle$	$\langle 0.2, 0.8 \rangle$	1
U_2	$\langle 0.1, 0.7 \rangle$	$\langle 0.1, 0.8 \rangle$	$\langle 0.3, 0.6 \rangle$	$\langle 0.5, 0.2 \rangle$	$\langle 0.2, 0.7 \rangle$	2
U_3	$\langle 0.1, 0.8 \rangle$	$\langle 0.1, 0.8 \rangle$	$\langle 0.2, 0.8 \rangle$	$\langle 0.5, 0.4 \rangle$	$\langle 0.6, 0.4 \rangle$	1
U_4	$\langle 0.1, 0.9 \rangle$	$\langle 0.6, 0.3 \rangle$	$\langle 0.2, 0.7 \rangle$	$\langle 0.2, 0.8 \rangle$	$\langle 0.6, 0.4 \rangle$	1
U_5	$\langle 0.4, 0.6 \rangle$	$\langle 0.2, 0.6 \rangle$	$\langle 0.2, 0.8 \rangle$	$\langle 0.2, 0.8 \rangle$	$\langle 0.2, 0.8 \rangle$	2
U_6	$\langle 0.1, 0.6 \rangle$	$\langle 0.2, 0.6 \rangle$	$\langle 0.2, 0.8 \rangle$	$\langle 0.2, 0.4 \rangle$	$\langle 0.2, 0.8 \rangle$	1
U_7	$\langle 0.6, 0.4 \rangle$	$\langle 0.6, 0.4 \rangle$	$\langle 0.6, 0.4 \rangle$	$\langle 0.7, 0.3 \rangle$	$\langle 0.4, 0.6 \rangle$	2
U_8	$\langle 0.6, 0.2 \rangle$	$\langle 0.6, 0.2 \rangle$	$\langle 0.8, 0.2 \rangle$	$\langle 0.4, 0.6 \rangle$	$\langle 0.4, 0.5 \rangle$	2
U_9	$\langle 0.6, 0.2 \rangle$	$\langle 0.6, 0.4 \rangle$	$\langle 0.8, 0.2 \rangle$	$\langle 0.1, 0.6 \rangle$	$\langle 0.8, 0.2 \rangle$	3
U_{10}	$\langle 0.6, 0.4 \rangle$	$\langle 0.6, 0.4 \rangle$	$\langle 0.8, 0.2 \rangle$	$\langle 0.8, 0.2 \rangle$	$\langle 0.6, 0.4 \rangle$	3

Table 3 Similarity relation between objects

$\mu_{R_{c_1}}(u_i, u_j)$						$\mu_{R_{c_2}}(u_i, u_j)$						$\mu_{R_{c_3}}(u_i, u_j)$					
1.0	1.0	0.699	0.0	0.0	0.0	1.0	0.0	0.568	1.0	1.0	0.0	1.0	0.0	0.036	0.0	0.0	0.0
1.0	1.0	0.699	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.137	0.0	1.0	0.036	0.518	0.518	0.518
0.699	0.699	1.0	0.0	0.0	0.0	0.568	0.0	1.0	0.568	0.568	0.0	0.036	0.036	1.0	0.0	0.0	0.0
0.0	0.0	0.0	1.0	0.699	0.699	1.0	0.0	0.568	1.0	1.0	0.0	0.0	0.518	0.0	1.0	1.0	1.0
0.0	0.0	0.0	0.699	1.0	1.0	1.0	0.0	0.568	1.0	1.0	0.0	0.0	0.518	0.0	1.0	1.0	1.0
0.0	0.0	0.0	0.699	1.0	1.0	0.0	0.137	0.0	0.0	0.0	1.0	0.0	0.518	0.0	1.0	1.0	1.0

Table 4 Fuzzy similarity relation $S_{c_1}(u_i, u_j)$ with respect to c_1

Objects	U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8	U_9	U_{10}
U_1	**1**	0.63	0.49	0.35	0.51	0.75	0.43	0.51	0.51	0.43
U_2	0.63	**1**	**0.86**	0.72	0.63	**0.86**	0.38	0.29	0.29	0.38
U_3	0.49	**0.86**	**1**	**0.86**	0.63	0.72	0.35	0.21	0.21	0.35
U_4	0.35	0.72	**0.86**	**1**	0.58	0.58	0.29	0.12	0.12	0.29
U_5	0.51	0.63	0.63	0.58	**1**	0.58	0.72	0.51	0.51	0.72
U_6	0.75	**0.86**	0.72	0.58	0.58	**1**	0.38	0.35	0.35	0.38
U_7	0.43	0.38	0.35	0.29	0.72	0.38	**1**	0.72	0.72	0
U_8	0.51	0.29	0.21	0.12	0.51	0.35	0.72	**1**	0	0.72
U_9	0.51	0.29	0.21	0.12	0.51	0.35	0.72	0	**1**	0.72
U_{10}	0.43	0.38	0.35	0.29	0.72	0.38	0	0.72	0.72	**1**

Bold values signify the similarity values between objects greater than threshold alpha

domain of d is $\{1, 2, 3\}$, where 1 means "Complete Examination", 2 means "Major Examination", and 3 means "No Examination".

Here, $D = \{d\}, U \backslash \ D = \{A_1, A_2, A_3\}, A_1 = \{u_1, u_3, u_4, u_6\}, A_2 = \{u_2, u_5, u_7, u_8\}, A_3 = \{u_9, u_{10}\}$

Calculating fuzzy similarity relation between two intuitionistic fuzzy numbers with respect to attribute $c_1 \in C$ by using Eq. (21), we get Table 4.

Taking $\alpha = 0.8$, R-foreset of each object with respect to attribute c_1 is given as:

$R_{u_1} = \{u_1\}, \quad R_{u_2} = \{u_2, u_3, u_6\}, \quad R_{u_3} = \{u_2, u_3, u_4\},$
$R_{u_4} = \{u_3, u_4\}, \quad R_{u_5} = \{u_5\},$
$R_{u_6} = \{u_2, u_6\}, \quad R_{u_7} = \{u_7\}, \quad R_{u_8} = \{u_8\}, R_{u_9} = \{u_9\},$
$R_{u_{10}} = \{u_{10}\}$

Table 5 Reduct sets for different parametric values

Datasets	Objects	Features	Reduct set		
			(0.3, 0.1), (0.2, 0.8)	(0.2, 0.9), (0.1, 0.7)	(0.3, 0.9), (0.1, 0.8)
Column_2C	310	7	5	5	5
Ionosphere	351	34	7	5	5
Iris	150	5	2	3	2
Glass	214	10	7	6	6
Hepatitis	155	20	2	2	2
Lung_Cancer	32	57	5	5	5
Soybean_small	47	22	2	2	2
Zoo	101	17	5	4	6

Table 6 Classification accuracies (in percentage) for parametric value (0.3, 0.1), (0.2, 0.8)

Datasets	Classifiers															
	Bayes Net		Naïve Bayes		IBk		JRip		PART		J48		LMT		Random tree	
	Unr.	Red.	Unr.	Red.	Unr.	Red.	Unr.	Red.	Unr.	Red.	Unr.	Red.	Unr.	Red.	Unr.	Red.
Column_2C	**76.45**	75.80	77.74	**77.74**	81.61	**82.25**	82.25	**82.90**	80.96	**81.61**	81.61	**82.25**	85.48	**85.48**	79.03	**81.61**
Ionosphere	89.45	**90.88**	82.62	**88.03**	86.32	**89.74**	89.17	**92.87**	91.73	**92.02**	91.45	**92.59**	92.87	**93.16**	86.60	**90.31**
Iris	92.66	**94.66**	96	**96**	95.33	**95.33**	95.33	92.66	94	**95.33**	96	**96**	94	93.33	92	**94**
Glass	**70.56**	65	**48.59**	47.19	**70.56**	68.69	**68.69**	67.18	**68.62**	65.42	**66.82**	64.48	68.69	**71.96**	**70.09**	67.71
Hepatitis	82.58	**84.51**	83.87	**84.51**	80.64	**85.16**	78.70	**84.51**	**86.45**	84.51	80.64	**84.51**	83.22	**84.51**	76.12	**84.51**
Lung_Cancer	43.75	**50**	65.62	**71.87**	50	**78.12**	50	**71.87**	37.5	**81.25**	40.62	**78.12**	59.37	**71.87**	28.12	**68.75**
Soybean_small	100	**100**	97.87	**100**	100	**100**	97.87	**100**	100	**100**	97.87	**100**	97.87	**100**	100	**100**
Zoo	94.05	**95.04**	**95.04**	90.09	**96.03**	95.04	88.11	**90.09**	92.07	**95.04**	92.07	**94.05**	95.04	**96.03**	**96.03**	95.04

Bold values signifies the larger value of classification accuracy of datasets for particular classifiers

Table 7 Classification accuracies (in percentage) for parametric value (0.2, 0.9), (0.1, 0.7)

Datasets	Classifiers															
	Bayes Net		Naïve Bayes		IBk		JRip		PART		J48		LMT		Random tree	
	Unr.	Red.	Unr.	Red.	Unr.	Red.	Unr.	Red.	Unr.	Red.	Unr.	Red.	Unr.	Red.	Unr.	Red.
Column_2C	**76.45**	75.80	77.74	**77.74**	81.61	**82.25**	82.25	**82.90**	80.96	**81.61**	81.61	**82.25**	85.48	**85.48**	79.03	**81.61**
Ionosphere	89.45	**90.59**	82.62	**90.31**	86.32	**90.88**	89.17	**90.88**	91.73	**92.59**	91.45	**92.30**	92.87	**92.30**	86.60	**90.02**
Iris	92.66	**93.33**	96	**96**	95.33	**95.33**	95.33	92.66	94	**95.33**	96	**96**	94	93.33	92	**94**
Glass	70.56	**71.96**	48.59	**48.59**	70.56	**76.63**	68.69	64.48	68.62	**74.29**	66.82	**69.15**	68.69	**68.69**	70.09	**71.02**
Hepatitis	82.58	**84.51**	83.87	**84.51**	80.64	**85.16**	78.70	**84.51**	**86.45**	84.51	80.64	**84.51**	83.22	**84.51**	76.12	**84.51**
Lung_Cancer	43.75	**50**	65.62	**71.87**	50	**78.12**	50	**71.87**	37.5	**81.25**	40.62	**78.12**	59.37	**71.87**	28.12	**68.75**
Soybean_small	100	**100**	97.87	**100**	100	**100**	97.87	**100**	100	**100**	97.87	**100**	97.87	**100**	100	**100**
Zoo	94.05	**94.05**	**95.04**	89.10	96.03	**98.01**	88.11	**91.06**	92.07	**96.03**	92.07	**94.05**	95.04	**97.02**	96.03	**98.01**

Bold values signifies the larger value of classification accuracy of datasets for particular classifiers

Now, proceeding in a similar manner as in Example 6.1, degree of dependencies of decision attribute D over conditional attributes are given as:

$$\Gamma_{c_1}(D) = 0.875, \quad \Gamma_{c_2}(D) = 0.469, \quad \Gamma_{c_3}(D) = 0.846,$$
$$\Gamma_{c_4}(D) = 0.808, \quad \Gamma_{c_5}(D) = 0.799$$

The degree of dependency in case of c_1 is highest; therefore, we add other attributes c_2, c_3 and c_4 to c_1. After

Table 8 Classification accuracies (in percentage) for parametric value (0.3, 0.9), (0.1, 0.8)

Datasets	Classifiers															
	Bayes Net		Naïve Bayes		IBk		JRip		PART		J48		LMT		Random tree	
	Unr.	Red.	Unr.	Red.	Unr.	Red.	Unr.	Red.	Unr.	Red.	Unr.	Red.	Unr.	Red.	Unr.	Red.
Column_2C	**76.45**	75.80	77.74	**77.74**	81.61	**82.25**	82.25	**82.90**	80.96	**81.61**	81.61	**82.25**	85.48	**85.48**	79.03	**81.61**
Ionosphere	89.45	**90.59**	82.62	**90.31**	86.32	**90.88**	89.17	**90.88**	91.73	**92.59**	91.45	**92.30**	**92.87**	92.30	86.60	**90.02**
Iris	92.66	**93.33**	96	**96**	95.33	**95.33**	**95.33**	92.66	94	**95.33**	96	**96**	**94**	93.33	92	**94**
Glass	70.56	**71.96**	48.59	**48.59**	70.56	**76.63**	**68.69**	64.48	68.62	**74.29**	66.82	**69.15**	68.69	**68.69**	70.09	**71.02**
Hepatitis	82.58	**84.51**	83.87	**84.51**	80.64	**85.16**	78.70	**84.51**	**86.45**	84.51	80.64	**84.51**	83.22	**84.51**	76.12	**84.51**
Lung_Cancer	43.75	**50**	65.62	**71.87**	50	**78.12**	50	**71.87**	37.5	**81.25**	40.62	**78.12**	59.37	**71.87**	28.12	**68.75**
Soybean_small	100	**100**	97.87	**100**	100	**100**	97.87	**100**	100	**100**	97.87	**100**	97.87	**100**	100	**100**
Zoo	**94.05**	88.11	**95.04**	88.11	**96.03**	90.09	88.11	**91.08**	**92.07**	88.11	**92.07**	88.11	**95.04**	87.12	**96.03**	91.08

Bold values signifies the larger value of classification accuracy of datasets for particular classifiers

applying the same process as above, we calculate degree of dependencies of decision attribute with respect to each pair of conditional attributes, we get

$$\Gamma_{\{c_1,c_2\}}(D) = 1, \quad \Gamma_{\{c_1,c_3\}}(D) = 0.86, \quad \Gamma_{\{c_1,c_4\}}(D) = 1,$$
$$\Gamma_{\{c_1,c_5\}}(D) = 0.86$$

Therefore, reduct set of above intuitionistic fuzzy information system is $\{\mathbf{c_1}, \mathbf{c_2}\}$ or $\{\mathbf{c_1}, \mathbf{c_4}\}$.

We have examined the proposed approach for feature selection on two example datasets. In Example 6.1, we have taken a decision system having three real-valued conditional attributes and one decision attribute. R-foreset of each object based on fuzzy similarity relation are calculated. Using quantifier-based lower approximation formula, positive region has been computed and it has been found that the degrees of dependency for attribute set $\{c_1, c_2\}$ and $\{c_2, c_3\}$ are highest amongst $\{c_1, c_2, c_3\}$. Therefore, $\{c_1, c_2\}$ or $\{c_2, c_3\}$ is the reduct set of the given decision system. After proceeding with the similar process, we have dealt with an intuitionistic fuzzy decision system in Example 6.2 and got $\{c_1, c_2\}$ or $\{c_1, c_4\}$ as the reduct set.

Example 6.3 In this example, proposed algorithm is applied on real datasets and classification accuracies are evaluated.

6.1 Experimental Setup

In this section, we apply proposed algorithm on eight benchmark datasets: Column_2C, Ionosphere, Iris, Glass, Hepatitis, Lung cancer, Soybean_small, Zoo [45]. First, we calculate reduct of these datasets by intuitionistic fuzzy quantified rough set approach. Reducts of different datasets are calculated by assigning different values to the parameters $(a_1, a_2), (b_1, b_2)$, and then, classification analysis is

performed for unreduced as well as reduced datasets. For classification part, eight different classifiers are used, namely JRip, PART, J48, Random Tree, IBK, Bayes Net, KSTAR, LMT. Classification accuracies for original and reduced datasets are calculated with these classifiers. Effects of the parameters on classification accuracies are also shown. All the experiments are performed in MATLAB 2017a.

6.2 Experimental Results

In Table 5, the reduct size of different datasets for different parametric values is compared. Table 6 shows the average classification accuracy for eight classifiers in the form of percentage obtained using 10-fold cross-validation. Initially, the classification process was performed on the unreduced (original) datasets and then on the reduced datasets obtained using the proposed technique. It can be observed that classification accuracies are either improving or remaining the same for most datasets. Effect of parametric values on classification accuracy is presented in Tables 6, 7 and 8. For dataset "Glass", classification accuracies are mostly decreasing for the parametric value (0.3, 1), (0.2, 0.8) and mostly increasing for the parametric value (0.2, 0.9), (0.1, 0.7) or (0.3, 0.9), (0.1, 0.8). For dataset Zoo, parametric value (0.2, 0.9), (0.1, 0.7) classification accuracy increases for all classifiers except Naïve Bayes and decreases for parametric value (0.3, 0.9), (0.1 ,0.8) for all classifiers except JRip. A slight change in parametric value can differ in accuracy a lot. So assigning values to the parameters plays an important role and can be chosen carefully for particular data domain.

7 Conclusion

In this paper, we have proposed a new approach for attribute reduction in an information system based on intuitionistic fuzzy quantified rough set. We have defined intuitionistic fuzzy quantifiers *most* and *some* and associated them with intuitionistic fuzzy rough set. Some results on lower and upper approximations are proven for justification of proposed rough set. Degree of dependency concept has been used for attribute reduction in a decision system with our model. Stepwise algorithm has been presented with two examples for better understanding of proposed work. Proposed algorithm has been implemented on some real benchmark datasets, and classification accuracies for different classifiers have been evaluated. It has been found that reduced datasets outperform unreduced datasets in terms of resulting classification performance.

In future, we intend to use intuitionistic fuzzy quantified rough set for attribute reduction in high-dimensional datasets by using discernibility matrix approach. We wish to define new intuitionistic fuzzy linguistic quantifier based on relational logic, and we will combine this with rough set to propose a new robust approach for attribute reduction in real datasets.

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Shivani Singh received her B.Sc. in 2013 and M.Sc. in Mathematics in 2015 from PPN PG College, Kanpur, UP. She is currently pursuing her Ph.D. in DST-Centre for Interdisciplinary Mathematical Sciences, Institute of Science, Banaras Hindu University, Varanasi, India. Her research area focuses on "Fuzzy Rough Set-based techniques for knowledge discovery process".

Shivam Shreevastava received his B.Sc. (Hons) from I.Sc., Banaras Hindu University, Varanasi, India, in 2011 and M.Sc. in Mathematics from Indian Institute of Technology, Kanpur, in 2013. He has completed his Ph.D. in the Department of Mathematical Sciences, Indian Institute of Technology (BHU), Varanasi, India, in 2018. His research area focuses on "Applications of Fuzzy Set Theory in Decision Making Problems".

Tanmoy Som is serving as Professor in the Department of Mathematical Sciences, Indian Institute of Technology (Banaras Hindu University), Varanasi, India, since last 10 years and has more than 30 years of teaching experience. He has done Ph.D. from Institute of Technology, Banaras Hindu University, Varanasi, India, in 1986. His main research areas are "Fixed Point Theory, Fuzzy and Soft Set Theory, Image Processing and Mathematical Modelling". Around 115 research publications are there to his credit in reputed journals/edited book chapters of National and International repute. He has delivered several invited talks at National and International conferences/workshops/refresher courses, which includes talks at Texas A&M University, Kingsville, USA, Naresuan University, Thailand, and University of California, Berkeley. Ten scholars have completed Ph.D. under his supervision. He is also Reviewer/Editorial Board member/Guest Editor of reputed journals. He has organized quite a few national/international level conferences/workshops.

Pankhuri Jain received her B.Sc. (Hons) Mathematics in 2015 and M.C.A in 2018 from Institute of Science, Banaras Hindu University, Varanasi, India. She is currently pursuing her Ph.D. in the Department of Mathematical Sciences, Indian Institute of Technology (BHU), Varanasi, India. Her research area focuses on "Machine Learning".