Int. J. Fuzzy Syst. (2019) 21(2):421–432
https://doi.org/10.1007/s40815-018-00599-2

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Weighted Interval-Valued Hesitant Fuzzy Sets and Its Application in Group Decision Making

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Received: 28 May 2018 / Revised: 16 November 2018 / Accepted: 13 December 2018 / Published online: 23 January 2019

Abstract The interval-valued hesitant fuzzy set, which allows decision makers to use several interval numbers to assess a variable, is a useful tool to deal with situations in which people are hesitant in providing their interval-valued assessments. In this paper, we introduce the concept of weighted interval-valued hesitant fuzzy set, in which different weights are designed to these possible membership degrees, and the weights indicate that the decision maker has different confidence in giving every possible assessment of the membership degree. Then we define some basic operations such as union, intersection, complement, multiplication and power operation of weighted interval-valued hesitant fuzzy sets and weighted interval-valued hesitant fuzzy elements, discuss their operation properties, and propose the score function of the weighted interval-valued hesitant fuzzy element to compare two weighted hesitant fuzzy elements. Furthermore, we introduce the concept of hesitance degree of weighted interval-valued hesitant fuzzy element, present four aggregation operators such as the weighted interval-valued hesitant fuzzy-weighted averaging operator, the weighted interval-valued hesitant fuzzy-weighted geometric operator, the generalized weighted interval-valued hesitant fuzzy-weighted averaging operator and the generalized weighted interval-valued hesitant fuzzy-weighted geometric operator to aggregate weighted interval-valued hesitant fuzzy information, and build the mathematical model of multi-criteria group decision making based on the expert weights (known and unknown). Finally, a numerical example is given to illustrate the effectiveness and feasibility of our proposed method.

Keywords Hesitant fuzzy sets · Interval-valued hesitant fuzzy sets · Weighted interval-valued hesitant fuzzy sets · Aggregation operator · Group decision making

1 Introduction

The theory of fuzzy set introduced by Zadeh [50] has achieved a great success in various field. Recently, hesitant fuzzy set (HFS) proposed by Torra [33, 34] is a generalization of traditional fuzzy set and has attracted a lot of researchers' interest. For example, Bedregal et al. [1] studied the aggregation operators for the class of hesitant fuzzy elements, Xia and Xu [40, 41] developed a series of aggregation operators for hesitant fuzzy information, and further discussed the correlations among the aggregation operators. Wei [38] investigated hesitant fuzzy-prioritized operators, Zhang and Wei [52] investigated the extension of VIseKriterijumska Optimizacija I Kompromisno Resenje (VIKOR) method based on hesitant fuzzy set, Zhang [53] investigated hesitant fuzzy power aggregation operators, Zhu et al. [54] investigated hesitant fuzzy geometric Bonferroni means, Yu et al. [49] investigated generalized hesitant fuzzy Bonferroni mean, Liao et al. [15] investigated the consistency and consensus of hesitant

This work is supported by Grants from the National Natural Science Foundation of China (10971243, 61472043).

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fuzzy preference relation and applied them in group decision making. Onar et al. [22] and Xu et al. [46] utilized hesitant fuzzy Technique for Order Preference by Similarity to an Ideal Solution (TOPSIS) method to select the best strategy in strategic decisions. Li et al. [12, 13] introduced the concept of hesitance degree of hesitant fuzzy element (HFE) which describes the decision maker's hesitance in decision-making process. Zeng et al. [51] proposed similarity measures between hesitant fuzzy sets based on the hesitance degree of hesitant fuzzy element and applied in pattern recognition. On the other hand, many researchers have paid attention on developing the extension of hesitant fuzzy set. Qian et al. [27] presented the generalized hesitant fuzzy set, Rodríguez et al. [29, 30] investigated hesitant fuzzy linguistic term sets for decision making. Wei et al. [37] introduced some aggregation operators for hesitant fuzzy linguistic term sets and applied them in multi-criteria decision making. Zhu and Xu [55] investigated the hesitant fuzzy linguistic preference relations. Lee and Chen [11] investigated the comparison method between hesitant fuzzy linguistic term sets. Liao et al. [16] presented a family of distance and similarity measures between two hesitant fuzzy linguistic term sets. Farhadinia [7] extended HFS to the higher-order hesitant fuzzy set (HOHFS). In addition, some researchers investigated the distance measures of hesitant fuzzy sets and investigated their related topics from different points of view [3, 6, 17, 21, 25, 31, 44, 45, 47].

In some practical decision-making problems, the precise membership degrees of an element to a set are sometimes hard to be specified because there exist too much complexity and uncertainty. To overcome the barrier, decision makers prefer to make an estimation across a range. Hence, many uncertainty formats are utilized to describe the uncertain information such as interval number [14, 24], fuzzy number [4, 32], linguistic value [9, 18, 20], and so on. Considering that using interval number to express the decision maker's judgment is much more intuitionistic, thus multiple attribute decision making (MCDM) under fuzzy environment is an important topic, and has received much attention from the scholars [24, 35]. One main characteristic of the aforementioned interval-valued approaches is that the decision maker usually provides his/her assessment by giving an interval number. However, when an expert is hesitant among several interval numbers, it is not easy for him/her to provide a single interval number as his/her evaluation. In order to model this situation, Chen et al. [2] and Wei et al. [39] investigated interval-valued hesitant fuzzy set (IVHFS) for multi-criteria decision making (MCDM), respectively, in which decision makers can provide their assessments with several interval values. Farhadinia [5] investigated some information measures for hesitant fuzzy sets and interval-valued hesitant fuzzy sets. Quirós et al. [28] proposed an entropy measure for finite interval-valued hesitant fuzzy sets. Wang et al. [36] extended linguistic term sets to interval-valued hesitant fuzzy linguistic sets. Peng et al. [26] investigated the continuous interval-valued hesitant fuzzy aggregation operators and applied in multi-criteria decision making. Xiong et al. [42] developed a series of generalized interval-valued hesitant fuzzy power geometric operators and applied in group decision making. Ju et al. [10] investigated interval-valued dual hesitant fuzzy aggregation operators and applied in multiple attribute decision making. Gitinavard et al. [8] investigated interval-valued hesitant fuzzy positive and negative ideal solutions based on group decision making and applied it to determine the weight of each decision maker or expert in the group decision-making process. Liu et al. [19] proposed some improved interval-valued hesitant fuzzy Hamacher aggregation operators to aggregate interval-valued hesitant fuzzy information, and applied in multiattribute group decision making. Ye [48] investigated interval-valued hesitant fuzzy-prioritized weighted aggregation operators for multiple attribute decision making.

Generally speaking, interval-valued hesitant fuzzy set (IVHFS) permits each of the possible interval values to be distributed with the same weight. However, when the decision maker hesitates among several interval numbers, he/she usually has different confidence in determining each interval number as the assessment. Namely, the possibilities of the possible interval numbers determined as the final assessments are different, or the weights of the possible interval numbers are different. For example, assume that ten experts are invited to anonymously evaluate the performance of two suppliers A and B according to some given criteria. Suppose the experts provide their assessments with interval numbers within [0, 1]. Each assessment that can indicate the satisfaction of the expert to the suppliers is provided to the decision maker. For supplier A, suppose eight experts give assessment [0, 8, 0.9], and two experts give the assessment [0.5, 0.6]. For supplier B, eight experts give assessment [0.5, 0.6], and two experts give the assessment [0.8, 0.9]. Assume the experts cannot persuade each other, therefore, according to the common approach of HFS provided by Chen et al. [2], the synthesized assessment for supplier A is [0.5, 0.6] or [0.8, 0.9], namely, the decision maker would hesitate between [0.5, 0.6] and [0.8, 0.9]. Similarly, for supplier B, the decision maker would also hesitate between [0.5, 0.6] and [0.8, 0.9]. By utilizing interval-valued hesitant fuzzy sets (IVHFSs), both the assessments of A and B are $\tilde{h} = \{[0.5, 0.6], [0.8, 0.9]\}$. However, for supplier A, [0.5, 0.6] is provided by two experts, [0.8, 0.9] is provided by eight experts, but for supplier B, eight experts provide the assessment of

[0.5, 0.6], and only two experts provide the assessment of [0.8, 0.9]. Apparently, IVHFSs cannot interpret this information. If we give the same evaluation $\tilde{h} = \{[0.5, 0.6], [0.8, 0.9]\}$ to A and B simultaneously, it is obviously an unreasonable result.

To overcome the above drawbacks, in this paper, we introduce the concept of weighted interval-valued hesitant fuzzy set (WIVHFS) which is the generalizations of interval-valued hesitant fuzzy set and the weights indicate that the decision maker has different confidence in giving every possible assessment of the membership degree, define some basic operations such as union, intersection, complement, multiplication and power operation of weighted interval-valued hesitant fuzzy sets and weighted interval-valued hesitant fuzzy elements, investigate their properties, and propose the score function of the weighted interval-valued hesitant fuzzy element to compare two weighted hesitant fuzzy elements. Furthermore, we introduce the concept of hesitance degree of weighted interval-valued hesitant fuzzy element, present four aggregation operators such as the weighted interval-valued hesitant fuzzy-weighted averaging (WIVHFWA) operator, the weighted interval-valued hesitant fuzzy-weighted geometric (WIVHFWG) operator, the generalized weighted interval-valued hesitant fuzzy-weighted averaging (GWIVHFWA) operator and the generalized weighted interval-valued hesitant fuzzy-weighted geometric (GWIVHFWG) operator to aggregate weighted interval-valued hesitant fuzzy information, and develop the mathematical model of multi-criteria group decision making based on the expert weights (known and unknown). Finally, a numerical example is given to illustrate the effectiveness and feasibility of our proposed method.

The organization of our work is as follows. In Sect. 2, we review some basic notions of hesitant fuzzy set and interval-valued hesitant fuzzy set, and some basic operations of hesitant fuzzy elements and interval numbers. In Sect. 3, we introduce the concept of weighted interval-valued hesitant fuzzy set (WIVHFS), define some operations of weighted interval-valued hesitant fuzzy elements (WIVHFEs), investigate their properties, and propose the score function of the weighted interval-valued hesitant fuzzy element to compare two weighted hesitant fuzzy elements. Meanwhile, we introduce the concept of hesitance degree of weighted interval-valued hesitant fuzzy element and propose the score function of the weighted interval-valued hesitant fuzzy element to compare two weighted hesitant fuzzy elements. In Sect. 4, we develop four aggregation operators such as weighted interval-valued hesitant fuzzy-weighted averaging (WIVHFWA) operator, weighted interval-valued hesitant fuzzy-weighted geometric (WIVHFWG) operator, generalized weighted interval-valued hesitant fuzzy-weighted averaging (GWIVHFWA) operator and generalized weighted interval-valued hesitant fuzzy-weighted geometric (GWIVHFWG) operator to aggregate weighted interval-valued hesitant fuzzy information, and propose a new group decision-making model based on weighted interval-valued hesitant fuzzy sets. In Sect. 5, the application of the air-conditioning system selection is provided to illustrate the effectiveness and applicability of our proposed method. The conclusion is given in the last section.

2 Preliminaries

Throughout this paper, we use $X = \{x_1, x_2, \ldots, x_n\}$ to denote the discourse set, HFS and HFE stand for hesitant fuzzy set and hesitant fuzzy element, respectively, IVHFS and WIVHFS stand for interval-valued hesitant fuzzy set and weighted interval-valued hesitant fuzzy set, respectively, IVHFE and WIVHFE stand for interval-valued hesitant fuzzy element and weighted interval-valued hesitant fuzzy element, respectively, $\tilde{A}$ and $\tilde{h}$ stand for an IVHFS and an IVHFE, respectively, $\tilde{A}^W$ and $\tilde{h}^w$ stand for a WIVHFS and a WIVHFE, respectively.

Definition 1 [33] Given a fixed set X, then a hesitant fuzzy set (HFS) on X is in terms of a function that when applied to X returns a subset of [0, 1].

For convenience, the HFS is often expressed simply by mathematical symbol in Xia and Xu [40].

$$E = (\langle x, h_E(x)\rangle | x \in X)$$

where $h_E(x)$ is a set of some values in [0, 1], denoting the possible membership degree of the element $x \in X$ to the set E. $h(x) = h_E(x)$ is called a hesitant fuzzy element (HFE).

After then, Torra [33], Xia and Xu [40] presented some operational laws on the hesitant fuzzy elements (HFEs), respectively.

In the process of decision making, when some criteria are qualitative, it is more suitable to evaluate them with interval values. The interval numbers enhance the flexibility and applicability of the decision-making models in dealing with qualitative information.

Definition 2 [43] Let $a = [a^-, a^+]$ and $b = [b^-, b^+]$ be two interval numbers, and $l(a) = a^+ - a^-$, $l(b) = b^+ - b^-$, then the possibility degree of $a \geq b$ is defined as follows:

$$P(a \geq b) = \max\left\{1 - \max\left(\frac{b^+ - a^-}{l(a) + l(b)}, 0\right), 0\right\} \quad (1)$$

Eq. (1) is used to compare two interval numbers. If $P(a \geq b) > 0.5$, then a is superior to b, denoted by $a > b$; if $P(a \geq b) = 0.5$, then a is equivalent to b, denoted by $a = b$.

Theorem 1 [43] *Let* $a = [a^-, a^+]$, $b = [b^-, b^+]$, *then we have*

(1) $0 \leq P(a \geq b) \leq 1$;
(2) $P(a \geq b) + P(b \geq a) = 1$.
Especially, $P(a \geq a) = 0.5$;
(3) $P(a \geq b) = 1$ *if and only if* $a^- \geq b^+$;
(4) $P(a \geq b) = 0$ *if and only if* $a^+ \leq b^-$;
(5) $P(a \geq b) \geq 0.5$ *if and only if* $a^- + a^+ \geq b^- + b^+$. *Especially*, $P(a \geq b) = 0.5$ *if and only if* $a^- + a^+ = b^- + b^+$.
(6) Let $a = [a^-, a^+]$, $b = [b^-, b^+]$ *and* $c = [c^-, c^+]$ *be three interval numbers, if* $P(a \geq b) \geq 0.5$ *and* $P(b \geq c) \geq 0.5$, *then we have* $P(a \geq c) \geq 0.5$.

Suppose that there are n interval numbers $a_1, a_2, \ldots, a_n$ to be ranked. Compare each a_i with all $a_j (j = 1, 2, \ldots, n)$ by utilizing Eq. (1) and let $p_{ij} = P(a_i \geq a_j)$, then we can establish a pairwise comparison matrix $P = (p_{ij})_{n \times n}$ which is also called the possibility degree matrix. Furthermore, we can establish a Boolean matrix $Q = (q_{ij})_{n \times n}$ by utilizing P, where

$$q_{ij} = \begin{cases} 1, & p_{ij} \geq 0.5 \\ 0. & p_{ij} < 0.5 \end{cases}$$

Q is also called ranking matrix of interval numbers $a_i (i = 1, 2, \ldots, n)$.

Let $\lambda_i = \sum_{j=1}^{n} q_{ij}$, we obtain the ranking vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. Then we can rank the interval numbers $a_1, a_2, \ldots, a_n$ in descending order in accordance with the value of λ_i.

Theorem 2 *Let* $a_1, a_2, \ldots, a_n$ *be* n *interval numbers. Suppose* $P = (p_{ij})_{n \times n}$ *and* $Q = (q_{ij})_{n \times n}$ *be the corresponding possibility degree matrix and the Boolean matrix of* $a_1, a_2, \ldots, a_n$, *respectively. Let* $\lambda_i = \sum_{j=1}^{n} q_{ij}$, *then* $\lambda_i \geq \lambda_j \Leftrightarrow p_{ij} \geq 0.5$.

Proof We first prove the sufficiency.

Sufficiency Here, we use the proof by contradiction. Suppose that the result does not hold, i.e., $\lambda_i \geq \lambda_j$ but $p_{ij} < 0.5$. Then, known by property (2) of Theorem 1, we have $p_{ji} \geq 0.5$. Thus, for any $k = 1, 2, \ldots, n$, if $p_{ik} \geq 0.5$, then known by property (6) of Theorem 1, we have $p_{jk} \geq 0.5$. It implies that $\lambda_i < \lambda_j$. It is contradictory.

Necessity Since $p_{ij} \geq 0.5$, then known by property (6) of Theorem 1, for any $k = 1, 2, \ldots, n$, if $p_{jk} \geq 0.5$, we have $p_{ik} \geq 0.5$. It implies that $\lambda_i \geq \lambda_j$.

Hence, we complete the proof of Theorem 2. □

Example 1 Suppose three interval numbers a_1, a_2, a_3, where $a_1 = [0, 0.08]$, $a_2 = [0.048, 0.09]$, and $a_3 = [0.01, 0.13]$.

Step 1 Calculate the possibility degrees based on Eq. (1) and establish the possibility degree matrix $P = (p_{ij})_{3 \times 3}$;

$$P = \begin{pmatrix} 0.5 & 0.2623 & 0.35 \\ 0.7377 & 0.5 & 0.4938 \\ 0.65 & 0.5062 & 0.5 \end{pmatrix}$$

Step 2 Establish the Boolean matrix $Q = (q_{ij})_{3 \times 3}$;

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Step 3 Calculate the ranking vector $\lambda = (1, 2, 3)$;
Step 4 Ranking the interval numbers $a_i (i = 1, 2, 3)$ in descending order in accordance with the value of $\lambda_i (i = 1, 2, 3)$, then we have $a_3 > a_2 > a_1$. The ranking result is consistent with the meaning of the possibility degree.

Remark 1 Theorem 2 verifies that our proposed method for ranking a set of interval numbers is scientific.

Definition 3 [2] Given a fixed set X, let $D[0, 1]$ be the set of all closed subintervals of [0, 1]. An interval-valued hesitant fuzzy set (IVHFS) on X is

$$\tilde{A} = \{(\langle x, h_{\tilde{A}}(x) \rangle | x \in X)$$

where $\tilde{h}_{\tilde{A}}(x) : X \to D[0, 1]$ denotes all possible interval-valued membership degrees of the element $x \in X$ to the set $\tilde{A}$. $\tilde{h}(x) = \tilde{h}_{\tilde{A}}(x) = \{\gamma | \gamma \in \tilde{h}_{\tilde{A}}(x)\}$ is called an interval-valued hesitant fuzzy element (IVHFE). Here $\gamma = [\gamma^-, \gamma^+]$ is an interval number.

Definition 4 [2] Let $\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_n$ be a collection of interval-valued hesitant fuzzy elements (IVHFEs), the interval-valued hesitant fuzzy-weighted averaging (IVHFWA) operator is defined as follows:

$$\text{IVHFWA}\ (\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_n) = \bigoplus_{j=1}^{n} (\omega_j \tilde{h}_j) = \bigcup_{\gamma_1 \in \tilde{h}_1, \gamma_2 \in \tilde{h}_2, \ldots, \gamma_n \in \tilde{h}_n} \left\{ 1 - \prod_{j=1}^{n} (1 - \gamma_j)^{\omega_j} \right\} \quad (2)$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ is the weight vector of $\tilde{h}_j (j = 1, 2, \ldots, n)$, and $\omega_j \geq 0, \sum_{j=1}^{n} \omega_j = 1$.

Definition 5 [2] Let $\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_n$ be a collection of IVHFEs, the interval-valued hesitant fuzzy-weighted geometric(IVHFWG) operator is defined as follows:

$$\text{IVHFWG}\ (\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_n) = \bigotimes_{j=1}^{n} (\tilde{h}_j)^{\omega_j} = \bigcup_{\gamma_1 \in \tilde{h}_1, \gamma_2 \in \tilde{h}_2, \ldots, \gamma_n \in \tilde{h}_n} \left\{ \prod_{j=1}^{n} \gamma_j^{\omega_j} \right\} \quad (3)$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ is the weight vector of $\tilde{h}_j (j = 1, 2, \ldots, n)$, and $\omega_j \geq 0, \sum_{j=1}^{n} \omega_j = 1$.

Definition 6 [2] For a given IVHFE $\tilde{h}(x)$,

$$s(\tilde{h}(x)) = \frac{1}{l(\tilde{h}(x))} \sum_{\gamma \in \tilde{h}(x)} \gamma \tag{4}$$

is called the score function of $\tilde{h}(x)$, where $l(\tilde{h}(x))$ is the number of the elements in $\tilde{h}(x)$. For two IVHFEs $\tilde{h}_1(x)$ and $\tilde{h}_2(x)$, if $s(\tilde{h}_1(x)) > s(\tilde{h}_2(x))$, then $\tilde{h}_1(x) > \tilde{h}_2(x)$; if $s(\tilde{h}_1(x)) = s(\tilde{h}_2(x))$, then $\tilde{h}_1(x) = \tilde{h}_2(x)$.

Li et al. [12, 13] introduced the concept of hesitance degree of hesitant fuzzy element as follows:

Definition 7 [12, 13] Let h be a hesitant fuzzy set on $X = \{x_1, x_2, \ldots, x_n\}$, and for any $x_i \in X$, $l(h(x_i))$ be the number of the elements in $h(x_i)$. Denote

$$u(h(x_i)) = 1 - \frac{1}{l(h(x_i))}$$

$$u(h) = \frac{1}{n} \sum_{i=1}^{n} u(h(x_i))$$

We call $u(h(x_i))$ the hesitance degree of $h(x_i)$, and $u(h)$ the hesitance degree of h, respectively.

For any hesitant fuzzy element $h(x)$, the value of $u(h(x))$ reflects the degree of hesitance for a decision maker when he/she determines the membership value for $h(x)$. The larger the value is, the more hesitant the decision maker will be. For example, if $l(h(x)) = 1$, then $u(h(x)) = 0$, it implies that the decision maker can determine the precise value of the membership confidently. Namely, there is no any hesitancy for decision maker to determine the value of membership. However, if $l(h(x))$ intends to infinite, then $u(h(x)) = 1$, it indicates that the decision maker is hesitant completely and can hardly determine the value of membership.

3 Weighted Interval-Valued Hesitant Fuzzy Set

Definition 8 Let $X = \{x_1, x_2, \ldots, x_n\}$ be a fixed set, then a weighted interval-valued hesitant fuzzy set (WIVHFS) on X is

$$\tilde{A}^W = \{\langle x, \tilde{h}_A^w(x)\rangle | x \in X\}$$

where $\tilde{h}_A^w(x) = \{\langle \gamma_1, w_{\gamma_1}\rangle, \langle \gamma_2, w_{\gamma_2}\rangle, \ldots, \langle \gamma_m, w_{\gamma_m}\rangle\}$, and $\gamma_j = [\gamma_j^-, \gamma_j^+] (j = 1, 2, \ldots, m)$ are interval numbers in [0, 1], denoting the possible membership degrees of the element $x \in X$ to the set $\tilde{A}^W$. $w_{\gamma_j} \in [0, 1] (j = 1, 2, \ldots, m)$ and $\sum_{j=1}^{m} w_{\gamma_j} = 1$, w_{γ_j} is called the weight of γ_j. The weight w_{γ_j} denotes the possible degree of γ_j being taken as the range of the membership degree of x, or the preference value that the decision maker takes γ_i as the range of the membership degree of x. For convenience, we call $\tilde{h}^w = \tilde{h}^w(x)$ a weighted interval-valued hesitant fuzzy element (WIVHFE).

In the following, we introduce the operations on the weighted interval-valued hesitant fuzzy elements (WIVHFEs) $\tilde{h}^w(x), \tilde{h}_1^w(x)$ and $\tilde{h}_2^w(x)$.

Definition 9 Given three WIVHFEs $\tilde{h}^w(x), \tilde{h}_1^w(x)$ and $\tilde{h}_2^w(x)$, for $\lambda > 0$, then

(1) $(\tilde{h}^w)^c(x) = \cup_{\langle \gamma, w_\gamma\rangle \in \tilde{h}^w(x)} \{\langle [1 - \gamma^+, 1 - \gamma^-], w_\gamma\rangle\}$;

(2) $\tilde{h}_1^w(x) \cup \tilde{h}_2^w(x) = \cup_{\langle \gamma_1, w_{\gamma_1}\rangle \in \tilde{h}_1^w(x), \langle \gamma_2, w_{\gamma_2}\rangle \in \tilde{h}_2^w(x)} \{\langle \max\{\gamma_1, \gamma_2\}, w'_{\max\{\gamma_1, \gamma_2\}}\rangle\}$, where $\max\{\gamma_1, \gamma_2\} = [\max\{\gamma_1^-, \gamma_2^-\}, \max\{\gamma_1^+, \gamma_2^+\}]$, and the collection of $\{w'_{\max\{\gamma_1, \gamma_2\}}\}$ is the normalization of the weights $\{w_{\max\{\gamma_1, \gamma_2\}}\}$, $w_{\max\{\gamma_1, \gamma_2\}} = \max\{w_{\gamma_1}, w_{\gamma_2}\}$;

(3) $\tilde{h}_1^w(x) \cap \tilde{h}_2^w(x) = \cup_{\langle \gamma_1, w_{\gamma_1}\rangle \in \tilde{h}_1^w(x), \langle \gamma_2, w_{\gamma_2}\rangle \in \tilde{h}_2^w(x)} \{\langle \min\{\gamma_1, \gamma_2\}, w'_{\min\{\gamma_1, \gamma_2\}}\rangle\}$, where $\min\{\gamma_1, \gamma_2\} = [\min\{\gamma_1^-, \gamma_2^-\}, \min\{\gamma_1^+, \gamma_2^+\}]$, and the collection of $\{w'_{\min\{\gamma_1, \gamma_2\}}\}$ is the normalization of the weights $\{w_{\min\{\gamma_1, \gamma_2\}}\}$, $w_{\min\{\gamma_1, \gamma_2\}} = \max\{w_{\gamma_1}, w_{\gamma_2}\}$;

(4) $(\tilde{h}^w)^\lambda(x) = \cup_{\langle \gamma, w_\gamma\rangle \in \tilde{h}^w(x)} \{\langle [(\gamma^-)^\lambda, (\gamma^+)^\lambda], w_\gamma\rangle\}$;

(5) $\lambda \tilde{h}^w(x) = \cup_{\langle \gamma, w_\gamma\rangle \in \tilde{h}^w(x)} \{\langle [1 - (1 - \gamma^-)^\lambda, 1 - (1 - \gamma^+)^\lambda], w_\gamma\rangle\}$;

(6) $\tilde{h}_1^w(x) \oplus \tilde{h}_2^w(x) = \cup_{\langle \gamma_1, w_{\gamma_1}\rangle \in \tilde{h}_1^w(x), \langle \gamma_2, w_{\gamma_2}\rangle \in \tilde{h}_2^w(x)} \{\langle [\gamma_1^- + \gamma_2^- - \gamma_1^- \gamma_2^-, \gamma_1^+ + \gamma_2^+ - \gamma_1^+ \gamma_2^+], w_{\gamma_1} w_{\gamma_2}\rangle\}$;

(7) $\tilde{h}_1^w(x) \otimes \tilde{h}_2^w(x) = \cup_{\langle \gamma_1, w_{\gamma_1}\rangle \in \tilde{h}_1^w(x), \langle \gamma_2, w_{\gamma_2}\rangle \in \tilde{h}_2^w(x)} \{\langle [\gamma_1^- \gamma_2^-, \gamma_1^+ \gamma_2^+], w_{\gamma_1} w_{\gamma_2}\rangle\}$.

Example 2 Let $X = \{x\}$ be a fixed set, and the weighted interval-valued hesitant fuzzy elements (WIVHEs) $\tilde{h}_1^w(x)$ and $\tilde{h}_2^w(x)$ are listed as follows:

$$\tilde{h}_1^w(x) = \{\langle [0.1, 0.3], 0.3\rangle, \langle [0.4, 0.5], 0.7\rangle\}$$

$$\tilde{h}_2^w(x) = \{\langle [0.1, 0.2], 0.1\rangle, \langle [0.3, 0.5], 0.8\rangle, \langle [0.7, 0.9], 0.1\rangle\}$$

then we have:

(1) $(\tilde{h}_1^w)^c(x) = \{\langle [0.7, 0.9], 0.3\rangle, \langle [0.5, 0.6], 0.7\rangle\}$;

(2) $\tilde{h}_1^w(x) \cup \tilde{h}_2^w(x) = \{\langle [0.1, 0.3], 0.111\rangle, \langle [0.3, 0.5], 0.296\rangle, \langle [0.4, 0.5], 0.519\rangle, \langle [0.7, 0.9], 0.074\rangle\}$;

(3) $\tilde{h}_1^w(x) \cap h_2^w(x) = \{\langle [0.1, 0.2], 0.278\rangle, \langle [0.1, 0.3], 0.306\rangle, \langle [0.3, 0.5], 0.222\rangle, \langle [0.4, 0.5], 0.194\rangle\}$;

(4) For $\lambda = 2$, $(\tilde{h}_1^w)^2(x) = \{\langle [0.01, 0.09], 0.3\rangle, \langle [0.16, 0.25], 0.7\rangle\}$;

(5) For $\lambda = 2$, $2\tilde{h}_1^w(x) = \{\langle[0.19, 0.51], 0.3\rangle, \langle[0.64, 0.75], 0.7\rangle\}$;
(6) $\tilde{h}_1^w(x) \oplus \tilde{h}_2^w(x) = \{\langle[0.19, 0.44], 0.03\rangle, \langle[0.37, 0.65], 0.24\rangle, \langle[0.73, 0.93], 0.03\rangle, \langle[0.46, 0.6], 0.07\rangle, \langle[0.58, 0.75], 0.56\rangle, \langle[0.92, 0.95], 0.07\rangle\}$;
(7) $\tilde{h}_1^w(x) \otimes \tilde{h}_2^w(x) = \{\langle[0.01, 0.06], 0.03\rangle, \langle[0.03, 0.15], 0.24\rangle, \langle[0.07, 0.27], 0.03\rangle, \langle[0.04, 0.1], 0.07\rangle, \langle[0.12, 0.25], 0.56\rangle, \langle[0.28, 0.4], 0.07\rangle\}$.

Theorem 3 *For three WIVHFEs $\tilde{h}^w(x), \tilde{h}_1^w(x)$ and $\tilde{h}_2^w(x)$, then $(\tilde{h}^w)^c(x), \tilde{h}_1^w(x) \cup \tilde{h}_2^w(x), \tilde{h}_1^w(x) \cap \tilde{h}_2^w(x), (\tilde{h}^w)^\lambda(x)$, $\lambda\tilde{h}^w(x)$, $\tilde{h}_1^w(x) \oplus \tilde{h}_2^w(x), \tilde{h}_1^w(x) \otimes \tilde{h}_2^w(x)$ are weighted interval-valued hesitant fuzzy elements.*

Known by Definitions 11 and 12, we can complete the proof of Theorem 3.

Theorem 4 *For three WIVHFEs $\tilde{h}^w(x), \tilde{h}_1^w(x)$ and $\tilde{h}_2^w(x)$, then we have:*

(1) $(\tilde{h}_1^w(x) \cup \tilde{h}_2^w(x))^c = (\tilde{h}_1^w)^c(x) \cap (\tilde{h}_2^w)^c(x)$;
(2) $(\tilde{h}_1^w(x) \cap \tilde{h}_2^w(x))^c = (\tilde{h}_1^w)^c(x) \cup (\tilde{h}_2^w)^c(x)$;
(3) $((\tilde{h}^w)^c(x))^\lambda = (\lambda\tilde{h}^w(x))^c$;
(4) $\lambda(\tilde{h}^w)^c(x) = ((\tilde{h}^w)^\lambda(x))^c$;
(5) $(\tilde{h}_1^w(x) \oplus \tilde{h}_2^w(x))^c = (\tilde{h}_1^w)^c(x) \otimes (\tilde{h}_2^w)^c(x)$;
(6) $(\tilde{h}_1^w(x) \otimes \tilde{h}_2^w(x))^c = ((\tilde{h}_1^w)^c(x) \oplus (\tilde{h}_2^w)^c(x))$.

Proof

(1)
$$\begin{aligned}&(\tilde{h}_1^w(x) \cup \tilde{h}_2^w(x))^c\\ &= \left(\cup_{\langle\gamma_1, w_{\gamma_1}\rangle \in \tilde{h}_1^w(x), \langle\gamma_2, w_{\gamma_2}\rangle \in \tilde{h}_2^w(x)} \{\langle 1 - \max\{\gamma_1, \gamma_2\}, w'_{\max\{\gamma_1,\gamma_2\}}\rangle\}\right)\\ &= \left(\cup_{\langle\gamma_1, w_{\gamma_1}\rangle \in \tilde{h}_1^w(x), \langle\gamma_2, w_{\gamma_2}\rangle \in \tilde{h}_2^w(x)} \{\langle \min\{1-\gamma_1, 1-\gamma_2\}, w_{\min\{1-\gamma_1, 1-\gamma_2\}}\rangle\}\right)\\ &= (\tilde{h}_1^w)^c(x) \cap (\tilde{h}_2^w)^c(x)\end{aligned}$$

(2) The proof is similar to that of (1).
(3) $((\tilde{h}^w)^c(x))^\lambda = \cup_{\langle\gamma, w_\gamma\rangle \in \tilde{h}^w(x)}\{\langle(1-\gamma)^\lambda, w_\gamma\rangle\}$, and $(\lambda\tilde{h}^w(x))^c = \cup_{\langle\gamma, w_\gamma\rangle \in \tilde{h}^w(x)}\{\langle 1-(1-(1-\gamma)^\lambda), w_\gamma\rangle\} = \cup_{\langle\gamma, w_\gamma\rangle \in \tilde{h}^w(x)}\{\langle(1-\gamma)^\lambda, w_\gamma\rangle\} = ((\tilde{h}^w)^c(x))^\lambda$
(4) The proof is similar to that of (3).
(5)
$$\begin{aligned}(\tilde{h}_1^w(x) \oplus \tilde{h}_2^w(x))^c &= \cup_{\langle\gamma_1, w_{\gamma_1}\rangle \in \tilde{h}_1^w(x), \langle\gamma_2, w_{\gamma_2}\rangle \in \tilde{h}_2^w(x)} \{\langle 1-(\gamma_1+\gamma_2-\gamma_1\gamma_2), w_{\gamma_1}w_{\gamma_2}\rangle\}\\ &= \cup_{\langle\gamma_1, w_{\gamma_1}\rangle \in \tilde{h}_1^w(x), \langle\gamma_2, w_{\gamma_2}\rangle \in \tilde{h}_2^w(x)} \{\langle (1-\gamma_1)(1-\gamma_2), w_{\gamma_1}w_{\gamma_2}\rangle\}\\ &= (\tilde{h}_1^w)^c(x) \otimes (\tilde{h}_2^w)^c(x)\end{aligned}$$

(6) The proof is similar to that of (5).
Hence, we complete the proof of Theorem 4.

□

To establish an order relation between WIVHFEs, we introduce the score function of WIVHFE in the following.

Definition 10 For a given weighted interval-valued hesitant fuzzy element $\tilde{h}^w(x)$,

$$s(\tilde{h}^w(x)) = \sum_{\langle\gamma, w_\gamma\rangle \in \tilde{h}^w(x)} w_\gamma \gamma \tag{5}$$

is called the score function of WIVHFE $\tilde{h}^w(x)$.

Applying the score function, we propose a law to compare any two weighted interval-valued hesitant fuzzy elements $\tilde{h}_1^w(x)$ and $\tilde{h}_2^w(x)$ as follows:

If $s(\tilde{h}_1^w(x)) > s(\tilde{h}_2^w(x))$, then $\tilde{h}_1^w(x) > \tilde{h}_2^w(x)$;
If $s(\tilde{h}_1^w(x)) = s(\tilde{h}_2^w(x))$, then $\tilde{h}_1^w(x) = \tilde{h}_2^w(x)$.

To describe the hesitant extent of the decision maker when he/she determines the possible membership degree by using interval numbers, we introduce the concept of hesitance degree of WIVHFE by applying information entropy as follows:

Definition 11 Let $\tilde{h}^w(x) = \{\langle\gamma_1, w_{\gamma_1}\rangle, \langle\gamma_2, w_{\gamma_2}\rangle, \ldots, \langle\gamma_m, w_{\gamma_m}\rangle\}$ be a weighted interval-valued hesitant fuzzy element on $X = \{x_1, x_2, \ldots, x_n\}$. For any $x \in X$, denote

$$u(\tilde{h}^w(x)) = 1 - \exp\left(\sum_{i=1}^m w_i \ln w_i\right)$$

$$u(\tilde{h}^w) = \frac{1}{n}\sum_{i=1}^n u(\tilde{h}^w(x_i))$$

We call $u(\tilde{h}^w(x))$ the hesitance degree of $\tilde{h}^w(x)$, and $u(\tilde{h}^w)$ the hesitance degree of $\tilde{h}^w$, respectively.

For any weighted interval-valued hesitant fuzzy element $\tilde{h}^w(x)$, the value of $u(\tilde{h}^w(x))$ reflects the degree of hesitance for a decision maker when he/she determines the membership degree for $\tilde{h}^w(x)$. The larger the value is, the more hesitant the decision maker will be. For example, if $w_i = 1$, then $u(\tilde{h}^w(x)) = 0$. Namely, there is no any hesitancy for decision maker to determine the interval number as the range of the membership degree. If the number of the elements in $\tilde{h}^w(x)$ is m that m intends to infinite and $w_i = \frac{1}{m}$, then $u(\tilde{h}^w(x)) = 1 - \exp(-\ln m) \to 1(m \to +\infty)$, it indicates that the decision maker is hesitant completely and can hardly determine the interval number as the range of the membership degree.

4 Multi-criteria Group Decision Making with Weighted Interval-Valued Hesitant Fuzzy Information

Firstly, we will introduce four aggregation operators for WIVHFEs and investigate their related properties.

Definition 12 Let $\tilde{h}_1^w(x), \tilde{h}_2^w(x), \ldots, \tilde{h}_n^w(x)$ be a collection of WIVHFEs, the weighted interval-valued hesitant fuzzy-weighted averaging (WIVHFWA) operator is defined as follows:

$$\text{WIVHFWA}\ (\tilde{h}_1^w(x), \tilde{h}_2^w(x), \ldots, \tilde{h}_n^w(x)) = \bigoplus_{j=1}^n (\omega_j \tilde{h}_j^w(x)) = \bigcup_{\langle \gamma_1, w_{\gamma_1}\rangle \in \tilde{h}_1^w(x), \langle \gamma_2, w_{\gamma_2}\rangle \in \tilde{h}_2^w(x), \ldots, \langle \gamma_n, w_{\gamma_n}\rangle \in \tilde{h}_n^w(x)} \left\{ \left\langle 1 - \prod_{j=1}^n (1-\gamma_j)^{\omega_j}, w_{\gamma_1} w_{\gamma_2} \ldots w_{\gamma_n} \right\rangle \right\} \quad (6)$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ is the weight vector of $\tilde{h}_j^w(x)(j = 1, 2, \ldots, n)$, and $\omega_j \geq 0, \sum_{j=1}^n \omega_j = 1$.

Definition 13 Let $\tilde{h}_1^w(x), \tilde{h}_2^w(x), \ldots, \tilde{h}_n^w(x)$ be a collection of WIVHFEs, the weighted interval-valued hesitant fuzzy-weighted geometric (WIVHFWG) operator is defined as follows:

$$\text{WIVHFWG}\ (\tilde{h}_1^w(x), \tilde{h}_2^w(x), \ldots, \tilde{h}_n^w(x)) = \bigotimes_{j=1}^n (\tilde{h}_j^w(x))^{\omega_j} = \bigcup_{\langle \gamma_1, w_{\gamma_1}\rangle \in \tilde{h}_1^w(x), \langle \gamma_2, w_{\gamma_2}\rangle \in \tilde{h}_2^w(x), \ldots, \langle \gamma_n, w_{\gamma_n}\rangle \in \tilde{h}_n^w(x)} \left\{ \left\langle \prod_{j=1}^n \gamma_j^{\omega_j}, w_{\gamma_1} w_{\gamma_2} \ldots w_{\gamma_n} \right\rangle \right\} \quad (7)$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ is the weight vector of $\tilde{h}_j^w(x)(j = 1, 2, \ldots, n)$, and $\omega_j \geq 0, \sum_{j=1}^n \omega_j = 1$.

Lemma 1 [37] *Let* $x_j > 0, \lambda_j > 0, j = 1, 2, \ldots, n,$ *and* $\sum_{j=1}^n \lambda_j = 1$, *then*

$$\prod_{j=1}^n x_j^{\lambda_j} \leq \sum_{j=1}^n \lambda_j x_j$$

Theorem 5 *Let* $\tilde{h}_1^w(x), \tilde{h}_2^w(x), \ldots, \tilde{h}_n^w(x)$ *be a collection of WIVHFEs and* $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ *be the weight vector with* $\omega_j > 0, \sum_{j=1}^n \omega_j = 1$, *then we have*

$$\text{WIVHFWG}\ (\tilde{h}_1^w(x), \tilde{h}_2^w(x), \ldots, \tilde{h}_n^w(x)) \leq \text{WIVHFWA}\ (\tilde{h}_1^w(x), \tilde{h}_2^w(x), \ldots, \tilde{h}_n^w(x))$$

Proof For any $\langle \gamma_1, w_{\gamma_1}\rangle \in \tilde{h}_1^w(x), \langle \gamma_2, w_{\gamma_2}\rangle \in \tilde{h}_2^w\ (x), \ldots, \langle \gamma_n, w_{\gamma_n}\rangle \in \tilde{h}_n^w(x)$, Known by Lemma 1, we have

$$\prod_{j=1}^n (\gamma_j^-)^{\omega_j} \leq \sum_{j=1}^n \omega_j \gamma_j^- = 1 - \sum_{j=1}^n \omega_j (1 - \gamma_j^-) \leq 1 - \prod_{j=1}^n (1-\gamma_j^-)^{\omega_j},$$

$$\prod_{j=1}^n (\gamma_j^+)^{\omega_j} \leq \sum_{j=1}^n \omega_j \gamma_j^+ = 1 - \sum_{j=1}^n \omega_j (1 - \gamma_j^+) \leq 1 - \prod_{j=1}^n (1-\gamma_j^+)^{\omega_j},$$

Hence

$$P\left(\prod_{j=1}^n \gamma_j^{\omega_j} \geq 1 - \prod_{j=1}^n (1-\gamma_j)^{\omega_j}\right) \leq 0.5$$

Thus, we have

$$\prod_{j=1}^n \gamma_j^{\omega_j} \leq 1 - \prod_{j=1}^n (1-\gamma_j)^{\omega_j}$$

Since the weights corresponding to $\prod_{j=1}^n \gamma_j^{\omega_j}$ and $1 - \prod_{j=1}^n (1-\gamma_j)^{\omega_j}$ are the same as $w_{\gamma_1} w_{\gamma_2} \ldots w_{\gamma_n}$, so we have

$$s(\text{WIVHFWG}\ (\tilde{h}_1^w(x), \tilde{h}_2^w(x), \ldots, \tilde{h}_n^w(x))) \leq s(\text{WIVHFWA}\ (\tilde{h}_1^w(x), \tilde{h}_2^w(x), \ldots, \tilde{h}_n^w(x)))$$

Hence, we complete the proof of Theorem 5. □

Definition 14 Let $\tilde{h}_1^w(x), \tilde{h}_2^w(x), \ldots, \tilde{h}_n^w(x)$ be a collection of WIVHFEs, the generalized weighted interval-valued hesitant fuzzy-weighted averaging (GWIVHFWA) operator is defined as follows:

$$\text{GWIVHFWA}\ (\tilde{h}_1^w(x), \tilde{h}_2^w(x), \ldots, \tilde{h}_n^w(x)) = \left(\bigoplus_{j=1}^n (\omega_j (\tilde{h}_j^w(x))^\lambda)\right)^{1/\lambda} = \bigcup_{\langle \gamma_1, w_{\gamma_1}\rangle \in \tilde{h}_1^w(x), \ldots, \langle \gamma_n, w_{\gamma_n}\rangle \in \tilde{h}_n^w(x)} \left\{ \left\langle \left(1 - \prod_{j=1}^n (1-\gamma_j^\lambda)^{\omega_j}\right)^{1/\lambda}, w_{\gamma_1} w_{\gamma_2} \ldots w_{\gamma_n} \right\rangle \right\} \quad (8)$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ is the weight vector of $\tilde{h}_j^w(x)(j = 1, 2, \ldots, n)$, and $\omega_j \geq 0, \sum_{j=1}^n \omega_j = 1$, $\lambda > 0$.

Especially, in the case where $\lambda = 1$, the GWIVHFWA operator reduces to the weighted interval-valued hesitant fuzzy-weighted averaging (WIVHFWA) operator.

Definition 15 Let $\tilde{h}_1^w(x), \tilde{h}_2^w(x), \ldots, \tilde{h}_n^w(x)$ be a collection of WIVHFEs, the generalized weighted interval-valued hesitant fuzzy-weighted geometric (GWIVHFWG) operator is defined as follows:

$$\text{GWIVHFWG}\ (\tilde{h}_1^w(x),\tilde{h}_2^w(x),\ldots,\tilde{h}_n^w(x)) = \frac{1}{\lambda}\bigotimes_{j=1}^{n}(\lambda\tilde{h}_j^w(x))^{\omega_j}$$
$$= \bigcup_{\langle\gamma_1,w_{\gamma_1}\rangle\in\tilde{h}_1^w(x),\ldots,\langle\gamma_n,w_{\gamma_n}\rangle\in\tilde{h}_n^w(x)} \left\{\left\langle 1-\left(1-\prod_{j=1}^{n}\left(1-(1-\gamma_j)^{\lambda}\right)^{\omega_j}\right)^{1/\lambda}, w_{\gamma_1}w_{\gamma_2}\ldots w_{\gamma_n}\right\rangle\right\} \tag{9}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ is the weight vector of $\tilde{h}_j^w(x)(j = 1, 2, \ldots, n)$, and $\omega_j \geq 0, \sum_{j=1}^{n}\omega_j = 1$, $\lambda > 0$.

Especially, in the case where $\lambda = 1$, the GWIVHFWG operator reduces to the weighted interval-valued hesitant fuzzy-weighted geometric (WIVHFWG) operator.

In the following, we will apply the above-mentioned operators to deal with the multi-criteria group decision making under the weighted interval-valued hesitant fuzzy environment.

Let $A = \{A_1, A_2, \ldots, A_n\}$ be a collection of alternatives, $C = \{C_1, C_2, \ldots, C_m\}$ a set of criteria, and $E = \{E_1, E_2, \ldots, E_p\}$ the set of experts who will give their evaluated values under each criteria for every alternative. Let interval number $r_{ij}^k \subseteq [0, 1]$ denote the assessments provided by expert E_k under the criteria C_j for the alternative A_i. The decision maker's goal is to obtain the ranking order of the alternatives or to get the best alternative.

Applying the weighted interval-valued hesitant fuzzy sets, we present a ranking method to these alternatives under the weighted interval-valued hesitant fuzzy environment according to the following steps.

Step 1 For every alternative A_i under each criteria C_j, considering two kinds of different cases, we construct a weighted interval-valued hesitant fuzzy element $(\tilde{h}^w)^{ij}$ by incorporating the experts' assessments, respectively:

Case 1 The weights of the experts are unknown, then

$$(\tilde{h}^w)^{ij} = \{\langle r_{ij}, w_{r_{ij}}\rangle | w_{r_{ij}} = l/p\} \tag{10}$$

where $r_{ij} \in \cup_k\{r_{ij}^k\}$ and l is the real number of the experts who give the assessment of r_{ij}.

Case 2 The weight vector of the experts, $v = (v_1, v_2, \ldots, v_t)^T$ with $v_k \geq 0$ and $\sum_{i=1}^{t} v_k = 1$, is given, then

$$(\tilde{h}^w)^{ij} = \left\{\langle r_{ij}, w_{ij}\rangle | w_{ij} = \sum\nolimits_{E_k\in N(r_{ij})} v_k\right\} \tag{11}$$

where $r_{ij} \in \cup_k\{r_{ij}^k\}$ and $N(r_{ij})$ denotes the collection of the experts who give the assessment of r_{ij}.

Step 2 Assume that $\omega = (\omega_1, \omega_2, \ldots, \omega_m)^T$ with $\omega_j \geq 0$ and $\sum_{j=1}^{m}\omega_j = 1$ is the weight vector of the criteria. For every alternative A_i, we aggregate the WIVHFEs $(\tilde{h}^w)^{ij}(j = 1, 2, \ldots, m)$ by applying the above-mentioned operators such as WIVHFWA operator, WIVHFWG operator, GWIVHFWA operator and GWIVHFWG operator to derive the overall aggregation value $(\tilde{h}^w)^i$. For example, if we utilize WIVHFWA operator, then

$$(\tilde{h}^w)^i = \text{WIVHFWA}\ ((\tilde{h}^w)^{i1}, (\tilde{h}^w)^{i2}, \ldots, (\tilde{h}^w)^{im}) \tag{12}$$

Step 3 Utilize Eq. (5) to calculate the score function $s((\tilde{h}^w)^i)$ of $(\tilde{h}^w)^i$.

Step 4 Compare each pair of $s((\tilde{h}^w)^i)(i = 1, 2, \ldots, n)$ by utilizing Eq. (1) to construct the possibility degree matrix $P = (p_{ij})_{n\times n}$, where $p_{ij} = P(s((\tilde{h}^w)^i) \geq s((\tilde{h}^w)^j))$.

Step 5 Establish the Boolean matrix $Q = (q_{ij})_{n\times n}$ of $s((\tilde{h}^w)^i)(i = 1, 2, \ldots, n)$.

Step 6 Let $\lambda_i = \sum_{j=1}^{n} q_{ij}$, we obtain the ranking vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$.

Step 7 Rank the alternatives $s((\tilde{h}^w)^i)(i = 1, 2, \ldots, n)$ in descending order in accordance with the value of λ_i.

5 Numerical Examples

To illustrate the effectiveness of our proposed approach, we will present two numerical examples in this section.

Example 3 Reconsidering the problem of performance evaluation given in Introduction.

Assuming that ten experts are invited to anonymously evaluate the performance of two suppliers A and B according to some given criteria. The experts provide their assessments by interval numbers. Each assessment that indicates the satisfaction of the expert to the suppliers is provided to the decision maker. For supplier A, eight experts give assessment [0, 8, 0.9], and two experts give the assessment [0.5, 0.6]. For supplier B, eight experts give assessment [0.5, 0.6], and two experts give the assessment [0, 8, 0.9]. Assume the experts can not persuade each other. Therefore, if using IVHFEs to express all the experts' evaluation, according to the common approach provided by Chen et al. [2], the synthesized assessment for supplier A is an IVHFE $\tilde{h}(A) = \{[0.5, 0.6], [0, 8, 0.9]\}$. Similarly, for supplier B, the synthesized assessment is also an IVHFE $\tilde{h}(B) = \{[0.5, 0.6], [0, 8, 0.9]\}$. By using Eq. (4), we have $s(\tilde{h}(A)) = s(\tilde{h}(B))$. Hence, the decision maker cannot discriminate between A and B.

If we use WIVHFEs to express all the experts' evaluation, then we have that the synthesized assessment for supplier A is a WIVHFE, $\tilde{h}^w(A) = \{\langle[0.5, 0.6], 0.2\rangle, \langle[0, 8, 0.9], 0.8\rangle\}$ and the synthesized assessment for supplier B is a WIVHFE, $\tilde{h}^w(B) = \{\langle[0.5, 0.6], 0.8\rangle, \langle[0, 8, 0.9], 0.2\rangle\}$. To select the best one, we calculate the score function by utilizing Eq. (5), then $s(\tilde{h}^w(A)) = 0.2[0.5, 0.6] + 0.8[0.8, 0.9] = [0.74, 0.84]$, and $s(\tilde{h}^w(B)) = 0.8[0.5, 0.6] + 0.2[0.8, 0.9] = [0.56, 0.66]$. By Eq. (1), we have $P(s(\tilde{h}^w(A)) \geq s(\tilde{h}^w(B))) = 1$.

Thus, the best supplier is A.

Remark 2 Example 3 verifies that WIVHFE is more effective and applicable than IVHFE under the interval-valued hesitant fuzzy environment.

Example 4 [23] A city is planning to build a library. One of the problems faced by the city development committee is how to determine what kind of air-conditioning systems should be installed in the library. The contractor offers five feasible alternatives, which might be adapted to the physical structure of the library. The offered air-conditioning system must take a decision according to the following four attributes: (1) C_1 is performance; (2) C_2 is maintainability; (3) C_3 is flexibility; and (4) C_4 is safety. The weight vector of the attributes $C_j (j = 1, 2, 3, 4)$ is $\omega = (0.2, 0.1, 0.3, 0.4)^T$. The five possible alternatives A_1, A_2, A_3, A_4, A_5 are to be evaluated using the interval numbers by three experts $E_k (k = 1, 2, 3)$(whose weight vector is $v = (0.4, 0.3, 0.3)^T$). The experts' assessments construct three interval number decision matrices $A^{(k)} = \left(r_{ij}^{(k)}\right)_{5\times4} (k = 1, 2, 3)$ as listed in Tables 1, 2 and 3, where $r_{ij}^{(k)} \subseteq [0, 1]$ denotes the satisfaction of the expert E_k for the possible values of the alternative A_i under the attribute C_j. In the following, we will use the above-mentioned approach to determine what kind of air-conditioning systems should be installed.

Table 1 The decision matrix provided by E_1

	C_1	C_2	C_3	C_4
A_1	[0.5, 0.7]	[0.6, 0.8]	[0.4, 0.5]	[0.6, 0.8]
A_2	[0.4, 0.6]	[0.7, 0.8]	[0.7, 0.9]	[0.6, 0.7]
A_3	[0.7, 0.8]	[0.5, 0.7]	[0.6, 0.7]	[0.5, 0.8]
A_4	[0.7, 0.9]	[0.6, 0.7]	[0.8, 0.9]	[0.5, 0.6]
A_5	[0.8, 0.9]	[0.6, 0.7]	[0.7, 0.8]	[0.5, 0.7]

Table 2 The decision matrix provided by E_2

	C_1	C_2	C_3	C_4
A_1	[0.6, 0.8]	[0.5, 0.7]	[0.5, 0.7]	[0.6, 0.9]
A_2	[0.5, 0.6]	[0.7, 0.9]	[0.6, 0.8]	[0.6, 0.7]
A_3	[0.6, 0.8]	[0.4, 0.7]	[0.5, 0.7]	[0.6, 0.8]
A_4	[0.6, 0.7]	[0.6, 0.8]	[0.7, 0.9]	[0.5, 0.7]
A_5	[0.7, 0.8]	[0.6, 0.8]	[0.6, 0.8]	[0.6, 0.8]

Table 3 The decision matrix provided by E_3

	C_1	C_2	C_3	C_4
A_1	[0.5, 0.7]	[0.5, 0.7]	[0.4, 0.5]	[0.6, 0.8]
A_2	[0.4, 0.6]	[0.7, 0.8]	[0.6, 0.8]	[0.6, 0.7]
A_3	[0.7, 0.8]	[0.4, 0.7]	[0.6, 0.7]	[0.5, 0.8]
A_4	[0.7, 0.9]	[0.6, 0.8]	[0.7, 0.9]	[0.5, 0.7]
A_5	[0.7, 0.8]	[0.6, 0.7]	[0.7, 0.8]	[0.5, 0.7]

Since the weights of the experts are given as $v = (0.4, 0.3, 0.3)^T$, then

Step 1 Utilizing Eq. (11), we obtain the weighted interval-valued hesitant fuzzy elements $A = (\langle r_{ij}, w_{r_{ij}} \rangle)$ as follows:

For A_1, we have

$$\begin{aligned}
(\tilde{h}^w)^{11} &= \{\langle[0.5, 0.7], 0.7\rangle, \langle[0.6, 0.8], 0.3\rangle\},\\
(\tilde{h}^w)^{12} &= \{\langle[0.5, 0.7], 0.6\rangle, \langle[0.6, 0.8], 0.4\rangle\}\\
(\tilde{h}^w)^{13} &= \{\langle[0.4, 0.5], 0.7\rangle, \langle[0.5, 0.7], 0.3\rangle\},\\
(\tilde{h}^w)^{14} &= \{\langle[0.6, 0.8], 0.7\rangle, \langle[0.6, 0.9], 0.3\rangle\}
\end{aligned}$$

For A_2, we have

$$\begin{aligned}
(\tilde{h}^w)^{21} &= \{\langle[0.4, 0.6], 0.7\rangle, \langle[0.5, 0.6], 0.3\rangle\},\\
(\tilde{h}^w)^{22} &= \{\langle[0.7, 0.8], 0.7\rangle, \langle[0.7, 0.9], 0.3\rangle\}\\
(\tilde{h}^w)^{23} &= \{\langle[0.6, 0.8], 0.6\rangle, \langle[0.7, 0.9], 0.4\rangle\},\\
(\tilde{h}^w)^{24} &= \{\langle[0.6, 0.7], 1\rangle\}
\end{aligned}$$

For A_3, we have

$$\begin{aligned}
(\tilde{h}^w)^{31} &= \{\langle[0.6, 0.8], 0.3\rangle, \langle[0.7, 0.8], 0.7\rangle\},\\
(\tilde{h}^w)^{32} &= \{\langle[0.4, 0.7], 0.6\rangle, \langle[0.5, 0.7], 0.4\rangle\}\\
(\tilde{h}^w)^{33} &= \{\langle[0.5, 0.7], 0.3\rangle, \langle[0.6, 0.7], 0.7\rangle\},\\
(\tilde{h}^w)^{34} &= \{\langle[0.5, 0.8], 0.7\rangle, \langle[0.6, 0.8], 0.3\rangle\}
\end{aligned}$$

For A_4, we have

$$\begin{aligned}
(\tilde{h}^w)^{41} &= \{\langle[0.6, 0.7], 0.3\rangle, \langle[0.7, 0.9], 0.7\rangle\},\\
(\tilde{h}^w)^{42} &= \{\langle[0.6, 0.7], 0.4\rangle, \langle[0.6, 0.8], 0.6\rangle\}\\
(\tilde{h}^w)^{43} &= \{\langle[0.8, 0.9], 0.4\rangle, \langle[0.7, 0.9], 0.6\rangle\},\\
(\tilde{h}^w)^{44} &= \{\langle[0.5, 0.7], 0.6\rangle, \langle[0.5, 0.6], 0.4\rangle\}
\end{aligned}$$

For A_5, we have

$$\begin{aligned}
(\tilde{h}^w)^{51} &= \{\langle[0.7, 0.8], 0.6\rangle, \langle[0.8, 0.9], 0.4\rangle\},\\
(\tilde{h}^w)^{52} &= \{\langle[0.6, 0.7], 0.7\rangle, \langle[0.6, 0.8], 0.3\rangle\}\\
(\tilde{h}^w)^{53} &= \{\langle[0.6, 0.8], 0.3\rangle, \langle[0.7, 0.8], 0.7\rangle\},\\
(\tilde{h}^w)^{54} &= \{\langle[0.5, 0.7], 0.7\rangle, \langle[0.6, 0.8], 0.3\rangle\}
\end{aligned}$$

Step 2 Utilize WIVHFWA operator, i.e., Eq. (6), we aggregate all of the preference values $(\tilde{h}^w)^{ij}(j=1,2,3,4)$ and derive the overall preference value $\tilde{h}^w(A_i)(i=1,2,3,4,5)$ as follows:

$$\begin{aligned}\tilde{h}^w(A_1)=\{&\langle[0.5170,0.7027],0.2058\rangle,\langle[0.5170,0.7747],0.0882\rangle,\\&\langle[0.5427,0.7449],0.0882\rangle,\langle[0.5427,0.8067],0.0378\rangle,\\&\langle[0.5276,0.7145],0.1372\rangle,\langle[0.5276,0.7836],0.0588\rangle,\\&\langle[0.5528,0.7551],0.0588\rangle,\langle[0.5528,0.8144],\\&0.0252\rangle,\langle[0.5381,0.7258],0.0882\rangle,\langle[0.5381,0.7922],0.0378\rangle,\\&\langle[0.5627,0.7648],0.0378\rangle,\langle[0.5627,0.8217],0.0162\rangle,\\&\langle[0.5483,0.7367],0.0588\rangle,\langle[0.5483,0.8005],0.0252\rangle,\\&\langle[0.5723,0.7741],0.0252\rangle,\langle[0.5723,0.8288],0.0108\rangle\}\\
\tilde{h}^w(A_2)=\{&\langle[0.5785,0.7298],0.2940\rangle,\langle[0.6134,0.7805],0.1960\rangle,\\&\langle[0.5785,0.7479],0.1260\rangle,\langle[0.6134,0.7952],0.0840\rangle,\\&\langle[0.5936,0.7298],0.1260\rangle,\langle[0.6272,0.7805],0.0840\rangle,\\&\langle[0.5936,0.7479],0.0540\rangle,\langle[0.6272,0.7952],0.0360\rangle\}\\
\tilde{h}^w(A_3)=\{&\langle[0.5130,0.7648],0.0378\rangle,\langle[0.5546,0.7648],0.0162\rangle,\\&\langle[0.5446,0.7648],0.0882\rangle,\langle[0.5834,0.7648],0.0378\rangle,\\&\langle[0.5218,0.7648],0.0252\rangle,\langle[0.5627,0.7648],0.0108\rangle,\\&\langle[0.5528,0.7648],0.0588\rangle,\langle[0.5910,0.7648],0.0252\rangle,\\&\langle[0.5403,0.7648],0.0882\rangle,\langle[0.5795,0.7648],0.0378\rangle,\\&\langle[0.5700,0.7648],0.2058\rangle,\langle[0.6067,0.7648],0.0882\rangle,\\&\langle[0.5486,0.7648],0.0588\rangle,\langle[0.5871,0.7648],0.0252\rangle,\\&\langle[0.5778,0.7648],0.1372\rangle,\langle[0.6138,0.7648],0.0588\rangle\}\\
\tilde{h}^w(A_4)=\{&\langle[0.6448,0.7842],0.0288\rangle,\langle[0.6448,0.7579],0.0192\rangle,\\&\langle[0.5988,0.7842],0.0432\rangle,\langle[0.5988,0.7579],0.0288\rangle,\\&\langle[0.6448,0.7928],0.0432\rangle,\langle[0.6448,0.7675],0.0288\rangle,\\&\langle[0.5988,0.7928],0.0648\rangle,\langle[0.5988,0.7675],0.0432\rangle,\\&\langle[0.6646,0.8268],0.0672\rangle,\langle[0.6646,0.8057],0.0448\rangle,\\&\langle[0.6212,0.8268],0.1008\rangle,\langle[0.6212,0.8057],0.0672\rangle,\\&\langle[0.6646,0.8337],0.1008\rangle,\langle[0.6646,0.8134],0.0672\rangle,\\&\langle[0.6212,0.8337],0.1512\rangle,\langle[0.6212,0.8134],0.1008\rangle\}\\
\tilde{h}^w(A_5)=\{&\langle[0.5871,0.7551],0.0882\rangle,\langle[0.6224,0.7917],0.0378\rangle,\\&\langle[0.6212,0.7551],0.2058\rangle,\langle[0.6536,0.7917],0.0882\rangle,\\&\langle[0.5871,0.7648],0.0378\rangle,\langle[0.6224,0.8000],0.0162\rangle,\\&\langle[0.6212,0.7648],0.0882\rangle,\langle[0.6536,0.8000],0.0378\rangle,\\&\langle[0.6193,0.7868],0.0588\rangle,\langle[0.6518,0.8187],0.0252\rangle,\\&\langle[0.6508,0.7868],0.1372\rangle,\langle[0.6806,0.8187],0.0588\rangle,\\&\langle[0.6193,0.7952],0.0252\rangle,\langle[0.6518,0.8259],0.0108\rangle,\\&\langle[0.6508,0.7952],0.0588\rangle,\langle[0.6806,0.8259],0.0252\rangle\}\end{aligned}$$

Step 3 Utilize Eq. (5) to calculate the score function values of $\tilde{h}^w(A_i)(i=1,2,3,4)$, we have

$$s(\tilde{h}^w(A_1))=[0.5351,0.7463],$$
$$s(\tilde{h}^w(A_2))=[0.5968,0.7551],$$
$$s(\tilde{h}^w(A_3))=[0.5679,0.7648],$$
$$s(\tilde{h}^w(A_4))=[0.6322,0.8096],$$
$$s(\tilde{h}^w(A_5))=[0.6328,0.7808].$$

Step 4 Compare each pair of $s((\tilde{h}^w)^i)(i=1,2,\ldots,5)$ by utilizing Eq. (1) to construct the possibility degree matrix $P=(p_{ij})_{5\times5}$, where $p_{ij}=P(s((\tilde{h}^w)^i)\geq s((\tilde{h}^w)^j)$.

$$P=\begin{pmatrix}0.5 & 0.4045 & 0.4371 & 0.2936 & 0.316\\ 0.5955 & 0.5 & 0.5271 & 0.3662 & 0.3994\\ 0.5629 & 0.4729 & 0.5 & 0.3542 & 0.3827\\ 0.7064 & 0.6338 & 0.6458 & 0.5 & 0.5433\\ 0.684 & 0.6006 & 0.6173 & 0.4567 & 0.5\end{pmatrix}$$

Step 5 Establish the Boolean matrix Q of $s((\tilde{h}^w)^i)(i=1,2,\ldots,5)$.

$$Q=\begin{pmatrix}1&0&0&0&0\\1&1&1&0&0\\1&0&1&0&0\\1&1&1&1&1\\1&1&1&0&1\end{pmatrix}$$

Step 6 Let $\lambda_i=\sum_{j=1}^{5}q_{ij}$, we obtain the ranking vector $\lambda=(1,3,2,5,4)$.

Step 7 Rank the alternatives $s((\tilde{h}^w)^i)(i=1,2,\ldots,5)$ in descending order in accordance with the value of $\lambda_i(i=1,2,\ldots,5)$, then we obtain the rank of the alternatives as follows:

$$A_4\succ A_5\succ A_2\succ A_3\succ A_1$$

Hence, the best alternative is A_4.

6 Conclusion

In this paper, we introduce the concept of weighted interval-valued hesitant fuzzy set (WIVHFS), in which its main characteristic of weighted interval-valued hesitant fuzzy element is that the interval numbers are distributed with different weights. Then we define some basic operations such as union, intersection, complement, multiplication and power operation of weighted interval-valued hesitant fuzzy sets and weighted interval-valued hesitant fuzzy elements, investigate their operation properties, and propose the score function of the weighted interval-valued hesitant fuzzy element to compare two weighted hesitant fuzzy elements. Furthermore, we introduce the concept of hesitance degree of weighted interval-valued hesitant fuzzy element, present four aggregation operators such as the weighted interval-valued hesitant fuzzy-weighted averaging (WIVHFWA) operator, the weighted interval-valued hesitant fuzzy-weighted geometric (WIVHFWG) operator, the generalized weighted interval-valued hesitant fuzzy-weighted averaging (GWIVHFWA) operator and the generalized weighted interval-valued hesitant fuzzy-weighted geometric (GWIVHFWG) operator to aggregate weighted interval-valued hesitant fuzzy information, and develop a new mathematical model of multi-criteria group decision making based on the expert weights (known and unknown).

Finally, a numerical example is used to illustrate the effectiveness and feasibility of our proposed method.

The following work is to enhance the study of the aggregation operators of weighted interval-valued hesitant fuzzy sets, the weighted interval-valued hesitant fuzzy linguistic terms and their aggregation operators, and to deeply develop the group decision-making model based on the weighted interval-valued hesitant fuzzy sets theory. We hope that it will enrich and provide more new idea and new methods for group decision making based on weighted interval-valued hesitant fuzzy environment.

Acknowledgements The authors are grateful to the anonymous reviewers, for their excellent comments and valuable suggestions, and the Editor-in-Chief, Professor Shun-Feng Su, for his kind help, that help us improve this paper.

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