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Multiple Criteria Decision Making with Probabilities in Interval-Valued Pythagorean Fuzzy Setting

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Abstract Interval-valued Pythagorean fuzzy set (IVPFS), as an extension of interval-valued intuitionistic fuzzy set (IVIFS) and Pythagorean fuzzy set to deal with uncertainty, has attracted much attention since its introduction, in both theory and application aspects. The present work aims at investigating new distance measures in the IVPFSs and then employing them into multiple criteria decision-making application. To begin with, generalized interval-valued Pythagorean fuzzy weighted distance measure and generalized interval-valued Pythagorean fuzzy ordered weighted distance measure are firstly introduced in the IVPFSs. Afterward, we propose generalized probabilistic interval-valued Pythagorean fuzzy weighted averaging distance (P-GIVPFWAD) operator, generalized probabilistic interval-valued Pythagorean fuzzy order weighted averaging distance (P-GIVPFOWAD) operator and immediate generalized probabilistic interval-valued Pythagorean fuzzy ordered weighted averaging distance (IP-GIVPFOWAD) operator which are new distance measures and are able to integrate the ordered weighted averaging operator, probabilistic weight and individual distance of two interval-valued Pythagorean fuzzy numbers (IVPFNs) in the same formulation. These distance measures are very suitable to deal with the situation where the input data are represented in IVPFNs. Then we present a kind of multiple criteria decision-making method with interval-valued Pythagorean fuzzy information based on the developed distance measures. Finally, a numerical example is provided to explain the feasibility of the proposed method and the validity of the developed method is also analyzed according to the validity criterion of multiple criteria decision making.

Keywords Interval-valued Pythagorean fuzzy set · Probability · Distance measures · Multi-criteria decision making

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1 Introduction

Multiple criteria decision making (MCDM), as an effective framework for comparison, has always been used to find the most desirable one from a finite set of alternatives on the predefined criteria or attributes. Due to the intrinsic complexity of natural objects, there exists much uncertain information in many real-world problems. So, it is difficult for experts or decision makers (DMs) to give their assessments on criteria or attributes with precise values and the performance ranking. Fortunately, Zadeh [1] introduced fuzzy set (FS) which is a generalization of classical set theory and has been found to be particularly suitable to describe the uncertain information when one assesses decision alternatives for MCDM problems. Since the fuzzy set was introduced, it has also drawn the attention of many researchers who have extended the fuzzy sets to interval-valued fuzzy sets (IVFSs), intuitionistic fuzzy sets (IFSs), interval-valued intuitionistic fuzzy sets (IVIFSs), hesitant fuzzy sets (HFSs), and so on; various fuzzy decision-making methods [2–15]based on them have been also constructed to handle some fuzzy and uncertainty information.

As an important extension of fuzzy set, IFS [16] is characterized by three parameters, namely, a membership degree, a non-membership degree and an indeterminacy degree. That is, an IFS A in a finite universe of discourse X has such a structure $A = \{\langle x, (\mu_A(x), v_A(x))\rangle | x \in X\}$, where μ_A represents the membership degree and v_A is the non-membership degree with the condition that $0 \leq \mu_A(x) + v_A(x) \leq 1$. Since IFS's appearance, it becomes a powerful tool to deal with some information with imprecision, uncertainty and vagueness. However, Yager [12] and Yager and Abbasoc [17] pointed out that there exists such a kind of useful extension of IFS $A = \{\langle x, (\mu_A(x), v_A(x))\rangle | x \in X\}$ which satisfies the condition $0 \leq \mu_A^2(x) + v_A^2(x) \leq 1$. Such a useful extension of IFSs is called a Pythagorean fuzzy set (PFS). The main difference between the IFSs and PFSs focuses on the membership degrees and the non-membership degrees of them. Therefore, it follows from the above analysis of IFS and PFS that PFS has more powerful ability than IFS to deal with uncertain information in MCDM problems. After introduction of PFSs, related weighted averaging operators [10, 18, 19], Pythagorean fuzzy aggregation operators [20], related distance and similarity of PFNs [19, 21], correlation coefficient of two PFNs [22], and fundamental properties of Pythagorean fuzzy functions [23] have been developed to deal with some decision-making problems and other aspects [11, 24, 25]. After introduction of PFSs, Peng and Yang [26] and Zhang [21] further proposed the interval-valued Pythagorean fuzzy set (IVPFS) whose membership function and non-membership function are represented by an interval number, and some operations and relations of IVPFS are also investigated. Since then, some researches on theories [27] and applications [14, 19] of IVPFSs have been developed.

Information measures have played the vital roles in the development of various fuzzy sets theories and their applications. However, distance measure, one of information measures, is used to compare the alternatives of the problem with some ideal results and obtain the optimal choice in decision-making problems. Therefore, the variety of distance measures have been proposed, such as the Hamming distance, Euclid distance, Hausdorff distance, and so on. Recently, on the basis of the idea of the ordered weighted averaging (OWA) operator, Xu and Chen [28] introduced ordered weighted distance measure, and Merigo [29] introduced an ordered weighted averaging distance (OWAD) operator, which is also a new distance measure by combining the OWA operator and a distance measure. Since its introduction, the OWAD has been further developed and extended, such as, ordered weighted averaging (OWAWA) operator [29], probabilistic OWA (POWA) operator [30], uncertain probabilistic OWA (UPOWA) operator [31], fuzzy probabilistic OWA (FPOWA) operator [31], and probabilistic ordered weighted averaging distance (UPOWAD) [32]; they were gradually introduced and applied to different decision-making problems.

Based on the existing work as reviewed above about the distance measures-related OWA operator, motivated by the ideas of OWAD operator and UPOWAD operator, which are two distance measures using the OWA operator to calculate the Hamming distance, in the present work, we propose some new interval-valued Pythagorean fuzzy distances, namely, generalized probabilistic interval-valued Pythagorean fuzzy weighted averaging distance (P-GIVPFWAD) operator, generalized probabilistic interval-valued Pythagorean fuzzy order weighted averaging distance (P-GIVPFOWAD) operator and immediate generalized probabilistic interval-valued Pythagorean fuzzy ordered weighted averaging distance (IP-GIVPFOWAD) operator, by applying related OWA operators, probabilistic weighted (PW) information and individual distance of IVPFNs. They are also extensions of UPOWA operator [32]. Compared with some existing weighted distance measures, these new interval-valued Pythagorean fuzzy distance measures can deal with more complex decision-making problems which include uncertain information evaluated with the IVPFNs, the probability information and the OWA operator. The main contributions of the present work are summarized as following: (1) introduce generalized interval-valued Pythagorean fuzzy order weighted distance GIVPFOWD measure; (2) introduce generalized probabilistic interval-valued Pythagorean fuzzy ordered weighted averaging distance P-GIVPFOWAD operator and immediate generalized probabilistic interval-valued Pythagorean fuzzy ordered weighted averaging distance IP-GIVPFOWAD operator, which are new distance measures that unifies interval-valued Pythagorean fuzzy information with OWA operator and individual distance measures of two IVPFNs; (3) based on the P-GIVPFOWAD and IP-GIVPFOWAD, we extend the traditional TOPSIS method to construct a MCDM method under IVPF environment.

The rest of the paper is organized as follows. In Sect. 2, we review some definitions on IVPFSs, score function and accuracy function of IVPFNs, which are used in the analysis throughout this paper. Section 3 is devoted to the main results concerning the distances of IVPFSs: GIVPFWD and GIVPFOWD. Section 4 is focused on P-GIVPFOWAD operator and IP-GIVPFOWAD operator. In Sect. 5, we construct MCDM approach based on some proposed distance operators in Sect. 4. Consequently, a practical example is provided in Sect. 6 to illustrate this method and analyze the validity of the proposed MCDM methods. This paper is concluded in Sect. 7.

2 Interval-Valued Pythagorean Fuzzy Sets

In this section, firstly some basic concepts related to IFS and PFS have been given and then IVPFS is recapped, which are the basis of this work.

2.1 Pythagorean Fuzzy Set

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite universe of discourse, an intuitionistic fuzzy set (IFS) [16] A in X characterized by a membership function $\mu_A : X \to [0,1]$ and a non-membership function $v_A : X \to [0,1]$, which satisfy the condition $0 \le \mu_A(x) + v_A(x) \le 1$. An IFS A can be expressed as

$$A = \{\langle x, (\mu_A(x), v_A(x))\rangle | x \in X\}.$$

$\pi_A(x) = 1 - \mu_A(x) - v_A(x)$ is called the degree of indeterminacy. For the convenience, called $(\mu_A(x), v_A(x))$ is an intuitionistic fuzzy number (IFN) and denoted by (μ_A, v_A).

However, there are some decision-making problems in which the DMs or the experts' attitudes are possibly not suitable to be described by applying an IFS. Under such situations, Pythagorean fuzzy set (PFS), introduced by Yager and Abbasov [17], is a novel concept to deal with this situation and also an extension of IFS:

In a finite universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$, a PFS P with the structure

$$P = \{\langle x, (\mu_P(x), v_P(x))\rangle | x \in X\}.$$

where $\mu_P : X \to [0,1]$ denotes the membership degree and $v_P : X \to [0,1]$ denotes the non-membership degree of the element $x \in X$ to the set P, respectively, with the condition that $0 \le (\mu_P(x))^2 + (v_P(x))^2 \le 1$. $\pi_P(x) = \sqrt{1 - (\mu_P(x))^2 - (v_P(x))^2}$ is called the degree of indeterminacy. For the convenience, Zhang and Xu [15] called $p = (\mu_p(x), v_p(x))$ a Pythagorean fuzzy number (PFN) denoted by $p = (\mu_p, v_p)$.

From the definitions of IFS and PFS, we can easily see that the main difference between PFN and IFN is their corresponding constraint conditions. Obviously, an IFN must be a PFN, but the converse is not true in generally. For instance, $p = (0.5, 0.8)$ is a PFN but not an IFN because $0.5 + 0.8 > 1$. In order to compare two PFNs, Zhang and Xu [15] introduced the concept of score function. For a PFN $p = (\mu_p, v_p)$, the score function $s(p)$ of p be defined as follows:

$$s(p) = (\mu_p)^2 - (v_p)^2, \tag{1}$$

where $s(p) \in [-1, 1]$.

For any two PFNs p_1, p_2,

(1) if $s(p_1) < s(p_2)$, then $p_1 \prec p_2$;
(2) if $s(p_1) > s(p_2)$, then $p_1 \succ p_2$;
(3) if $s(p_1) = s(p_2)$, then $p_1 \sim p_2$.

Peng and Yang [20] pointed out that the score function defined in above is not reasonable. For instance, for two PFNs $p_1 = (0.4, 0.4)$ and $p_2 = (0.7, 0.7)$, we have $p_1 \sim p_2$ according to definition of score function of a PFN. However, it is obviously unreasonable. In order to overcome the drawback of score function, Peng and Yang [20] introduced the concept of accuracy function of a PFN and modified the comparison rules, which are showed as follows:

For any PFN $p = (\mu_p, v_p)$, the accuracy function $a(p)$ of p is defined as follows:

$$a(p) = (\mu_p)^2 + (v_p)^2, \tag{2}$$

where $a(p) \in [0, 1]$.

For any two PFNs p_1, p_2,

(1) if $s(p_1) < s(p_2)$, then $p_1 \prec p_2$;
(2) if $s(p_1) = s(p_2)$, then,
   (a) if $a(p_1) < a(p_2)$, then $p_1 \prec p_2$;
   (b) if $a(p_1) = a(p_2)$, then $p_1 \sim p_2$.

2.2 Interval-Valued Pythagorean Fuzzy Set

In many real decision-making problems, there exist some situations where expert's or DM's opinions may be represented by a subinterval of [0,1] not a crisp number. In order to describe such decision-making problems, Peng and Yang [26] and Zhang [14] introduced the concept of interval-valued Pythagorean fuzzy sets (IVPFSs), which can describe such a problem whose membership degree and non-membership degree are all expressed in a subinterval of [0,1], respectively.

Let *Int*([0, 1]) denote the set of all closed subintervals of [0, 1], and X be a universe of discourse. An IVPFS [14, 26] in X has such a structure

$$\tilde{P} = \{\langle x, (\mu_{\tilde{P}}(x), v_{\tilde{P}}(x))\rangle | x \in X\}.$$

where $\mu_{\tilde{P}} : X \to Int([0,1])$ denotes the membership degree and $v_{\tilde{P}} : X \to Int([0,1])$ denotes the non-membership degree of the element $x \in X$ to the set $\tilde{P}$, respectively, with the condition that $0 \le sup(\mu_{\tilde{P}}(x))^2 + sup(v_{\tilde{P}}(x))^2 \le 1$.

For each $x \in X$, $\mu_{\tilde{P}}(x)$ and $v_{\tilde{P}}(x)$ denote $\mu_{\tilde{P}}(x) = [\mu^-_{\tilde{P}}(x), \mu^+_{\tilde{P}}(x)]$, $v_{\tilde{P}}(x) = [v^-_{\tilde{P}}(x), v^+_{\tilde{P}}(x)]$, respectively. Therefore, $\tilde{P}$ can also be expressed in another style as follows:

$$\tilde{P} = \left\{\left\langle x, \left(\left[\mu^-_{\tilde{P}}(x), \mu^+_{\tilde{P}}(x)\right], \left[v^-_{\tilde{P}}(x), v^+_{\tilde{P}}(x)\right]\right)\right\rangle | x \in X\right\}. \tag{3}$$

where Eq. (3) satisfies the condition $(\mu_{\tilde{P}}^{+}(x))^2 + (v_{\tilde{P}}^{+}(x))^2 \leq 1$.

$$\pi_{\tilde{P}}(x) = \left[\pi_{\tilde{P}}^{-}(x), \pi_{\tilde{P}}^{+}(x)\right] = \left[\sqrt{1-\left(\mu_{\tilde{P}}^{+}(x)\right)^2-\left(v_{\tilde{P}}^{+}(x)\right)^2}, \sqrt{1-\left(\mu_{\tilde{P}}^{-}(x)\right)^2-\left(v_{\tilde{P}}^{-}(x)\right)^2}\right]$$

is called the indeterminacy degree. For the convenience, $\tilde{p} = ([\mu_{\tilde{p}}^{-}, \mu_{\tilde{p}}^{+}], [v_{\tilde{p}}^{-}, v_{\tilde{p}}^{+}])$ is called an interval-valued Pythagorean fuzzy number (IVPFN).

Hereafter, $\mathcal{IVPFN}$ denotes the set of all IVPFNs of a IVPFS on X.

For two IVPFNs $\tilde{p}_1 = ([\mu_{\tilde{p}_1}^{-}, \mu_{\tilde{p}_1}^{+}], [v_{\tilde{p}_1}^{-}, v_{\tilde{p}_1}^{+}])$ and $\tilde{p}_2 = ([\mu_{\tilde{p}_2}^{-}, \mu_{\tilde{p}_2}^{+}], [v_{\tilde{p}_2}^{-}, v_{\tilde{p}_2}^{+}])$, a relation $\leq$ on the IVPFNs is defined as follows:

$$\mu_{\tilde{p}_1}^{-} \leq \mu_{\tilde{p}_2}^{-}, \mu_{\tilde{p}_1}^{+} \leq \mu_{\tilde{p}_2}^{+} \text{ and } v_{\tilde{p}_1}^{-} \geq v_{\tilde{p}_2}^{-}, v_{\tilde{p}_1}^{+} \geq v_{\tilde{p}_2}^{+}. \quad (4)$$

In order to compare two IVPFNs, Peng and Yang [26] introduced the concepts of score function and accuracy function of an IVPFN, which are defined as follows:

For any IVPFN $\tilde{p} = ([\mu_{\tilde{p}}^{-}, \mu_{\tilde{p}}^{+}], [v_{\tilde{p}}^{-}, v_{\tilde{p}}^{+}])$, the score function $\tilde{s}(p)$ of $\tilde{p}$ is defined as follows:

$$\tilde{s}(\tilde{p}) = \frac{1}{2}\left((\mu_{\tilde{p}}^{-})^2 + (\mu_{\tilde{p}}^{+})^2 - (v_{\tilde{p}}^{-})^2 - (v_{\tilde{p}}^{+})^2\right), \quad (5)$$

where $\tilde{s}(\tilde{p}) \in [-1, 1]$.

For any IVPFN $\tilde{p} = ([\mu_{\tilde{p}}^{-}, \mu_{\tilde{p}}^{+}], [v_{\tilde{p}}^{-}, v_{\tilde{p}}^{+}])$, the accuracy function $\tilde{a}(p)$ of $\tilde{p}$ is defined as follows:

$$\tilde{a}(\tilde{p}) = \frac{1}{2}\left((\mu_{\tilde{p}}^{-})^2 + (\mu_{\tilde{p}}^{+})^2 + (v_{\tilde{p}}^{-})^2 + (v_{\tilde{p}}^{+})^2\right), \quad (6)$$

where $\tilde{s}(\tilde{p}) \in [0, 1]$.

Based on the score function and accuracy function of an IVPFN, the comparison rules [26] between two IVPFNs are given as follows:

For any two IVPFNs $\tilde{p}_1, \tilde{p}_2$,

(1) if $\tilde{s}(\tilde{p}_1) < \tilde{s}(\tilde{p}_2)$, then $\tilde{p}_1 \prec \tilde{p}_2$.
(2) if $\tilde{s}(\tilde{p}_1) = \tilde{s}(\tilde{p}_2)$, then,
 (a) if $\tilde{a}(\tilde{p}_1) < \tilde{a}(\tilde{p}_2)$, then $\tilde{p}_1 \prec \tilde{p}_2$;
 (b) if $\tilde{a}(\tilde{p}_1) = \tilde{a}(\tilde{p}_2)$, then $\tilde{p}_1 \sim \tilde{p}_2$.

3 Generalized Interval-Valued Pythagorean Fuzzy Ordered Weighted Distance Measures

Ordered weighted averaging (OWA) operator [33] provides a powerful tool to aggregate multiple inputs that lie between the max and min operators, it has been used in many ranges of applications. In this section, we will introduce a kind of new distance measure, namely, generalized interval-valued Pythagorean fuzzy ordered weighted distance measure (GIVPFOWD). Before the introduction of GIVPFOWD, we first review the concepts related to the OWA operator.

Definition 1 ([33]) An OWA operator of dimension n is a mapping OWA: $R^n \to R$ that has an associated weighting $\omega = (\omega_1, \ldots, \omega_n)$ with $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$, such that

$$\mathrm{OWA}(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} \omega_j b_j \quad (7)$$

where $(a_1, a_2, \ldots, a_n) \in R^n$ and b_j is the jth largest of a_i.

Definition 2 ([30]) Let $A = (a_1, \ldots, a_n), B = (b_1, \ldots, b_n)$ be two sets of arguments. An ordered weighted averaging distance (OWAD) operator of dimension n is a mapping OWAD: $R^n \times R^n \to R$ that has an associated weighting $\omega = (\omega_1, \ldots, \omega_n)$ with $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$, such that

$$\mathrm{OWAD}(A, B) = \sum_{j=1}^{n} \omega_j d_j \quad (8)$$

where d_j is the jth largest of $|a_i - b_i|$.

Remark 1 In Definition 2, if we do not consider the order of individual distance, then the OWAD is called the weighted Haming distance measure [32]. That is, a weighted distance measure is a mapping WHD: $R^n \times R^n \to R$ that has an associated weighting $\omega = (\omega_1, \ldots, \omega_n)$ with $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$, such that

$$\mathrm{WHD}(A, B) = \sum_{j=1}^{n} \omega_j |a_j - b_j|.$$

It follows from Definitions 1 and 2 that OWAD operator is an extension of the Hamming distances by applying the OWA operator and is also a distance measure. The main difference between the OWAD operator and the normalized Hamming distance is that the orders of arguments of the individual distances are arranged according to their values.

Motivated by the OWAD operator, we propose the concept of generalized interval-valued Pythagorean fuzzy ordered weighted distance GIVPFOWD measure below. Let $\tilde{A} = \{\tilde{\alpha}_i = (\mu_{\tilde{\alpha}_i}, v_{\tilde{\alpha}_i}) | i = 1, 2, \ldots, n\}$ and $\tilde{B} = \{\tilde{\beta}_i = (\mu_{\tilde{\beta}_i}, v_{\tilde{\beta}_i}) | i = 1, 2, \ldots, n\}$ be two collections of IVPFNs, where $(\mu_{\tilde{\alpha}_i}, v_{\tilde{\alpha}_i}) = ([\mu_{\tilde{\alpha}_i}^{-}, \mu_{\tilde{\alpha}_i}^{+}], [v_{\tilde{\alpha}_i}^{-}, v_{\tilde{\alpha}_i}^{+}])$, $(\mu_{\tilde{\beta}_i}, v_{\tilde{\beta}_i}) = ([\mu_{\tilde{\beta}_i}^{-}, \mu_{\tilde{\beta}_i}^{+}], [v_{\tilde{\beta}_i}^{-}, v_{\tilde{\beta}_i}^{+}])$. Before the GIVPFOWAD is given, we first define the p-distance between two IVPFNs $\tilde{p}_1 = ([\mu^-_{\tilde{p}_1}, \mu^+_{\tilde{p}_1}], [v^-_{\tilde{p}_1}, v^+_{\tilde{p}_1}])$ and $\tilde{p}_2 = ([\mu^-_{\tilde{p}_2}, \mu^+_{\tilde{p}_2}], [v^-_{\tilde{p}_2}, v^+_{\tilde{p}_2}])$.

Definition 3 For two IVPFNs $\tilde{p}_1$ and $\tilde{p}_2$, the p-distance $d^p(\tilde{p}_1, \tilde{p}_2)$ between $\tilde{p}_1$ and $\tilde{p}_2$ is defined as follows, where $p > 0$:

$$\begin{aligned} d^p(\tilde{p}_1, \tilde{p}_2) = \frac{1}{4}(&|(\mu^-_{\tilde{p}_1})^2 - (\mu^-_{\tilde{p}_2})^2|^p + |(\mu^+_{\tilde{p}_1})^2 - (\mu^+_{\tilde{p}_2})^2|^p \\ &+ |(v^-_{\tilde{p}_1})^2 - (v^-_{\tilde{p}_2})^2|^p + |(v^+_{\tilde{p}_1})^2 - (v^+_{\tilde{p}_2})^2|^p \\ &+ |(\pi^-_{\tilde{p}_1})^2 - (\pi^-_{\tilde{p}_2})^2|^p + |(\pi^+_{\tilde{p}_1})^2 - (\pi^+_{\tilde{p}_2})^2|^p). \end{aligned} \tag{9}$$

Hereafter, $d^p(\tilde{p}_1, \tilde{p}_2)$ always denotes the interval-valued Pythagorean fuzzy p-distance (IVPF p-distance) between two IVPFNs $\tilde{p}_1, \tilde{p}_2$ if not specified. In this paper, the symbol $d^p(\tilde{p}_1, \tilde{p}_2)$ is just a distance defined in Eq. (9), not p powers of $d(\tilde{p}_1, \tilde{p}_2)$.

Theorem 1 Let $d^p(\tilde{p}_1, \tilde{p}_2)$ be the IVPF p-distance between $\tilde{p}_1 = ([\mu^-_{\tilde{p}_1}, \mu^+_{\tilde{p}_1}], [v^-_{\tilde{p}_1}, v^+_{\tilde{p}_1}])$ and $\tilde{p}_2 = ([\mu^-_{\tilde{p}_2}, \mu^+_{\tilde{p}_2}], [v^-_{\tilde{p}_2}, v^+_{\tilde{p}_2}])$. Then the following properties hold.

(1) $d^p(\tilde{p}_1, \tilde{p}_2) \geq 0$;
(2) $d^p(\tilde{p}_1, \tilde{p}_2) = d^p(\tilde{p}_2, \tilde{p}_1)$;
(3) $d^p(\tilde{p}_1, \tilde{p}_2) = 0$ if and only if $\tilde{p}_1 = \tilde{p}_2$;
(4) If $\tilde{p}_1 \leq \tilde{p}_2 \leq \tilde{p}_3$, then $d^p(\tilde{p}_1, \tilde{p}_2) \leq d^p(\tilde{p}_1, \tilde{p}_3)$ and $d^p(\tilde{p}_2, \tilde{p}_3) \leq d^p(\tilde{p}_1, \tilde{p}_3)$.

Proof It is obviously that (1), (2), (3) hold. Therefore, we only need to prove (4).

(4). Let $\tilde{p}_i = ([\mu^-_{\tilde{p}_i}, \mu^+_{\tilde{p}_i}], [v^-_{\tilde{p}_1}, v^+_{\tilde{p}_1}])(i = 1, 2, 3)$ and $\tilde{p}_1 \leq \tilde{p}_2 \leq \tilde{p}_3$, we have $\mu^-_{\tilde{p}_1} \leq \mu^-_{\tilde{p}_2} \leq \mu^-_{\tilde{p}_3}$, $\mu^+_{\tilde{p}_1} \leq \mu^+_{\tilde{p}_2} \leq \mu^+_{\tilde{p}_3}$, $v^-_{\tilde{p}_1} \geq v^-_{\tilde{p}_2} \geq v^-_{\tilde{p}_3}$ and $v^+_{\tilde{p}_1} \geq v^+_{\tilde{p}_2} \geq v^+_{\tilde{p}_3}$. Therefore

$$\begin{aligned} &|(\mu^-_{\tilde{p}_1})^2 - (\mu^-_{\tilde{p}_2})^2| \leq |(\mu^-_{\tilde{p}_1})^2 - (\mu^-_{\tilde{p}_3})^2|, \\ &|(\mu^+_{\tilde{p}_1})^2 - (\mu^+_{\tilde{p}_2})^2| \leq |(\mu^+_{\tilde{p}_1})^2 - (\mu^+_{\tilde{p}_3})^2|, \\ &|(v^-_{\tilde{p}_1})^2 - (v^-_{\tilde{p}_2})^2| \leq |(v^-_{\tilde{p}_1})^2 - (v^-_{\tilde{p}_3})^2|, \\ &|(v^+_{\tilde{p}_1})^2 - (v^+_{\tilde{p}_2})^2| \leq |(v^+_{\tilde{p}_1})^2 - (v^+_{\tilde{p}_3})^2|. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} &|(\mu^-_{\tilde{p}_1})^2 - (\mu^-_{\tilde{p}_2})^2|^p \leq |(\mu^-_{\tilde{p}_1})^2 - (\mu^-_{\tilde{p}_3})^2|^p, \\ &|(\mu^+_{\tilde{p}_1})^2 - (\mu^+_{\tilde{p}_2})^2|^p \leq |(\mu^+_{\tilde{p}_1})^2 - (\mu^+_{\tilde{p}_3})^2|^p, \\ &|(v^-_{\tilde{p}_1})^2 - (v^-_{\tilde{p}_2})^2|^p \leq |(v^-_{\tilde{p}_1})^2 - (v^-_{\tilde{p}_3})^2|^p, \\ &|(v^+_{\tilde{p}_1})^2 - (v^+_{\tilde{p}_2})^2|^p \leq |(v^+_{\tilde{p}_1})^2 - (v^+_{\tilde{p}_3})^2|^p. \end{aligned}$$

and

$$\begin{aligned} |(\pi^-_{\tilde{p}_1})^2 - (\pi^-_{\tilde{p}_2})^2| &= |(\mu^+_{\tilde{p}_2})^2 - (\mu^+_{\tilde{p}_1})^2 + (v^+_{\tilde{p}_2})^2 - (v^+_{\tilde{p}_1})^2| \\ &\leq |1 - (\mu^+_{\tilde{p}_3})^2 - (\mu^+_{\tilde{p}_1})^2 - (1 - (v^+_{\tilde{p}_3})^2 - (v^+_{\tilde{p}_1})^2)| \\ &= |(\pi^-_{\tilde{p}_1})^2 - (\pi^-_{\tilde{p}_3})^2|. \end{aligned}$$

Thus, $|(\pi^-_{\tilde{p}_1})^2 - (\pi^-_{\tilde{p}_2})^2|^p \leq |(\pi^-_{\tilde{p}_1})^2 - (\pi^-_{\tilde{p}_3})^2|^p$. Similarly, we have $|(\pi^+_{\tilde{p}_1})^2 - (\pi^+_{\tilde{p}_2})^2|^p \leq |(\pi^+_{\tilde{p}_1})^2 - (\pi^+_{\tilde{p}_3})^2|^p$.

Therefore,

$$\begin{aligned} d^p(\tilde{p}_1, \tilde{p}_2) = &\frac{1}{4}(|(\mu^-_{\tilde{p}_1})^2 - (\mu^-_{\tilde{p}_2})^2|^p + |(\mu^+_{\tilde{p}_1})^2 - (\mu^+_{\tilde{p}_2})^2|^p \\ &+ |(v^-_{\tilde{p}_1})^2 - (v^-_{\tilde{p}_2})^2|^p + |(v^+_{\tilde{p}_1})^2 - (v^+_{\tilde{p}_2})^2|^p \\ &+ |(\pi^-_{\tilde{p}_1})^2 - (\pi^-_{\tilde{p}_2})^2|^p + |(\pi^+_{\tilde{p}_1})^2 - (\pi^+_{\tilde{p}_2})^2|^p) \\ \leq &\frac{1}{4}(|(\mu^-_{\tilde{p}_1})^2 - (\mu^-_{\tilde{p}_3})^2|^p + |(\mu^+_{\tilde{p}_1})^2 - (\mu^+_{\tilde{p}_3})^2|^p \\ &+ |(v^-_{\tilde{p}_1})^2 - (v^-_{\tilde{p}_3})^2|^p + |(v^+_{\tilde{p}_1})^2 - (v^+_{\tilde{p}_3})^2|^p \\ &+ |(\pi^-_{\tilde{p}_1})^2 - (\pi^-_{\tilde{p}_3})^2|^p + |(\pi^+_{\tilde{p}_1})^2 - (\pi^+_{\tilde{p}_3})^2|^p) \\ = &d^p(\tilde{p}_1, \tilde{p}_3). \end{aligned}$$

Analogously, we can also prove $d^p(\tilde{p}_2, \tilde{p}_3) \leq d^p(\tilde{p}_1, \tilde{p}_3)$, which completes the proof of (4).

Now we introduce the generalized interval-valued Pythagorean fuzzy weighted distance GIVPFWD measure based on the IVPF p-distance.

Hereafter, $\tilde{A}$ and $\tilde{B}$ will represent two sets $(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n)$, $(\tilde{\beta}_1, \tilde{\beta}_2, \ldots, \tilde{\beta}_n)$ of IVPFNs, respectively, if not further specified. □

Definition 4 Let $\tilde{A}$ and $\tilde{B}$ be two collections of IVPFNs. A generalized interval-valued Pythagorean fuzzy weighted distance GIVPFWD measure is a function

$$\text{GIVPFWD} : \mathcal{IVPFN}^n \times \mathcal{IVPFN}^n \to \mathbf{R},$$

which has associated weighting vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^{\mathrm{T}}$ with $\omega_i > 0, \sum_{i=1}^n \omega_i = 1$ such that

$$\text{GIVPFWD}(\tilde{A}, \tilde{B}) = \left(\sum_{i=1}^n \omega_i (d^p(\tilde{\alpha}_i, \tilde{\beta}_i))\right)^{\frac{1}{p}}, \tag{10}$$

where $d^p(\tilde{\alpha}_i, \tilde{\beta}_i)$ is the IVPF p-distance between IVPFNs $\tilde{\alpha}_i$ and $\tilde{\beta}_i$.

If $p = 1$ in Eq. (10), GIVPFWD will be degenerated the interval-valued Pythagorean fuzzy weighted averaging distance IVPFWD measure

$$\text{IVPFWD}(\tilde{A}, \tilde{B}) = \sum_{i=1}^n \omega_i d(\tilde{\alpha}_i, \tilde{\beta}_i). \tag{11}$$

If $p = 2$, GIVPFWD will be degenerated the interval-valued Pythagorean fuzzy weighted Euclid distance IVPFWED measure

$$\mathrm{IVPFWED}(\tilde{A},\tilde{B}) = \sqrt{\sum_{i=1}^{n} \omega_i d^2(\tilde{\alpha}_i,\tilde{\beta}_i)}. \tag{12}$$

Example 1 Let

$$\begin{aligned}\tilde{A} =& \{\tilde{\alpha}_1 = ([0.7,0.9],[0.2,0.3]), \tilde{\alpha}_2 = ([0.7,0.8],[0.1,0.2]),\\ &\tilde{\alpha}_3 = ([0.2,0.3],[0.8,0.9]), \tilde{\alpha}_4 = ([0.5,0.6],[0.35,0.45])\},\\ \tilde{B} =& \{\tilde{\beta}_1 = ([0.4,0.7],[0.2,0.6]), \tilde{\beta}_2 = ([0.5,0.6],[0.4,0.5]),\\ &\tilde{\beta}_3 = ([0.2,0.4],[0.8,0.9]), \tilde{\beta}_4 = ([0.4,0.7],[0.5,0.6])\}\end{aligned}$$

be two sets of IVPFNs on the reference set $X = \{x_1, x_2, \ldots, x_4\}$ and the weight vector is $\omega = (0.2, 0.3, 0.1, 0.4)$. Firstly, we can calculate the $d^p(\tilde{\alpha}_i,\tilde{\beta}_i)$ by employing Eq. (9):

(1) When $p = 1$,

$$d(\tilde{\alpha}_1,\tilde{\beta}_1) = 0.585, d(\tilde{\alpha}_2,\tilde{\beta}_2) = 0.540, d(\tilde{\alpha}_3,\tilde{\beta}_3) = 0.660, d(\tilde{\alpha}_4,\tilde{\beta}_4) = 0.2075.$$

According to Eq. (12),

$$\begin{aligned}\mathrm{IVPFWD}(\tilde{A},\tilde{B}) &= 0.2 \times 0.325 + 0.3 \times 0.26 + 0.1 \times 0.035 + 0.4 \times 0.2075\\ &= 0.2295.\end{aligned}$$

(2) When $p = 2$,

$$d^2(\tilde{\alpha}_1,\tilde{\beta}_1) = 0.0989, d^2(\tilde{\alpha}_2,\tilde{\beta}_2) = 0.0539,$$
$$d^2(\tilde{\alpha}_3,\tilde{\beta}_3) = 0.0025, d^2(\tilde{\alpha}_4,\tilde{\beta}_4) = 0.0375.$$

According to Eq. (12),

$$\begin{aligned}\mathrm{IVPFWED}(\tilde{A},\tilde{B}) &= (0.2 \times 0.0989 + 0.3 \times 0.0539 + 0.1 \times 0.0025 + 0.4 \times 0.0375)^{\frac{1}{2}}\\ &= 0.2263.\end{aligned}$$

Definition 5 Let $\tilde{A}$ and $\tilde{B}$ be two collections of IVPFNs. A generalized interval-valued Pythagorean fuzzy order weighted distance GIVPFOWD measure is a function

$$\mathrm{GIVPFOWD} : \mathcal{IVPFN}^n \times \mathcal{IVPFN}^n \to \mathbf{R},$$

which has an associated weighting vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^{\mathrm{T}}$ with $\omega_i > 0, \sum_{i=1}^{n} \omega_i = 1 (i = 1, 2, \ldots, n)$, such that

$$\mathrm{GIVPFOWD}(\tilde{A},\tilde{B}) = \left(\sum_{i=1}^{n} \omega_i d^p(\tilde{\alpha}_i,\tilde{\beta}_i)\right)^{\frac{1}{p}}, \tag{13}$$

where $d^p(\tilde{\alpha}_i,\tilde{\beta}_i)$ is the ith largest of $d^p(\tilde{\alpha}_j,\tilde{\beta}_j)$ and $d^p(\tilde{\alpha}_j,\tilde{\beta}_j)$ is IVPF p-distance between two IVPFNs $\tilde{\alpha}_i$ and $\tilde{\beta}_i$.

In Definition 5, if $p = 1$, GIVPFOWD will be degenerated to interval-valued Pythagorean fuzzy order weighted averaging distance IVPFOWD

$$\mathrm{IVPFOWD}(\tilde{A},\tilde{B}) = \sum_{i=1}^{n} \omega_i d(\tilde{\alpha}_i,\tilde{\beta}_i). \tag{14}$$

If $p = 2$, GIVPFOWD will be degenerated to interval-valued Pythagorean fuzzy order weighted averaging Euclid distance IVPFOWED

$$\mathrm{IVPFOWED}(\tilde{A},\tilde{B}) = \left(\sum_{i=1}^{n} \omega_i d^2(\tilde{\alpha}_i,\tilde{\beta}_i)\right)^{\frac{1}{2}}. \tag{15}$$

The GIVPFOWD is a generalization of the interval-valued Pythagorean fuzzy distance measures [19] by applying the OWA operators. The main difference between GIVPFOWD and GIVPFWD is that is that the orders of the individual distances of IVPFNs are arranged according to the values of individual distances.

4 Generalized Probabilistic Interval-Valued Pythagorean Fuzzy OWA Distance Operators

In this section, we introduce the generalized probabilistic interval-valued Pythagorean fuzzy order weighted averaging distance P-GIVPFOWAD operator and immediate generalized probabilistic interval-valued Pythagorean fuzzy order weighted averaging distance IP-GIVPFOWAD operator, which are two new distances that combining OWA operator, probabilistic weights and individual distances. Therefore, they can evaluate the more complex information which is imprecise and cannot be expressed with exact numbers, but all this imprecision information may be evaluated by IVPFNs.

Definition 6 Let $\tilde{A}$ and $\tilde{B}$ be two collections of IVPFNs. A generalized probabilistic interval-valued Pythagorean fuzzy weighted distance P-GIVPFWAD operator is a function

$$\mathrm{P\text{-}GIVPFWAD} : \mathcal{IVPFN}^n \times \mathcal{IVPFN}^n \to \mathbf{R}$$

that has an associated weighting vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^{\mathrm{T}}$ with $\omega_i > 0, \sum_{i=1}^{n} \omega_i = 1 (i = 1, 2, \ldots, n)$, such that

$$\mathrm{P\text{-}GIVPFWAD}((\tilde{A},\tilde{B}) = \left(\sum_{i=1}^{n} \hat{\rho}_i (d^p(\tilde{\alpha}_i,\tilde{\beta}_i))\right)^{\frac{1}{p}}, \tag{16}$$

where $d^p(\tilde{\alpha}_i,\tilde{\beta}_i)$ is the IVPF p-distance between IVPFNs $\tilde{\alpha}_i, \tilde{\beta}_i$. $\hat{\rho}_i = \xi\omega_i + (1-\xi)p_i$ with $\xi \in [0,1]$, p_i is the associated probability of $d^p(\tilde{\alpha}_i,\tilde{\beta}_i)$.

In Definition 6, if $\xi = 0$, then P-GIVPFWAD will be reduced to generalized probabilistic interval-valued Pythagorean fuzzy distance P-GIVPFD measure. If $\xi = 1$, it will be generalized interval-valued Pythagorean fuzzy weighted distance GIVPFWD measure. If $p = 1$, P-GIVPFWAD will reduced to probabilistic interval-valued Pythagorean fuzzy weighted averaging distance (P-IVPFWAD) operator

$$\text{P-IVPFWAD}(\tilde{A}, \tilde{B}) = \sum_{i=1}^{n} \hat{\rho}_i d(\tilde{\alpha}_i, \tilde{\beta}_i).$$

As the parameter ξ as concerned, it represents the important degree of weight, while $1 - \xi$ represents the important degree of probabilistic information. In Eax.1, assume the probabilistic weight vector (0.3, 0.2, 0.4, 0.1). Note that weight important degree of 30% while probabilistic information important degree of 70%, then we have

$\tilde{\rho}_1 = 0.3 \times 0.2 + 0.7 \times 0.3 = 0.27,$
$\tilde{\rho}_2 = 0.3 \times 0.3 + 0.7 \times 0.2 = 0.23,$
$\tilde{\rho}_3 = 0.3 \times 0.1 + 0.7 \times 0.4 = 0.31,$
$\tilde{\rho}_4 = 0.3 \times 0.4 + 0.7 \times 0.1 = 0.19.$

Therefore, when $p = 1$, it follows from Eq. (16) that

$$\begin{aligned}\text{P-IVPFWAD}(\tilde{A}, \tilde{B}) &= 0.27 \times 0.325 + 0.23 \times 0.26 + 0.31 \\ &\quad \times 0.035 + 0.19 \times 0.2075 \\ &= 0.1978.\end{aligned}$$

When $p = 2$,
$d^2(\tilde{\alpha}_1, \tilde{\beta}_1) = 0.0989, d^2(\tilde{\alpha}_2, \tilde{\beta}_2) = 0.0539,$
$d^2(\tilde{\alpha}_3, \tilde{\beta}_3) = 0.0025, d^2(\tilde{\alpha}_4, \tilde{\beta}_4) = 0.0375.$

It follows from Eq. (16) that

$$\begin{aligned}\text{P-GIVPFWAD}(\tilde{A}, \tilde{B}) &= (0.27 \times 0.0989 + 0.23 \times 0.0539 \\ &\quad + 0.31 \times 0.0025 + 0.19 \times 0.0375)^{\frac{1}{2}} \\ &= 0.2168.\end{aligned}$$

Definition 7 Let $\tilde{A}$ and $\tilde{B}$ be two collections of IVPFNs. A generalized probabilistic interval-valued Pythagorean fuzzy ordered weighted averaging distance P-GIVPFOWAD operator is a function

$$\text{P-GIVPFOWAD} : \mathcal{IVPFN}^n \times \mathcal{IVPFN}^n \to \mathbf{R}$$

that has an associated weighting vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^{\mathrm{T}}$ with $\omega_i > 0, \sum_{i=1}^{n} \omega_i = 1 (i = 1, 2, \ldots, n)$, such that

$$\text{P-GIVPFOWAD}(\tilde{A}, \tilde{B})) = \left(\sum_{i=1}^{n} \hat{\rho}_i (d^p(\tilde{\alpha}_i, \tilde{\beta}_i)) \right)^{\frac{1}{p}}, \tag{17}$$

where $d^p(\tilde{\alpha}_i, \tilde{\beta}_i)$ is the ith largest of $d^p(\tilde{\alpha}_j, \tilde{\beta}_j)$ and $d^p(\tilde{\alpha}_j, \tilde{\beta}_j)$ is IVPF p-distance between two IVPFNs $\tilde{\alpha}_j, \tilde{\beta}_j$.

$$\hat{\rho}_i = \xi \omega_i + (1 - \xi) p_i \tag{18}$$

with $\xi \in [0, 1]$ and p_i is the probabilistic p_j according to $d^p(\alpha_i, \beta_i)$, that is, according to the ith largest of the $d^p(\tilde{\alpha}_j, \tilde{\beta}_j)$.

In Definition 7, if $p = 1$, P-GIVPFOWAD will reduced to probabilistic interval-valued Pythagorean fuzzy ordered weighted averaging distance P-IVPFOWAD operator

$$\text{P-IVPFOWAD}(\tilde{A}, \tilde{B}) = \sum_{i=1}^{n} \hat{\rho}_i d(\tilde{\alpha}_i, \tilde{\beta}_i). \tag{19}$$

If $\xi = 0$, then P-GIVPFOWAD will be generalized probabilistic interval-valued Pythagorean fuzzy weighted distance P-GIVPFWAD operator; if $\xi = 1$, it will be generalized interval-valued Pythagorean fuzzy ordered weighted distance GIVPFOWD operator.

In the following example, we will present a numerical example to show how to use the above distance operator.

Example 2 In Example 1, we calculated the $d^p(\tilde{\alpha}_i, \tilde{\beta}_i)$ by employing Eq. (9) as follows, when $p = 1$,

$$\begin{aligned} d(\tilde{\alpha}_1, \tilde{\beta}_1) &= 0.325, d(\tilde{\alpha}_2, \tilde{\beta}_2) = 0.26, d(\tilde{\alpha}_3, \tilde{\beta}_3) \\ &= 0.035, d(\tilde{\alpha}_4, \tilde{\beta}_4) = 0.2075. \end{aligned}$$

According to the above interval-valued Pythagorean fuzzy distance, we obtain the new probabilistic weight (0.3, 0.2, 0.1, 0.4) by reordering the probabilistic weight (0.3, 0.2, 0.4, 0.1), the new weight vector as follows:

$\tilde{\rho}_1 = 0.3 \times 0.2 + 0.7 \times 0.3 = 0.27,$
$\tilde{\rho}_2 = 0.3 \times 0.3 + 0.7 \times 0.2 = 0.23,$
$\tilde{\rho}_3 = 0.3 \times 0.1 + 0.7 \times 0.1 = 0.10,$
$\tilde{\rho}_4 = 0.3 \times 0.4 + 0.7 \times 0.4 = 0.40.$

Therefore, when $p = 1$, it follows from Eq. (19) that

$$\begin{aligned}\text{P-IVPFOWAD}(\tilde{A}, \tilde{B}) &= 0.27 \times 0.325 + 0.23 \times 0.26 + 0.10 \\ &\quad \times 0.2075 + 0.40 \times 0.035 = 0.1823.\end{aligned}$$

When $p = 3$, it follows from Eq. (17) that

$$\begin{aligned} d^3(\alpha_1, \beta_1) &= 0.0311, d^3(\alpha_2, \beta_2) = 0.0124, \\ d^3(\alpha_3, \beta_3) &= 0.0002, d^3(\alpha_4, \beta_4) = 0.0082. \end{aligned}$$

$$\begin{aligned}\text{P-GIVPFOWAD}(\tilde{A}, \tilde{B}) &= (0.27 \times 0.0311 + 0.23 \times 0.0124 \\ &\quad + 0.1 \times 0.0002 + 0.40 \times 0.0082)^{\frac{1}{3}} \\ &= 0.2441.\end{aligned}$$

Now, we can also develop the generalized immediate probabilistic interval-valued Pythagorean fuzzy order weighted averaging distance IP-GIVPFOWAD operator by applying interval-valued Pythagorean fuzzy information, individual distance, and immediate probability [34].

Definition 8 Let $\tilde{A}$ and $\tilde{B}$ be two collections of IVPFNs. A generalized immediate probabilistic interval-valued Pythagorean fuzzy order weighted averaging distance IP-GIVPFOWAD operator is a function IP-GIVPFOWAD: $\mathcal{IVPFN}^n \times \mathcal{IVPFN}^n \to \mathbf{R}$ which has an associated weighting vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the with $\omega_i > 0, \sum_{i=1}^n \omega_i = 1 (i = 1, 2, \ldots, n)$, such that

$$\text{IP-GIVPFOWAD}(\tilde{A}, \tilde{B}) = \left(\sum_{i=1}^{n} \hat{\rho}_i (d^p(\tilde{\alpha}_i, \tilde{\beta}_i)) \right)^{\frac{1}{p}}, \tag{20}$$

where $d^p(\tilde{\alpha}_i, \tilde{\beta}_i)$ is the ith largest of $d^p(\tilde{\alpha}_j, \tilde{\beta}_j)$ and $d^p(\tilde{\alpha}_j, \tilde{\beta}_j)$ is the IVPF p-distance between IVPFNs $\tilde{\alpha}_j, \tilde{\beta}_j$ and a probabilistic weight $p_i > 0, \sum_{i=1}^n p_i = 1$.

$$\hat{\rho}_i = \frac{\omega_i p_i}{\sum_{i=1}^{n} \omega_i p_i} \tag{21}$$

and p_i is the probabilistic p_j according to $d^p(\tilde{\alpha}_i, \tilde{\beta}_i)$, that is, according to the ith largest of the $d^p(\tilde{\alpha}_j, \tilde{\beta}_j)$.

In Definition 8, when $p = 1$, IP-GIVPFOWAD will be degenerated to immediate probabilistic interval-valued Pythagorean fuzzy ordered weighted averaging distance IP-IVPFOWAD operator, i.e.,

$$\text{IP-IVPFOWAD}(\tilde{A}, \tilde{B}) = \sum_{i=1}^{n} \hat{\rho}_i d(\tilde{\alpha}_i, \tilde{\beta}_i). \tag{22}$$

In Eq. (21), if $\omega_i = \frac{1}{n}$, IP-GIVPFOWAD will be degenerated to generalized interval-valued Pythagorean fuzzy ordered weighted distance GIVPFOWD measure; if $p_i = \frac{1}{n}$, IP-GIVPFOWAD will degenerated to generalized interval-valued Pythagorean fuzzy ordered weighted distance measure GIVPFOWD. In the following example, we will present a numerical example to show how to use the above distance operator.

Example 3 In Example 2, since the following weight vector $\omega = (0.2, 0.3, 0.1, 0.4)$ and the probabilistic weight vector (0.3, 0.2, 0.4, 0.1). We calculate the $d^3(\tilde{\alpha}_i, \tilde{\beta}_i)$ when $p = 3$ by employing Eq. (9) as follows,

$$d^3(\tilde{\alpha}_1, \tilde{\beta}_1) = 0.0311, d^3(\tilde{\alpha}_2, \tilde{\beta}_2) = 0.0124, d^3(\tilde{\alpha}_3, \tilde{\beta}_3) = 0.0002, d^3(\tilde{\alpha}_4, \tilde{\beta}_4) = 0.0082.$$

According to the above distances of IVPFNs, we obtain the new probabilistic weight (0.3, 0.2, 0.1, 0.4) by reordering the probabilistic weight (0.3, 0.2, 0.4, 0.1),

$$\sum_{i=1}^{5} \omega_i p_i = (0.2, 0.3, 0.1, 0.4)(0.3, 0.2, 0.1, 0.4)^T = 0.29.$$

Therefore, $\hat{\rho_1} = \frac{\omega_1 p_1}{\sum_{i=1}^{5} \omega_i p_i} = \frac{0.2 \times 0.3}{0.29} = 0.2069$. Similarly, we have $\hat{\rho_2} = 0.2069$, $\hat{\rho_3} = 0.0345$, $\hat{\rho_4} = 0.5517$. Therefore, when $p = 3$, we have

$$\begin{aligned}\text{IP-GIVPFOWAD}(\tilde{A}, \tilde{B}) &= (0.2069 \times 0.0311 + 0.2069 \times 0.0124 \\ &\quad + 0.0345 \times 0.0002 \quad + 0.5517 \times 0.082)^{\frac{1}{3}} \\ &= 0.3786.\end{aligned}$$

Remark 2 P-GIVPFOWAD and IP-GIVPFOWAD operator, which are distance aggregation operators and also new distance measures by using OWA operator and probability information. P-GIVPFOWAD operator is a distance measure which unifies the probability and the OWA operator in the same formulation considering the degree of importance of each concept in the aggregation, it also uses information represented in the form of IVPFNs. However, IP-GIVPFOWAD operator also unifies both concepts, but it is more strict because it does not allow different degrees of importance between the OWA operators and probability. Compared with some existing weighted distance measures, these new interval-valued Pythagorean fuzzy distance measures can deal with more complex decision-making problems which include uncertain information evaluated with the IVPFNs, the probability information and OWA operator.

In the process of decision making, the aggregation results would be more reliable if the selected operator is monotonic, the lack of monotonicity may debase the reliability and dependability of the final decision-making results. However, we can prove P-GIVPFOWAD and IP-GIVPFOWAD are bounded, monotonic and reflexivity. The proof of this result is similar to Theorems 1–3 in [13].

Theorem 2 Let D be a P-GIVPFOWAD operator or an IP-GIVPFOWAD operator, then D satisfies the following properties:

(1) (**Boundary**) $min_i\{d^p(\tilde{\alpha}_i, \tilde{\beta}_i)\} \le D(\tilde{A}, \tilde{B}) \le max_i\{d^p(\tilde{\alpha}_i, \tilde{\beta}_i)\}$ $(i = 1, 2, \ldots, n)$.

(2) (**Monotonicity**) Let $\tilde{C} = \{\tilde{\gamma}_1, \ldots, \tilde{\gamma}_n\}$ be a set of PFNs, if $d^p(\tilde{\alpha}_i, \tilde{\beta}_i) \ge d^p(\tilde{\alpha}_i, \tilde{\gamma}_i)$, then $D(\tilde{A}, \tilde{B}) \ge D(\tilde{A}, \tilde{C})$.

(3) (**Reflexivity**) $D(\tilde{A}, \tilde{A}) = 0$.

5 An Approach to MCDM with Interval-Valued Pythagorean Fuzzy Information

Multiple criteria decision-making (MCDM) is considered as a complex decision-making tool involving both quantitative and qualitative factors. The practical purpose of the proposed method in this section aims to handle the MCDM problem under interval-valued Pythagorean fuzzy environment, especially on MCDM problems with the subjective information and the attitudinal character of the decision maker(s). A multi-criteria decision-making problems with interval-valued Pythagorean fuzzy information can be described as follows:

Let $X = \{x_1, x_2, \ldots, x_m\}$ be a set of m alternatives, $C = \{C_1, C_2, \ldots, C_n\}$ the set of criteria and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^{\mathrm{T}}$ be the weight vector of all criteria, which satisfy $0 \le \omega_i \le 1$ with $\sum_i^n \omega_i = 1$. Assume that the performance of alternative $x_i(i = 1, 2, \ldots, m)$ with respect to the criteria $C_j(j = 1, 2, \ldots, n)$ are measured by IVPFNs $C_j(x_i) = ([\mu_{ij}^-, \mu_{ij}^+], [v_{ij}^-, v_{ij}^+])(j = 1, 2, \ldots, n; i =, 2, \ldots, m)$ and $R_{m\times n} = (C_j(x_i))_{m\times n}$ is an IVPF decision matrix. On the basis of the P-GIVPFOWAD or IP-GIVPFOWAD, we develop a new MCDM approach, in which both the subjective information and the attitudinal character of the decision maker(s) are considered. The method can be described by the following steps:

Step 1 According to the decision matrix $R_{m\times n} = (C_j(x_i))_{m\times n}$, compute the scores and accuracies of IVPFNs by applying Eqs. (5) and (6).

Step 2 Determine the IVPF PIS and the IVPF NIS according to the scores and accuracies of IVPFNs which given in Step 1. We denote the IVPF PIS by x^+ and IVPF NIS by x^-, which can be determined by the following formula:

$$x^+ = \{\langle x_i, \max_i s(C_j(x_i))\rangle | j = 1, 2, \ldots, n; i = 1, 2, \ldots, m\}, \tag{23}$$

$$x^+ = \{\langle x_i, \min_i s(C_j(x_i))\rangle | j = 1, 2, \ldots, n; i = 1, 2, \ldots, m\}. \tag{24}$$

Step 3 Calculate the distance between two IVPFNs, which are in X and IVPFNs in $x^+(x^-)$ according to Eq. (9), respectively.

Step 4 According to the results of Step 3, rearrange the order of probability weight and compute the new weights according to Eqs. (19) or (21).

Step 5 According to Step 4, compute the P-GIVPFOWAD or IP-GIVPFOWAD of the positive ideal IVPFS x^+ and alternative x_i and the negative ideal IVPFS x^- and alternative x_i.

Step 6 Similar to classical TOPSIS method, we need to calculate the relative closeness of the alternative x_i as below:

$$\theta(x_i) = \frac{D(x_i, x^-)}{D(x_i, x^-) + D(x_i, x^+)} \tag{25}$$

where $D(\ .\ )$ is an IP-GIVPFOWAD or P-GIVPFOWAD.

Step 7 According to the relative closeness index $\theta(x_i)$, the ranking orders of all alternatives and the optimal alternatives can be determined. And the priority of the alternative $x_i(i = 1, 2, \ldots, m)$ are gotten by ranking $\theta(x_i)(i = 1, 2, \ldots, m)$.

6 Numerical Example

In this section, we show the application of developed IP-IVPFOWAD and P-IVPFOWAD through a practical example about the optimal production strategy, the background is from [35]. Assume a company wants to create a new product and they are analyzing the optimal target in order to obtain the highest benefits. What kind of customers should the new product exactly oriented to?

After analyzing the market, there are five possible strategies to be followed: (1) x_1: the new product oriented to the high-income customers; (2) x_2: the new product oriented to the mid-income customers; (3) x_3: the new product oriented to the low-income customers; (4) x_4: the new product adapted to all customers; (5) x_5: Not creating any products. After careful review of the information, the decision maker establishes the following general information about the production strategy. He (she) has summarized the information of the strategies in five general characteristics: (1) S_1 Benefits in the short term; (2) S_2 Benefits in the mid-term; (3) S_3 Benefits in the long term; (4) S_4 Risk of the production strategy; (5) S_5 Other factors. The five possible strategies $x_i(i = 1, 2, \ldots, 5)$ are to be evaluated applying the IVPFNs by the decision makers under the above five general characteristics, and construct the decision matrix as shown in Table 1.

With respect to this problem, the experts in the company find probabilistic information given as follows: $p = (0.3, 0.3, 0.2, 0.1, 0.1)$. They assume that the WA, that represents the degree of importance of each characteristics, is $\omega = (0.2, 0.25, 0.15, 0.3, 0.1)$. Now we make decision by applying the proposed method in the Sect. 5.

6.1 Process of MCDM Based on the Proposed Method

- (1) Process of MCDM based on IP-IVPFOWAD.

Determine the interval-valued Pythagorean fuzzy PIS x^+ and the interval-valued Pythagorean fuzzy NIS x^- by the score function and accuracy function. The result is shown Table 2.

From Table 2, we can see that $s_j(S_1)(j = 1, 2, \ldots, 5)$ all are different, so do $s_j(S_2), s_j(S_3), s_j(S_4), s_j(S_5)(j = 1, 2, \ldots, 5)$. Therefore, it is not necessary to calculate the accuracy

Table 1 Interval-valued Pythagorean fuzzy decision matrix

	S_1	S_2	S_3	S_4	S_5
x_1	([0.7,0.8], [0.2,0.3])	([0.7,0.8], [0.2,0.3])	([0.8,0.9], [0.3,0.4])	([0.6,0.7], [0.3,0.4])	([0.7,0.8], [0.4,0.5])
x_2	([0.4,0.5], [0.6,0.7])	([0.3,0.4], [0.6,0.7])	([0.6,0.7], [0.4,0.5])	([0.7,0.8], [0.3,0.5])	([0.6,0.7], [0.4,0.6])
x_3	([0.7,0.8], [0.2,0.4])	([0.5,0.7], [0.4,0.5])	([0.6,0.8], [0.3,0.4])	([0.4,0.5], [0.5,0.6])	([0.5,0.6], [0.4,0.5])
x_4	([0.5,0.7], [0.5,0.6])	([0.5,0.6], [0.3,0.4])	([0.5,0.6], [0.4,0.6])	([0.5,0.6], [0.4,0.5])	([0.5,0.7], [0.3,0.4])
x_5	([0.6,0.8], [0.2,0.4])	([0.5,0.7], [0.2,0.3])	([0.6,0.8], [0.2,0.3])	([0.3,0.5], [0.5,0.6])	([0.6,0.7], [0.2,0.4])

of the IVPFNs. And so, we can obtain the interval-valued Pythagorean fuzzy PIS x^+ and the interval-valued Pythagorean fuzzy NIS x^- as follows:

$$\begin{aligned} x^+ = \{ & \langle S_1, ([0.7,0.8],[0.2,0.3])\rangle, \langle S_2, ([0.7,0.8],[0.2,0.3])\rangle, \\ & \langle S_3, ([0.8,0.9],[0.3,0.4])\rangle, \langle S_4, ([0.7,0.8],[0.3,0.5])\rangle, \\ & \langle S_5, ([0.7,0.8],[0.4,0.5])\rangle\}, \\ x^- = \{ & \langle S_1, ([0.4,0.5],[0.6,0.7])\rangle, \langle S_2, ([0.3,0.4],[0.6,0.7])\rangle, \\ & \langle S_3, ([0.5,0.6],[0.4,0.6])\rangle, \langle S_4, ([0.3,0.5],[0.5,0.6])\rangle, \\ & \langle S_5, ([0.5,0.6],[0.4,0.5])\rangle\}. \end{aligned}$$

We can use Eqs. (9), (21) and (20) to calculate the IP-GIVPFOWAD$(x^+, x_i)(i = 1,2,3,4,5)$. For convenience, we denote IP-GIVPFOWAD(x_i, x^+) and IP-GIVPFOWAD(x_i, x^-) as $D(x_i, x^+)$ and $D(x_i, x^-)$, respectively. The results are found in Tables 3 and 4.

Calculate the relative closeness of $x_i (i = 1,2,3,4,5)$ according to Eq. (25). The results are found in Table 5.

From Table 5, we can see that the ranking are the same by applying the IP-IVPFOWAD when parameter p changes. All of the results show that x_1 is the best alternative.

- (2) Process of MCDM based on P-IVPFOWAD.

If we can consider the WA an importance of 40% and the probabilistic information has an importance of 60%. Applying the generalized probabilistic interval-valued Pythagorean fuzzy ordered weighted averaging distance to decision making. We can use Eqs. (9), (18) and (17) to calculate the P-GIVPFOWAD$(x^+, x_i)(i = 1,2,3,4,5)$. For convenience, we denote P-GIVPFOWAD(x_i, x^+) and P-GIVPFOWAD(x_i, x^-) as $D(x_i, x^+)$ and $D(x_i, x^-)$, respectively. The results are found in Tables 6 and 7.

Table 2 The results by applying score function

	S_1	S_2	S_3	S_4	S_5
s_1	0.5	0.5	0.6	0.3	0.36
s_2	−0.22	−0.3	0.22	0.395	0.165
s_3	0.465	0.165	0.375	−0.1	0.1
s_4	0.065	0.18	0.045	0.1	0.245
s_5	0.4	0.305	0.435	−0.135	0.325

Calculate the relative closeness of $x_i (i = 1,2,3,4,5)$ according to Eq. (26). The results are found in Table 8.

From Table 8, we can see that the ranking are the same by applying the generalized probabilistic interval-valued Pythagorean fuzzy distance when parameter changes. All of the results show that x_1 is the best alternative.

6.2 Validity Test of the Proposed Method

Since practically it is not possible to determine which one is the best suitable alternative for a given decision problem, Wang and Triantaphyllou [36] established the following testing criteria to evaluate the validity of MCDM methods.

Test criterion 1 An effective MCDM method should not change the indication of the best alternative on replacing a non-optimal alternative by another worse alternative without changing the relative importance of each decision criteria.

Test criterion 2 An effective MCDM method should follow transitive property.

Test criterion 3 When a MCDM problem is decomposed into some smaller problems and the same MCDM method is applied on these smaller problems to rank the alternatives, combined ranking of the alternatives should be identical to the original ranking of un-decomposed problem.

Now we choose the schedule obtained by the IP-IVPFOWAD distance operators and the satisfaction degree under $p = 2$ to be analyzed. In this situation, the ranking of original MCDM is $x_1 \succ x_3 \succ x_5 \succ x_4 \succ x_2$.

6.2.1 Validity Test of Proposed Method Applying Criterion 1

We have obtained the ranking of all alternatives in this section, x_1 is the desirable alternative and the ranking $x_1 \succ x_3 \succ x_5 \succ x_4 \succ x_2$. In order to test the validity of the proposed IP-IVPFOWAD method under criterion 1, the following interval-valued Pythagorean fuzzy decision-making matrix (Table 9.) is used. This decision making is obtained by interchanging the intervals of membership and non-membership grades of

Table 3 IP-GIVPFOWAD between x_i and x^+

	$D(x_1, x^+)$	$D(x_2, x^+)$	$D(x_3, x^+)$	$D(x_4, x^+)$	$D(x_5, x^+)$
$p = 1$	0.0176	0.3267	0.2012	0.2985	0.2262
$p = 2$	0.0582	0.2925	0.2081	0.2715	0.2269
$p = 4$	0.1055	0.3350	0.2302	0.2860	0.2484
$p = 6$	0.1345	0.360	0.2652	0.30708	0.2688
$p = 10$	0.1679	0.3895	0.2812	0.3432	0.3003

Table 4 IP-GIVPFOWAD between x_i and x^-

	$D(x_1, x^-)$	$D(x_2, x^-)$	$D(x_3, x^-)$	$D(x_4, x^-)$	$D(x_5, x^-)$
$p = 1$	0.3804	0.0814	0.2360	0.1808	0.2828
$p = 2$	0.3616	0.1344	0.3018	0.2261	0.3237
$p = 4$	0.3709	0.2022	0.3538	0.2457	0.3585
$p = 6$	0.3826	0.2457	0.3832	0.2615	0.3819
$p = 10$	0.4009	0.2944	0.4136	0.2810	0.4096

Table 5 Relative closeness obtained by IP-GIVPFOWAD

	$\theta(x_1)$	$\theta(x_2)$	$\theta(x_3)$	$\theta(x_4)$	$\theta(x_5)$	Ranking
$p = 1$	0.9558	0.1995	0.5398	0.3772	0.5556	$x_1 \succ x_5 \succ x_3 \succ x_4 \succ x_2$
$p = 2$	0.8613	0.3149	0.5919	0.4543	0.5879	$x_1 \succ x_3 \succ x_5 \succ x_4 \succ x_2$
$p = 4$	0.7786	0.3764	0.6058	0.4621	0.5907	$x_1 \succ x_3 \succ x_5 \succ x_4 \succ x_2$
$p = 6$	0.7399	0.4056	0.5910	0.4599	0.5869	$x_1 \succ x_3 \succ x_5 \succ x_4 \succ x_2$
$p = 10$	0.7048	0.4305	0.5953	0.4502	0.5770	$x_1 \succ x_3 \succ x_5 \succ x_4 \succ x_2$

Table 6 Distances between x_i and x^+ obtained by P-GIVPFOWAD

	$D(x_1, x^+)$	$D(x_2, x^+)$	$D(x_3, x^+)$	$D(x_4, x^+)$	$D(x_5, x^+)$
$p = 1$	0.0259	0.2862	0.2002	0.2933	0.2279
$p = 2$	0.0653	0.30469	0.2130	0.2720	0.2288
$p = 4$	0.1118	0.3387	0.2388	0.2865	0.2537
$p = 6$	0.1397	0.3593	0.2670	0.3075	0.2763
$p = 10$	0.1719	0.3852	0.2892	0.3433	0.3086

Table 7 The Distances between x_i and x^- obtained by P-GIVPFOWAD

	$D(x_1, x^-)$	$D(x_2, x^-)$	$D(x_3, x^-)$	$D(x_4, x^-)$	$D(x_5, x^-)$
$p = 1$	0.3667	0.1049	0.2365	0.192	0.2802
$p = 2$	0.3510	0.1544	0.2830	0.2046	0.3020
$p = 4$	0.3625	0.2180	0.3429	0.2331	0.3457
$p = 6$	0.3754	0.2586	0.3755	0.2523	0.3730
$p = 10$	0.3955	0.3037	0.4087	0.2749	0.4043

Table 8 Relative closeness obtained by P-GIVPFOWAD

	$\theta(x_1)$	$\theta(x_2)$	$\theta(x_3)$	$\theta(x_4)$	$\theta(x_5)$	Ranking
$p=1$	0.9340	0.2682	0.5416	0.3956	0.5515	$x_1 \succ x_5 \succ x_3 \succ x_4 \succ x_2$
$p=2$	0.8431	0.3363	0.5705	0.4293	0.5690	$x_1 \succ x_3 \succ x_5 \succ x_4 \succ x_2$
$p=4$	0.7643	0.3916	0.5894	0.4486	0.5767	$x_1 \succ x_3 \succ x_5 \succ x_4 \succ x_2$
$p=6$	0.7288	0.4185	0.5844	0.4507	0.5745	$x_1 \succ x_3 \succ x_5 \succ x_4 \succ x_2$
$p=10$	0.6971	0.4408	0.5856	0.4447	0.5671	$x_1 \succ x_3 \succ x_5 \succ x_4 \succ x_2$

Table 9 Modified Interval-valued Pythagorean fuzzy decision matrix

	S_1	S_2	S_3	S_4	S_5
x_1	([0.7,0.8], [0.2,0.3])	([0.7,0.8], [0.2,0.3])	([0.8,0.9], [0.3,0.4])	([0.6,0.7], [0.3,0.4])	([0.7,0.8], [0.4,0.5])
x_2	([0.4,0.5], [0.6,0.7])	([0.3,0.4], [0.6,0.7])	([0.6,0.7], [0.4,0.5])	([0.7,0.8], [0.3,0.5])	([0.6,0.7], [0.4,0.6])
x_3	([0.2,0.4], [0.7,0.8])	([0.4,0.5], [0.5,0.7])	([0.3,0.4], [0.6,0.8])	([0.5,0.6], [0.4,0.5])	([0.4,0.5], [0.5,0.6])
x_4	([0.5,0.6], [0.5,0.7])	([0.3,0.4], [0.5,0.6])	([0.4,0.6], [0.5,0.6])	([0.4,0.5], [0.5,0.6])	([0.3,0.4], [0.5,0.7])
x_5	([0.6,0.8], [0.2,0.4])	([0.5,0.7], [0.2,0.3])	([0.6,0.8], [0.2,0.3])	([0.3,0.5], [0.5,0.6])	([0.6,0.7], [0.2,0.4])

alternative x_3 (non-optimal)and x_4 (less desirable than x_3) in the original decision-making matrix (Table 1).

Since, the relative importance of the criteria remains unchanged in the modified MCDM problems. It follows from the score function and accuracy function that the modified interval-valued Pythagorean fuzzy PIS x^+ and the interval-valued Pythagorean fuzzy NIS x^- as follows:

$$x^+ = \{\langle S_1, ([0.7,0.8],[0.2,0.3])\rangle, \langle S_2, ([0.7,0.8],[0.2,0.3])\rangle, \langle S_3, ([0.8,0.9],[0.3,0.4])\rangle, \langle S_4, ([0.7,0.8],[0.3,0.5])\rangle, \langle S_5, ([0.7,0.8],[0.4,0.5])\rangle\},$$
$$x^- = \{\langle S_1, ([0.2,0.4],[0.7,0.8])\rangle, \langle S_2, ([0.3,0.4],[0.6,0.7])\rangle, \langle S_3, ([0.3,0.4],[0.6,0.8])\rangle, \langle S_4, ([0.3,0.5],[0.5,0.6])\rangle, \langle S_5, ([0.3,0.4],[0.5,0.7])\rangle\}.$$

Applying Eqs. (9), (20), and (21) of IP-IVPFOWAD method, the following distance measures of the alternatives $x_i (i = 1, 2, \ldots, 5)$ are calculated:

$$D(x_1, x^+) = 0.0582, D(x_2, x^+) = 0.3313, D(x_3, x^+) = 0.4369, D(x_4, x^+) = 0.3659, D(x_5, x^+) = 0.2269.$$
$$D(x_1, x^-) = 0.4562, D(x_2, x^-) = 0.2179, D(x_3, x^-) = 0.2701, D(x_4, x^-) = 0.4318, D(x_5, x^-) = 0.7231.$$

According to Eq. (26), we have

$$\theta(x_1) = 0.8868, \theta(x_2) = 0.3967, \theta(x_3) = 0.3820, \theta(x_4) = 0.5413, \theta(x_5) = 0.7611.$$

Therefore, the alternatives are ranked as $x_1 \succ x_5 \succ x_4 \succ x_2 \succ x_3$, which also showed x_1 is the most desirable alternative. That is, according to the test criterion 1, the indication of the best alternative not change when the alternatives are ranked again by the same method. The same should also be true for the relative rankings of the rest of the unchanged alternatives.

6.2.2 Validity Test of Proposed Method Using Criterion 2 and Criterion 3

In order to test validity of proposed method using test criterion 2 and test criterion 3, original MCDM problem is decomposed into a set of smaller MCDM problems $\{x_1, x_2, x_3, x_4\}$ and $\{x_1, x_3, x_4, x_5\}$. Following the steps of proposed method, rankings of these two sub-problems are $x_1 \succ x_3 \succ x_4 \succ x_2$ and $x_3 \succ x_5 \succ x_4 \succ x_2$, respectively.

We combined the ranking of alternatives of sub-problems $\{x_1, x_2, x_3, x_4\}$ and $\{x_2, x_3, x_4, x_5\}$, the final ranking $x_1 \succ x_3 \succ x_5 \succ x_4 \succ x_2$ is obtained which is identical to the ranking of un-decomposed MCDM problem and also exhibits transitive property. Hence the proposed method is valid under test criterion 2 and test criterion 3.

7 Conclusions

The uncertainty probabilistic OWA distance operators [32] are generally suitable for dealing with the information taking the form of interval values, and yet they will fail in dealing with interval-valued Pythagorean fuzzy information. In this paper, with respect to probabilistic decision-making problems with interval-valued Pythagorean fuzzy information, new multiple criteria decision-making method is developed. Specifically, GIVPFWD and GIVPFOWD are developed first. In addition, we introduced new P-GIVPFOWAD and IP-GIVPFOWAD operators which unify the OWA operators, the probability information and also use information represent in the form of interval-valued Pythagorean fuzzy numbers. Consequently, the method for MCDM problem with interval-valued Pythagorean fuzzy information is constructed based on the some proposed distance operators. Finally, some illustrative examples have been given to show the developed method and analyzed the validity of the proposed MCDM method. In future research, we expect to further develop the theories of (interval-valued) Pythagorean fuzzy sets and its relative applications in business decision-making.

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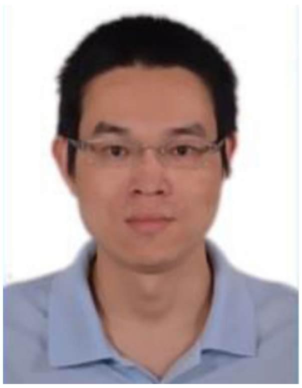
Authors' Contributions YL proposed the ideas of decision making, defined the relevant distance operators, constructed the decision-making method and wrote the paper. YQ main contribution is that made the validity analysis of the decision-making method. YH checked some computing and ensured the correctness of calculation, and also checked the final reversion.

References

1. Zadeh, L.A.: Fuzzy sets. Inf. Comput. **8**, 338–353 (1965)
2. Gao, J., Xu, Z.S., Liao, H.C.: A dynamic reference point method for emergency response under hesitant probabilistic fuzzy environment. Int. J. Fuzzy Syst. **5**, 12–15 (2017). doi:10.1007/s40815-017-0311-4
3. Garg, H.: A novel accuracy function under interval-valued Pythagorean fuzzy environment for solving multi-criteria decision making problem. J. Intell. Fuzzy Syst. **31**, 529–540 (2016a)
4. Liao, H., Xu, Z.S., Zeng, X.J.: Distance and similarity measures for hesitant fuzzy linguistic term sets and their application in multi-criteria decision making. Inf. Sci. **271**, 125–142 (2014)
5. Liao, H., Xu, Z.S.: Intuitionistic fuzzy hybrid weighted aggregation operators. Int. J. Intell. Syst. **29**, 971–993 (2014)
6. Liao, H., Xu, Z.S.: Approaches to manage hesitant fuzzy linguistic information based on the cosine distance and similarity measures for HFLTSs and their application in qualitative decision making. Expert Syst. Appl. **42**, 5328–5336 (2015a)
7. Liao, H., Xu, Z.S.: Consistency of the fused intuitionistic fuzzy preference relation in group intuitionistic fuzzy analytic hierarchy process. Appl. Soft Comput. **35**, 812–826 (2015b)
8. Liao, H., Xu, Z.S., Zeng, X.J., Xu, D.L.: An enhanced consensus reaching process in group decision making with intuitionistic fuzzy preference relations. Inf. Sci. **329**, 274–286 (2016)
9. Liu, W.S., Liao, H.: A bibliometric analysis of fuzzy decision research during 1970–2015. Int. J. Fuzzy Syst. **19**, 1–14 (2017)
10. Ma, Z., Xu, Z.S.: Symmetric Pythagorean fuzzy weighted geometric/averaging operators and their application in multi-criteria decision-making problems. Int. J. Intell. Syst. **31**, 1198–1219 (2016)
11. Ren, P., Xu, Z.S., Gou, X.: Pythagorean fuzzy TODIM approach to multi-criteria decision making. Appl. Soft Comput. **42**, 246–259 (2016)
12. Yager, R.R.: Pythagorean membership grades in multi-criteria decision making. IEEE Trans. Fuzzy Syst. **22**, 958–965 (2014)
13. Zeng, S., Chen, J., Li, X.: A hybrid method for Pythagorean fuzzy multiple-criteria decision making. Int. J. Inf. Technol. Decis. Mak. **15**, 403–422 (2016)
14. Zhang, X.: Multi-criteria Pythagorean fuzzy decision analysis: A hierarchical QUALIFLEX approach with the closeness index-based ranking methods. Inf. Sci. **330**, 104–124 (2016)
15. Zhang, X.L., Xu, Z.S.: Extension of TOPSIS to multiple criteria decision making with Pythagorean fuzzy sets. Int. J. Intell. Syst. **29**, 1061–1078 (2014)
16. Attanassov, K.T.: Intuitionistic fuzzy sets. Fuzzy Sets Syst. **20**, 87–96 (1986)
17. Yager, R.R., Abbasov, A.M.: Pythagorean membership grades, complex numbers, and decision making. Int. J. Intell. Syst. **28**, 436–452 (2013)
18. Garg, H.: A new generalized Pythagorean fuzzy information aggregation using Einstein operations and its application to decision making. Int. J. Intell. Syst. **31**, 886–920 (2016b)
19. Peng, X., Yuan, H., Yang, Y.: Pythagorean fuzzy information measures and their applications. Int. J. Intell. Syst. **5**, 15–19 (2017). doi:10.1002/int.21880
20. Peng, X., Yang, Y.: Some results for Pythagorean fuzzy sets. Int. J. Intell. Syst. **30**, 1133–1160 (2015)
21. Zhang, X.L.: A novel approach based on similarity measure for Pythagorean fuzzy multiple criteria group decision making. Int. J. Intell. Syst. **31**, 593–611 (2016)
22. Garg, H.: A novel correlation coefficients between Pythagorean fuzzy sets and its applications to decision-making processes. Int. J. Intell. Syst. **31**, 1234–1252 (2016c)
23. Gou, X., Xu, Z.S., Ren, P.: The properties of continuous Pythagorean fuzzy information. Int. J. Intell. Syst. **31**, 401–424 (2016)
24. Dick, S., Yager, R.R., Yazdanbakhsh, O.: On Pythagorean and complex fuzzy set operations. IEEE Trans. Fuzzy Syst. **24**, 1009–1021 (2016)
25. Peng, X., Yang, Y.: Pythagorean fuzzy Choquet integral based MABAC method for multiple attribute group decision making. Int. J. Intell. Syst. **31**, 989–1020 (2016a)
26. Peng, X., Yang, Y.: Fundamental properties of interval-valued Pythagorean fuzzy aggregation operators. Int. J. Intell. Syst. **31**, 444–487 (2016b)
27. Du, Y., Hou, F., Zafar, W., Yu, Q.: A novel method for multi-attribute decision making with interval-valued Pythagorean fuzzy linguistic information. Int. J. Intell. Syst. (2017). doi:10.1002/int.21881
28. Xu, Z.S., Chen, J.: Ordered weighted distance measure. J. Syst. Sci. Syst. Eng. **17**, 432–445 (2008)
29. Merigo, J.M.: A unified model between the weighted average and the induced OWA operator. Expert Syst. Appl. **38**, 11560–11572 (2011a)
30. Merigo, J.M., Wei, G.: Probabilistic aggregation operators and their application in uncertain multi-person decision-making. Technol. Econ. Dev. Econ. **17**, 335–351 (2011)
31. Merigo, J.M.: Fuzzy multi-person decision making with fuzzy probabilistic aggregation operators. Int. J. Fuzzy Syst. **13**, 163–173 (2011b)
32. Zeng, S., Merigó, J.M., Su, W.: The uncertain probabilistic OWA distance operator and its application in group decision making. Appl. Math. Model. **37**, 6266–6275 (2013)

33. Yager, R.R.: On ordered weighted averaging aggregation operators in multi-criteria decision making. IEEE Trans. Syst. Man Cybern. B **18**, 183–190 (1988)
34. Yager, R.R., Engemann, K.J., Filev, D.P.: On the concept of immediate probabilitie. Int. J. Intell. Syst. **10**, 373–397 (1995)
35. Wei, G.W., Merigó, J.M.: Methods for strategic decision-making problems with immediate probabilities in intuitionistic fuzzy setting. Sci. Iran **19**, 1936–1946 (2012)
36. Wang, X., Triantaphyllou, E.: Ranking irregularities when evaluating alternatives by using some ELECTRE methods. Omega **36**, 45–63 (2008)



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