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Solving Fully Fuzzy Multi-objective Linear Programming Problem Using Nearest Interval Approximation of Fuzzy Number and Interval Programming

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Abstract This paper focuses on Fully Fuzzy Multi-Objective Linear Programming (FFMOLP) problem in which all the coefficients and decision variables are *LR* flat fuzzy numbers, the more generalized version of fuzzy numbers and all the constraints are fuzzy inequalities. A new algorithm is proposed for solving FFMOLP problem which first converts it into the Multi-Objective Interval Linear Programming (MOILP) problem. Further, taking the help of fuzzy slack variable, fuzzy surplus variables, nearest interval approximation of fuzzy numbers and scalarization technique, MOILP is then converted into the Crisp Linear Programming (CLP) problem. It is shown that the optimal solution of CLP problem is the fuzzy Pareto optimal solution of FFMOLP problem. The main advantage of the proposed algorithm is that it transforms FFMOLP problem into Crisp Linear Programming problem. Moreover, to apply algorithm, only the knowledge of arithmetic operations of *LR* flat fuzzy numbers, centre and width of the closed intervals are required. At the end, to illustrate the proposed method and its effectiveness over the existing method, numerical examples are solved and compared.

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1 Introduction

Linear Programming (LP) has several applications in the area of engineering and management. But in the real world, most of the times we do not know the precise value of the decision parameters of the LP problem. This motivates several authors to develop interest in fuzzy linear programming problems. Zimmerman [33] was the first one who incorporated the concept of fuzziness in the multi-objective LP problem. In the literature, various types of fuzzy linear programming problems have been studied by several researchers [4–7, 9, 14, 17, 22, 26, 27] in which either parameters or variables are fuzzy in nature.

A fuzzy linear programming in which all the decision parameters and variables are fuzzy is called the Fully Fuzzy Linear Programming (FFLP) problem. Lotfi et al. [23] solved the FFLP problem using lexicographical method and found the fuzzy approximation solution. Kumar et al. [19] proposed a new method for solving FFLP problem. Khan et al. [20] gave a simplified novel technique for solving FFLP problems. Kaur and Kumar [21] introduced a new method called Mehar's method for solving FFLP problem in which fuzzy numbers were *LR* flat fuzzy numbers. Cheng et al. [8] solved the FFLP problem through compromise programming problems. Nasseri et al. [25] used the membership function to solve the FFLP problem. Ezzati et al. [12] transformed the FFLP problem into multi-objective linear programming problem and found the exact optimal solution.

Mohanaselvi and Ganesan [24] solved the FFMOLP problem in which triangular fuzzy numbers were considered. They found the ideal and nonideal solution of each objective function. The fuzzy Pareto optimal solution of FFMOLP problem was obtained using the linear membership function and max–min approach. Aggarwal and Sharma [3] solved fully fuzzy multi-objective multi-choice linear programming problem in which variables and decision parameters were triangular fuzzy number and right-hand side of each constraint have two choices. Using the ranking function and deviation degree of two triangular fuzzy numbers, they obtained the δ-fuzzy Pareto optimal solution of fully fuzzy multi-objective multi-choice linear programming problem. Hadi-Vencheh et al. [15] solved the FFMOLP problem in which constraints having crisp equality and fuzzy numbers were triangular fuzzy numbers. Scalarization technique was used to transform the FFMOLP problem into the single objective fuzzy linear programming problem. Further, the fuzzy objective function was converted into the crisp one, and component wise fuzzy constraints were compared. It was shown that the optimal solution of the CLP problem is the fuzzy Pareto optimal solution of FFMOLP problem. Jayalakshmi and Pandian [18] found the proper efficient solution of FFMOLP problem using "Total objective-segregation" method. In 2015, Das [11] solved the FFMOLP problem using triangular fuzzy numbers and fuzzy inequality constraints. He [11] converted the k fuzzy objective function into the $3k$ crisp objective functions, m fuzzy constraints into the $3m$ crisp constraints and n fuzzy variables into n constraints to obtain the fuzzy optimal solution of FFMOLP problem. Aggarwal and Sharma [1] solved fully fuzzy multi-objective multi-choice linear programming problem in which all the coefficients and decision variables were trapezoidal fuzzy numbers and all the constraints were fuzzy equality or inequality. A new similarity measure was introduced for trapezoidal fuzzy numbers. They used the magnitude of trapezoidal fuzzy number to obtain the fuzzy Pareto optimal solution of fully fuzzy multi-objective multi-choice linear programming problem. Aggarwal and Sharma [2] solved the FFMOLP problem in which fuzzy numbers were triangular fuzzy numbers and converted the FFMOLP problem into the nonlinear programming problem using the deviation degree of two closed intervals. Aggarwal and Sharma [1, 2] solved $(2k+1)$ nonlinear programming problems in order to find the fuzzy Pareto optimal solution of k objective FFMOLP problem.

Aggarwal and Sharma [3] developed a method by replacing fuzzy numbers into real numbers using ranking function, due to which most of the information is lost. Decision Maker (DM) have to solve $(2k+1)$ linear or nonlinear programming problems by the method given by Mohanaselvi et al. [24] and Aggarwal et al. [2], respectively, to solve k objective FFMOLP problem. Methods proposed by Aggarwal and Sharma [1] and Das [11] can solve only those FFMOLP problems in which fuzzy numbers are triangular and trapezoidal fuzzy numbers, respectively. All the methods which are mentioned above cannot solve the FFMOLP problem in which fuzzy numbers are *LR* flat fuzzy numbers. This motivates us to solve the FFMOLP problem having decision variables and parameters as *LR* flat fuzzy numbers. The aim of this paper is to introduce an algorithm for solving FFMOLP problem in which it is first converted into the MOILP problem using nearest interval approximation of fuzzy numbers. Then, using scalarization technique, MOILP problem is transformed into linear programming problem. The proposed algorithm has following advantages:

1. FFMOLP problem transforms into linear programming problem,
2. DM has to solve only one linear programming problem,
3. it avoids the pitfall of fuzziness,
4. it is easy to apply as DM needs to know only arithmetic operations of *LR* flat fuzzy numbers and fuzzy nearest interval approximation.

At the end, numerical examples are solved and compared with the existing method [18]. This paper is organized as follows: in Sect. 2, we have given some basic definitions related to closed interval and fuzzy set theory; Sect. 3 provides a new method for solving FFMOLP problem and obtaining fuzzy Pareto optimal solution; in Sect. 4, numerical examples are solved and compared with the existing method [18]; finally, the conclusion is drawn in Sect. 5.

2 Preliminaries

In this section, some basic definitions of closed intervals, *LR* flat fuzzy numbers and arithmetic operations of *LR* flat fuzzy numbers related to fuzzy set theory are reviewed.

Definition 1 [28] Let $A=[a^l,a^u]$ be a closed interval. The centre and width of A are defined as $m(A)=\frac{a^l+a^u}{2}$ and $w(A)=\frac{a^u-a^l}{2}$ respectively.

Remark 1 Let $A=[a^l,a^u]$ be a closed interval. The closed interval can also be represented by its centre and width as $A=\langle m(A),w(A)\rangle$.

Definition 2 [16] Let $A=[a^l,a^u]$ and $B=[b^l,b^u]$ be the closed interval. The order relations between two closed intervals A and B are defined as follows:

1. $A \le_{mw} B$ iff $m(A) \le m(B)$ and $w(A) \ge w(B)$,
2. $A <_{mw} B$ iff $m(A) < m(B)$ and $w(A) > w(B)$,
3. $A =_{mw} B$ iff $m(A) = m(B)$ and $w(A) = w(B)$.

Definition 3 [19] The characteristic function μ_A of a crisp set $A \subseteq X$ assigns a value either 0 or 1 to each member in X. This function can be generalized to a function $\mu_{\widetilde{A}}$ such that the value assigned to the element of the universal set X fall within a specified range, i.e. $\mu_{\widetilde{A}} : X \to [0, 1]$. The assigned value indicates the membership grade of the element in the set A. The function $\mu_{\widetilde{A}}$ is called the membership function, and the set

$$\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x)) : x \in X\}$$

defined by $\mu_{\widetilde{A}}(x)$ for each $x \in X$ is called a fuzzy set.

Definition 4 [13] A fuzzy subset $\widetilde{A}$ of the real line R with membership function $\mu_{\widetilde{A}} : X \to [0, 1]$ is called a fuzzy number if

1. $\widetilde{A}$ is normal, i.e. there exist an element x such that $\mu_{\widetilde{A}}(x) = 1$;
2. $\widetilde{A}$ is fuzzy convex, i.e. $\mu_{\widetilde{A}}(\lambda x + (1-\lambda)y) \ge \mu_{\widetilde{A}}(x) \wedge \mu_{\widetilde{A}}(y)\ \forall\, x, y \in R,\ \forall\, \lambda \in [0, 1]$;
3. $\mu_{\widetilde{A}}$ is upper semi-continuous;
4. $\mathrm{supp}\widetilde{A}$ is bounded, where $\mathrm{supp}(\widetilde{A}) = \overline{\{x \in R : \mu_{\widetilde{A}}(x) > 0\}}$, i.e. closure of the set $\{x \in R : \mu_{\widetilde{A}}(x) > 0\}$.

Definition 5 [10] A function $L : [0, \infty) \to [0, 1]\,(or\, R : [0, \infty) \to [0, 1])$ is said to be reference function of fuzzy number if and only if

1. $L(0) = 1\,(or\, R(0) = 1)$.
2. $L(or\, R) is\, nonincreasing\, function\, on\, [0, \infty)$.

Definition 6 [10] A fuzzy number $\widetilde{A}$, defined on universal set of real numbers $\Re$, denoted by $(m, n, \alpha_1, \alpha_2)_{LR}$ is said to be an LR flat fuzzy number if its membership function is given by

$$\mu_{\widetilde{A}}(x) = \begin{cases} L\left(\dfrac{m-x}{\alpha_1}\right), & x \le m, \alpha_1 > 0, \\ R\left(\dfrac{x-n}{\alpha_2}\right), & x \ge n, \alpha_2 > 0, \\ 1, & \text{otherwise}. \end{cases}$$

Remark 2 Three nonlinear reference functions which are commonly used in the literature using the real number q are as follows:

1. $L(x) = R(x) = \max(0, 1 - x^q), q \ge 0$,
2. $L(x) = R(x) = e^{-x^q}$,
3. $L(x) = R(x) = \frac{1}{(1+x^q)}, q \ge 0$.

Remark 3 If $L(x) = R(x) = \max(0, 1 - x)$ then LR flat fuzzy numbers become trapezoidal fuzzy numbers and it is also denoted by (a_1, a_2, a_3, a_4) where $a_1 \le a_2 \le a_3 \le a_4$.

Remark 4 $F(R)$ denotes the set of all LR flat fuzzy numbers.

Definition 7 [10] An LR flat fuzzy number $\widetilde{A} = (m, n, \alpha_1, \alpha_2)_{LR}$ is said to be nonnegative LR flat fuzzy number if $m - \alpha_1 \ge 0$ and is said to be nonpositive LR flat fuzzy number if $n + \alpha_2 \le 0$.

Definition 8 [10] Let $\widetilde{A} = (m, n, \alpha_1, \alpha_2)_{LR}$ be an LR flat fuzzy number and α be real number in the interval [0, 1] then a crisp set,

$$\begin{aligned}\tilde{A}_\alpha &= \{x \in X : \mu_{\widetilde{A}} \ge \alpha\} \\ &= \{x \in X : [m - \alpha_1 L^{-1}(\alpha), n + \alpha_2 R^{-1}(\alpha)]\},\end{aligned}$$

is said to be α-cut of $\widetilde{A}$.

Definition 9 [10] Let $\widetilde{A}_1 = (m_1, n_1, \alpha_{11}, \alpha_{12})_{LR}$ and $\widetilde{A}_2 = (m_2, n_2, \alpha_{21}, \alpha_{22})_{LR}$ be any LR flat fuzzy numbers then $\widetilde{A}_1 = \widetilde{A}_2$ iff $m_1 = m_2, n_1 = n_2, \alpha_{11} = \alpha_{21}, \alpha_{12} = \alpha_{22}$.

Definition 10 [10] Let $\widetilde{A}_1 = (m_1, n_1, \alpha_{11}, \alpha_{12})_{LR}$, $\widetilde{A}_2 = (m_2, n_2, \alpha_{21}, \alpha_{22})_{LR}$ be any LR flat fuzzy numbers and $\widetilde{A}_3 = (m_3, n_3, \alpha_{31}, \alpha_{32})_{RL}$ be any RL flat fuzzy number. The arithmetic operations on be any LR flat fuzzy number are given by as follows:

1. $\widetilde{A}_1 \oplus \widetilde{A}_2 = (m_1 + m_2, n_1 + n_2, \alpha_{11} + \alpha_{21}, \alpha_{12} + \alpha_{22})_{LR}$
2. $\widetilde{A}_1 \ominus \widetilde{A}_3 = (m_1 - n_3, n_1 - m_3, \alpha_{11} + \alpha_{32}, \alpha_{12} + \alpha_{31})_{LR}$
3. If $\widetilde{A}_1$ and $\widetilde{A}_2$ both are nonnegative, then
$$\begin{aligned}\widetilde{A}_1 \otimes \widetilde{A}_3 \simeq (&m_1 m_2, n_1 n_2, m_1\alpha_{21} + \alpha_{11} m_2 - \alpha_{11}\alpha_{21}, n_1\alpha_{22} \\ &+ \alpha_{12} n_2 - \alpha_{12}\alpha_{22})_{LR}\end{aligned}$$
4. If $\widetilde{A}_1$ is nonpositive and $\widetilde{A}_2$ is nonnegative, then
$$\widetilde{A}_1 \otimes \widetilde{A}_3 \simeq (m_1 n_2, n_1 m_2, n_2\alpha_{11} - \alpha_{22} m_1, m_2\alpha_{21} - n_1\alpha_{21})_{LR}$$
5. If $\widetilde{A}_1$ is nonnegative and $\widetilde{A}_2$ is nonpositive, then
$$\widetilde{A}_1 \otimes \widetilde{A}_3 \simeq (n_1 m_2, m_1 n_2, n_1\alpha_{21} - \alpha_{12} m_2, m_1\alpha_{22} - n_2\alpha_{11})_{LR}$$
6. If $\widetilde{A}_1$ and $\widetilde{A}_2$ both are nonpositive, then
$$\begin{aligned}\widetilde{A}_1 \otimes \widetilde{A}_3 \simeq (&n_1 n_2, m_1 m_2, -n_1\alpha_{22} - \alpha_{12} n_2, \\ &- m_1\alpha_{21} - \alpha_{11} m_2)_{LR}\end{aligned}$$
7. The scalar multiplication is defined as

$$\lambda\widetilde{A}_1 = \begin{cases} (\lambda m_1, \lambda n_1, \lambda\alpha_{11}, \lambda\alpha_{12})_{LR}, & \lambda \geq 0, \\ (\lambda n_1, \lambda m_1, -\lambda\alpha_{12}, -\lambda\alpha_{11})_{RL}, & \lambda < 0. \end{cases}$$

Definition 11 [13] Let $\widetilde{A}$ and $\widetilde{B}$ be two fuzzy numbers with α-cut $[A_l(\alpha), A_u(\alpha)]$ and $[B_l(\alpha), B_u(\alpha)]$ respectively. The distance or metric d between $\widetilde{A}$ and $\widetilde{B}$ is defined as

$$d(\widetilde{A}, \widetilde{B}) = \sqrt{\begin{pmatrix} \int_0^1 (\tilde{A}_l(\alpha) - \tilde{B}_l(\alpha))^2 \mathrm{d}\alpha \\ + \int_0^1 (\tilde{A}_l(\alpha) - \tilde{B}_u(\alpha))^2 \mathrm{d}\alpha \end{pmatrix}}$$

The metric d is a particular member of the family of distance $\delta_{p,q}$ defined as follows:

$$\delta_{p,q}(\widetilde{A}, \widetilde{B}) = \begin{pmatrix} \int_0^1 (1-q)|\tilde{A}_l(\alpha) - \tilde{B}_l(\alpha)|^p \mathrm{d}\alpha \\ + \int_0^1 q|\tilde{A}_l(\alpha) - \tilde{B}_u(\alpha)|^p \mathrm{d}\alpha \end{pmatrix}^{\frac{1}{p}},$$

where $1 \leq p < \infty$ and $0 \leq q \leq 1$.

Definition 12 [13] The Nearest Interval Approximation (NIA) of fuzzy numbers $\widetilde{A}$ with respect to metric d is defined as

$$C_d(\widetilde{A}) = \left[\int_0^1 \widetilde{A}_l(\alpha)\mathrm{d}\alpha, \int_0^1 \widetilde{A}_u(\alpha)\mathrm{d}\alpha\right] = \left[(C_d(\widetilde{A}))_l, (C_d(\widetilde{A}))_u\right].$$

We can easily observe the following fuzzy order relation on $F(R)$ using Definition 2 and Definition 12:

Let $\widetilde{A}, \widetilde{B} \in F(R)$ then

1. $\widetilde{A} \preceq_{mw} \widetilde{B}$ iff $m(C_d(\widetilde{A})) \leq m(C_d(\widetilde{B}))$ and $w(C_d(\widetilde{A})) \geq w(C_d(\widetilde{B}))$,
2. $\widetilde{A} \prec_{mw} \widetilde{B}$ iff $m(C_d(\widetilde{A})) < m(C_d(\widetilde{B}))$ and $w(C_d(\widetilde{A})) > w(C_d(\widetilde{B}))$.
3. $\widetilde{A} \succeq_{mw} \widetilde{B}$ iff $m(C(\widetilde{A})) \geq m(C_d(\widetilde{B}))$ and $w(C_d(\widetilde{A})) \leq w(C_d(\widetilde{B}))$,
4. $\widetilde{A} \succ_{mw} \widetilde{B}$ iff $m(C_d(\widetilde{A})) > m(C_d(\widetilde{B}))$ and $w(C_d(\widetilde{A})) < w(C_d(\widetilde{B}))$.

Both of the above order relations $\preceq_{mw}$ and $\succeq_{mw}$are reflexive, antisymmetric and transitive, hence define partial ordering between fuzzy numbers.

Remark 5 Linearity property of NIA of fuzzy numbers:

Let $\tilde{A} = (m_1, n_1, \alpha_{11}, \alpha_{12})_{L_1R_1}$, and $\tilde{B} = (m_2, n_2, \alpha_{21}, \alpha_{22})_{L_2R_2}$ be two LR flat fuzzy numbers and k_1, k_2 be two nonnegative real numbers. Then using Definition 8, the λ cut of $\tilde{A}$, $\tilde{B}$ and $k_1\tilde{A} \oplus k_2\tilde{B}$ are as follows:

1. $A_\alpha = [m_1 - \alpha_{11}L_1^{-1}(\alpha), n_1 + \alpha_{12}R_1^{-1}(\alpha)]$
2. $B_\alpha = [m_2 - \alpha_{21}L_2^{-1}(\alpha), n_2 + \alpha_{22}R_2^{-1}(\alpha)]$
3. $(k_1A + k_2B)_\alpha = [k_1m_1 + k_2m_2 - k_1\alpha_{11}L_1^{-1}(\alpha) - k_2\alpha_{21}L_2^{-1}(\alpha), k_1n_1 + k_2n_2 + k_1\alpha_{12}R_1^{-1}(\alpha) + k_2\alpha_{22}R_2^{-1}(\alpha)]$.

Then

$$\begin{aligned} C_d(k_1\tilde{A} \oplus k_2\tilde{B}) &= \left[\int_0^1 \left(k_1m_1 + k_2m_2 - k_1\alpha_{11}L_1^{-1}(\alpha) - k_2\alpha_{21}L_2^{-1}(\alpha)\right)\mathrm{d}\alpha,\right. \\ &\quad \left.\int_0^1 \left(k_1n_1 + k_2n_2 + k_1\alpha_{12}R_1^{-1}(\alpha) + k_2\alpha_{22}R_2^{-1}(\alpha)\right)\mathrm{d}\alpha\right] \\ &= \left[\int_0^1 \left(k_1m_1 - k_1\alpha_{11}L_1^{-1}(\alpha)\right)\mathrm{d}\alpha, \int_0^1 \left(k_1n_1 + k_1\alpha_{21}R_1^{-1}(\alpha)\right)\mathrm{d}\lambda\right] \\ &\quad + \left[\int_0^1 \left(k_2m_2 - k_2\alpha_{21}L_2^{-1}(\alpha)\right)\mathrm{d}\alpha, \int_0^1 \left(k_2n_2 + k_2\alpha_{22}R_2^{-1}(\alpha)\right)\mathrm{d}\alpha\right] \\ &= k_1\left[\int_0^1 \left(m_1 - \alpha_{11}L_1^{-1}(\alpha)\right)\mathrm{d}\alpha, \int_0^1 \left(n_1 + \alpha_{12}R_1^{-1}(\alpha)\right)\mathrm{d}\alpha\right] \\ &\quad + k_2\left[\int_0^1 \left(m_2 - \alpha_{21}L_2^{-1}(\alpha)\right)\mathrm{d}\alpha, \int_0^1 \left(n_2 + \alpha_{22}R_2^{-1}(\alpha)\right)\mathrm{d}\alpha\right] \end{aligned}$$

Hence $C_d(k_1\tilde{A} \oplus k_2\tilde{B}) = k_1C_d(\tilde{A}) + k_2C_d(\tilde{B})$

Definition 13 [18] Let $\tilde{A} = (a_1, a_2, a_3)$ and $\tilde{B} = (b_1, b_2, b_3)$ are triangular fuzzy number then:

1. $\tilde{A} \approx \tilde{B}$ if and only if $a_i = b_i \, \forall \, i = 1, 2, 3$.
2. $\tilde{A} \preceq \tilde{B}$ if and only if $a_i \leq b_i \, \forall \, i = 1, 2, 3$.
3. $\tilde{A} \succeq \tilde{B}$ if and only if $a_i \geq b_i \, \forall \, i = 1, 2, 3$.

3 Fully Fuzzy Multi-Objective Linear Programming Problem

In this section, we propose a new method to solve the FFMOLP problem having parameters and variables as *LR* flat fuzzy numbers. FFMOLP problem with k objectives, m inequality constraints and n variables can be formulated as follows:

$$\left.\begin{array}{l} \text{Max } \widetilde{Z}_1(\widetilde{X}) = \sum_{j=1}^n \widetilde{c}_{1j} \otimes \widetilde{x}_j \\ \text{Max } \widetilde{Z}_2(\widetilde{X}) = \sum_{j=1}^n \widetilde{c}_{2j} \otimes \widetilde{x}_j \\ \quad \vdots \\ \text{Max } \widetilde{Z}_k(\widetilde{X}) = \sum_{j=1}^n \widetilde{c}_{kj} \otimes \widetilde{x}_j \\ \text{subject to} \\ \sum_{j=1}^n \widetilde{a}_{ij} \otimes \widetilde{x}_j \preceq \widetilde{b}_i, \quad i = 1, 2, \ldots, m_1, \\ \sum_{j=1}^n \widetilde{a}_{ij} \otimes \widetilde{x}_j \succeq \widetilde{b}_i, \quad i = m_1 + 1, m_1 + 2, \ldots, m, \\ \widetilde{X} \geq \widetilde{0}, \end{array}\right\} \quad (1)$$

where $\widetilde{C}_q = [\widetilde{c}_{qj}]_{n\times1}$, $\widetilde{X} = [\widetilde{x}_j]_{n\times1}$, $\widetilde{x}_j, \widetilde{a}_{ij}, \widetilde{b}_j, \widetilde{c}_{qj} \in F(R)$, $q = 1, 2, \ldots, k$, $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, m$. $\preceq$ and $\succeq$ are fuzzy less than equal to and fuzzy greater than equal to inequalities, respectively.

In multi-objective linear programming problem, the DM wants to achieve the optimal value of the all the objective functions simultaneously but it is very unlikely that all the objective functions will attain their optimal values concurrently due to the conflicting behaviour of the objective functions. Therefore, in the literature Pareto optimal solution is defined for multi-objective linear programming problem. In the fuzzy environment decision making, several authors [1, 3, 7, 24] have defined the fuzzy Pareto optimal solution for fuzzy multi-objective linear programming problem on the basis of various ranking functions, membership function etc. In this paper, we will define a fuzzy Pareto optimal solution based on the concept of fuzzy order relations for FFMOLP problem as follows:

Definition 14 A vector $\widetilde{X}^* = [\widetilde{x}_j^*]_{n\times 1}$ where $\widetilde{x}_j^* \in$F(R), $j = 1, 2, \ldots, n$, is said to be fuzzy Pareto optimal solution of FFMOLP problem if it satisfies the following conditions

1. $\widetilde{X}^*$ satisfies the constraints of (1),
2. There does not exist any $\widetilde{X} = [\widetilde{x}_j]_{n\times 1}$, where $\widetilde{x}_j \in$F(R), $j = 1, 2, \ldots, n$, satisfying constraints of (1) such that

$$\widetilde{Z}_q(\widetilde{X}^*) \preceq_{mw} \widetilde{Z}_q(\widetilde{X}) \quad \text{forall q} = 1, 2, \ldots, \text{k}$$
and
$$\widetilde{Z}_r(\widetilde{X}^*) \prec_{mw} \widetilde{Z}_r(\widetilde{X}) \quad \text{foratleastone r} = 1, 2, \ldots, \text{k}.$$

3.1 Procedure to Solve Fully Fuzzy Multi-Objective Linear Programming Problem

In this subsection, an algorithm to solve the fully fuzzy multi-objective linear programming problem is described. The steps of the proposed algorithm are given as follows:

Step 1: Add the fuzzy slack and fuzzy surplus variable to convert the fuzzy inequality constraints into the fuzzy equality constraints.

$$\left.\begin{array}{l}
\text{Max } \widetilde{Z}_1(\widetilde{X}) = \sum_{j=1}^n \widetilde{c}_{1j} \otimes \widetilde{x}_j \\
\text{Max } \widetilde{Z}_2(\widetilde{X}) = \sum_{j=1}^n \widetilde{c}_{2j} \otimes \widetilde{x}_j \\
\quad \vdots \\
\text{Max } \widetilde{Z}_k(\widetilde{X}) = \sum_{j=1}^n \widetilde{c}_{kj} \otimes \widetilde{x}_j \\
\text{subject to} \\
\sum_{j=1}^n \widetilde{a}_{ij} \otimes \widetilde{x}_j \oplus \tilde{s}_i \cong \widetilde{b}_i, \\
\qquad\qquad i = 1, 2, \ldots, m_1, \\
\sum_{j=1}^n \widetilde{a}_{ij} \otimes \widetilde{x}_j \ominus \tilde{s}_i \cong \widetilde{b}_i, \\
\qquad\qquad i = m_1 + 1, m_1 + 2, \ldots, m, \\
\widetilde{X} \geq \widetilde{0}, \\
\widetilde{S} \geq \widetilde{0},
\end{array}\right\} \quad (2)$$

where $\tilde{S} = [\tilde{s}_i]_{m\times 1}, \tilde{s}_i \in F(R) \forall i = 1, 2, \ldots, m.$

Step 2: Assuming $\sum_{j=1}^n \widetilde{c}_{qj} \otimes \widetilde{x}_j = (u_q', v_q', \gamma_q', \delta_q')_{LR}$, $\sum_{j=1}^n \tilde{a}_{ij} \otimes \tilde{x}_j \oplus s_i = (m_i', n_i', \alpha_i', \beta_i')_{LR}$ and $\sum_{j=1}^n \tilde{a}_{ij} \otimes \tilde{x}_j \ominus s_i = (m_i'', n_i'', \alpha_i'', \beta_i'')_{LR}$, using Definition 10, (2) obtained in step 1 can be written as :

$$\left.\begin{array}{l}
\text{Max } \widetilde{Z}_1(\widetilde{X}) = \left(u_1', v_1', \gamma_1', \delta_1'\right)_{LR} \\
\text{Max } \widetilde{Z}_2(\widetilde{X}) = \left(u_2', v_2', \gamma_2', \delta_2'\right)_{LR} \\
\qquad\qquad \vdots \\
\text{Max } \widetilde{Z}_k(\widetilde{X}) = \left(u_k', v_k', \gamma_k', \delta_k'\right)_{LR} \\
\text{subject to} \\
\left(m_i', n_i', \alpha_i', \beta_i'\right)_{LR} \cong \left(b_{i1}', b_{i2}', \zeta_i', \eta_i'\right)_{LR}, \\
\qquad\qquad i = 1, 2, \ldots, m_1, \\
\left(m_i'', n_i'', \alpha_i'', \beta_i''\right)_{LR} \cong \left(b_{i1}'', b_{i2}'', \zeta_i'', \eta_i''\right)_{LR}, \\
\qquad\qquad i = m_1 + 1, m_1 + 2, \ldots, m, \\
\widetilde{X} \geq \widetilde{0}, \\
\widetilde{S} \geq \widetilde{0}.
\end{array}\right\} \quad (3)$$

Step 3: Converting (3) obtained in Step 2, into MOLP problem with the help of Definition 12, obtain the following problem:

$$\left.\begin{array}{l}
\text{Max } \widetilde{Z}_1(\tilde{X}) = [Z_1^l, Z_1^u] \\
\text{Max } \widetilde{Z}_2(\tilde{X}) = [Z_2^l, Z_2^u] \\
\quad \vdots \\
\text{Max } \widetilde{Z}_k(\tilde{X}) = [Z_k^l, Z_k^u] \\
\text{subject to} \\
\left[A_i^l, A_i^u\right] = \left[b_i^l, b_i^u\right], \quad i = 1, 2, \ldots, m_1, \\
\left[A_i^l, A_i^u\right] = \left[b_i^l, b_i^u\right], \quad i = m_1 + 1, m_1 + 2, \ldots, m, \\
x_{j1} - \alpha_j, x_{j2} - x_{j1} \geq 0, \quad j = 1, 2, \ldots, n, \\
\alpha_j, \beta_j > 0, \quad j = 1, 2, \ldots, n, \\
s_{i1}, s_{i2} - s_{i1} \geq 0, \quad i = 1, 2, \ldots, m, \\
\gamma_i, \delta_i > 0, \quad i = 1, 2, \ldots, m,
\end{array}\right\} \quad (4)$$

where

$$\left[Z_q^l, Z_q^u\right] = C_d\left(\left(u_q', v_q', \gamma_q', \delta_q'\right)_{LR}\right) \qquad \forall q = 1, 2, \ldots, k,$$
$$\left[A_i^l, A_i^u\right] = C_d\left((m_i', n_i', \alpha_i', \beta_i')_{LR}\right) \qquad \forall i = 1, 2, \ldots, m,$$
$$\left[b_i^l, b_i^u\right] = C_d\left((b_{i1}', b_{i2}', \zeta_i', \eta_i')_{LR}\right) \qquad \forall i = 1, 2, \ldots, m.$$

Step 4: Regarding to Definition 1, (4) can be written as follows:

$$\left.\begin{array}{l}
\text{Max } \tilde{Z}_1(\tilde{X}) = [m(\tilde{Z}_1(\tilde{X})), w(\tilde{Z}_1(\tilde{X}))] \\
\text{Max } \tilde{Z}_2(\tilde{X}) = [m(\tilde{Z}_2(\tilde{X})), w(\tilde{Z}_2(\tilde{X}))] \\
\quad \vdots \\
\text{Max } \tilde{Z}_k(\tilde{X}) = [m(\tilde{Z}_k(\tilde{X})), w(\tilde{Z}_k(\tilde{X}))] \\
\text{subject to} \\
\left[\frac{A_i^l + A_i^u}{2}, \frac{A_i^u - A_i^l}{2}\right] =_{mw} \left[\frac{b_i^l + b_i^u}{2}, \frac{b_i^U - b_i^L}{2}\right], \\
\qquad\qquad i = 1,2,\ldots,m_1, \\
\left[\frac{A_i^l + A_i^u}{2}, \frac{A_i^u - A_i^l}{2}\right] =_{mw} \left[\frac{b_i^l + b_i^u}{2}, \frac{b_i^u - b_i^l}{2}\right] \\
\qquad\qquad i = m_1+1, m_1+2,\ldots,m, \\
x_{j1} - \alpha_j, x_{j2} - x_{j1} \geq 0, \quad j = 1,2,\ldots,n, \\
\qquad \alpha_j, \beta_j > 0, \quad j = 1,2,\ldots,n, \\
s_{i1}, s_{i2} - s_{i1} \geq 0, \quad i = 1,2,\ldots,m, \\
\gamma_i, \delta_i > 0, \quad i = 1,2,\ldots,m.
\end{array}\right\} \quad (5)$$

Step 5: Regarding to Definition 2 and using scalarization technique, (5) can be written as:

$$\left.\begin{array}{l}
\text{Max } \sum_{q=1}^{k} \lambda_q m(\tilde{Z}_q(\tilde{X})) - \sum_{q=1}^{k} \mu_q w(\tilde{Z}_q(\tilde{X})) \\
\text{subject to} \\
\left[\frac{A_i^l + A_i^u}{2}\right] = \left[\frac{b_i^l + b_i^u}{2}\right], \; i = 1,2,\ldots,m_1, \\
\left[\frac{A_i^u - A_i^l}{2}\right] = \left[\frac{b_i^u - b_i^l}{2}\right], \; i = 1,2,\ldots,m_1, \\
\left[\frac{A_i^l + A_i^u}{2}\right] = \left[\frac{b_i^l + b_i^u}{2}\right], \; i = m_1+1, m_1+2,\ldots,m, \\
\left[\frac{A_i^u - A_i^l}{2}\right] = \left[\frac{b_i^u - b_i^l}{2}\right], \; i = m_1+1, m_1+2,\ldots,m, \\
\sum_{q=1}^{k} \lambda_q = 1, \\
\sum_{q=1}^{k} \mu_q = 1, \\
\lambda_q, \mu_q \geq 0, \quad q = 1,2,\ldots,k \\
x_{j1} - \alpha_j, x_{j2} - x_{j1} \geq 0, \quad j = 1,2,\ldots,n, \\
\alpha_j, \beta_j > 0, \quad j = 1,2,\ldots,n, \\
s_{i1}, s_{i2} - s_{i1} \geq 0, \quad i = 1,2,\ldots,m, \\
\gamma_i, \delta_i > 0, \quad i = 1,2,\ldots,m.
\end{array}\right\} \quad (6)$$

Step 6: Solve (6) using LINGO 14.0 and obtain the optimal solution.

Now, we will show that the optimal solution of (6) is the fuzzy Pareto optimal solution of (1).

Theorem 1 *Let* $(x^*_{jp}, \alpha^*_j, \beta^*_j, s^*_{ip}, \gamma^*_i, \delta^*_i)$ $(p = 1,2;\; j = 1,2,\ldots,n;\; i = 1,2,\ldots m)$ *be the optimal solution of* (6) *then* $\tilde{X}^* = [\tilde{x}^*_j]_{n\times 1}$, *where* $\tilde{x}^*_j = (x^*_{j1}, x^*_{j2}, \alpha^*_j, \beta^*_j)$ $(j = 1,2\ldots n)$ *will be the fuzzy Pareto optimal solution of* (1).

Proof Let if possible, $\tilde{X}^*$ not be a fuzzy Pareto optimal solution of (1), then there exist a feasible solution $\tilde{X}^\circ$, where $\tilde{X}^\circ = [\tilde{x}^\circ_j]_{n\times 1}$, $\tilde{x}^\circ_j = (x^\circ_{j1}, x^\circ_{j2}, \alpha^\circ_j, \beta^\circ_j)(j = 1,2, \ldots, n)$ of (1) such that

$$\left.\begin{array}{l}
m(\tilde{Z}_q(\tilde{X}^*)) \leq m(\tilde{Z}_q(\tilde{X}^\circ)) \\
w(\tilde{Z}_q(\tilde{X}^*)) \geq w(\tilde{Z}_q(\tilde{X}^\circ)) \\
\qquad\qquad \forall \text{q} = 1,2,\ldots,\text{k}, \\
\qquad\quad \text{and} \\
m(\tilde{Z}_r(\tilde{X}^*)) < m(\tilde{Z}_r(\tilde{X}^\circ)) \\
w(\tilde{Z}_r(\tilde{X}^*)) > w(\tilde{Z}_r(\tilde{X}^\circ)) \\
\qquad\qquad \text{for at least one } r = 1,2,\ldots,k.
\end{array}\right\} \quad (7)$$

Corresponding to the feasible solution $\tilde{X}^\circ$ of (1), there exist $\tilde{S}^\circ = [\tilde{s}^\circ_i]_{m\times 1}$, where $\tilde{s}^\circ_i = (s^\circ_{i1}, s^\circ_{i2}, \gamma^\circ_i, \delta^\circ_i)$ $i = 1,2,\ldots,n$ such that $(\tilde{X}^\circ, \tilde{S}^\circ)$ is the feasible solution of (2).

Since (3) is obtained from (2) using Definition 10, $(\tilde{X}^\circ, \tilde{S}^\circ)$ is also the feasible solution of (3).

Let S and S' be the set of all feasible solutions of (3) and (4), respectively.

We claim if $(\tilde{X}^\circ, \tilde{S}^\circ) \in$S then

$(x^\circ_{jp}, \alpha^\circ_j, \beta^\circ_j, s^\circ_{ip}, \gamma^\circ_i, \delta^\circ_i) \in S'$ $(p = 1,2;\;\; j = 1,2,\ldots,n;$ $i = 1,2,\ldots m)$.

Let $(\tilde{X}^\circ, \tilde{S}^\circ) \in S$

$\Rightarrow (\tilde{X}^\circ, \tilde{S}^\circ)$ satisfy

$(m'_u, n'_u, \alpha'_u, \beta'_u)_{LR} \cong (b'_{u1}, b'_{u2}, \zeta'_u, \eta'_u)_{LR}$, and $(m''_v, n''_v, \alpha''_v, \beta''_v)_{LR} \cong (b''_{v1}, b''_{v2}, \zeta''_v, \eta''_v)_{LR}$, for all $u = 1,2,\ldots,m_1$, and $v = m_1+1, m_1+2,\ldots,m$. $\Rightarrow$ $(x^\circ_{jp}, \alpha^\circ_j, \beta^\circ_j, s^\circ_{ip}, \gamma^\circ_i, \delta^\circ_i) \in S'$, satisfy $C_d((m'_u, n'_u, \alpha'_u, \beta'_u)_{LR}) = C_d((b'_{u1}, b'_{u2}, \zeta'_u, \eta'_u)_{LR})$, and $C_d((m''_v, n''_v, \alpha''_v, \beta''_v)_{LR}) = C_d((b''_{v1}, b''_{v2}, \zeta''_v, \eta''_v)_{LR})$, for all $u = 1,2,\ldots,m_1$, and $v = m_1+1, m_1+2,\ldots,m$.

$\Rightarrow (x^\circ_{jp}, \alpha^\circ_j, \beta^\circ_j, s^\circ_{ip}, \gamma^\circ_i, \delta^\circ_i) \in S'$,

Hence, $(x^\circ_{jp}, \alpha^\circ_j, \beta^\circ_j, s^\circ_{ip}, \gamma^\circ_i, \delta^\circ_i) \in S'$ is the feasible solution of (4).

As (4) and (5) are equivalent, $(x^\circ_{jp}, \alpha^\circ_j, \beta^\circ_j, s^\circ_{ip}, \gamma^\circ_i,$ $\delta^\circ_i)$ $(p = 1,2;\; j = 1,2,\ldots,n;\; i = 1,2,\ldots m)$ should be the feasible solution of (5).

Now, both (5) and (6) have the same constraint sets, therefore, $(x^\circ_{jp}, \alpha^\circ_j, \beta^\circ_j, s^\circ_{ip}, \gamma^\circ_i, \delta^\circ_i)$ $(p = 1,2;\; j = 1,2,\ldots,n;$ $i = 1,2,\ldots m)$ should be the feasible solution of (6).

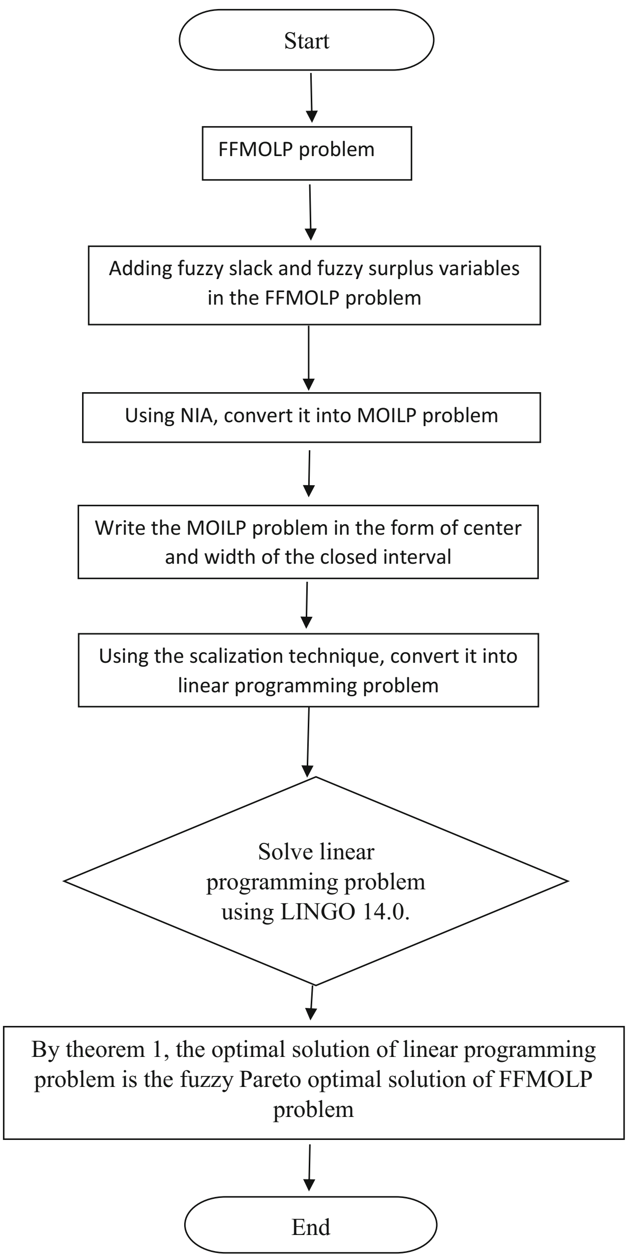
Since$(x^\circ_{jp}, \alpha^\circ_j, \beta^\circ_j, s^\circ_{ip}, \gamma^\circ_i, \delta^\circ_i)$ and $(x^*_{jp}, \alpha^*_j, \beta^*_j, s^*_{ip}, \gamma^*_i, \delta^*_i)$ $(p = 1,2;\; j = 1,2,\ldots,n;\; i = 1,2,\ldots m)$ are feasible solution of (6), hence using (7), we get,

$$\sum_{q=1}^{k}\lambda_q m(\tilde{Z}_q(\tilde{X}^*)) - \sum_{q=1}^{k}\mu_q w(\tilde{Z}_q(\tilde{X}^*)) < \sum_{q=1}^{k}\lambda_q m(\tilde{Z}_q(\tilde{X}^\circ)) - \sum_{q=1}^{k}\mu_q w(\tilde{Z}_q(\tilde{X}^\circ))$$

This is a contradiction as $(x^*_{jp}, \alpha^*_j, \beta^*_j, s^*_{ip}, \gamma^*_i, \delta^*_i)$ $(p = 1, 2;$ $j = 1, 2, \ldots, n;$ $i = 1, 2, \ldots m)$ is the optimal solution of (6).

Hence, $\tilde{X}^* = [\tilde{x}^*_j]_{n\times 1}$ is the fuzzy Pareto optimal solution of (1). □

Next, we show the steps of the proposed algorithm with help of flow chart as follows:



4 Examples

This section illustrates the proposed algorithm through examples. The effectiveness of the method is shown by comparing one example with the existing method [18].

Example 1 In a firm two variety of products, product 1 (X) and product 2 (Y) are manufactured with the help of two type of machines, machine 1 (A) and machine 2 (B), respectively. The time taken for manufacturing product X and Y on machine A and B is taken to be fuzzy data as it alters due to electricity supply, efficiency of labour and machine. The firm exports product Y and imports a raw material for product X. Due to this export and import, the firm generates a fuzzy revenue as it depends upon transportation charges. The profit is also taken to be fuzzy data as it varies from season to season. The firm desires to maximize the profit and the export–import balance, i.e. maximizing export and minimizing the import. All the data of profit, revenue, total time and time taken by machine A and B for making products X and Y are listed in Tables 1 and 2.

Let $\tilde{x}_1$ and $\tilde{x}_2$ be the units of products X and Y, respectively, manufactured with the help of machines A and B. The problem is formulated as follows:

$$\left.\begin{array}{l}\text{Max}\tilde{Z}_1(\tilde{X}) = (20,21,22,23)\tilde{x}_1 \oplus (21,23,24,25)\tilde{x}_2\\ \text{Max}\tilde{Z}_2(\tilde{X}) = \ominus(8,9,10,11)\tilde{x}_1 \oplus (12,13,14,15)\tilde{x}_2\\ \text{subject to}\\ (0.1,0.2,0.3,0.4)\tilde{x}_1 \oplus (0.2,0.3,0.4,0.5)\tilde{x}_2 \preceq (8,9,10,11)\\ (0.2,0.3,0.4,0.5)\tilde{x}_1 \oplus (0.1,0.2,0.3,0.4)\tilde{x}_2 \preceq (7,8,9,10)\\ \tilde{x}_i \succeq \tilde{0}, \qquad i = 1,2.\end{array}\right\} \tag{8}$$

where $\widetilde{X} = [\widetilde{x}_i]_{2\times 1}$, $\widetilde{x}_i = (x_i, y_i, u_i, v_i)$ is a *LR* flat fuzzy number, $\forall i = 1, 2$ (See Remark 3).

The proposed algorithm is used to solve the above FFMOLPP. Steps of the algorithm are described as follows:

Step 1: Adding the trapezoidal fuzzy slack variables $\widetilde{s}_j$, j = 1, 2. in the constraints as follows:

$$\left.\begin{array}{l}\text{Max}\tilde{Z}_1(\tilde{X}) = (20,21,22,23)\otimes\tilde{x}_1 \oplus (21,23,24,25)\otimes\tilde{x}_2\\ \text{Max}\tilde{Z}_2(\tilde{X}) = \ominus(8,9,10,11)\otimes\tilde{x}_1 \oplus (12,13,14,15)\otimes\tilde{x}_2\\ \text{subject to}\\ (0.1,0.2,0.3,0.4)\otimes\tilde{x}_1 \oplus (0.2,0.3,0.4,0.5)\otimes\tilde{x}_2 \oplus \tilde{s}_1\\ \cong (8,9,10,11)\\ (0.2,0.3,0.4,0.5)\otimes\tilde{x}_1 \oplus (0.1,0.2,0.3,0.4)\otimes\tilde{x}_2 \oplus \tilde{s}_2\\ \cong (7,8,9,10)\\ \tilde{s}_j, \tilde{x}_i \succeq \tilde{0}, \quad i,j = 1,2.\end{array}\right\} \tag{9}$$

Step 2: Converting (9) into MOLIP problem, with the help of Definitions 10 and 12, the problem obtained in step 1 is rewritten as:

Table 1 Time taken by machine A and B to manufacture X and Y

Product	Machine A (in h)	Machine B (in h)
X	(0.1, 0.2, 0.3, 0.4)	(0.2, 0.3, 0.4, 0.5)
Y	(0.2, 0.3, 0.4, 0.5)	(0.1, 0.2, 0.3, 0.4)
Total available time	(8, 9, 10, 11)	(7, 8, 9, 10)

$$\left.\begin{array}{l}
\mathrm{Max}\tilde{Z}_1(\tilde{X}) = \frac{1}{2}P_1 \\
\mathrm{Max}\tilde{Z}_2(\tilde{X}) = \frac{1}{2}Q_1 \\
\text{subject to} \\
A_1 = [17, 21] \\
\qquad A_2 = [15, 19] \\
s_{j1}, s_{j2} - s_{j1} \geq 0, \qquad j = 1,2, \\
s_{j3} - s_{j2}, s_{j4} - s_{j3} \geq 0, \qquad j = 1,2, \\
x_i, y_i - x_i \geq 0, \qquad i = 1,2, \\
u_i - y_i, v_i - u_i \geq 0, \qquad i = 1,2,
\end{array}\right\} \tag{10}$$

where

$$\begin{aligned}
P_1 &= \frac{1}{2}[20x_1 + 21y_1 + 21x_2 + 23y_2, 22u_1 + 23v_1 \\
&\quad + 24u_2 + 25v_2] \\
Q_1 &= \frac{1}{2}[-11v_1 - 10u_1 + 12x_2 + 13y_2, -9y_1 - 8x_1 \\
&\quad + 14u_2 + 15v_2] \\
A_1 &= [0.1x_1 + 0.2y_1 + 0.2x_2 + 0.3y_2 + s_{11} + s_{12}, 0.3u_1 \\
&\quad + 0.4v_1 + 0.4u_2 + 0.5v_2 + s_{13} + s_{14}] \\
A_2 &= [0.2x_1 + 0.3y_1 + 0.1x_2 + 0.2y_2 + s_{21} + s_{22}, 0.4u_1 \\
&\quad + 0.5v_1 + 0.3u_2 + 0.4v_2 + s_{23} + s_{24}]
\end{aligned}$$

Step 3: Regarding to Definition 2, the problem (10) in step 2 is rewritten as follows:

$$\left.\begin{array}{l}
\mathrm{Max}\tilde{Z}_1(\tilde{X}) = [m(\tilde{Z}_1(\tilde{X})), w(\tilde{Z}_1(\tilde{X}))] \\
\mathrm{Max}\tilde{Z}_2(\tilde{X}) = [m(\tilde{Z}_2(\tilde{X})), w(\tilde{Z}_2(\tilde{X}))] \\
\text{subject to} \\
[m(A_1), w(A_1)] =_{mw} [17, 21] \\
[m(A_2), w(A_2)] =_{mw} [15, 19] \\
s_{j1}, s_{j2} - s_{j1} \geq 0, \qquad j = 1,2, \\
s_{j3} - s_{j2}, s_{j4} - s_{j3} \geq 0, \qquad j = 1,2, \\
x_i, y_i - x_i \geq 0, \qquad i = 1,2, \\
u_i - y_i, v_i - u_i \geq 0, \qquad i = 1,2,
\end{array}\right\} \tag{11}$$

where

$$\begin{aligned}
m(\tilde{Z}_1(\tilde{X})) &= \frac{1}{4}(20x_1 + 21y_1 + 21x_2 + 23y_2 + 22u_1 \\
&\quad + 23v_1 + 24u_2 + 25v_2), \\
w(\tilde{Z}_1(\tilde{X})) &= \frac{1}{4}(22u_1 + 23v_1 + 24u_2 + 25v_2 - 20x_1 \\
&\quad - 21y_1 - 21x_2 - 23y_2), \\
m(\tilde{Z}_2(\tilde{X})) &= \frac{1}{4}(-11v_1 - 10u_1 + 12x_2 + 13y_2 - 9y_1 \\
&\quad - 8x_1 + 14u_2 + 15v_2), \\
w(\tilde{Z}_2(\tilde{X})) &= \frac{1}{4}(-9y_1 - 8x_1 + 14u_2 + 15v_2) + 11v_1 \\
&\quad + 10u_1 - 12x_2 - 13y_2), \\
m(A_1) &= (0.1x_1 + 0.2y_1 + 0.2x_2 + 0.3y_2 + s_{13} + s_{14} \\
&\quad + 0.3u_1 + 0.4v_1 + 0.4u_2 + 0.5v_2 + s_{13} + s_{14}), \\
w(A_1) &= (0.3u_1 + 0.4v_1 + 0.4u_2 + 0.5v_2 + s_{13} + s_{14} \\
&\quad - 0.1x_1 - 0.2y_1 - 0.2x_2 - 0.3y_2 - s_{11} - s_{12}), \\
m(A_2) &= (0.1x_1 + 0.2y_1 + 0.2x_2 + 0.3y_2 + s_{21} \\
&\quad + s_{22}0.3u_1 + 0.4v_1 + 0.4u_2 + 0.5v_2 + s_{23} + s_{24}), \\
w(A_2) &= (0.3u_1 + 0.4v_1 + 0.4u_2 + 0.5v_2 + s_{23} + s_{24} \\
&\quad - 0.1x_1 - 0.2y_1 - 0.2x_2 - 0.3y_2 - s_{21} - s_{22}).
\end{aligned}$$

Step 4: Combining all the centre and width of objective function of (11) obtained in step 3 with the help of scalarization method, we get:

$$\left.\begin{array}{l}
\mathrm{Min}\frac{1}{2}A \\
\text{subject to} \\
m(A_1) = 17, \\
w(A_1) = 21, \\
m(A_2) = 15, \\
w(A_2) = 19, \\
s_{j1}, s_{j2} - s_{j1} \geq 0, \qquad j = 1,2, \\
s_{j3} - s_{j2}, s_{j4} - s_{j3} \geq 0, \qquad j = 1,2, \\
x_i, y_i - x_i \geq 0, \qquad i = 1,2, \\
u_i - y_i, v_i - u_i \geq 0, \qquad i = 1,2,
\end{array}\right\} \tag{12}$$

where

$$A = (m(\tilde{Z}_1(\tilde{X})) + m(\tilde{Z}_2(\tilde{X})) - w(\tilde{Z}_1(\tilde{X})) - w(\tilde{Z}_2(\tilde{X}))).$$

Solving (12) using LINGO 14.0, the optimal value is 129.375. By using Theorem 1, we obtain the fuzzy Pareto optimal solution of (8). The fuzzy Pareto optimal solution

Table 2 Data of the profit and export–import for product A and B

Product	Profit per piece (Rs in 100)	Import per piece (Rs in 100)	Export per piece (Rs in 100)
X	(20, 21, 22, 23)	(8, 9, 10, 11)	-
Y	(21, 23, 24, 25)	-	(12, 13, 14, 15)

Table 3 Fuzzy Pareto optimal solution of problem (8)

Fuzzy variable of (8)	Fuzzy Pareto optimal solution of (8)
$\tilde{x}_1$	(0, 0, 0, 0)
$\tilde{x}_2$	(7.5, 7.5, 7.5, 7.5)

Table 4 Fuzzy objective function values of problem of (8)

Fuzzy objective function of (8)	Fuzzy objective functions value of (8)
$\tilde{Z}_1(\tilde{X})$	(157.5, 172.5, 180, 187.5)
$\tilde{Z}_2(\tilde{X})$	(90, 97.5, 105, 112.5)

and fuzzy objective functions value of problem (8) are given in Tables 3 and 4.

Example 2 [18]

$$\left.\begin{array}{l}\text{Max}\tilde{Z}_1(\tilde{X}) = (1,2,3)\tilde{x}_1 \oplus (2,4,5)\tilde{x}_2\\ \text{Max}\tilde{Z}_2(\tilde{X}) = (2,3,4)\tilde{x}_1 \oplus (3,4,5)\tilde{x}_2\\ \text{subject to}\\ (0,1,2)\tilde{x}_1 \oplus (1,2,3)\tilde{x}_2 \preceq (1,10,27)\\ (1,2,3)\tilde{x}_1 \oplus (0,1,2)\tilde{x}_2 \preceq (2,11,28)\\ \tilde{x}_1,\tilde{x}_2 \text{are non negative fuzzy number,}\end{array}\right\} \tag{13}$$

Step 1: Adding the fuzzy slack variables in the constraints of (13) as follows:

$$\left.\begin{array}{l}\text{Max}\tilde{Z}_1(\tilde{X}) = (1,2,3)\tilde{x}_1 \oplus (2,4,5)\tilde{x}_2\\ \text{Max}\tilde{Z}_2(\tilde{X}) = (2,3,4)\tilde{x}_1 \oplus (3,4,5)\tilde{x}_2\\ \text{subject to}\\ (0,1,2)\otimes\tilde{x}_1 \oplus (1,2,3)\otimes\tilde{x}_2 \oplus \tilde{s}_{11} \cong (1,10,27)\\ (1,2,3)\otimes\tilde{x}_1 \oplus (0,1,2)\otimes\tilde{x}_2 \oplus \tilde{s}_{21} \cong (2,11,28)\\ \tilde{x}_1,\tilde{x}_2,\tilde{s}_{11},\tilde{s}_{21} \text{ are non negative fuzzy number}\end{array}\right\} \tag{14}$$

Step 2: Using Definition 10, (15) can be written as:

$$\left.\begin{array}{l}\text{Max}\tilde{Z}_1(\tilde{X}) = (x_1+2x_2, 2y_1+4y_2, 3z_1+5z_2)\\ \text{Max}\tilde{Z}_2(\tilde{X}) = (2x_1+3x_2, 2y_1+4y_2, 3z_1+5z_2)\\ \text{subject to}\\ (x_2+s_{11}, y_1+2y_2+s_{12}, 2z_1+3z_2+s_{13}) \cong (1,10,27)\\ (x_1+s_{21}, 2y_1+y_2+s_{22}, 3z_1+2z_2+s_{23}) \cong (2,11,28)\\ x_j, (y_j-x_j), (z_j-y_j) \geq 0,\ \forall j=1,2\\ s_{i1}, (s_{i2}-s_{i1}), (s_{i3}-s_{i2}) \geq 0,\ \forall i=1,2.\end{array}\right\} \tag{15}$$

Step 3: Converting (15) into MOLIP problem, using Definition 12, the problem in step 2 can be rewritten as:

$$\left.\begin{array}{l}\text{Max}\tilde{Z}_1(\tilde{X}) = \frac{1}{2}P_2\\ \text{Max}\tilde{Z}_2(\tilde{X}) = \frac{1}{2}Q_2\\ \text{subject to}\\ A_3 = [38, 26]\\ A_4 = [41, 26]\\ x_j, (y_j-x_j), (z_j-y_j) \geq 0,\ \forall j=1,2,\\ s_{i1}, (s_{i2}-s_{i1}), (s_{i3}-s_{i2}) \geq 0,\ \forall i=1,2,\end{array}\right\} \tag{16}$$

where

$$\begin{aligned} P_2 =& \frac{1}{2}[x_1+2x_2+4y_1+8y_2+3z_1+5z_2,\\ & 3z_1+5z_2-x_1-2x_2]\\ Q_2 =& \frac{1}{2}[2x_1+3x_2+4y_1+8y_2+3z_1+5z_2,\\ & 3z_1+5z_2-2x_1-3x_2]\\ A_3 =& [x_1+s_{11}+2y_1+4y_2+s_{12}+2z_1+3z_2+s_{13},\\ & 2z_1+3z_2+s_{13}-x_2-s_{11}]\\ A_4 =& [x_1+s_{21}+4y_1+2y_2+2s_{22}+3z_1+2z_2+s_{23},\\ & 3z_1+2z_2+s_{23}-x_1-s_{21}].\end{aligned}$$

Step 4: Regarding to Definition 2, the problem (16) in step 3 can be rewritten as follows:

$$\left.\begin{array}{l}\text{Max}\tilde{Z}_1(\tilde{X}) = [m(\tilde{Z}_1(\tilde{X})), w(\tilde{Z}_1(\tilde{X}))]\\ \text{Max}\tilde{Z}_2(\tilde{X}) = [m(\tilde{Z}_2(\tilde{X})), w(\tilde{Z}_2(\tilde{X}))]\\ \text{subject to}\\ [m(A_3), w(A_3)] =_{mw} [38, 26]\\ [m(A_4), w(A_4)] =_{mw} [41, 26]\\ x_j, (y_j-x_j), (z_j-y_j) \geq 0,\ \forall j=1,2,\\ s_{i1}, (s_{i2}-s_{i1}), (s_{i3}-s_{i2}) \geq 0,\ \forall i=1,2,\end{array}\right\} \tag{17}$$

where,

$$\begin{aligned} m(\tilde{Z}_1(\tilde{X})) &= \frac{1}{4}(x_1+2x_2+4y_1+8y_2+3z_1+5z_2),\\ w(\tilde{Z}_1(\tilde{X})) &= \frac{1}{4}(3z_1+5z_2-x_1-2x_2)\\ m(\tilde{Z}_2(\tilde{X})) &= \frac{1}{4}(2x_1+3x_2+6y_1+8y_2+4z_1+5z_2),\\ w(\tilde{Z}_2(\tilde{X})) &= \frac{1}{4}(3z_1+5z_2-x_1-3x_2)\\ m(A_3) &= x_2+s_{11}+2y_1+4y_2+2s_{12}+2z_1+3z_2+s_{13},\\ w(A_3) &= 2z_1+3z_2+s_{13}-x_2-s_{11}\\ m(A_4) &= x_1+s_{21}+4y_1+2y_2+2s_{22}+3z_1+2z_2+s_{23}\\ w(A_2) &= 3z_1+2z_2+s_{23}-x_1-s_{21}.\end{aligned}$$

Step 5: Combine all the centre and width of objective function of (17) obtained in step 4 using scalarization method, we get:

$$\left.\begin{aligned}
&\text{Max}\,\frac{1}{2}B\\
&\text{subject to}\\
&m(A_3) = 38,\\
&w(A_3) = 26,\\
&m(A_4) = 41,\\
&w(A_4) = 26,\\
&x_j, (y_j - x_j), (z_j - y_j) \geq 0, \ \forall j = 1, 2\\
&s_{i1}, (s_{i2} - s_{i1}), (s_{i3} - s_{i2}) \geq 0, \ \forall i = 1, 2
\end{aligned}\right\} \tag{18}$$

where,

$$B = (m(\tilde{Z}_1(\tilde{X})) + m(\tilde{Z}_2(\tilde{X})) - w(\tilde{Z}_1(\tilde{X})) - w(\tilde{Z}_2(\tilde{X}))).$$

Solving (18) with the help of LINGO 14.0, the optimal value is 12.67. By Theorem 1, the fuzzy Pareto optimal solution and the fuzzy objective function value of (13) are given in Tables 5 and 6, respectively.

Table 5 Fuzzy Pareto optimal solution of (13)

Fuzzy variable of (13)	Fuzzy Pareto optimal solution of (13)
$\tilde{x}_1$	(1.78, 1.78, 1.78)
$\tilde{x}_2$	(1.17, 7.86, 7.86)

Table 6 Fuzzy objective function values of (13)

Fuzzy objective function of (13)	Fuzzy objective functions value of (13)
$\tilde{Z}_1(\tilde{X})$	(4.42, 59.42, 81.27)
$\tilde{Z}_2(\tilde{X})$	(7.67, 73.41, 95.26)

Table 7 Fuzzy efficient solution of (13) using Total objective-segregation method [18]

Fuzzy variable of (13)	Fuzzy Pareto optimal solution of (13)
$\tilde{x}_1$	(2, 4, 6)
$\tilde{x}_2$	(1, 3, 5)

Table 8 Fuzzy objective function values of problem of (13) using Total objective-segregation method [18]

Fuzzy objective function of (13)	Fuzzy objective functions value of (13)
$\tilde{Z}_1(\tilde{X})$	(4, 20, 43)
$\tilde{Z}_2(\tilde{X})$	(7, 24, 49)

Now, using, Total objective-segregation method [18], the fuzzy efficient solution and the fuzzy objective function values of (13) are given in Tables 7 and 8, respectively.

Using Definition 13, to compare the respective Tables 6 and 8, we observe that the fuzzy optimal solution obtained from the proposed method give better result than the Total objective-segregation method [18].

5 Conclusion

In the recent years, several authors have applied ranking function in the fuzzy linear programming. The main drawback of the ranking function is that it converts the fuzzy number into real number and most of the imprecise information is lost. Therefore, in this paper, a new method has been proposed to solve the FFMOLP problem which first converts the fuzzy problem into MOILP problem using nearest interval approximations of fuzzy numbers in order to avoid pitfalls of the essential information. Then, with the help of centre, width and scalarization technique, interval programming problem is converted into LP problem. We have shown that optimal solution of LP problem is the fuzzy Pareto optimal solution of FFMOLP problem. Numerical examples are solved and compared with the existing method in order to show the applicability of the proposed method in day to day life. The proposed method has the following advantages:

1. Instead of solving $(2k + 1)$ nonlinear problem in [1, 2], the proposed method reduces the FFMOLP problem into one linear programming problem in order to obtain the fuzzy Pareto optimal solution of FFMOLP problem.
2. It is not difficult to convert the FFMOLP problem into the final crisp linear programming problem because the arithmetic operations on *LR* flat fuzzy numbers are well defined and simple to operate. Moreover, the procedure for converting it to interval programming problem and then to find crisp linear programming problem is also simple as we have described in the paper. Therefore, the computational complexity of the proposed algorithm is very less.
3. It captures basic features of the original fuzzy quantities and avoid pitfalls of fuzziness.

We are further investigating the new developments and their applicability related to fuzzy systems, linguistic variables and fuzzy random variables etc. see [28] – [31] as part of our future work.

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