Int. J. Fuzzy Syst. (2018) 20(1):163–173
https://doi.org/10.1007/s40815-017-0325-y

Development of FDEA Models to Measure the Performance Efficiencies of DMUs

Alka Arya[1] · Shiv Prasad Yadav[1]

Received: 6 August 2016 / Revised: 20 April 2017 / Accepted: 27 April 2017 / Published online: 10 May 2017

Abstract Data envelopment analysis (DEA) is a linear programming-based methodology to measure the relative performance efficiencies of DMUs which produce multiple outputs by consuming multiple inputs. The input data and output data can be considered as a linguistic variables characterized by fuzzy numbers. So, in the present study, we extend DEA to fuzzy DEA (FDEA) in which the input and output data are taken as fuzzy numbers (FNs), in particular triangular FNs. In this paper, we develop two FDEA models to measure the left hand and right hand relative performance efficiencies of each DMU using α-cut approach. Further, we propose a ranking method to rank the DMUs based on left hand and right hand efficiencies. Finally, the developed FDEA models and proposed ranking models are illustrated with an example and then compared the results with geometric average efficiency index ranking method. The proposed model is also experimented with a real-life problem.

Keywords Hospitals · FDEA · Ranking methods · Health sector efficiencies

1 Introduction

Charnes et al. [7] developed DEA approach to measure the relative performance efficiencies of decision making units (DMUs), and Banker et al. [5] extended to study returns to scale (RTS). DMUs can be any governmental agencies and nonprofitable organizations like hospitals, educational institutions, banks, transportation. The relative performance efficiency of DMU is defined as the ratio of its performance efficiency to the largest performance efficiency. The relative performance efficiency of a DMU lies in the range (0, 1].

DEA is useful due to the following reasons:

- DEA determines the efficient and inefficient DMUs and also finds the most efficient DMUs and the inefficient DMUs for which efficiency improvements are possible.
- DEA improves the productivity of the inefficient DMUs by reducing inputs and/or increasing outputs.
- DEA calculates the efficiency of DMUs which have multiple inputs and multiple outputs.

The most important role in the economy of any country is health care of rural and urban areas. Health care is of three types: primary (in which individuals and families are directly connected to health system), secondary (in which patients from primary health care are referred to specialists in higher hospitals for treatment) and tertiary health care (in which specialized consultative care is provided usually on referral from primary and secondary medical care).

Real-world problems have some input and/or output data which possess some degree of fluctuation or imprecision or uncertainties such as quality of input resources, quality of treatment, the satisfaction level of patients, quality of medicines in health sector. The fluctuation can take the form of intervals, ordinal relations and fuzzy numbers etc. Therefore, to deal with such type of real-life situations, we propose to extend crisp DEA to Fuzzy DEA (FDEA) by making the use of fuzzy numbers in DEA. FDEA models represent real-world applications more realistically than the conventional DEA models.

✉ Alka Arya
alka1dma@gmail.com

Shiv Prasad Yadav
spyorfma@gmail.com

[1] Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee 247667, India

Despotis et al. [8] proposed left hand and right hand DEA models with imprecise input–output data. These models are not suitable for fuzzy data. In real-life applications, the input–output data fluctuate. In this paper, we have taken the fluctuation in the form of fuzzy numbers (FNs), particularly triangular fuzzy numbers (TFNs). So, we have propose the left and right hand fuzzy DEA (FDEA) models using α-cut.

The rest of the paper is organized as follows: Sect. 2 presents literature review. Section 3 presents the preliminary, Sect. 4 presents the proposed FDEA models and proposed ranking approach. Section 5 presents an example and an application of the proposed approach to the health sector. Conclusions of the study are given in the last section of the paper.

2 Literature Review

In this section, we will give literature review for FDEA and ranking.

2.1 Literature Review for FDEA

There exist various mathematical programs in DEA such as fractional DEA program, output maximization (input oriented) and input minimization (output oriented), SBM DEA programs. Mogha et al. [17] proposed a new slack model (NSM) of DEA, which directly deals with input and output slacks, to assess the efficiency of public sector hospitals of State of Uttarakhand (India). Tyagi et al. [24] applied DEA to determine the performance efficiencies of academic departments of IIT Roorkee (India).

There are some studies of DEA with applications to health care both public and private sector in Indian context [17–20]. Hollingsworth et al. [12] described the economic theory of efficiency and productivity, and their measurement in health care using DEA and found the potential benefits and problems of measuring efficiency. Adang and Wensing [1] presented the potential discrepancy between long-run and short-run efficiencies of innovative technologies in health care and explored dis-economy of scale in Dutch hospitals. Barnum and Walton [6] presented the effect of the conflict between DEA and hospital applications and found efficiency indicators that assume nonsubstitutability rather than substitutability. Akdag et al. [3] evaluated the service quality of some Turkish hospitals by fuzzy MCDM approach.

Several researches have been done in different areas both from DEA and FDEA perspective. Alizadeh et al. [4] presented the location–allocation models of fuzzy multi-objective nonlinear programming in FDEA and found the solution procedure based on a modification of fuzzy parametric programming (FPP) and minimum deviation method. Tsai et al. [23] proposed the fuzzy analytic hierarchy process (FAHP) and fuzzy sensitive analysis-based approach to resolve the uncertainty and imprecision of service evaluations during pre-negotiation stages, where the comparison judgments of a decision maker are represented as triangular fuzzy numbers (TFNs). Dotoli et al. [9] developed a novel cross-efficiency fuzzy DEA model for evaluating different elements under uncertainty with application to health-care system. Jahanshahloo et al. [13] proposed a slack-based measure (SBM) model which is employed for evaluation and ranking of all DMUs and extended this model to an FDEA model for evaluating efficiency and ranking of DMUs with fuzzy data. Kao and Liu [14] reduced the crisp DEA model into the fuzzy DEA model using α-cut method and expressed performance efficiency as a fuzzy number.

2.2 Literature Review for Ranking

Khodabakhshi et al. [15] developed a ranking method to rank all DMUs. Lotfi et al. [16] proposed a ranking method by using generalized variance for inputs and outputs. Wen et al. [26] transformed fuzzy model into linear programming with the help of new approach combined with fuzzy simulation and genetic algorithm. Guo and Wu [10] proposed a unique ranking method by using undesirable outputs through new Maximal Balance Index (MBI) based on the optimal shadow prices. Adler et al. [2] discussed different ranking methods in DEA. Puri and Yadav [21] proposed a ranking method for crisp as well as fuzzy numbers for optimistic and pessimistic models and finally proposed a complete ranking method using super-efficiency technique. Wang et al. [25] proposed the geometric average efficiency index to determine the overall performance efficiency and ranking of the DMUs in crisp DEA.

3 Preliminary

This section includes some basic definitions and notions of fuzzy set theory which are required to develop the concept of FDEA for handling fuzzy inputs/outputs in real-life problems. It includes the definitions of fuzzy set (FS), fuzzy number (FN) and triangular fuzzy number (TFN).

3.1 Performance Efficiency

The performance efficiency of a DMU is defined as the ratio of the weighted sum of outputs (called virtual output) to the weighted sum of inputs (called virtual input). Thus,

$$\text{Performance efficiency} = \frac{\text{Virtual output}}{\text{Virtual input}}.$$

DEA evaluates the relative performance efficiency of a DMU in a set of homogeneous DMUs. The relative performance efficiency of a DMU lies in the range (0, 1].

3.2 Fuzzy Number (FN)

Zadeh [27] A FN $\tilde{A}$ is a convex fuzzy set $\tilde{A}$ of the real line $\mathbb{R}$ such that

- there exists exactly one $x_0 \in \mathbb{R}$ with $\mu_{\tilde{A}}(x_0) = 1$.
- $\mu_{\tilde{A}}(x)$ is a piecewise continuous.

3.3 Triangular Fuzzy Number (TFN)

The TFN $\tilde{A}$ is a fuzzy number denoted by $\tilde{A} = (a, b, c)$ and is defined by the membership function $\mu_{\tilde{A}}$ given by (see Fig. 1)

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x-a}{b-a}, & a < x \le b, \\ \dfrac{c-x}{c-b}, & b \le x < c, \\ 0, & \text{elsewhere}, \end{cases}$$

for all $x \in \mathbb{R}$.

α **cut:** The α-cut of a fuzzy set $\tilde{A}$ is denoted by A_α and defined by

$A_\alpha = \{x \in X: \mu_{A(x)} \ge \alpha\}$, where $\alpha \in [0, 1]$.

4 Proposed Methodology for Fuzzy Efficiencies and Ranks of the DMUs

4.1 Description of DEA Model

Let us suppose that the performance of a set of n homogeneous DMUs (DMUj; $j = 1, \ldots, n$) is to be measured. Suppose that m inputs produce s outputs. Let us assume

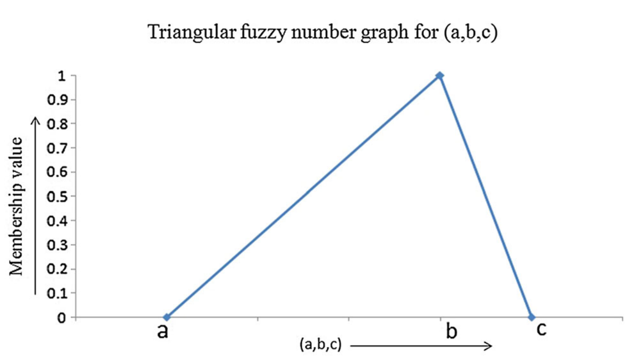
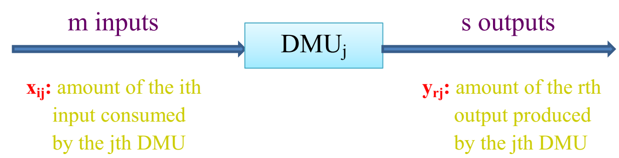


Fig. 1 TFN (a,b,c)

that the amount of the rth output produced by the jth DMU is $y_{rj}, r = 1, 2, 3, \ldots, s$ and the amount of the ith input consumed by the jth DMU is $x_{ij}, i = 1, 2, 3, \ldots, m$.



The CCR (ratio) fractional DEA program [7] for DMU_{j_o} is given by the following model:

Model 1

$$\max E_{j_o} = \frac{\sum_{r=1}^{s} v_r y_{rj_o}}{\sum_{i=1}^{m} u_i x_{ij_o}}$$

subject to

$$\frac{\sum_{r=1}^{s} v_r y_{rj_o}}{\sum_{i=1}^{m} u_i x_{ij_o}} \le 1,$$
$$\frac{\sum_{r=1}^{s} v_r y_{rj}}{\sum_{i=1}^{m} u_i x_{ij}} \le 1, \quad \forall j, j \ne j_o$$
$$u_i, v_r \ge \varepsilon, \ \forall i, r,$$

where u_i is the weight corresponding to x_{ij} and v_r is the weight corresponding to y_{rj}.

The fractional program in Model 1 is reduced to the following LPP (Model 2)

Model 2

$$\min E_{j_o} = \sum_{i=1}^{m} u_i x_{ij_o}$$

subject to

$$\sum_{r=1}^{s} v_r y_{rj_o} = 1$$
$$\sum_{r=1}^{s} v_r y_{rj_o} - \sum_{i=1}^{m} u_i x_{ij_o} \le 0,$$
$$\sum_{r=1}^{s} v_r y_{rj} - \sum_{i=1}^{m} u_i x_{ij} \le 0,$$
$$j = 1, 2, \ldots, n, \ j \ne j_o$$
$$u_i, v_r \ge \varepsilon, \ \forall i, r$$

In real-world problems, input and output data x_{ij} and y_{rj} cannot be obtained exactly due to vagueness/fluctuation. Suppose that they lie the intervals $\left[x_{ij}^L, x_{ij}^U\right]$ and $\left[y_{rj}^L, y_{rj}^U\right]$, ($i = 1, 2, 3, \ldots, m$; $j = 1, 2, 3, \ldots, n$; $r = 1, 2, 3, \ldots, s$), where $x_{ij}^L, y_{rj}^L > 0$. To deal with such type of situations, the CCR DEA model [25] in left limit and right limit is given below:

Model 3

$$\max E_{j_o} = \frac{\sum_{r=1}^{s} v_r \left[y_{rj_o}^L, y_{rj_o}^U\right]}{\sum_{i=1}^{m} u_i \left[x_{ij_o}^L, x_{ij_o}^U\right]}$$

subject to

$$\frac{\sum_{r=1}^{s} v_r \left[y_{rj_o}^L, y_{rj_o}^U\right]}{\sum_{i=1}^{m} u_i \left[x_{ij_o}^L, x_{ij_o}^U\right]} \leq 1$$

$$\frac{\sum_{r=1}^{s} v_r \left[y_{rj}^L, y_{rj}^U\right]}{\sum_{i=1}^{m} u_i \left[x_{ij}^L, x_{ij}^U\right]} \leq 1, \quad j = 1, 2, 3, \ldots, n, j \neq j_o$$

$$u_i, v_r \geq \varepsilon, \ \forall i, r.$$

4.2 Proposed Input-Oriented Minimization FDEA Model

Let $E_j = \frac{\sum_{r=1}^{s} v_r [y_{rj}^L, y_{rj}^U]}{\sum_{i=1}^{m} u_i [x_{ij}^L, x_{ij}^U]}$, $j = 1, 2, 3, \ldots, n$.

Obviously, E_j should also be an interval form. Let $E_j = \left[E_j^L, E_j^U\right]$. Then $\left[E_j^L, E_j^U\right] = \frac{\sum_{r=1}^{s} v_r \left[y_{rj}^L, y_{rj}^U\right]}{\sum_{i=1}^{m} u_i \left[x_{ij}^L, x_{ij}^U\right]} \subseteq (0, 1]$.
The left and right hand performance efficiencies of DMU_{j_o} are given by the following fractional DEA models for DMU_{j_o}:

$$\max E_{j_o}^L = \frac{\sum_{r=1}^{s} v_r y_{rj_o}^L}{\sum_{i=1}^{m} u_i x_{ij_o}^U}$$

subject to

$$\frac{\sum_{r=1}^{s} v_r y_{rj_o}^L}{\sum_{i=1}^{m} u_i x_{ij_o}^U} \leq 1$$

$$\frac{\sum_{r=1}^{s} v_r y_{rj}^U}{\sum_{i=1}^{m} u_i x_{ij}^L} \leq 1, j = 1, 2, 3, \ldots, n; j \neq j_o$$

$$u_i, v_r \geq \varepsilon, \ \forall i, r.$$

$$\max E_{j_o}^U = \frac{\sum_{r=1}^{s} v_r y_{rj_o}^U}{\sum_{i=1}^{m} u_i x_{ij_o}^L}$$

subject to

$$\frac{\sum_{r=1}^{s} v_r y_{rj_o}^U}{\sum_{i=1}^{m} u_i x_{ij_o}^L} \leq 1$$

$$\frac{\sum_{r=1}^{s} v_r y_{rj}^L}{\sum_{i=1}^{m} u_i x_{ij}^U} \leq 1, j = 1, 2, 3, \ldots, n; j \neq j_o$$

$$u_i, v_r \geq \varepsilon, \ \forall i, r.$$

The above pair of fractional DEA models can be simplified as the equivalent LPP models. The CCR input minimization left hand DEA model for DMU_{j_o} is minimization of right hand of virtual input subject to the conditions that left hand of virtual output is equal to 1, and difference between left hand of virtual output and right hand of virtual input is less than or equal to 0. Thus, the CCR input minimization left hand DEA model (in LPP form) is given by the following model:

Model 4

$$\min E_{j_o}^L = \sum_{i=1}^{m} u_i x_{ij_o}^U$$

subject to

$$\sum_{r=1}^{s} v_r y_{rj_o}^L = 1,$$

$$\sum_{r=1}^{s} v_r y_{rj_o}^L - \sum_{i=1}^{m} u_i x_{ij_o}^U \leq 0,$$

$$\sum_{r=1}^{s} v_r y_{rj}^U - \sum_{i=1}^{m} u_i x_{ij}^L \leq 0, \quad j = 1, 2, 3, \ldots, n, j \neq j_o$$

$$u_i, v_r \geq \varepsilon, \quad \forall\, i, \ r.$$

The CCR input minimization right hand DEA model for DMU_{j_o} is minimization of left hand of virtual input subject to the conditions that right hand of virtual output is equal to 1, and difference between right hand of virtual output and left hand of virtual input is less than or equal to 0. Thus, the CCR input minimization right hand DEA model (in LPP form) is given by the following model:

Model 5

$$\min E_{j_o}^U = \sum_{i=1}^{m} u_i x_{ij_o}^L$$

subject to

$$\sum_{r=1}^{s} v_r y_{rj_o}^U = 1,$$

$$\sum_{r=1}^{s} v_r y_{rj_o}^U - \sum_{i=1}^{m} u_i x_{ij_o}^L \leq 0,$$

$$\sum_{r=1}^{s} v_r y_{rj}^L - \sum_{i=1}^{m} u_i x_{ij}^U \leq 0, \quad j = 1, \ldots, n, j \neq j_o$$

$$u_i, v_r \geq \varepsilon, \quad \forall\, i \quad \text{and} \quad r.$$

where $E_{j_o}^L$ stands for the left hand of the best possible relative performance efficiency of DMU_{j_o} when all the DMUs are in the state of best production activity, while $E_{j_o}^U$ stands for the right hand of the best possible relative efficiency achieved by DMU_{j_o}. When they constitute the best possible relative efficiency interval $\left[E_{j_o}^L, E_{j_o}^U\right]$.

The CCR DEA model in fuzzy environment, in which inputs and outputs are considered to be fuzzy numbers, we need CCR fuzzy DEA model. Let us suppose that fuzzy input is $\tilde{x}_{ij}$ and fuzzy output is $\tilde{y}_{rj}$. Then linear FDEA model is given by

Model 6

$$\max \tilde{E_{j_o}} = \sum_{i=1}^{m} u_i \tilde{x}_{ij_o}$$

subject to

$$\sum_{r=1}^{s} v_r \tilde{y}_{rj_o} = \tilde{1},$$
$$\sum_{r=1}^{s} v_r \tilde{y}_{rj_o} - \sum_{i=1}^{m} u_i \tilde{x}_{ij_o} \le \tilde{0},$$
$$\sum_{r=1}^{s} v_r \tilde{y}_{rj} - \sum_{i=1}^{m} u_i \tilde{x}_{ij} \le \tilde{0}, j = 1, 2, 3, \ldots, n, \; j \ne j_o$$
$$u_i, v_r \ge \varepsilon, \; \forall i, r.$$

where $\tilde{1} = (1, 1, 1)$ and $\tilde{0} = (0, 0, 0)$.

Assume that fuzzy input $\tilde{x}_{ij}$ and fuzzy output $\tilde{y}_{rj}$ are TFNs $(x_{ij}^L, x_{ij}^M, x_{ij}^U)$ and $(y_{rj}^L, y_{rj}^M, y_{rj}^U)$, respectively. Then, CCR FDEA model is given by

Model 7

$$\max \tilde{E}_{j_o} = \sum_{i=1}^{m} u_i \left(x_{ij_o}^L, x_{ij_o}^M, x_{ij_o}^U \right)$$

subject to

$$\sum_{r=1}^{s} v_r \left(y_{rj_o}^L, y_{rj_o}^M, y_{rj_o}^U \right) = (1, 1, 1),$$
$$\sum_{r=1}^{s} v_r \left(y_{rj_o}^L, y_{rj_o}^M, y_{rj_o}^U \right) - \sum_{i=1}^{m} u_i \left(x_{ij_o}^L, x_{ij_o}^M, x_{ij_o}^U \right) \le (0, 0, 0),$$
$$\sum_{r=1}^{s} v_r \left(y_{rj}^L, y_{rj}^M, y_{rj}^U \right) - \sum_{i=1}^{m} u_i \left(x_{ij}^L, x_{ij}^M, x_{ij}^U \right) \le (0, 0, 0),$$
$$j = 1, 2, 3, \ldots, n, \; j \ne j_o$$
$$u_i, v_r \ge \varepsilon, \; \forall i, r.$$

Replacing the fuzzy input and output by their α-cuts; $\tilde{x}_{ij} = [\alpha x_{ij}^M + (1-\alpha) x_{ij}^L, \alpha x_{ij}^M + (1-\alpha) x_{ij}^U]$ and $\tilde{y}_{rj} = [\alpha y_{rj}^M + (1-\alpha) y_{rj}^L, \alpha y_{rj}^M + (1-\alpha) y_{rj}^U]$, we get Model 8 from Model 7 as given below:

Model 8

$$\max \left[E_{j_o}^L, E_{j_o}^U \right] = \sum_{i=1}^{m} u_i \left[\alpha x_{ij_o}^M + (1-\alpha) x_{ij_o}^L, \alpha x_{ij_o}^M + (1-\alpha) x_{ij_o}^U \right]$$

subject to

$$\sum_{r=1}^{s} v_r \left[\alpha y_{rj_o}^M + (1-\alpha) y_{rj_o}^L, y_{rj_o}^M + (1-\alpha) y_{rj_o}^U \right] = [1, 1],$$
$$\sum_{r=1}^{s} v_r \left[\alpha y_{rj_o}^M + (1-\alpha) y_{rj_o}^L, \alpha y_{rj_o}^M + (1-\alpha) y_{rj_o}^U \right]$$
$$- \sum_{i=1}^{m} u_i \left[\alpha x_{ij_o}^M + (1-\alpha) x_{ij_o}^L, \right.$$
$$\left. \alpha x_{ij_o}^M + (1-\alpha) x_{ij_o}^U \right] \le [0, 0],$$
$$\sum_{r=1}^{s} v_r \left[\alpha y_{rj}^M + (1-\alpha) y_{rj}^L, \alpha y_{rj}^M + (1-\alpha) y_{rj}^U \right]$$
$$- \sum_{i=1}^{m} u_i \left[\alpha x_{ij}^M + (1-\alpha) x_{ij}^L, \right.$$
$$\left. \alpha x_{ij}^M + (1-\alpha) x_{ij}^U \right] \le [0, 0], \quad j = 1, 2, 3, \ldots, n, \; j \ne j_o$$
$$u_i, v_r \ge \varepsilon, \; \forall i, r, 0 < \alpha \le 1.$$

From Model 8, we get left hand DEA model (Model 9) and right hand DEA model (Model 10) as given below:

Model 9

$$\min E_{j_o}^L = \sum_{i=1}^{m} u_i \left(\alpha x_{ij_o}^M + (1-\alpha) x_{ij_o}^U \right)$$

subject to

$$\sum_{r=1}^{s} v_r \left(\alpha y_{rj_o}^M + (1-\alpha) y_{rj_o}^L \right) = 1$$
$$\sum_{r=1}^{s} v_r \left(\alpha y_{rj_o}^M + (1-\alpha) y_{rj_o}^L \right)$$
$$- \sum_{i=1}^{m} u_i \left(\alpha x_{ij_o}^M + (1-\alpha) x_{ij_o}^U \right) \le 0,$$
$$\sum_{r=1}^{s} v_r \left(\alpha y_{rj}^M + (1-\alpha) y_{rj}^U \right)$$
$$- \sum_{i=1}^{m} u_i \left(\alpha x_{ij}^M + (1-\alpha) x_{ij}^L \right) \le 0,$$
$$j = 1, 2, 3, \ldots, n, \; j \ne j_o$$
$$u_i, v_r \ge \varepsilon, \; \forall i, r$$

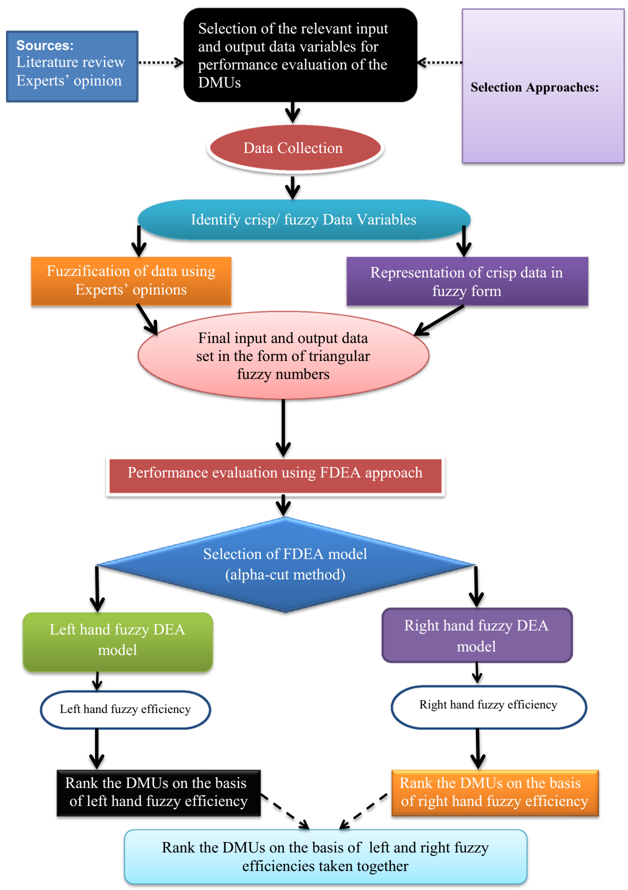


Fig. 2 Flow chart for stepwise procedure

Model 10

$$\min E_{j_o}^U = \sum_{i=1}^{m} u_i\left(\alpha x_{ij_o}^M + (1-\alpha)x_{ij_o}^L\right)$$

subject to

$$\sum_{r=1}^{s} v_r\left(\alpha y_{rj_o}^M + (1-\alpha)y_{rj_o}^U\right) = 1$$

$$\sum_{r=1}^{s} v_r\left(\alpha y_{rj}^M + (1-\alpha)y_{rj}^U\right) - \sum_{i=1}^{m} u_i\left(\alpha x_{ij}^M + (1-\alpha)x_{ij}^L\right) \le 0,$$

$$\sum_{r=1}^{s} v_r\left(\alpha y_{rj}^M + (1-\alpha)y_{rj}^L\right) - \sum_{i=1}^{m} u_i\left(\alpha x_{ij}^M + (1-\alpha)x_{ij}^U\right) \le 0,$$

$$j = 1, 2, 3, \ldots, n,\ j \ne j_o$$

$$u_i, v_r \ge \varepsilon,\ \forall i, r$$

Models 9 and 10 are the proposed FDEA models.

4.3 Proposed Ranking Approach

In this section, we propose a new ranking method based on left hand and right hand efficiencies of DMUs. The method is as follows:

Table 1 Input and output data. *Source* Guo and Tanaka [11]

DMU	Inputs		Outputs	
	Input 1	Input 2	Output 1	Output 2
DMU1	(3.5, 4.0, 4.5)	(1.9, 2.1, 2.3)	(2.4, 2.6, 2.8)	(3.8, 4.1, 4.4)
DMU2	(2.9, 2.9, 2.9)	(1.4, 1.5, 1.6)	(2.2, 2.2, 2.2)	(3.3, 3.5, 3.7)
DMU3	(4.4, 4.9, 5.4)	(2.2, 2.6, 3.0)	(2.7, 3.2, 3.7)	(4.3, 5.1, 5.9)
DMU4	(3.4, 4.1, 4.8)	(2.1, 2.3, 2.5)	(2.5, 2.9, 3.3)	(5.5, 5.7, 5.9)
DMU5	(5.9, 6.5, 7.1)	(3.6, 4.1, 4.6)	(4.4, 5.1, 5.8)	(6.5, 7.4, 8.3)

Table 2 Left and right hand efficiency of DMUs for α values

DMUs	Left hand					Right hand				
	$\alpha = 0.1$	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$	$\alpha = 0.1$	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$
DMU1	0.65	0.70	0.76	0.81	0.85	1	1	1	0.92	0.85
DMU2	0.86	0.988	0.99	1	1	1	1	1	1	1
DMU3	0.59	0.64	0.71	0.79	0.86	1	1	1	0.93	0.86
DMU4	0.88	0.93	1	1	1	1	1	1	1	1
DMU5	0.67	0.73	0.84	0.97	1	1	1	1	1	1

Step 1 Obtain the left hand efficiency $E^L_{j_o}$ and right hand efficiency $E^U_{j_o}$ for each DMU_{j_o} using left hand model (Model 9) and right hand model (Model 10), respectively.

Step 2 Let $I = \{1, 2, \ldots, n\}$ be an index set. Suppose that $T_1 = \min\left\{E^L_{j_o} | j_o \in I\right\}$ and $T_2 = \min\left\{E^U_{j_o} | j_o \in I\right\}$. T_1 and T_2 are the minimum efficiencies obtained from left hand and right hand models, respectively.

Step 3 Let us suppose that $R_1 = \max\left\{E^L_{j_o} | j_o \in I\right\}$ and $R_2 = \max\left\{E^U_{j_o} | j_o \in I\right\}$. R_1 and R_2 are the maximum efficiencies obtained from left hand and right hand models, respectively.

Step 4 Find the deviations ${Dj_o}^L$ and ${Dj_o}^U$ in the left hand and right hand efficiencies for all DMUs, where ${Dj_o}^L = \dfrac{R_1 - E^L_{j_o}}{R_1 - T_1}$ and ${Dj_o}^U = \dfrac{R_2 - E^U_{j_o}}{R_2 - T_2}, j_o \in I$.

Step 5 Find the total deviation $D_{j_o} = {Dj_o}^U + {Dj_o}^L$ for each $\mathrm{DMU}_{j_o}, j_o \in I$.

Step 6 Rank all DMUs according to the increasing values of D_{j_o}.

This proposed ranking method is compared with geometric ranking method developed by Wang et al. [25]. Further, in order to validate the proposed ranking method, we provide two different numerical examples and also apply the proposed method to the health sector (Subsection 5.2). Figure 2 gives the flow chart representing the stepwise procedure.

5 Numerical Examples

To ascertain the validity of the proposed methodology, we consider the following examples.

5.1 Example (Guo and Tanaka)

The fuzzy input and fuzzy output data are listed in Table 1. There are 5 DMUs having two fuzzy inputs and two fuzzy outputs which are represented as TFNs. The left hand and right hand efficiencies of the DMUs are evaluated using Models 9 and 10, respectively, for different α values, and the results are shown in Table 2. The ranks of the DMUs are obtained by using the proposed ranking method discussed in Subsection 4.3 and are listed in Table 3. These left hand and right hand efficiencies also make a TFN. By the proposed ranking method, DMUs are ranked in the order of DMU4 > DMU2 > DMU5 > DMU1 > DMU3 for $\alpha = 0.1$, whereas for $\alpha = 0.25, 0.5$, they are ranked in

Table 3 Total deviation and rank

DMUs	$\alpha = 0.1$			$\alpha = 0.25$			$\alpha = 0.5$			$\alpha = 0.75$				$\alpha = 1.0$		
	D_{j_o}	$E^{\text{geometric}}_{jo}$	Rank	D_{j_o}	$E^{\text{geometric}}_{jo}$	Rank	D_{j_o}	$E^{\text{geometric}}_{jo}$	Rank	D_{j_o}	Rank D_{j_o}	$E^{\text{geometric}}_{jo}$	Rank	D_{j_o}	$E^{\text{geometric}}_{jo}$	Rank
DMU1	0.79	0.81	4	0.824	0.84	4	0.86	0.87	4	1.90	4	0.86	3	2	0.85	3
DMU2	0.069	0.93	2	0	0.99	1	0.36	0.994	2	0	1	1	1	0	1	1
DMU3	1	0.77	5	1	0.799	5	1	0.85	5	1.87	3	0.86	4	1.86	0.86	2
DMU4	0	0.94	1	0.15	0.97	2	0	1	1	0	1	1	1	0	1	1
DMU5	0.72	0.82	3	0.73	0.86	3	0.55	0.92	3	0.14	2	0.98	2	0	1	1

the order of DMU4 > DMU2 > DMU5 > DMU1 > DMU3 and for $\alpha = 0.75$, they are ranked in the order of DMU4 = DMU2 > DMU5 > DMU3 > DMU1 for D_{j_o} and DMU4 = DMU2 > DMU5 > DMU1 > DMU3 for $E_{jo}^{\text{geometric}}$, and for $\alpha = 1$, they are ranked in the order of DMU2 = DMU4 = DMU5 > DMU3 > DMU1. For comparison, the geometric average efficiency index $E_{jo}^{\text{geometric}}$ due to Wang et al. [25] for every DMU is also measured, and the results are listed in Table 3. We observe that the ranking results by the proposed ranking method and by geometric average efficiency index are similar. It seems to be the same.

5.2 Health Sector Application

This is a real-life application with two inputs and two outputs for 12 hospitals. The Uttar Pradesh (U.P) state is one of the largest states of India. It has 75 districts. Meerut is one of them which has 12 blocks, and each block has a Community Health Center or Community Health Care (CHC). Each CHC has some Primary Health Center or Primary Health Care (PHC), where the population is near about 30,000. Total number of PHCs in Meerut district is 31. At present, in UP PHCs are working under CHCs. CHCs and PHCs are mainly in rural areas. In this paper, we discuss the performance efficiency of CHCs/PHCs which are in Meerut district. The following tables (Table 4) give CHCs/PHCs in Meerut district: In this paper, the following inputs and outputs are taken:

- **Input 1** Number of doctors (x_1)
- **Input 2** Sum of number of superintendents and number of health workers (x_2)
- **Output 1** Number of inpatients (y_1)
- **Output 2** Number of outpatients (y_2)

In this application, hospitals are DMUs. Because these input and output data are measured by the human being, vagueness and uncertainty are present in the data. Number of doctors, and sum of number of superintendents and number of health workers are taken as two fuzzy inputs; number of inpatients and number of outpatients as two fuzzy outputs in the present study are represented by TFNs on the basis of thorough discussions with the officers during the data collection phase. The input–output data are considered as TFNs (See Table 5). The left hand and right hand efficiencies of the DMUs are evaluated using Models 9 and 10, respectively for different α

Table 4 CHCs and PHCs in Meerut district

Name of CHCs	Mawana (H1)	Sardhana (H2)	Daurala (H3)	Bhudbharal (H4)	Janikhurd (H5)	Rohta (H6)	Kharkhoda (H7)
(a)							
Number of PHCs	3	2	2	1	3	2	1

Name of CHCs	Hastinapur (H8)	Parikshit garh (H9)	Bhawanpur (H10)	Machra (H11)	Sarurpurkhurd (H12)
(b)					
Number of PHCs	4	3	3	4	3

Table 5 Input and output data of 12 hospitals

DMU	Input		Output	
	x_{1j}	x_{2j}	y_{1j}	y_{1j}
H1	(5, 10, 14)	(17, 20, 32)	(3640, 3650, 3655)	(134,130, 134,137, 134,140)
H2	(4, 9, 15)	(12, 15, 27)	(4150, 4160, 4170)	(116,060, 116,062, 116,070)
H3	(8, 11, 13)	(10, 17, 29)	(4360, 4370, 4380)	(94,060, 94,066, 94,070)
H4	(3, 8, 12)	(19, 24, 28)	(485, 492, 500)	(24,325, 24,329, 24,334)
H5	(6, 10, 15)	(14, 19, 25)	(2460, 2464, 2470)	(99,745, 99,748, 99,750)
H6	(7, 11, 16)	(15, 20, 28)	(1360, 1368, 1375)	(49,398, 49,401, 49,405)
H7	(8, 10, 13)	(12, 15, 22)	(1055, 1062, 1070)	(37,769, 37,772, 37,776)
H8	(7, 11, 15)	(9, 11, 20)	(1295, 1302, 1310)	(82,838, 82,841, 82,845)
H9	(8, 12, 17)	(8, 16, 29)	(1660, 1671, 1680)	(100,590, 100,596, 100,600)
H10	(9, 15, 19)	(15, 25, 31)	(1010, 1018, 1025)	(64,349, 64,351, 64,358)
H11	(6, 11, 16)	(18, 23, 29)	(1500, 1504, 1510)	(80,050, 80,056, 80,060)
H12	(4, 8, 11)	(12, 17, 28)	(1960, 1965, 1972)	(58,160, 58,167, 58,170)

Table 6 Left and right hand efficiencies of 12 hospitals for α values

DMUs	Left hand efficiencies					Right hand efficiencies				
	$\alpha = 0.1$	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$	$\alpha = 0.1$	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$
H1	0.45	0.50	0.62	0.81	0.99	1	1	1	1	0.99
H2	0.53	0.61	0.77	0.98	1	1	1	1	1	1
H3	0.46	0.51	0.61	0.75	0.93	1	1	1	1	0.93
H4	0.09	0.10	0.14	0.18	0.23	0.70	0.55	0.39	0.29	0.23
H5	0.40	0.44	0.53	0.60	0.75	1	1	1	0.94	0.75
H6	0.18	0.20	0.24	0.27	0.34	0.72	0.64	0.53	0.43	0.34
H7	0.17	0.20	0.24	0.28	0.32	0.68	0.61	0.51	0.41	0.32
H8	0.40	0.47	0.62	0.77	0.97	1	1	1	1	0.97
H9	0.38	0.43	0.52	0.64	0.81	1	1	1	1	0.81
H10	0.21	0.23	0.27	0.30	0.33	0.89	0.76	0.58	0.44	0.33
H11	0.28	0.31	0.36	0.43	0.54	1	1	0.84	0.67	0.54
H12	0.23	0.26	0.34	0.44	0.56	1	1	0.97	0.73	0.56

Table 7 Total deviation and rank

DMUs	$\alpha = 0.1$			$\alpha = 0.25$			$\alpha = 0.5$			$\alpha = 0.75$			$\alpha = 1.0$		
	D_{j_o}	$E_{j_o}^{\text{geometric}}$	Rank	D_{j_o}	$E_{j_o}^{\text{geometric}}$	Rank	D_{j_o}	$E_{j_o}^{\text{geometric}}$	Rank	D_{j_o}	$E_{j_o}^{\text{geometric}}$	Rank	D_{j_o}	$E_{j_o}^{\text{geometric}}$	Rank
H1	0.18	0.67	3	0.22	0.71	3	0.242	0.79	3	0.21	0.9	2	0.02	0.99	2
H2	0	0.728	1	0	0.781	1	0	0.877	1	0.02	0.989	1	0	1	1
H3	0.16	0.678	2	0.20	0.714	2	0.25	0.781	4	0.29	0.866	4	0.18	0.927	4
H4	1.94	0.25	12	2.00	0.234	12	2.00	0.235	12	2.00	0.228	12	2	0.23	12
H5	0.28	0.70	4	0.31	0.67	5	0.38	0.725	5	0.47	0.751	6	0.64	0.75	6
H6	1.66	0.36	10	1.60	0.357	10	1.61	0.356	10	1.69	0.341	10	1.72	0.34	9
H7	1.82	0.34	11	1.67	0.349	11	1.64	0.349	11	1.70	0.338	11	1.76	0.32	11
H8	0.29	0.632	5	0.27	0.685	4	0.24	0.787	2	0.26	0.877	3	0.08	0.97	3
H9	0.34	0.616	6	0.35	0.656	6	0.40	0.721	6	0.42	0.80	5	0.50	0.81	5
H10	1.07	0.432	9	1.27	0.418	9	1.48	0.395	9	1.54	0.363	9	1.74	0.33	10
H11	0.57	0.529	7	0.59	0.567	7	0.91	0.549	8	1.15	0.537	8	1.20	0.54	8
H12	0.68	0.479	8	0.69	0.51	8	0.73	0.574	7	1.05	0.567	7	1.14	0.56	7

values and the results are listed in Table 6. The ranks of the DMUs obtained by applying the proposed ranking method discussed in Subsection 4.3 are presented in Table 7. These left hand and right hand efficiencies also make a TFN. By the proposed ranking method, DMUs are ranked in the order of H2 > H3 > H1 > H5 > H8 > H9 > H11 > H12 > H10 > H6 > H7 > H4 for $\alpha = 0.1$; H2 > H3 > H1 > H8 > H5 > H9 > H11 > H12 > H10 > H6 > H7 > H4 for $\alpha = 0.25$; H2 > H8 > H1 > H3 > H5 > H9 > H11 > H12 > H10 > H6 > H7 > H4 for $\alpha = 0.5$; H2 > H1 > H8 > H3 > H9 > H5 > H12 > H11 > H10 > H6 > H7 > H4 for $\alpha = 0.75$; and H2 > H1 > H8 > H3 > H9 > H5 > H12 > H11 > H6 > H10 > H7 > H4 for $\alpha = 1$. In this example, the ranking results by the proposed ranking methods and by geometric average efficiency index are similar. Hospital H2 is the best efficient, and H4 is the worst efficient for all α values ($\alpha = 0.1, 0.25, 0.5, 0.75, 1$).

6 Conclusion

In this paper, we have developed the left hand and right hand FDEA models using α-cut approach to measure the efficiencies of the DMUs (Sect. 4.2). Also, we have proposed a ranking method, which presents not only a full ranking but also the information that to what degree a fuzzy efficiency is bigger/lesser than that of other DMUs (Sect. 4.3). The proposed ranking approach is compared with geometric average method [25]. The proposed methods have been applied to two examples (i) to determine the performance efficiencies of 5 DMUs which is taken from Guo and Tanaka [11] and (ii) to determine the performance efficiencies of 12 hospitals of Meerut district, India. The inefficiency percentage of 12 hospitals [inefficiency percentage $= \left(1 - \frac{D_{j_o}}{\text{Highest}\, D_{j_o}}\right) \times 100\%$] for $\alpha = 1$ is $H2(100\%) > H1(99\%) > H8(96\%) > H3(91\%) > H9(75\%) > H5(68\%) > H12(43\%) > H11(40\%) > H6(14\%) > H10(13\%) > H7(12\%) > H4(0\%)$. According to inefficiency percentage of 12 hospitals for $\alpha = 1$, the most inefficient DMU is Sardhana (H2) and worst inefficient DMU is Bhudbharal (H4). The input inefficiency percentage represents the degree to which the input should change to become fully efficient.

7 Future Research Plan

We are working to develop the left and right hand FDEA models and ranking methodology which can take care of public and private sector hospitals, etc. where input–output data are available in subjective and linguistic forms. Further we plan to apply integrated fuzzy VIKOR method developed by Shekarian et al. [22] to assess and rank the quality of public and private sector hospitals.

Acknowledgements The authors are thankful to the learned reviewers for rigorous comments which improved the original manuscript. The authors are thankful to the Ministry of Human Resource Development (MHRD), Govt. of India, India, for financial support in pursuing this research. The authors are also thankful to Mr. Deen Bandhu, ARO, Chief Medical Office, Department Head Office, Meerut, India, for providing the valuable data of the hospitals.

References

1. Adang, E.M.M., Wensing, M.: Economic barriers to implementation of innovations in health care: is the long run.short run. Health Policy **88**, 236–242 (2008)
2. Adler, N., Friedman, L., Stern, Z.S.: Review of ranking methods in the data envelopment analysis context. Eur. J. Oper. Res. **140**, 249–265 (2002)
3. Akdag, H., Kalayci, T., Karagoz, S., Zulfikar, H., Giz, D.: The evaluation of hospital service quality by fuzzy MCDM. Appl. Soft Comput. **23**, 239–248 (2014)
4. Alizadeh, H.M., Rasouli, S.M., Moghaddam, R.T.: The use of multicriteria data envelopment analysis (MCDEA) for location-allocation problems in a fuzzy environment. Expert Syst. Appl. **38**, 5687–5695 (2011)
5. Banker, R.D., Charnes, A., Cooper, W.W.: Some models for the estimation of technical and scale inefficiencies in data envelopment analysis. Manag. Sci. **30**, 1078–1092 (1984)
6. Barnum, D.T., Walton, S.M.: Measuring hospital efficiency with data envelopment analysis: nonsubstitutable vs. substitutable inputs and outputs. J. Med. Syst. **35**, 1393–1401 (2011)
7. Charnes, A., Cooper, W., Rhodes, E.: Measuring the efficiency of decision making units. Eur. J. Oper. Res. **2**(6), 428–444 (1978)
8. Despotis, D.K., Smirlis, Y.G.: Data envelopment analysis with imprecise data. Eur. J. Oper. Res. **140**, 24–36 (2002)
9. Dotoli, M., Epicoco, N., Falagario, M., Sciancalepore, F.: A cross-efficiency fuzzy data envelopment analysis technique for performance evaluation of decision making units under uncertainty. Comput. Ind. Eng. **79**, 103–114 (2015)
11. Guo, P., Tanaka, H.: Fuzzy DEA: a perceptual evaluation method. Fuzzy Sets Syst. **119**, 149–160 (2001)
10. Guo, D., Wu, J.: A complete ranking of DMUs with undesirable outputs using restrictions in DEA models. Math. Comput. Model. **58**, 1102–1109 (2013)
12. Hollingsworth, B., Dawson, P.J., Maniadakis, N.: Efficiency measurement of health care: a review of non-parametric methods and applications. Health Care Manag. Sci. **2**, 161–172 (1999)
13. Jahanshahloo, G.R., Damaneh, M.S., Nasrabadi, E.: Measure of efficiency in DEA with fuzzy input–output levels: a methodology for assessing, ranking and imposing of weights restrictions. Appl. Math. Comput. **156**, 175–187 (2004)
14. Kao, C., Liu, S.T.: Fuzzy efficiency measures in data envelopment analysis. Fuzzy Sets Syst. **113**, 427–437 (2000)
15. Khodabakhshi, M., Aryavash, K.: Ranking units with fuzzy data in DEA. Data Envel. Anal. Decis. Sci. **2014**, 1–10 (2014)
16. Lotfi, F.H., Jahanshahloo, G.R., Balf, F.R., Rezai, H.Z.: Ranking of DMUs on Interval Data by DEA. Int. J. Contemp. Math. Sci. **2**(4), 159–166 (2007)
17. Mogha, S.K., Yadav, S.P., Singh, S.P.: Technical and relative efficiency assessment of some private sector hospitals in India. In: Proceedings of the International Conference on SocProS. AISC 130, pp. 657–666 (2011)
18. Mogha, S.K., Yadav, S.P., Singh, S.P.: New slack model based efficiency assessment of public sector hospitals of Uttarakhand: state of India. Int. J. Syst. Assur. Eng. Manag. **5**(1), 32–42 (2014)
19. Mogha, S.K., Yadav, S.P., Singh, S.P.: Performance evaluation of Indian private hospitals using DEA approach with sensitivity analysis. Int. J. Adv. Manag. Econ. **1**, 2278–3369 (2014)
20. Mogha, S.K., Yadav, S.P., Singh, S.P.: Estimating technical and scale efficiencies of private hospitals using a non-parametric approach: case of India. Int. J. Oper. Res. **20**(1), 21–40 (2014)
21. Puri, J., Yadav, S.P.: Intuitionistic fuzzy data envelopment analysis: an application to the banking sector in India. Expert Syst. Appl. **42**, 4982–4998 (2015)
22. Shekarian, E., Abdul-Rashid, S.H., Olugu, E.U.: An integrated fuzzy VIKOR method for performance management in healthcare. Organizational Productivity and Performance Measurements Using Predictive Modeling and Analytics. IGI Global, Pennsylvania, pp. 40–61 (2016)
23. Tsai, H.Y., Chang, C.W., Lin, H.L.: Fuzzy hierarchy sensitive with Delphi method to evaluate hospital organization performance. Expert Syst. Appl. **37**, 5533–5541 (2010)
24. Tyagi, P., Yadav, S.P., Singh, S.P.: Relative performance of academic departments using DEA with sensitivity analysis. Eval. Program Plan. **32**, 168–177 (2009)
25. Wang, Y.M., Chin, K.S., Yang, J.B.: Measuring the performance of decision making units using geometric average efficiency. J. Oper. Res. Soc. **58**, 929–937 (2007)

26. Wen, M., Li, H.: Fuzzy data envelopment analysis (DEA): model and ranking method. J. Comput. Appl. Math. **223**, 872–878 (2009)
27. Zadeh, L.A.: Fuzzy sets. Inf. Control **8**, 338–353 (1965). doi:10.1016/S0019-9958(65)90241-X

Alka Arya is a Research Scholar in the Department of Mathematics, Indian Institute of technology Roorkee, Roorkee, India. She is working on Performance measurement of public sector hospitals in Uttar Pradesh through FDEA. She has interest in fuzzy optimization and Intuitionistic fuzzy optimization also.

Shiv Prasad Yadav is a Professor of Mathematics at the Indian Institute of Technology Roorkee, Roorkee, India. He holds a Ph.D. in Optimal Control Theory from the Applied Mathematics Department, Institute of Technology, Banaras Hindu University, Varansi, India. His current research areas include optimal control theory, operations research, data envelopment analysis, fuzzy optimization, Intuitionistic fuzzy optimization and fuzzy reliability studies.