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ELECTRE-Based Outranking Method for Multi-criteria Decision Making Using Hesitant Intuitionistic Fuzzy Linguistic Term Sets

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Abstract An outranking method is developed within the environment of hesitant intuitionistic fuzzy linguistic term sets (HIFLTSs), where the membership degree and the non-membership degree of the element are subsets of linguistic term set. The directional Hausdorff distance, which uses HIFLTSs, is proposed, and the dominance relations are subsequently defined using this distance. Moreover, some interesting characteristics of the proposed directional Hausdorff distance are further discussed in detail. In this context, a collective decision matrix is obtained in the form of hesitant intuitionistic fuzzy linguistic elements and analyzes the collective data by using proposed ELECTRE-based outranking method. The linguistic scale functions are employed in this paper to conduct the transformation between qualitative information and quantitative data. Furthermore, based on the proposed method, we also investigate the ranking of the alternatives based on a new proposed definition of HIFLTS. The feasibility and applicability of the proposed method are illustrated with an example, and a comparative analysis is performed with other approaches to validate the effectiveness of the proposed methodology.

Keywords Directional Hausdorff distance · Hesitant fuzzy linguistic term sets · Hesitant intuitionistic fuzzy linguistic term sets · Multi-criteria decision making · Outranking method

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1 Introduction

In many cases, it is difficult for decision makers (DMs) to precisely express a preference when solving MCDM problems with inaccurate, uncertain or incomplete information. Under these situations, Zadeh's fuzzy set [48], where the membership degree is represented by a real number between [0, 1], are observed as an important tool for solving MCDM problems [5, 45], fuzzy logic and approximate reasoning [49] and pattern recognition [22]. In practice, however, the information about an alternative in the universe corresponding to a fuzzy concept may be imperfect, i.e., the sum of the membership and non-membership degrees of an element corresponding to the fuzzy concept can be less than one. The fuzzy set theory fails in dealing with the insufficient understanding to the membership degree, while Atanassov's intuitionistic fuzzy set (IFS), an extension of Zadeh's fuzzy set which were introduced by Atanassov [1], effectively handled the problems by adding a non-membership degree. Therefore, IFSs were expected to be applicable to simulate human decision making process and activities that require corresponding expertise and knowledge. MCDM problems with IFSs have received appropriate attentions [44, 47]. Some further extensions of fuzzy sets have been developed, including type-2 fuzzy sets, hesitant fuzzy set (HFS), interval-valued hesitant fuzzy set (IVHFS) and generalized hesitant fuzzy set (GHFSs) [6, 27, 32]. HFSs permit the membership degree of an element to be a set of several possible values between [0, 1] and can be highly useful in handling the situation where people hesitate to provide their preferences over objects in a decision making process. Recently, HFSs have been the subject of a great deal of research and have been widely applied to MCDM or multi-criteria group decision making (MCGDM) problems [36, 41, 50, 53]. Zhou et al. [51] proposed an idea of hesitant intuitionistic fuzzy set (HIFS) and developed the corresponding group decision making approach by presenting the hesitant intuitionistic fuzzy preference relation and its hesitant intuitionistic fuzzy complementary preference relation for uncertain preference information. Peng et al. [24] developed an extension of ELECTRE method with multi-hesitant fuzzy sets (MHFSs), and they also studied the fuzzy cross-entropy of IHFSs [23]. Although GHFSs, IVHFSs, IHFSs and MHFSs have enlarged the applications of HFSs, all of them are limited to quantitative information and weak in dealing with qualitative data.

When an expert is hesitant and thinking of several terms at the same time to assess an alternative, it is not easy for him/her to provide a single term as an expression of his/her understanding. In order to handle this situation, Rodríguez et al. [28] used Torra's idea in defining HFSs [32] to introduce the concept of hesitant fuzzy linguistic term sets (HFLTSs). Beg and Rashid [3] extended the fuzzy TOPSIS(the technique for order preference by similarity to the ideal solution) for HFLTS to solve the multi-criteria group decision making problems. MCDM problems with HFLTSs are most prominent and have been successfully used in many fields [17, 19, 20, 54]. Recently, the studies on MCDM problems in the context of HFLTSs are growing. For example, Peng and Wang [26] proposed hesitant uncertain linguistic Z-numbers and their application in MCGDM problems, Zhou et al. [56] discussed MCDM approaches based on distance measures for linguistic hesitant fuzzy sets, while Wang et al. [38] presented a likelihood-based TODIM approach based on multi-hesitant fuzzy linguistic information for evaluation in logistics outsourcing.

Rodríguez et al. [28] gave a method for ranking HFLTSs which is conducted by interval values constructed by the indexes of the HFLTSs' envelopes. However, the comparison results that have been derived by this method may not accord with common sense, because it seems to be unreasonable to say one HFLTS is absolutely superior to another if these two HFLTSs have some common elements. To remove this deficiency, Wei et al. [40] developed comparison methods and studied the aggregation theory for HFLTSs. Wang et al. [34] further developed the directional Hausdorff distance and dominance relations together with some properties and propositions and constructed an outranking approach to solve MCDM problems in the context of HFLTSs. Recently, Wei et al. [39] developed some aggregation operators for aggregating hesitant fuzzy linguistic information: hesitant fuzzy linguistic weighted average operator, hesitant fuzzy linguistic ordered weighted average operator and hesitant fuzzy linguistic hybrid average operator.

In fuzzy decision making environment, decision makers may hesitate to choose appropriate linguistic term or linguistic interval to assess alternatives. Thus to manage this type of situation, Beg and Rashid [4] introduced the concept of hesitant intuitionistic fuzzy linguistic term set (HIFLTS) which is more suitable for dealing with fuzziness and uncertainty than the HFLTS. They extended fuzzy TOPSIS for HIFLTSs with the opinion of finite decision makers about the criteria of alternatives. Furthermore, Liu et al. [21] developed the hesitant intuitionistic fuzzy linguistic set and the hesitant intuitionistic fuzzy uncertain linguistic set which permit the possible membership degree and non-membership degree of an element to a linguistic term and an uncertain linguistic term having sets of intuitionistic fuzzy numbers. Zhou et al. [52] proposed a new definition of HIFLTS which they named as intuitionistic hesitant linguistic set (IHLS) and introduced the corresponding operations of intuitionistic hesitant linguistic numbers and developed some aggregation operators and used them to solve MCDM problems. The family of ELECTRE methods [14, 30, 46] have been selected for widespread and extensive use in real-world decision situations. In general, the ELECTRE methods are based on a common rule: With knowledge of the concordance and discordance sets for all ordered pairs of alternatives, one can exploit the outranking relation, which is specific for a particular choice or a ranking problem. ELECTRE and its extensions have been widely studied [12, 31, 33] and applied in various MCDM problems [7, 24, 36]. Recently, numerous studies have been conducted on the extended ELECTRE methods in a fuzzy context, such as an ELECTRE-based outranking method for MCGDM using HIFLTSs [11], stochastic MCDM approach based on SMAA-ELECTRE with extended gray numbers [55], ELECTRE approach with multi-valued neutrosophic information [25] and an extended outranking approach to rough stochastic MCDM problems [37].

Motivated by the development of directional Hausdorff distance and dominance relations of HFLTSs by Wang et al. [34], an outranking method to solve MCDM problems is developed in the context of HIFLTSs, which is based upon the features of the directional Hausdorff distance and the elicitation of the ELECTRE methods used by Devi and Yadav [8] and Figueira et al. [13]. This method will be further tested and compared with the TOPSIS method employing HIFLTSs as well as a new proposed definition of HIFLTS.

The rest of the paper is organized as follows: the preliminary concepts related to the work are briefly reviewed in Sect. 2. In Sect. 3, the dominance relations between HIFLTSs and some useful properties are defined together with a new proposed definition of HIFLTSs. Two MCDM approaches for decision making based on outranking and TOPSIS methods with HIFLTSs along with a methodology of determining the criteria weights are developed in Sect. 4. A numerical example is given to illustrate the proposed method in Sect. 5, and it is also validated through a comparative analysis. The conclusion is presented in the last section.

2 Preliminaries

Some basic concepts and operations of linguistic term sets HFSs, HIFSs, HFLTSs and HIFLTSs are briefly reviewed and discussed in this section.

2.1 Linguistic Term Sets

The linguistic approach is an approximate technique which represents qualitative data using linguistic variables whose values are not numbers but words or sentences in a natural or artificial language [15, 16]. Let $S = \{s_0, s_1, \ldots, s_g\}$ be a finite linguistic term set with odd cardinality, where each $s_i\,(0 \le i \le g)$ represents a possible value for a linguistic variable. The following characteristics need to be satisfied:

1. There is a negation operator: $\mathrm{neg}(s_i) = s_j$, such that $i + j = g$;
2. The set is ordered: $s_i \le s_j \Leftrightarrow i \le j$. Therefore, there exist a maximization operator: $\max(s_i, s_j) = s_i$, if $s_j \le s_i$, and a minimization operator: $\min(s_i, s_j) = s_i$, if $s_i \le s_j$.

To preserve all of the given information, Xu [42, 43] extended the discrete linguistic term set S to a continuous linguistic term set $\overline{S} = \{s_\alpha | s_0 \le s_\alpha \le s_g\}$, where if $s_\alpha \in S$, then s_α is called an original linguistic term; otherwise, s_α is called a virtual linguistic term. In general, the decision maker uses the original linguistic terms to evaluate alternatives, and the virtual linguistic terms can only appear in operation.

For any two linguistic terms $s_\alpha, s_\beta \in \overline{S}$ and $\lambda \in [0, 1]$, the following operational laws were introduced by Xu [42, 43] as follows:

1. $s_\alpha \oplus s_\beta = s_\beta \oplus s_\alpha = s_{\alpha+\beta}$;
2. $s_\alpha \otimes s_\beta = s_\beta \otimes s_\alpha = s_{\alpha\beta}$;
3. $\lambda s_\alpha = s_{\lambda\alpha}$;
4. $s_\alpha^\lambda = s_{\alpha^\lambda}$.

2.2 Hesitant Fuzzy Set

Torra [32] proposed a HFS, which is a more general fuzzy set and permits the membership to include a set of possible values.

Definition 2.1 [32] Let X be a fixed set, a hesitant fuzzy set A on X is defined in terms of a function $h_A(x)$ that when applied to X returns a finite subset of [0, 1].

To be easily understood, Xia and Xu [41] express the HFS by a mathematical symbol:

$$E = \{\langle x, h_E(x)\rangle \quad | \, x \in X\},$$

where $h_E(x)$ is a set of some values in [0, 1], denoting the possible membership degrees of the element $x \in X$ to the set E.

2.3 Hesitant Intuitionistic Fuzzy Set

Zhou et al. [51] developed the idea of HIFS and hesitant intuitionistic fuzzy number (HIFN) as follows:

Definition 2.2 [51] Let X be a fixed set, then a HIFS K on X can be defined as follows:

$$K = \{(x, h_K(x), v_K(x)) | x \in X\}$$

where $h_K(x)$ and $v_K(x)$ represent the membership degree and the non-membership degree, respectively, of element x to K, and $h_K(x)$ is a HFE with $h_K(x) \subseteq [0, 1]$ and $\max\{h_K(x)\} + \; v_K(x) \le 1$. In addition, the pair $(h_K(x), v_K(x))$ represents the HIFN.

2.4 Hesitant Fuzzy Linguistic Term Set

Motivated by the idea of HFS, Rodríguez et al. [29] introduced the HFLTS as follows:

Definition 2.3 [29] Let $S = \{s_0, s_1, \ldots, s_g\}$ be a linguistic term set. A HFLTS, H_S, is an ordered finite subset of the consecutive linguistic terms of S.

It should be noted that Definition 2.3 does not give any mathematical form for HFLTS. To overcome this incompleteness, Liao et al. [18] refined the definition of HFLTS as follows:

Let X be fixed and $S = \{s_0, s_1, \ldots, s_g\}$ be a linguistic term set. A HFLTS H_S on X is

$$H_S = \{\langle x, h_S(x)\rangle \; | \, x \in X\}$$

where $h_S(x)$ is a set of some values in the linguistic term set S and can be expressed as

$h_S(x) = \{s_{\phi_l}(x) \mid s_{\phi_l} \in S, l = 1, \ldots, L\}$

with L being the number of linguistic terms in $h_S(x)$. $h_S(x)$ denotes the possible degrees of linguistic variable x to the linguistic term set S. For convenience, $h_S(x)$ is called the hesitant fuzzy linguistic element.

2.5 Hesitant Intuitionistic Fuzzy Linguistic Term Set

To extend the HFLTS [29], a HIFLTS that provides a powerful tool to address vagueness has been proposed by Beg and Rashid [4] . A prominent characteristic of the HIFLTS is the simultaneous consideration of membership degree and non-membership degree. Beg and Rashid [4] proposed the definition of HIFLTS as follows:

Definition 2.4 [4] A HIFLTS on X are functions $M(x)$ and $N(x)$ that when applied to X return ordered finite subsets of the consecutive linguistic term set, $S = \{s_0, s_1, \ldots, s_g\}$, which can be represented as the following mathematical symbol:

$E_S = \{(x, M_S(x), N_S(x)) \mid x \in X\}$,

where $M_S(x)$ and $N_S(x)$ are the subsets of the consecutive linguistic term set S, denoting the possible membership degrees and non-membership degrees of the element $x \in X$ to the set E_S, with the conditions that

$s_0 \leq \max(M_S(x)) \oplus \min(N_S(x)) \leq s_g$
and $s_0 \leq \min(M_S(x)) \oplus \max(N_S(x)) \leq s_g$.

For convenience, we let $E_S = \{M_S, N_S\}$ be a hesitant intuitionistic fuzzy linguistic element (HIFLE) and the set $\mathbb{C}$ a family of all HIFLEs. Further, the cardinality of E_S denoted by $|E_S|$ is defined to be the sum of cardinalities of M_S and N_S i.e., $|E_S| = |M_S| + |N_S|$.

Definition 2.5 Let $E_S = \{M_S, N_S\}$be an HIFLTS. Let $M_S^- = \min_{l=1,\ldots,\#M_S}(s_{m_l})$, $M_S^+ = \max_{l=1,\ldots,\#M_S}(s_{m_l})$, $N_S^- = \min_{l=1,\ldots,\#N_S}(s_{n_l})$ and $N_S^+ = \max_{l=1,\ldots,\#N_S}(s_{n_l})$. Then the linguistic terms $\bar{s}_m$ and $\bar{s}_n$ can be inserted into M_S and N_S respectively, as follows:

$$\bar{s}_m = \xi M_S^+ + (1-\xi)M_S^- \text{ and } \mathrm{s_n} = \xi \mathrm{N}_S^+ + (1-\xi)\mathrm{N}_S^-, \quad 0 \leq \xi \leq 1 \tag{2.1}$$

M_S^-, M_S^+, N_S^- and N_S^+ are, respectively, called the lower and upper bounds of the membership function M_S and non-membership function N_S of the HIFLTS E_S. Without loss of generality, in rest of the paper, we assume $\xi = \frac{1}{2}$.

Definition 2.6 Let $S = \{s_0, s_1, \ldots, s_g\}$ be a linguistic term set and $E_S^1 = \{M_S^1, N_S^1\}$ and $E_S^2 = \{M_S^2, N_S^2\}$ be two arbitrary HIFLTSs on S, where $M_S^k = \cup_{s_{i_k}} \{s_{i_k} | i_k = 1, \ldots, \#M_S^k\}$, $N_S^k = \cup_{s_{j_k}} \{s_{j_k} | j_k = 1, \ldots, \#N_S^k\}$, $k = 1, 2$. We suppose that $L = \#M_S^1 = \#N_S^1 = \#M_S^2 = \#N_S^2$ (otherwise, we can extend the shorter one by inserting suitable linguistic terms given as Eq. 2.1).The generalized distance d_{gd} between E_S^1 and E_S^2 for any $\lambda > 0$ can be defined as follows:

$$d_{gd}\left(E_S^1, E_S^2\right) = \left(\frac{1}{L}\sum_{l=1}^{L}\left(\left(\left|m_{i_1}^1 - m_{i_2}^2\right|\right)^{\lambda} + \left(\left|n_{j_1}^1 - n_{j_2}^2\right|\right)^{\lambda}\right)\right)^{\frac{1}{\lambda}}. \tag{2.2}$$

For $\lambda = 1$ and $\lambda = 2$, the above generalized distance becomes, respectively, the Hamming distance and the Euclidean distance between E_S^1 and E_S^2.

One can easily prove that the generalized distance d_{gd} satisfies the following for any $\lambda > 0$:

1. $0 \leq d_{gd}\left(E_S^1, E_S^2\right) \leq 1$;
2. $d_{gd}\left(E_S^1, E_S^2\right) = 0$ if and only if $E_S^1 = E_S^2$;
3. $d_{gd}\left(E_S^1, E_S^2\right) = d_{gd}\left(E_S^2, E_S^1\right)$.

Definition 2.7 [28, 29] Let $S = \{s_0, s_1, \ldots, s_g\}$ be a linguistic term set, and E_{G_H} be a function that transforms the linguistic expression II, obtained by a context-free grammar G_H, into a HFLTS E_S:

$E_{G_H} : II \longrightarrow E_S$.

The linguistic expressions, generated by using the production rules, are then transformed into HFLTSs in different ways as below:

1. $E_{G_H} = \{s_i | s_i \in S\}$;
2. E_{G_H}(less than s_i) $= \{s_j | s_j \in S$ and $s_j \leq s_i$ };
3. E_{G_H} (greater than s_i) $= \{s_j | s_j \in S$ and $s_j \geq s_i$ };
4. E_{G_H}(between s_i and s_j) $= \{s_k | s_k \in S$ and $s_k \geq s_i$ and $s_k \leq s_j$ }.

3 Dominance Relations Between HIFLTSs

3.1 Linguistic Scale Function

To transform the qualitative data properly into quantitative data, we will use the same linguistic scale functions (LSFs) as used in [2, 35].

Definition 3.1 [2, 35] Given the linguistic term s_i in S, the LSF $f : s_i \longrightarrow \theta_i$ conducting the mapping from s_i to θ_i $(i = 0, 1, \ldots, g)$, where $0 \le \theta_0 < \theta_1 < \cdots < \theta_g \le 1$. The function f is a strictly monotonically increasing function. On the basis of the label function [35] of linguistic terms, the simplest form of f is

$$f : s_i \longrightarrow \frac{i}{g} \quad (i = 0, 1, \ldots, g). \tag{3.1}$$

The evaluation scale of the linguistic information given in Formula 3.1 is divided on average. A scale function that can achieve a bidirectional increase in the geometric progression of the scale value is defined as

$$f(s_i) = \theta_i = \begin{cases} \dfrac{a^g - a^{g-i}}{2a^g - 2} & (i = 0, 1, \ldots, \frac{g}{2}) \\ \dfrac{a^g + a^{i-g} - 2}{2a^g - 2} & (i = \frac{g}{2} + 1, \frac{g}{2} + 2, \ldots, g) \end{cases}. \tag{3.2}$$

Considerable experimental research has determined that a usually lies in the interval [1.36, 1.4] (see [2]).

If $S = \{s_0, s_1, \ldots, s_g\}$ is a linguistic term set, then the threshold established for Formulae 3.1 and 3.2 is $\min_{s_i \in S}(f(s_{i+1}) - f(s_i)) = \frac{1}{g}$.

In terms of the comparison rules, Faizi et al. [11] defined a comparison method of HIFLTSs as follows:

Definition 3.2 [11] Let $E_S = \{M_S, N_S\}$ an HIFLE. The score function $Sc(E_S)$ and the accuracy function $Ac(E_S)$ of E_S can be defined as follows:

$$Sc(E_S) = sc(M_S) - sc(N_S) \tag{3.3}$$

$$Ac(E_S) = sc(M_S) + sc(N_S) \tag{3.4}$$

here, $sc(M_S) = \frac{1}{g}\sum_{s_{m_i} \in M_S} \frac{f^*(s_{m_i})}{\#M_S}$, $sc(N_S) = \frac{1}{g}\sum_{s_{n_j} \in N_S} \frac{f^*(s_{n_j})}{\#N_S}$ and f^* is a LSF defined as before. $\#M_S$ and $\#N_S$ indicate the number of the elements in M_S and N_S respectively.

Definition 3.3 [11] For any two HIFLEs $E_S^1 = \{M_S^1, N_S^1\}$ and $E_S^2 = \{M_S^2, N_S^2\}$,

1. if $Sc(E_S^1) > Sc(E_S^2)$, then $E_S^1 \succ E_S^2$;
2. if $Sc(E_S^1) = Sc(E_S^2)$, and

(a) $Ac(E_S^1) > Ac(E_S^2)$, then $E_S^1 \succ E_S^2$;
(b) $Ac(E_S^1) = Ac(E_S^2)$, then $E_S^1 = E_S^2$.

The score values of two HIFLEs E_S^1 and E_S^2 do not indicate the degree to which E_S^1 outranks E_S^2. Wang et al. [34] achieved this goal by defining the directional Hausdorff distance for HFLTSs. Therefore, motivated by this idea, we now propose the directional Hausdorff distance for HIFLTSs as follows:

Definition 3.4 Let $S = \{s_0, s_1, \ldots, s_g\}$ be a linguistic term set and $E_S^1 = \{M_S^1, N_S^1\}$ and $E_S^2 = \{M_S^2, N_S^2\}$ be two arbitrary HIFLTSs on S, where $M_S^k = \cup_{s_{i_k}} \{s_{i_k} | i_k = 1, \ldots, \#M_S^k\}$, $N_S^k = \cup_{s_{j_k}} \{s_{j_k} | j_k = 1, \ldots, \#N_S^k\}$, $k = 1, 2$. The directional Hausdorff distance d_{dhd} from E_S^1 to E_S^2 can be defined as follows:

$$d_{dhd}(E_S^1, E_S^2) = \begin{cases} \dfrac{1}{|E_S^2|}\left(\begin{array}{l} \sum_{s_{i_2} \in M_S^2} \min_{s_{i_1} \in M_S^1} \left(\max\left(0, f(s_{i_1}) - f(s_{i_2})\right)\right) \\ + \sum_{s_{j_1} \in N_S^1} \min_{s_{j_2} \in N_S^2} \left(\max\left(0, f(s_{j_2}) - f(s_{j_1})\right)\right) \end{array} \right) & \text{if } M_{S^+}^1 = M_{S^+}^2 \text{ and } N_{S^+}^1 = N_{S^+}^2 \\ \dfrac{1}{|E_S^1|}\left(\begin{array}{l} \sum_{s_{i_1} \in M_S^1} \min_{s_{i_2} \in M_S^2} \left(\max\left(0, f(s_{i_1}) - f(s_{i_2})\right)\right) \\ + \sum_{s_{j_2} \in N_S^2} \min_{s_{j_1} \in N_S^1} \left(\max\left(0, f(s_{j_2}) - f(s_{j_1})\right)\right) \end{array} \right) & \text{otherwise} \end{cases} \tag{3.5}$$

Example 3.5 Let $S = \{s_0, s_1, \ldots, s_6\}$ be a linguistic term set and $E_S^1 = \{(s_0, s_1, s_2), (s_4, s_5)\}$ and $E_S^2 = \{(s_4, s_5), (s_0, s_1)\}$ be two HIFLTSs and f be a function that defined in Eq. 3.1. The directional Hausdorff distances between E_S^1 and E_S^2 using Formulae 3.1 and 3.5 can be given as follows: $d_{dhd}(E_S^1, E_S^2) = 0$, $d_{dhd}(E_S^2, E_S^1) = 0.7071$

Similarly, the directional Hausdorff distances between E_S^1 and E_S^2 using Formulae 3.2 and 3.5 are given as follows: $d_{dhd}(E_S^1, E_S^2) = 0$, $d_{dhd}(E_S^2, E_S^1) = 0.6244$

Definition 3.6 Let $S = \{s_0, s_1, \ldots, s_g\}$ be a linguistic term set, E_S^1, E_S^2 and E_S^3 be three HIFLTSs, then the directional Hausdorff distance d_{dhd} from E_S^1 to E_S^2 defined in Definition 3.4 satisfy the following properties:

1. $d_{dhd}(E_S^1, E_S^1) = 0$;
2. $0 \le d_{dhd}(E_S^1, E_S^2) \le 1$;
3. $d_{dhd}(E_S^1, E_S^2) \ne d_{dhd}(E_S^2, E_S^1)$.

It is apparent that Properties 1–3 are true, and the proof is therefore omitted here.

Definition 3.7 The generalized directional Hausdorff distance d_{gdhd} from E_S^1 to E_S^2 can also be defined as follows:

$$d_{gdhd}(E_S^1,E_S^2)=\begin{cases}\left(\frac{1}{|E_S^2|}\left(\sum_{s_{i_2}\in M_S^2}\left(\min_{s_{i_1}\in M_S^1}\left(\max\left(0,f(s_{i_1})-f(s_{i_2})\right)\right)\right)^{\lambda}+\sum_{s_{j_1}\in N_S^1}\left(\min_{s_{j_2}\in N_S^2}\left(\max\left(0,f(s_{j_2})-f(s_{j_1})\right)\right)\right)^{\lambda}\right)\right)^{\frac{1}{\lambda}} & \text{if } M_{S^+}^1=M_{S^+}^2 \text{ and } N_{S^+}^1=N_{S^+}^2\\ \left(\frac{1}{|E_S^1|}\left(\sum_{s_{i_1}\in M_S^1}\left(\min_{s_{i_2}\in M_S^2}\left(\max\left(0,f(s_{i_1})-f(s_{i_2})\right)\right)\right)^{\lambda}+\sum_{s_{j_2}\in N_S^2}\left(\min_{s_{j_1}\in N_S^1}\left(\max\left(0,f(s_{j_2})-f(s_{j_1})\right)\right)\right)^{\lambda}\right)\right)^{\frac{1}{\lambda}} & \text{otherwise}\end{cases} \tag{3.6}$$

In particular, for $\lambda=1$ and $\lambda=2$, we can obtain the directional Hausdorff distance d_{dhd} and the directional Euclidean–Hausdorff distance d_{dehd} between E_S^1 and E_S^2 respectively. It should be noted that both distances defined in Eqs. 2.2 and 3.6 are different in nature. The distance defined in Eq. 2.2 represents a traditional distance between E_S^1 and E_S^2 while distance defined in Eq. 3.6 determines a degree that E_S^1 outranks E_S^2.

Definition 3.8 Let $S=\{s_0,s_1,\ldots,s_6\}$ be a linguistic term set and E_S^1, E_S^2 be two HIFLTSs, then the strong dominance relation, weak dominance relation, and indifference relation of HIFLTSs can be defined as follows:

1. E_S^1 strongly dominates E_S^2 (or E_S^2 is strongly dominated by E_S^1):$E_S^1\succ_s E_S^2$(or $E_S^2\prec_s E_S^1$) $\Leftrightarrow d_{dhd}(E_S^1,E_S^2)\geq\frac{1}{g}$ and $d_{dhd}(E_S^2,E_S^1)=0$;
2. E_S^1 weekly dominates E_S^2 (or E_S^2 is weekly dominated by E_S^1):$E_S^1\succ_w E_S^2$(or $E_S^2\prec_w E_S^1$) $\Leftrightarrow d_{dhd}(E_S^1,E_S^2)<\frac{1}{g}$ and $d_{dhd}(E_S^2,E_S^1)=0$;
3. Indifference: if $d_{dhd}(E_S^1,E_S^2)=d_{dhd}(E_S^2,E_S^1)=0$, then E_S^1 is indifferent to E_S^2, denoted by $E_S^1\sim_{id}E_S^2$.
4. Incomparable relation: If none of the relations mentioned above exist between E_S^1 and E_S^2, then E_S^1 and E_S^2 are incomparable, denoted by $E_S^1\perp_{ic}E_S^2$.

In continuation of Example 3.5 and using Definition 3.8, we conclude that $E_S^2\succ_s^{0.5164}E_S^1$, $E_S^3\succ_s^{0.7071}E_S^1$ and $E_S^3\succ_s^{0.2887}E_S^2$.

Proposition 3.9 *Let $E_S^1=\{M_S^1,N_S^1\}$ and $E_S^2=\{M_S^2,N_S^2\}$ be two arbitrary HIFLTSs on S. If $M_{S^-}^1>M_{S^+}^2$ and $N_{S^-}^2>N_{S^+}^1$, then $E_S^1\succ_s E_S^2$.*

Proof Since $M_{S^-}^1>M_{S^+}^2$ and $N_{S^-}^2>N_{S^+}^1$, then $\min\{s_{i_1}|s_{i_1}\in M_S^1\}>\max\{s_{i_2}|s_{i_2}\in M_S^2\}$ and $\max\{s_{j_2}|s_{j_2}\in N_S^2\}>\min\{s_{j_1}|s_{j_1}\in N_S^1\}$, which indicates that

$$f(s_{i_1})-f(s_{i_2})>0,\ s_{i_1}\in M_S^1 \text{ and } s_{i_2}\in M_S^2$$
$$f(s_{j_2})-f(s_{j_1})>0,\ s_{j_1}\in N_S^1 \text{ and } s_{j_2}\in N_S^2.$$

Therefore, by our proposed definition

$$\begin{aligned}d_{dhd}(E_S^1,E_S^2)&=\frac{1}{|E_S^1|}\left(\sum_{s_{i_1}\in M_S^1}\min_{s_{i_2}\in M_S^2}\left(\max\left(0,f(s_{i_1})-f(s_{i_2})\right)\right)+\sum_{s_{j_2}\in N_S^2}\min_{s_{j_1}\in N_S^1}\left(\max\left(0,f(s_{j_2})-f(s_{j_1})\right)\right)\right)\\&=\frac{1}{|E_S^1|}\left(\sum_{s_{i_1}\in M_S^1}\min_{s_{i_2}\in M_S^2}\left(f(s_{i_1})-f\left(s_{i_2}\right)\right)+\sum_{s_{j_2}\in N_S^2}\min_{s_{j_1}\in N_S^1}\left(f(s_{j_2})-f(s_{j_1})\right)\right)\\&=\frac{1}{|E_S^1|}\left(\sum_{s_{i_1}\in M_S^1}\left(f(s_{i_1})-f(M_{S^+}^2)\right)+\sum_{s_{j_2}\in N_S^2}\left(f(s_{j_2})-f(N_{S^+}^1)\right)\right)\\&=\frac{1}{|E_S^1|}\begin{pmatrix}(f(M_{S^-}^1)-f(M_{S^+}^2))+\cdots+(f(M_{S^+}^1)-f(M_{S^+}^2))+\\(f(N_{S^-}^2)-f(N_{S^+}^1))+\cdots+(f(N_{S^+}^2)-f(N_{S^+}^1))\end{pmatrix}\\&\geq\frac{|E_S^1|((f(M_{S^-}^1)-f(M_{S^+}^2))+(f(N_{S^-}^2)-f(N_{S^+}^1)))}{|E_S^1|}\\&\geq\min_{s_i\in S}(f(s_{i+1})-f(s_i))=\frac{1}{g}\end{aligned}$$

Further, the directional Hausdorff distance between E_S^2 and E_S^1 is given as

$$\begin{aligned}d_{dhd}(E_S^2,E_S^1)&=\frac{1}{|E_S^2|}\left(\sum_{s_{i_2}\in M_S^2}\min_{s_{i_1}\in M_S^1}\left(\max\left(0,f(s_{i_1})-f(s_{i_2})\right)\right)+\sum_{s_{j_1}\in N_S^1}\min_{s_{j_2}\in N_S^2}\left(\max\left(0,f(s_{j_2})-f(s_{j_1})\right)\right)\right)\\&=\frac{1}{|E_S^2|}\left(\min_{s_{i_1}\in M_S^1}(0)+\min_{s_{j_2}\in N_S^2}(0)\right)=0. \text{ Hence } E_S^1\succ_s E_S^2.\end{aligned}$$

□

Proposition 3.10 *Let $E_S^1=\{M_S^1,N_S^1\}$ and $E_S^2=\{M_S^2,N_S^2\}$ be two arbitrary HIFLTSs on S. If $M_{S^+}^1>M_{S^+}^2\geq M_{S^-}^1$ and $N_{S^+}^1<N_{S^+}^2\leq N_{S^-}^1$, then $E_S^1\succ_s E_S^2$ or $E_S^1\succ_w E_S^2$.*

Proof Suppose some $s_{i_1}\leq M_{S^+}^2$ and $s_{j_2}\leq N_{S^+}^1$ where $s_{i_1}\in M_S^1$, $s_{j_2}\in N_S^2$. Then

$$\max\left(0,f(s_{i_1})-f\left(M_{S^+}^2\right)\right)=\min_{s_{i_2}\in M_S^2}\left\{\max\left(0,f(s_{i_1})-f(s_{i_2})\right)\right\}=0;$$
$$\max\left(0,f(s_{j_2})-f\left(N_{S^+}^1\right)\right)=0 \text{ and } \min_{s_{j_1}\in N_S^1}\left\{\max\left(0,f(s_{j_2})-f(s_{j_1})\right)\right\}=0.$$

Further, suppose $s_{i_1}>M_{S^+}^2$, where $s_{i_1}\in M_S^1$ and $s_{i_1}>s_{i_2}$ for any $s_{i_2}\in M_S^2$, then

$$\min_{s_{i_2}\in M_S^2}\left\{\max\left(0,f(s_{i_1})-f(s_{i_2})\right)\right\}=\min_{s_{i_2}\in M_S^2}\left\{f(s_{i_1})-f(s_{i_2})\right\}$$
$$=f(s_{i_1})-f\left(M_{S^+}^2\right)$$

Similarly, suppose $s_{j_2} > N_{S^+}^1$, where $s_{j_2} \in N_{S^+}^2$ and $s_{j_2} > s_{j_1}$ for some $s_{j_1} \in N_S^1$, then

$$\min_{s_{j_1}\in N_S^1}\left\{\max\left(0,f(s_{j_2})-f(s_{j_1})\right)\right\}=\min_{s_{j_1}\in N_S^1}\left\{f(s_{j_2})-f(s_{j_1})\right\}$$
$$=f(s_{j_2})-f\left(N_{S^+}^1\right)$$

So by our definition

$$d_{dhd}(E_S^1,E_S^2)=\frac{1}{|E_S^1|}\left(\begin{array}{l}\sum_{s_{i_1}\in M_S^1}\min_{s_{i_2}\in M_S^2}\left(\max\left(0,f(s_{i_1})-f(s_{i_2})\right)\right)\\+\sum_{s_{j_2}\in N_S^2}\min_{s_{j_1}\in N_S^1}\left(\max\left(0,f(s_{j_2})-f(s_{j_1})\right)\right)\end{array}\right)$$
$$=\frac{1}{|E_S^1|}\left(\sum_{s_{i_1}\in M_S^1}\left(f(s_{i_1})-f(M_{S^+}^2)\right)+\sum_{s_{j_2}\in N_S^2}\left(f(s_{j_2})-f(N_{S^+}^1)\right)\right)>0$$

For any $s_{i_2} \in M_S^2$, if $s_{i_2} \le M_{S^+}^2 < M_{S^+}^1$, then $f(s_{i_2}) - f(M_{S^+}^1) < 0$.

Similarly, for any $s_{j_1} \in N_S^1$, if $s_{j_1} \le N_{S^+}^1 < N_{S^+}^2$, then $f(s_{j_1}) - f(N_{S^+}^2) < 0$.

Therefore, $\max\left(0,f(s_{i_2})-f(M_{S^+}^1)\right)=0$ and $\max\left(0,f(s_{j_1})-f(N_{S^+}^2)\right)=0$

$$d_{dhd}(E_S^2,E_S^1)=\frac{1}{|E_S^2|}\left(\begin{array}{l}\sum_{s_{i_2}\in M_S^2}\min_{s_{i_1}\in M_S^1}\left(\max\left(0,f(s_{i_1})-f(s_{i_2})\right)\right)\\+\sum_{s_{j_1}\in N_S^1}\min_{s_{j_2}\in N_S^2}\left(\max\left(0,f(s_{j_2})-f(s_{j_1})\right)\right)\end{array}\right)$$
$$=\frac{1}{|E_S^2|}\left(\sum_{s_{i_2}\in M_S^2}\min_{s_{i_1}\in M_S^1}(0)+\sum_{s_{j_1}\in N_S^1}\min_{s_{j_2}\in N_S^2}(0)\right)=0$$

Therefore, $d_{dhd}(E_S^1,E_S^2)>0, d_{dhd}(E_S^2,E_S^1)=0$ which further implies that $E_S^1 \succ_s E_S^2$ or $E_S^1 \succ_w E_S^2$ by the statement related to strong and week dominance relations given in Definition 3.8. □

Proposition 3.11 *Let $E_S^1=\{M_S^1,N_S^1\}$ and $E_S^2=\{M_S^2,N_S^2\}$ be two arbitrary HIFLTSs on S. If $M_{S^+}^1 > M_{S^+}^2$ and $N_{S^+}^1 > N_{S^+}^2$, then $E_S^1 \perp_{ic} E_S^2$ i.e. $d_{dhd}(E_S^1,E_S^2)\neq 0$ and $d_{dhd}(E_S^2,E_S^1)\neq 0$.*

Proof The proof is similar to that of Proposition 3.9 and is therefore omitted here. □

Properties Let $S=\{s_0,s_1,\ldots,s_g\}$ be a linguistic term set and $E_S, E_S^1, E_S^2 \in \mathbb{C}$. The followings are true.

1. The strong dominance relations are categorized as:
   (a) Irreflexivity: $\forall E_S \in \mathbb{C}$, $E_S \not\succ_s E_S$;
   (b) Asymmetry: $\forall E_S^1, E_S^2 \in \mathbb{C}$, $E_S^1 \succ_s E_S^2 \not\Rightarrow E_S^2 \succ_s E_S^1$;
   (c) Transitivity: $\forall E_S, E_S^1, E_S^2 \in \mathbb{C}$, $E_S \succ_s E_S^1$ and $E_S^1 \succ_s E_S^2 \Rightarrow E_S \succ_s E_S^2$.
2. The weak dominance relations are categorized as:
   (a) Irreflexivity: $\forall E_S \in \mathbb{C}$, $E_S \not\succ_w E_S$;
   (b) Asymmetry: $\forall E_S^1, E_S^2 \in \mathbb{C}$, $E_S^1 \succ_w E_S^2 \not\Rightarrow E_S^2 \succ_w E_S^1$;
   (c) Non-transitivity: $\forall E_S, E_S^1, E_S^2 \in \mathbb{C}$, $E_S \succ_w E_S^1$ and $E_S^1 \succ_w E_S^2 \not\Rightarrow E_S \succ_w E_S^2$.
3. The indifference relations are categorized as:
   (a) Irreflexivity: $\forall E_S \in \mathbb{C}$, $E_S \sim_{id} E_S$;
   (b) Asymmetry: $\forall E_S^1, E_S^2 \in \mathbb{C}$, $E_S^1 \sim_{id} E_S^2 \not\Rightarrow E_S^2 \sim_{id} E_S^1$;
   (c) Transitivity: $\forall E_S, E_S^1, E_S^2 \in \mathbb{C}$, $E_S \sim_{id} E_S^1$ and $E_S^1 \sim_{id} E_S^2 \Rightarrow E_S \sim_{id} E_S^2$.
4. The incomparable relations are categorized as:
   (a) Irreflexivity: $\forall E_S \in \mathbb{C}$, $E_S \not\vdash E_S$;
   (b) Asymmetry: $\forall E_S^1, E_S^2 \in \mathbb{C}$, $E_S^1 \perp_{ic} E_S^2 \Rightarrow E_S^2 \perp_{ic} E_S^1$;
   (c) Non-transitivity: $\forall E_S, E_S^1, E_S^2 \in \mathbb{C}$, $E_S \perp_{ic} E_S^1$ and $E_S^1 \perp_{ic} E_S^2 \not\Rightarrow E_S \perp_{ic} E_S^2$.

To avoid the difficulty of providing the membership and non-membership information simultaneously in the decision making process under the HIFN environment, Zhou et al. [51] further provided a simplified HIFN in Definition 2.2. Motivated by this idea, we further propose a simplified HIFLTS as follows:

Definition 3.12 Let X be a fixed set and $S=\{s_0,s_1,\ldots,s_g\}$ be a linguistic term set. Then a HIFLTS $\mathbf{E}_S$ on X can also be defined as follows:

$$\mathbf{E}_S=\{(x,\mathbf{M}_S(x),\mathbf{n}_S(x))\mid x\in X\},$$

where $\mathbf{M}_S$ and $\mathbf{n}_S$ represent the membership degree and the non-membership degree, respectively, of element x to $\mathbf{E}_S$, and $\mathbf{M}_S$ is a HFE of the consecutive linguistic term set S i.e.$\mathbf{M}_S \subseteq S$ and $\mathbf{n}_S=\{s_j\}$ is a singleton set satisfying $s_0 \le \max\{\mathbf{M}_S\}\oplus\ s_j \le s_g$. For convenience, $\mathbf{E}_S=\{\mathbf{M}_S(x),\mathbf{n}_S(x)\}$ a hesitant intuitionistic fuzzy linguistic element (HIFLE). We can see that the HIFLTS $\mathbf{E}_S$ satisfies the basic condition of HIFLTS which was initially proposed by Beg and Rashid [4].

We can also modify the generalized directional Hausdorff distance formula between HIFLTSs $\mathbf{E}_S^1$ and $\mathbf{E}_S^2$ as follows:

Definition 3.13 Let $S = \{s_0, s_1, \ldots, s_g\}$ be a linguistic term set and $\mathbf{E}_S^1 = \{\mathbf{M}_S^1, \mathbf{n}_S^1\}$ and $\mathbf{E}_S^2 = \{\mathbf{M}_S^2, \mathbf{n}_S^2\}$ be two arbitrary HIFLTSs on S, where $\mathbf{M}_S^k = \cup_{s_{i_k}} \{s_{i_k} \mid i_k = 1, \ldots, \# \quad \mathbf{M}_S^k\}$, $\mathbf{n}_S^k = \{s_{j_k}\}$, $s_0 \le \max\{\mathbf{M}_S^k\} \oplus s_{j_k} \le s_g$, $k = 1, 2$. The generalized directional Hausdorff distance $\mathbf{d}_{gdhd}$ from $\mathbf{E}_S^1$ to $\mathbf{E}_S^2$ can be defined as follows:

$$\mathbf{d}_{gdhd}(\mathbf{E}_S^1, \mathbf{E}_S^2) = \begin{cases} \left(\dfrac{1}{|\mathbf{E}_S^2|} \left(\displaystyle\sum_{s_{i_2} \in \mathbf{M}_S^2} \left(\min_{s_{i_1} \in \mathbf{M}_S^1} \left(\max\left(0, f(s_{i_1}) - f(s_{i_2})\right)\right)\right)^{\lambda} + \left(\max\left(0, f(s_{j_2}) - f(s_{j_1})\right)\right)^{\lambda} \right) \right)^{\frac{1}{\lambda}} & \text{if } \mathbf{M}_{S+}^1 = \mathbf{M}_{S+}^2 \text{ and } s_{j_1} = s_{j_2} \\ \left(\dfrac{1}{|\mathbf{E}_S^1|} \left(\displaystyle\sum_{s_{i_1} \in \mathbf{M}_S^1} \left(\min_{s_{i_2} \in \mathbf{M}_S^2} \left(\max\left(0, f(s_{i_1}) - f(s_{i_2})\right)\right)\right)^{\lambda} + \left(\max\left(0, f(s_{j_2}) - f(s_{j_1})\right)\right)^{\lambda} \right) \right)^{\frac{1}{\lambda}} & \text{otherwise} \end{cases} \tag{3.7}$$

For $\lambda = 1$, we can obtain the directional Hausdorff distance $\mathbf{d}_{dhd}$ between $\mathbf{E}_S^1$ and $\mathbf{E}_S^2$. Similarly for $\lambda = 2$, we can obtain the directional Euclidean–Hausdorff distance $\mathbf{d}_{dehd}$ between $\mathbf{E}_S^1$ and $\mathbf{E}_S^2$.

Proposition 3.14 *Let* $\mathbf{E}_S^1 = \{\mathbf{M}_S^1, \mathbf{n}_S^1\}$ *and* $\mathbf{E}_S^2 = \{\mathbf{M}_S^2, \mathbf{n}_S^2\}$ *be two arbitrary HIFLTSs on S as defined in Definition 3.12. Then the following can be easily proven.*

1. *If* $\mathbf{M}_{S-}^1 > \mathbf{M}_{S+}^2$ *and* $s_{j_2} > s_{j_1}$, *then* $\mathbf{E}_S^1 \succ_s \mathbf{E}_S^2$;
2. *If* $\mathbf{M}_{S+}^1 > \mathbf{M}_{S+}^2 \ge \mathbf{M}_{S-}^1$ *and* $s_{j_1} < s_{j_2}$, *then* $\mathbf{E}_S^1 \succ_s \mathbf{E}_S^2$ *or* $\mathbf{E}_S^1 \succ_w \mathbf{E}_S^2$;
3. *If* $\mathbf{M}_{S+}^1 > \mathbf{M}_{S+}^2$ *and* $s_{j_1} > s_{j_2}$, *then* $\mathbf{E}_S^1 \perp_{ic} \mathbf{E}_S^2$ *i.e.* $d_{dhd}(\mathbf{E}_S^1, \mathbf{E}_S^2) \ne 0$ *and* $d_{dhd}(\mathbf{E}_S^2, \mathbf{E}_S^1) \ne 0$.

4 MCDM Approaches with HIFLTSs

This section describes two MCDM approaches for decision making based on outranking and TOPSIS methods using HIFLTSs. It also includes a methodology of determining the criteria weights based on score values of HIFLEs.

In a multi-criteria linguistic ranking or selection problem, there are n alternatives, denoted by $A = \{a_1, a_2, \ldots, a_n\}$. Each alternative is assessed by means of m criteria, denoted by $C = \{c_1, c_2, \ldots, c_m\}$, and the evaluations are undertaken using linguistic expressions. The weight of the criterion c_j is w_j, where $w_j \ge 0$ $(j = 1, 2, \ldots, m)$, and $\sum_{j=1}^{m} w_j = 1$. The hesitant intuitionistic fuzzy linguistic value of a_i under c_j is in the form of HIFLTSs and denoted by a_{ij}.

4.1 Outranking Method Based on Directional Hausdorff Distance

The concordance and discordance indices.

Definition 4.1 The set of subscripts for all criteria is $J = \{j | j = 1, 2, \ldots, m\}$. Based on the relationship between the alternatives a_i and a_k, the following sets can be then defined:

1. The set of subscripts for all criteria that meet the constraint $a_{ij} \succ_s a_{kj}$ is $C_s(a_i, a_k) = \{j | 1 \le j \le m, a_{ij} \succ_s a_{kj}\}$.
2. The set of subscripts for all criteria that meet the constraint $a_{ij} \succ_w a_{kj}$ is $C_w(a_i, a_k) = \{j | 1 \le j \le m, a_{ij} \succ_w a_{kj}\}$.
3. The set of subscripts for all criteria that meet the constraint $a_{ij} \prec_s a_{kj}$ is $D_s(a_i, a_k) = \{j | 1 \le j \le m, a_{ij} \prec_s a_{kj}\}$.
4. The set of subscripts for all criteria that meet the constraint $a_{ij} \prec_w a_{kj}$ is $D_w(a_i, a_k) = \{j | 1 \le j \le m, a_{ij} \prec_w a_{kj}\}$.
5. The set of subscripts for all criteria that meet the constraint $a_{ij} \sim_{id} a_{kj}$ is $ID(a_i, a_k) = ID(a_k, a_i) = \{j | 1 \le j \le m, a_{ij} \sim_{id} a_{kj}\}$.

The integration of the ELECTRE method and HIFLTSs is utilized in this paper, and corresponding definitions are therefore required here.

Using the weight vector ω associated with criteria, we define the comprehensive concordance index c_{ik} between a_i and a_k as follows:

$$c_{ik} = \sum_{j \in R(a_i, a_k)} w_j + \sum_{j \in D_w(a_i, a_k)} w_j d_{hdh}(a_{kj}, a_{ij}), \tag{4.1}$$

where $R(a_i, a_k) = C_s(a_i, a_k) \cup C_w(a_i, a_k) \cup ID(a_i, a_k)$ and the concordance matrix is

$$C = \begin{bmatrix} - & c_{12} & c_{13} & \cdots & c_{1(n-1)} & c_{1n} \\ c_{21} & - & c_{23} & \cdots & c_{2(n-1)} & c_{2n} \\ \cdots & \cdots & - & \cdots & \cdots & \cdots \\ c_{(n-1)1} & c_{(n-1)2} & c_{(n-1)3} & \cdots & - & c_{(n-1)n} \\ c_{n1} & c_{n2} & c_{n3} & \cdots & c_{n(n-1)} & - \end{bmatrix}.$$

The discordance index d_{ik} between a_i and a_k is thus defined as follows:

$$d_{ik} = \frac{\max\limits_{j \in D_s(a_i, a_k) \cup D_w(a_i, a_k)} \{w_j d_{hdh}(a_{kj}, a_{ij})\}}{\max\limits_{j \in J} w_j (d_{hdh}(a_{ij}, a_{kj}) + d_{hdh}(a_{kj}, a_{ij}))}, \tag{4.2}$$

where the discordance matrix D is

$$D = \begin{bmatrix} - & d_{12} & d_{13} & \cdots & d_{1(n-1)} & d_{1n} \\ d_{21} & - & d_{23} & \cdots & d_{2(n-1)} & d_{2n} \\ \cdots & \cdots & - & \cdots & \cdots & \cdots \\ d_{(n-1)1} & d_{(n-1)2} & d_{(n-1)3} & \cdots & - & d_{(n-1)n} \\ d_{n1} & d_{n2} & d_{n3} & \cdots & d_{n(n-1)} & - \end{bmatrix}.$$

To rank all alternatives, the net dominance index of a_k is

$$c_k = \sum_{\substack{l=1 \\ l \neq k}}^{n} c_{kl} - \sum_{\substack{l=1 \\ l \neq k}}^{n} c_{lk}, \tag{4.3}$$

and the net disadvantage index of a_k is

$$d_k = \sum_{\substack{l=1 \\ l \neq k}}^{n} d_{kl} - \sum_{\substack{l=1 \\ l \neq k}}^{n} d_{lk}. \tag{4.4}$$

Obviously, c_k is the major factor that equals the sum of the concordance indices between a_k and others, minus the sum of the concordance indices between $a_l\ (l \neq k)$ and a_k. The dominance index c_k reflects the dominance degree of a_k among all alternatives. Similarly, d_k reflects the disadvantage degree of a_k . Therefore, while c_k is greater and d_k is less, a_k gets higher dominance among all alternatives. Now the ranking rule can be defined as follows.

Definition 4.2 The ranking rule of two alternatives is

1. if $c_l < c_k$ and $d_l > d_k$, then a_k is superior to a_l , denoted by $a_k \succ a_l$;
2. if $c_l = c_k$ and $d_l = d_k$, then a_l is indifferent to a_k, denoted by $a_l \sim a_k$;
3. if none of the relations mentioned above exists between a_l and a_k, then a_l and a_k are incomparable, denoted by $a_l \perp a_k$.

4.2 TOPSIS Method for MCDM with HIFLTSs Based on Generalized Euclidean Distance

In this subsection, we now describe MCDM approach using the TOPSIS method with HIFLTSs. With the aforementioned assumptions as given in the same Section, TOPSIS is performed using the following five steps:

Step 1 Construction of the hesitant intuitionistic fuzzy linguistic decision matrix $A = [E_{S_{ij}}]$ using the expert opinion where $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

Step 2 Determine the hesitant intuitionistic linguistic positive ideal solution x^+ and hesitant intuitionistic linguistic negative ideal solution x^- which can be defined, respectively, as follows:

$x^+ = \{E_{S_{1+}}, E_{S_{2+}}, \ldots, E_{S_{n+}}\}$ and $x^- = \{E_{S_{1-}}, E_{S_{2-}}, \ldots, E_{S_{n-}}\}$ where for $j = 1, 2, \ldots, n$,

$$E_{S_{j+}} = \begin{cases} \left\{ \left(\max\limits_{i=1,\ldots,n} M^{+}_{S_{ij}} \right), \left(\min\limits_{i=1,\ldots,n} N^{+}_{S_{ij}} \right) \right\}, & \text{forbenefitcriterionc}_j \\ \left\{ \left(\min\limits_{i=1,\ldots,n} M^{t-}_{S_{ij}} \right), \left(\max\limits_{i=1,\ldots,n} N^{t-}_{S_{ij}} \right) \right\}, & \text{forcostcriterionc}_j \end{cases},$$

$$E_{S_{j-}} = \begin{cases} \left\{ \left(\min\limits_{i=1,\ldots,n} M^{+}_{S_{ij}} \right), \left(\max\limits_{i=1,\ldots,n} N^{+}_{S_{ij}} \right) \right\}, & \text{forbenefitcriterionc}_j \\ \left\{ \left(\max\limits_{i=1,\ldots,n} M^{t-}_{S_{ij}} \right), \left(\min\limits_{i=1,\ldots,n} N^{t-}_{S_{ij}} \right) \right\}, & \text{forcostcriterionc}_j \end{cases}.$$

Step 3 Using the proposed generalized distance measure (Eq. 2.2) between two HIFLTSs for different values of λ, calculate the distance between each alternative a_i and the hesitant intuitionistic fuzzy linguistic positive ideal solution x^+, and the distance between each alternative x_i and the hesitant intuitionistic fuzzy linguistic negative ideal solution x^- which can be defined, respectively, as follows:

$$D_i^+ = \sum_{j=1}^{m} d_{gd}(A_{S_{ij}}, E_{S_{j+}}), \quad D_i^- = \sum_{j=1}^{m} d_{gd}(A_{S_{ij}}, E_{S_{j-}}), \quad i = 1, 2, \ldots, n.$$

Step 4 Calculate the relative closeness coefficient C_i of alternative a_i with respect to ideal solution x^+ using the following equation:

$$C_i = \frac{D_i^-}{D_i^+ + D_i^-}$$

Step 5 Rank the alternatives according to ranking of C_i. The larger the value of C_i, the higher priority of the alternative a_i.

4.3 Method of Determining the Criteria Weights

It is worth noticing that, sometimes, the information about criteria weights is completely unknown due to lack of information, incomplete knowledge of the studied system, time pressure or expert's hesitation. It is necessary, therefore to allocate weights to the criteria in order to give them their relative priority in a MCDM problem. The score-based weight generation method is given as follows:

Let $B = [Sc_{ij}]$ be a matrix obtained by calculating the score values (using Eq. 3.3) of each HIFLE $E_{S_{ij}}$ of the matrix $A = [E_{S_{ij}}]$, $(i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m)$. The weight index of each criterion can be determined as:

$$w_j = \frac{\sum_{i=1}^{n} \frac{1 + Sc_{ij}}{2}}{\sum_{j=1}^{m} \sum_{i=1}^{n} \frac{1 + Sc_{ij}}{2}} \tag{4.5}$$

where $w_j \geq 0\ (j = 1, 2, \ldots, m)$, $\sum_{j=1}^{m} w_j = 1$. Here, it should be noted that if the score value of a criterion is bigger across alternatives, then the criterion should be assigned bigger weight; otherwise, such a criterion will be judged unimportant by most decision makers.

5 An Illustrative Example

In this section, a practical example with given information about criteria weights is considered to illustrate the proposed method.

Due to the emergence of Mathematics as a rapidly growing discipline/field, a distance education university wants to initiate three more courses in the department. In this context, it is decided to use the internal resources for the recording of these courses. The Head of Department selected three members/alternatives a_1, a_2 and a_3 from his team for the accomplishment of this task. After the completion, the recorded courses will be evaluated on the basis of four criteria: the subject knowledge C_1, the communication skills C_2, the course presentation C_3 and the time management C_4. Assuming $W = (0.4, 0.2, 0.3, 0.1)$ as the weight vector for the above listed criteria. Here, it should be noted that if the criteria weights are completely unknown for the problem. The optimal weight vector for the matrix A can be determined using Eq. 4.5, and then, ranking order of alternatives will be changed accordingly.

Let $S = \{s_0 =$Very Poor$, s_1 =$Poor$, s_2 =$Medium Poor$, s_3 =$Fair$, s_4 =$Medium Good$, s_5 =$Good$, s_6 =$Very Good$\}$.

The procedures for obtaining the best alternative are now outlined.

Step 1 Transform the linguistic expressions into HIFLTSs according to Definition 2.7. The three alternatives are to be evaluated under the above four criteria in the form of HIFLTSs, as shown in the following hesitant intuitionistic fuzzy linguistic decision matrix A as follows:

$$A = \begin{bmatrix} \{(s_2, s_3), (s_0)\} & \{(s_4, s_5, s_6), (s_0)\} & \{(s_0, s_1, s_2), (s_4)\} & \{(s_4, s_5), (s_0, s_1)\} \\ \{(s_3), (s_2, s_3)\} & \{(s_2, s_3, s_4), (s_0, s_1)\} & \{(s_2, s_3), (s_3, s_4)\} & \{(s_6), (s_0)\} \\ \{(s_3, s_4), (s_0, s_1)\} & \{(s_3, s_4), (s_0, s_2)\} & \{(s_4), (s_0, s_2)\} & \{(s_0, s_1, s_2, s_3), (s_3)\} \end{bmatrix}$$

Step 2 Determine the sets of criteria labels in terms of Definition 4.1 and dominance relations between a_{ij} and a_{kj}.

The sets of criteria labels are given below:

$C_S(a_1, a_2) = \{1, 2\}$; $C_W(a_1, a_2) = \emptyset$; $ID(a_1, a_2) = \emptyset$; $D_S(a_1, a_2) = \{4\}$;
$D_W(a_1, a_2) = \{3\}$; $C_S(a_1, a_3) = \{2, 4\}$; $C_W(a_1, a_3) = \emptyset$;
$ID(a_1, a_3) = \{1\}$; $D_S(a_1, a_3) = \{3\}$; $D_W(a_1, a_3) = \emptyset$;
$C_S(a_2, a_3) = \{4\}$; $C_W(a_2, a_3) = \{2\}$; $ID(a_2, a_3) = \emptyset$;
$D_S(a_2, a_3) = \{1, 3\}$; $D_W(a_2, a_3) = \emptyset$.

The dominance relations are determined as follows:

$a_{11} \succ_s^{0.2778} a_{21}, a_{12} \succ_s^{0.1667} a_{22}, a_{13} \prec_w^{0.0417} a_{23}, a_{14} \prec_s^{0.1667} a_{24}$.

Similarly $a_{11} \perp_{ic} a_{31}, a_{12} \succ_s^{0.2083} a_{32}, a_{13} \prec_s^{0.2222} a_{33}, a_{14} \succ_s^{0.2083} a_{34}$,

$a_{21} \prec_s^{0.1667} a_{31}, a_{22} \succ_w^{0.0333} a_{32}, a_{23} \prec_s^{0.2222} a_{33}, a_{24} \succ_s^{0.5000} a_{34}$.

Step 3 Determine the concordance and discordance matrix.

Based on Formula 4.1, the concordance matrix C can be obtained as below:

$$C = \begin{bmatrix} 0 & 0.6125 & 0.7 \\ 0.4 & 0 & 0.3 \\ 0.7 & 0.7067 & 0 \end{bmatrix}$$

Based on Formula 4.2, the discordance matrix D is as follows:

$$D = \begin{bmatrix} 0 & 0.15 & 1 \\ 1 & 0 & 1 \\ 0.6250 & 0.7499 & 0 \end{bmatrix}$$

Step 4 Get the net dominance and disadvantage indices:

Based on Formulae 4.3 and 4.4, the net dominance index and the net disadvantage index for each alternative are, respectively, shown as follows:

$c_1 = 0.2125, c_2 = -0.6192, c_3 = 0.4067;$
$d_1 = -0.4749, d_2 = 1.1001, d_3 = -0.6252.$

Step 5 Rank the alternatives in accordance with c_i and d_i, referring to Definition 4.2.

As $c_3 > c_1 > c_2$ and $d_3 < d_1 < d_2$. Hence, $a_3 \succ a_1 \succ a_2$ and thus the best alternative is a_3.

In addition, if we use Definition 3.12 of HIFLTS $\mathbf{E}_S$ proposed in Sect. 3, the corresponding decision matrix is

$$\mathbf{A} = \begin{bmatrix} \{(s_2, s_3), (s_3)\} & \{(s_4, s_5, s_6), (s_0)\} & \{(s_0, s_1, s_2), (s_4)\} & \{(s_4, s_5), (s_1)\} \\ \{(s_3), (s_3)\} & \{(s_2, s_3, s_4), (s_2)\} & \{(s_2, s_3), (s_3)\} & \{(s_6), (s_0)\} \\ \{(s_3, s_4), (s_2)\} & \{(s_3, s_4), (s_2)\} & \{(s_4), (s_2)\} & \{(s_0, s_1, s_2, s_3), (s_3)\} \end{bmatrix}$$

Suppose that the hesitant intuitionistic fuzzy linguistic value of the alternative a_i under the criteria c_j in the form of HIFLTSs (Definition 3.12) is denoted by $\mathbf{a}_{ij}$. The concordance and discordance matrices $\mathbf{C}$ and $\mathbf{D}$ respectively, are obtained as follows:

$$\mathbf{C} = \begin{bmatrix} 0 & 0.6333 & 0.3444 \\ 0.8 & 0 & 0.3 \\ 0.7 & 0.9 & 0 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 & 0.1384 & 0.3114 \\ 0.1730 & 0 & 0.1558 \\ 0.1297 & 0.1557 & 0 \end{bmatrix}$$

The net dominance and disadvantage indices are, respectively,

$\mathbf{c}_1 = -0.5222, \mathbf{c}_2 = -0.4333, \mathbf{c}_3 = 0.9556;$
$\mathbf{d}_1 = 0.1471, \mathbf{d}_2 = 0.0346, \mathbf{d}_3 = -0.1817.$

As $\mathbf{c}_3 > \mathbf{c}_2 > \mathbf{c}_1$ and $\mathbf{d}_3 < \mathbf{d}_2 < \mathbf{d}_1$. Hence, $a_3 \succ a_2 \succ a_1$ and thus the best alternative is again a_3.

5.1 Comparison Analysis

For HIFLTS E_S proposed in Definition 2.4, the dominance relations between a_{11}, a_{21} and a_{31} are, respectively, $a_{11} >_s^{0.2778} a_{21}$, $a_{11} \perp_{ic} a_{31}$ and $a_{21} <_s^{0.16667} a_{31}$; between a_{22}, a_{32} is $a_{22} >_w^{0.0333} a_{32}$ for $\lambda = 1$. Similarly for HIFLTS $\mathbf{E}_S$ proposed in Definition 3.12, the dominance relations between $\mathbf{a}_{11}$, $\mathbf{a}_{21}$, $\mathbf{a}_{31}$ are, respectively, $\mathbf{a}_{11} \sim_{id} \mathbf{a}_{21}$, $\mathbf{a}_{11} <_w^{0.1111} \mathbf{a}_{31}$ and $\mathbf{a}_{21} <_w^{0.1111} \mathbf{a}_{31}$; between $\mathbf{a}_{22}$, $\mathbf{a}_{32}$ is $\mathbf{a}_{22} \sim_{id} \mathbf{a}_{32}$ for $\lambda = 1$.

As in the decision matrix based on HIFLTS E_S, $a_{11} = \{(s_2, s_3), (s_0)\}$, $a_{21} = \{(s_3), (s_2, s_3)\}$, $a_{31} = \{(s_3, s_4), (s_0, s_1)\}$, $a_{22} = \{(s_2, s_3, s_4), (s_0, s_1)\}$ and $a_{32} = \{(s_3, s_4), (s_0, s_2)\}$. After applying Definition 3.12, the corresponding HIFLEs are $\mathbf{a}_{11} = \{(s_2, s_3), (s_3)\}$, $\mathbf{a}_{21} = \{(s_3), (s_3)\}$, $\mathbf{a}_{31} = \{(s_3, s_4), (s_2)\}$, $\mathbf{a}_{22} = \{(s_2, s_3, s_4), (s_2)\}$ and $\mathbf{a}_{32} = \{(s_3, s_4), (s_2)\}$. The membership and non-membership functions of $\mathbf{a}_{11}$ and $\mathbf{a}_{21}$ have the linguistic variables of almost same weightage. That is why, $\mathbf{d}_{dhd}(\mathbf{a}_{11}, \mathbf{a}_{21}) = \mathbf{d}_{dhd}(\mathbf{a}_{21}, \mathbf{a}_{11}) = 0$ means $\mathbf{a}_{11} \sim_{id} \mathbf{a}_{21}$. In a_{11} and a_{31} based on HIFLTS E_S, $M_{11}^+ \leq M_{31}^+$ and $N_{11}^+ \leq N_{31}^+$ and as a result $a_{11} \perp_{ic} a_{31}$ while the result is different $(\mathbf{a}_{11} <_w^{0.1111} \mathbf{a}_{31})$ based on HIFLTS $\mathbf{E}_S$ due to getting strength in the non-membership function of $\mathbf{a}_{31}$ imposed by Definition 3.12. Similarly, a slight change also occurred while computing the dominance relation between $\mathbf{a}_{21}$ and$\mathbf{a}_{31}$. For a_{22} and a_{32} based on HIFLTS E_S, the dominance relation between them is $a_{22} >_w^{0.0333} a_{32}$ but after applying Definition 3.12, the non-membership functions of both $\mathbf{a}_{22}$ and $\mathbf{a}_{32}$ are similar and that is why $\mathbf{d}_{dhd}(\mathbf{a}_{22}, \mathbf{a}_{32}) = \mathbf{d}_{dhd}(\mathbf{a}_{32}, \mathbf{a}_{22}) = 0$ means $\mathbf{a}_{22} \sim_{id} \mathbf{a}_{32}$. All these four results effect the computation of decision analysis based on HIFLTS $\mathbf{E}_S$ and we get a slight change in the ranking order of alternatives a_1 and a_2. Therefore, we believe that Definition 2.4 is somehow stronger than Definition 3.12 because during the assessment based on Definition 2.4, DMs have liberty to analyze an alternative based on its non-membership linguistic variables. Here, it should be noted that the linguistic expressions provided by DMs in the form of HIFLTSs (Definitions 2.4 and 3.12) cannot be transformed into HIFLTSs proposed by Zhou et al. in Definition 3.3 by using the transformation function E_{G_H} (Definition 2.7) because of having different natures. In decision making problems, decision makers can assess an alternative according to his/her need and interests.

The ranking results of HIFLTS E_S (Definition 2.4) for different values of λ using Formulae 3.1, 3.6 and Formulae 3.2, 3.6 are summarized in Table 1 and Table 2, respectively. This indicates that the final ranking of the alternatives remains unchanged regardless of the values of parameter λ and the corresponding LSF f, which also verifies the strength of the proposed method. We can also see that $a_1 \perp a_2$ and $a_2 \perp a_3$ when $\lambda = 1$ using Formulae 3.2 and 3.6 but with the increase in the value of parameter λ, the ranking of alternatives gets stronger as shown in Table 2.

Similarly, the ranking results with HIFLTS $\mathbf{E}_S$(Definition 3.12) for different values of λ using Formulae 3.1, 3.7 and Formulae 3.2, 3.7 (by selecting $a = 1.4$) are summarized in Table 3 and Table 4, respectively. This also indicates that the final ranking of the alternatives remains unchanged regardless of the values of parameter λ and the corresponding LSF f, which also verifies the strength of the proposed method. The alternatives a_2 and a_3 are incomparable when $\lambda = 1$ using Formulae 3.2 and 3.7 but again with the increase in the value of parameter λ, the alternatives ranking get stronger as shown in Table 4.

The same numerical example is solved with the help of TOPSIS method by following steps 1–5 as mentioned in Sect. 4.2, and the ranking results are shown in Table 5 for different values of λ.

From Tables 1 and 5, we notice that the ranking order of alternatives obtained from the TOPSIS method agrees with the ranking order as obtained from the outranking method for $\lambda = 1, 2$ using the LSF 3.1. However, the ranking order is slightly different in both methods for $\lambda = 4$ and somehow unstable for $\lambda = 6$ and $\lambda = 10$. For example, the alternatives have the same ranking $a_3 \succ a_1 \succ a_2$ for $\lambda = 1, 2$ after utilizing both methods. For $\lambda = 4$, the ranking order of the alternatives using the TOPSIS method is $a_1 \succ a_3 \succ a_2$, which is slightly different from the ranking order as obtained in the outranking method for the same value of λ.

Similarly, from Tables 2 and 5 and using the LSF 3.2, we notice that the ranking order of alternatives obtained from the TOPSIS method is slightly different from the ranking order as obtained from the outranking method for

Table 1 The evaluation results of E_S using Formulae 3.1 and 3.6

	c_1	c_2	c_3	d_1	d_2	d_3	Ranking
$\lambda = 1$	0.2125	−0.6192	0.4067	−0.4749	1.1001	−0.6252	$a_3 \succ a_1 \succ a_2$
$\lambda = 2$	0.2000	−0.6149	0.4149	0.0877	1.2525	−1.3402	$a_3 \succ a_1 \succ a_2$
$\lambda = 4$	0.2000	−0.6223	0.4223	0.4988	0.9517	−1.4504	$a_3 \succ a_1 \succ a_2$
$\lambda = 6$	0.2000	−0.6255	0.4255	0.6418	0.7876	−1.4294	$a_3 \succ a_1 \succ a_2$
$\lambda = 10$	0.2000	−0.6284	0.4284	0.6728	0.7362	−1.4090	$a_3 \succ a_1 \succ a_2$

Table 2 The evaluation results of E_S using Formulae 3.2 and 3.6

	c_1	c_2	c_3	d_1	d_2	d_3	Ranking
$\lambda = 1$	0.2379	−0.5600	0.3221	−0.8244	0.9011	−0.0768	$a_3 \succ a_1$
$\lambda = 2$	0.2000	−0.6144	0.4144	−0.1456	1.3145	−1.1689	$a_3 \succ a_1 \succ a_2$
$\lambda = 4$	0.2000	−0.6215	0.4215	0.2796	1.0415	−1.3211	$a_3 \succ a_1 \succ a_2$
$\lambda = 6$	0.2000	−0.6245	0.4245	0.4221	0.9128	−1.3349	$a_3 \succ a_1 \succ a_2$
$\lambda = 10$	0.2000	−0.6273	0.4273	0.6315	0.7057	−1.3372	$a_3 \succ a_1 \succ a_2$

Table 3 The evaluation results of $\mathbf{E}_S$ using Formulae 3.1 and 3.7

	$\mathbf{c}_1$	$\mathbf{c}_2$	$\mathbf{c}_3$	$\mathbf{d}_1$	$\mathbf{d}_2$	$\mathbf{d}_3$	Ranking
$\lambda = 1$	−0.5222	−0.4333	0.9556	0.1471	0.0346	−0.1817	$a_3 \succ a_2 \succ a_1$
$\lambda = 2$	−0.6000	−0.4000	1.0000	1.2113	0.1250	−1.3363	$a_3 \succ a_2 \succ a_1$
$\lambda = 4$	−0.6000	−0.4000	1.0000	1.4161	0.1097	−1.5258	$a_3 \succ a_2 \succ a_1$
$\lambda = 6$	−0.6000	−0.4000	1.0000	1.4003	0.1057	−1.5059	$a_3 \succ a_2 \succ a_1$
$\lambda = 10$	−0.6000	−0.4000	1.0000	1.3852	0.1026	−1.4877	$a_3 \succ a_2 \succ a_1$

different values of λ. For example, the ranking order of alternatives is $a_3 \succ a_1 \succ a_2$ using the outranking method, whereas in the TOPSIS method, the ranking order is $a_1 \succ a_3 \succ a_2$ for different values of λ. The λ can be seen as the risk attitude of the decision maker. The larger the λ is, the more risk-seeking the decision maker is. This is due to that the large evaluation values play more and more important role in the results and the effects of relative small values have been reduced during aggregation process. The decision maker can select the corresponding λ according to his/her risk attitude and real needs.

6 Conclusion

This paper has presented an outranking method to solve MCDM problems with HIFLEs. In the method, a decision matrix is constructed by the DMs with HIFLTSs. There are two components in the evaluation information; one component expresses the probable priority intensity in which an alternative is preferred to another, whereas the other component calculates the non-priority intensity in which an alternative is non-preferred to another. Based on the decision matrix, the dominance relations between a_{ij} and a_{kj} are determined by using the proposed directional Hausdorff distance of HIFLTSs and found the net dominance and disadvantage indices based on the concordance and discordance matrices. The proposed directional Hausdorff distance formula for HIFLTSs has a clear logic and a simple computation process. Some comparisons between the dominance and disadvantage indices for different values of λ are determined, and their differences are analyzed in details through a numerical example. The accuracy and reliability of the proposed method was demonstrated with comparison of results obtained in the TOPSIS method employing HIFLTSs. It is important to highlight that, since the proposed definition of HIFLTS is new and different from the existing definitions, it can give experts or decision analysts one more choice for identifying the appropriate decision model(s) to solve MCDM problems. The method has large computation difficulties while increasing the number alternatives/criteria in MCDM problems. It is worth noting that the mathematical expression of HIFLTS looks more complex than other linguistic models and the corresponding operations are tedious. Nevertheless, considering the merits of HIFLTS in facilitating decision makers to generate linguistic assessment values under uncertainty, HIFLTS deserves wide recognition and further research. In view of this, the possible applications of the proposed approach include new product development, performance evaluation, emergency management evaluation, service quality evaluation and other similar linguistic decision making problems that require the hesitance in the original evaluation information to be retained using HIFLTSs. The following directions should be considered for our further research:

1. Dong et al. [9] presented a technique of connecting the linguistic hierarchy and the numerical scale for the 2-tuple linguistic model and its uses to deal with hesitant unbalanced linguistic information. We argue that it will be interesting in any future research to develop such technique and the numerical scale in the context of hesitant intuitionistic fuzzy linguistic GDM environment.
2. Dong et al. [10] defined a complex and dynamic MCGDM problem and proposed its resolution framework and proposed a selection process in the context of heterogeneous attributes that obtains the ranking of individual alternatives and a collective solution. It will be an interesting future study to workout in the complex and dynamic MCGDM problem under hesitant intuitionistic fuzzy linguistic environment.

Table 4 The evaluation results of $\mathbf{E}_S$ using Formulae 3.2 and 3.7

	$\mathbf{c}_1$	$\mathbf{c}_2$	$\mathbf{c}_3$	$\mathbf{d}_1$	$\mathbf{d}_2$	$\mathbf{d}_3$	Ranking
$\lambda = 1$	−0.5464	−0.3579	0.9044	−0.1836	0.1953	−0.0117	$a_2, a_3 \succ a_1$
$\lambda = 2$	−0.6000	−0.4000	1.0000	0.9200	0.2327	−1.1527	$a_3 \succ a_2 \succ a_1$
$\lambda = 4$	−0.6000	−0.4000	1.0000	1.2656	0.1746	−1.4402	$a_3 \succ a_2 \succ a_1$
$\lambda = 6$	−0.6000	−0.4000	1.0000	1.2858	0.1595	−1.4453	$a_3 \succ a_2 \succ a_1$
$\lambda = 10$	−0.6000	−0.4000	1.0000	1.2947	0.1483	−1.4430	$a_3 \succ a_2 \succ a_1$

Table 5 The evaluation results obtained in the TOPSIS method using Definition 2.4, Formulae 3.1 and 3.2

Alternatives	The results using LSF 3.1				The results using LSF 3.2			
	D_i^+	D_i^-	C_i	Rank	D_i^+	D_i^-	C_i	Rank
$\lambda = 1$								
a_1	0.4063	0.4482	0.5245	2	0.6413	0.6514	0.5039	1
a_2	0.4956	0.2628	0.3465	3	0.8632	0.4050	0.3193	3
a_3	0.3585	0.5255	0.5945	1	0.5826	0.4489	0.4352	2
$\lambda = 2$								
a_1	0.4784	0.4148	0.5185	2	0.3472	0.3440	0.4977	1
a_2	0.3852	0.4148	0.4070	3	0.5964	0.2053	0.2561	3
a_3	0.4158	0.2854	0.5410	1	0.2876	0.2200	0.4334	2
$\lambda = 4$								
a_1	0.4154	0.4275	0.5072	1	0.1137	0.1106	0.4931	1
a_2	0.4240	0.3468	0.4499	3	0.4077	0.0669	0.1410	3
a_3	0.4923	0.4634	0.4849	2	0.0886	0.0639	0.4193	2
$\lambda = 6$								
a_1	0.4559	0.4423	0.4924	1	0.0415	0.0380	0.4778	1
a_2	0.4527	0.3858	0.4601	2	0.3609	0.0234	0.0610	3
a_3	0.5827	0.4865	0.4550	3	0.0301	0.0206	0.4060	2
$\lambda = 10$								
a_1	0.5182	0.4608	0.4707	1	0.0070	0.0049	0.4082	1
a_2	0.5043	0.4261	0.4580	2	0.3459	0.0031	0.0088	3
a_3	0.7034	0.5288	0.4291	3	0.0038	0.0025	0.3938	2

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