Int. J. Fuzzy Syst. (2016) 18(6):1031–1038
DOI 10.1007/s40815-016-0203-z

Stability Analysis and Fuzzy Control for Uncertain Delayed T–S Nonlinear Systems

Jiali Yu[1] · **Zhang Yi**[2]

Received: 22 January 2016 / Revised: 22 April 2016 / Accepted: 16 May 2016 / Published online: 3 June 2016

Abstract This paper studies the global exponential stability and fuzzy control for Takagi–Sugeno (T–S) nonlinear systems with bounded uncertain delays. Most existing T–S methods represent global nonlinear systems by connecting local linear systems with linguistic description. However, many complex systems cannot be represented by linear systems. In this paper, some local nonlinear systems having nice dynamic properties are employed to represent a global complex system. Moreover, the delays are any uncertain bounded continuous functions. Sufficient conditions for global exponential stability of these delayed global complex systems are derived. Criteria for design of nonlinear fuzzy controllers to feedback control the stability of global nonlinear fuzzy systems are given.

Keywords Global exponential stability · Fuzzy control · Takagi–Sugeno (T–S) model · Nonlinear · Bounded uncertain delays

1 Introduction

Since Tanaka and Sugeno proposed Takagi–Sugeno (T–S) fuzzy model in 1985 [1], a great number of results have been reported for T–S systems [2–4]. The T–S model gives an effective method to combine some simple local systems with their linguistic description to represent complex nonlinear dynamic systems. Control design and stability analysis for T–S fuzzy systems has received increasing attention [5–7]. In [8], the T–S fuzzy model approach was extended to the stability analysis and control design for both continuous and discrete-time nonlinear systems with time delay. Some excellent and important works have been done in [9–11] for solving the control design problem for interconnected nonlinear systems with unmeasured states.

Time delays in dynamic systems have been studied for many years. It is well known that delays can affect dynamics of some nonlinear systems, a stable system may become unstable by introducing some delays [12]. In recent years, some authors have paid their attention to control of nonlinear systems with delays by T–S fuzzy models. There exist two kinds of delays: one is continuous, see, for example, [13, 14], and the other is discrete, see, for example, [15, 16]. The non-delayed systems are described by ordinary differential equations which are easy to analyze. Because of the characters of the delay, the time-delayed systems are represented by stochastic differential equations which do form a nonMarkovian process. Generally, there is no method to obtain the explicit solution of these stochastic differential equations. We can only use different approximate methods to analyze them theoretically. Budini et al. use variable transformation method [17], Frank use Novikov theorem [18], and perturbation theory [19] to discuss the dynamics of delayed nonlinear stochastic system. In control engineering, delays are difficult to be known exactly, so stability for systems with uncertain delays is quite interesting [20]. In this paper, the delays are assumed to be any uncertain bounded continuous functions. We do not require the delays to be differentiable, and it is also not necessary to know the bounds of

✉ Jiali Yu
yujiali@uestc.edu.cn

Zhang Yi
zhangyi@scu.edu.cn

1 School of Mathematics Science, University of Electronic Science and Technology of China, Chengdu, China

2 College of Computer Science, Sichuan University, Chengdu, China

the delays. By constructing a novel Lyapunov function and supposing the delays to be bounded, a time partitioning method has been developed to deal with the uncertain delays. It provides a useful idea to deal with this kind of future research.

Although the T–S fuzzy control design has achieved a great progress, in most reported stability results of T–S model, linear systems are used to form global nonlinear fuzzy systems [21, 22]. However, there are many complex nonlinear fuzzy systems cannot be connected by local linear systems. In this paper, unlike using local linear systems in previous study, a class of nonlinear systems with delays having nice dynamical properties [23] will be used as local systems to form some global complex nonlinear fuzzy systems by T–S method. Stability of T–S model fuzzy systems is quite important for practical applications. It has been widely studied by many authors, see, for example, [7]. It is well known in control engineering that the global exponential stability (GES) of nonlinear systems is more interesting than asymptotic stability and other stability. Our stability conditions will guarantee the global exponential stability of the global complex nonlinear fuzzy delayed systems.

So there are two differences between this work and the existing ones: one is that each local system in this paper is nonlinear system but not linear system, and the other is that the delay is uncertain in each local system. Many complex systems are described by the model in this paper. In the first aspect, the nonlinear function will introduce a lot of obstacles to the stability analysis. The existing Lyapunov function is not useful, a novel Lyapunov function which include this nonlinear function should be constructed. We will derive stability conditions: some of them will be represented in the form of Linear Matrix Inequalities (LMIs), which could be solved by numeric method efficiently, and others will be represented by simple algebraic inequalities and are easy to check.

This paper is organized as follows: In Sect. 2, some preliminaries for delayed fuzzy control systems are given. In Sect. 3, conditions for global exponential stability of fuzzy systems with delays are proposed and proved. In Sect. 4, state feedback stabilization of delayed fuzzy control systems are discussed. In Sect. 5, simulations are given. This paper is concluded in Sect. 6.

2 Preliminaries

Consider a T–S fuzzy time-delay model which is composed of r plant rules. For each $s = 1, \ldots, r$, the sth plant rule can be represented as follows:

Plant Rule s **:** IF $\alpha_1(t)$ is M_{1s} AND $\cdots$ AND $\alpha_p(t)$ is M_{ps}, THEN

$$\dot{x}(t) = -x(t) + W_s g(x(t)) + J_s g\big(x(t - \tau_s(t))\big) + P_s u(t) \tag{1}$$

for $t \geq 0$, where $x(t) = (x_1(t), \ldots, x_n(t))^T$ is the state vector, $\alpha_1(t), \ldots, \alpha_p(t)$ are the premise variables, and each $M_{is}(i = 1, \ldots, p)$ is the fuzzy set corresponding to $\alpha_i(t)$ and plant rule s. $W_s = \left(W_{ij}^s\right)_{n\times n}$, $J_s = \left(J_{ij}^s\right)_{n\times n}$, and $P_s = \left(P_{ij}^s\right)_{n\times m}$ are constant matrices. $u(t)$ is the control input vector, and $\tau_s(t)$ is the time delay which satisfies $0 \leq \tau_s(t) \leq \tau$.

For any $x \in R^n$, $g(x) = \left(g(x_1), \ldots, g(x_n)\right)^T$, and the function g is defined as follows:

$$g(s) = \frac{|s+1| - |s-1|}{2}, \quad s \in R.$$

The function g is continuous but nondifferentiable. So the local system is nonlinear, which is the main feature of this paper different from others.

Let $M_{is}(\alpha_i(t))$ be the membership function of the fuzzy set M_{is} at the position $\alpha_i(t)$ and denote

$$w_s(\alpha(t)) = \prod_{i=1}^{p} M_{is}(\alpha_i(t)),$$

$$h_s(\alpha(t)) = \frac{w_s(\alpha(t))}{\sum_{i=1}^{r} w_i(\alpha(t))} \geq 0, \quad \sum_{s=1}^{r} h_s(\alpha(t)) = 1.$$

Then the overall delayed fuzzy control system is inferred as

$$\begin{aligned}\dot{x}(t) = -x(t) + \sum_{s=1}^{r} h_s(\alpha(t))\Big[W_s g(x(t)) + J_s g\big(x(t - \tau_s(t))\big) \\ + P_s u(t)\Big].\end{aligned} \tag{2}$$

For each solution, the initial value is assumed to be $x(t) = \phi(t), t \in [-\tau, 0]$, where $\phi(t) = (\phi_1(t), \ldots, \phi_n(t))^T$ is a vector continuous function. We define

$$\|\phi\| = \sup_{-\tau \leq \theta \leq 0} \sqrt{\phi_1^2(\theta) + \cdots + \phi_n^2(\theta)}.$$

In this paper, for a matrix S, we will use $S > 0$ or $S < 0$ to denote that S is a symmetric positive matrix or a symmetric negative matrix, respectively.

Lemma 1 [23] *Let Q be any of a $n \times n$matrix, for all $x, y \in R^n$, we have for any constant $k > 0$ and any symmetric positive matrix $S > 0$ that*

$$2x^T Q y \leq k x^T Q S^{-1} Q^T x + \frac{1}{k} y^T S y.$$

Lemma 2 *For above function g, we have*

$$g^2(s) \le s \cdot g(s) \le 2\int_0^s g(\theta)d\theta \le s^2,$$
$$\text{and} \int_0^s g(\theta)d\theta < s \cdot g(s).$$

Proof *Three cases will be considered to complete the proof.*

Case 1: $s \ge 1$. Then, $g(s) = 1, g^2(s) = 1, s \cdot g(s) = s$. $2\int_0^s g(\theta)d\theta = 2\int_0^1 \theta d\theta + 2\int_1^s d\theta = 2s - 1$. Thus, $1 \le s \le 2s - 1 \le s^2$, and $s - \frac{1}{2} < s$. So, $g^2(s) \le s \cdot g(s) \le 2\int_0^s g(\theta) d\theta \le s^2$, and $\int_0^s g(\theta)d\theta < s \cdot g(s)$.

Case 2: $-1 < s < 1$. Then, $g(s) = s, g^2(s) = s^2, s \cdot g(s) = s^2$. $2\int_0^s g(\theta)d\theta = 2\int_0^s \theta d\theta = s^2$. Thus, $\frac{s^2}{2} < s^2$. So, $g^2(s) \le s \cdot g(s) \le 2\int_0^s g(\theta)d\theta \le s^2$, and $\int_0^s g(\theta)d\theta < s \cdot g(s)$.

Case 3: $s \le -1$. Then, $g(s) = -1, g^2(s) = 1, s \cdot g(s) = -s$. $2\int_0^s g(\theta)d\theta = -2\int_s^0 g(\theta)d\theta = 2\int_s^{-1} d\theta - 2\int_{-1}^0 \theta d\theta = -2s - 1$. Thus, $1 \le -s \le -2s - 1 \le s^2$, and $-s - \frac{1}{2} < -s$. So, $g^2(s) \le s \cdot g(s) \le 2\int_0^s g(\theta)d\theta \le s^2$, and $\int_0^s g(\theta)d\theta < s \cdot g(s)$.

The proof is complete. □

Then, from *Lemma* 1 and *Lemma* 2, it follows that

Lemma 3 *For above function g and* $x = (x_1, \ldots, x_n)^T \in R^n$ *, it holds that*

$$g^T(x)g(x) \le g^T(x)x = x^T g(x) \le 2\sum_{i=1}^n \int_0^{x_i} g(\theta)d\theta \le x^T x$$

and

$$\sum_{i=1}^n \int_0^{x_i} g(\theta)d\theta \le g^T(x)x.$$

D^+ *is used to denote the upper right-hand Dini derivative in this paper. For any continuous function* $g : R \to R$ *, the upper right-hand Dini derivative of g(t) is defined as* $D^+g(t) = \lim_{\theta \to 0^+} \sup \frac{g(t+\theta) - g(t)}{\theta}$ *. It is easy to see that if g(t) is locally Lipschitz then* $|D^+g(t)| < +\infty$.

3 Stability Analysis of Free Fuzzy Delayed Systems

We first introduce a class of fuzzy system with time delays

$$\dot{x}(t) = -x(t) + \sum_{s=1}^r h_s(\alpha(t))\Big[W_s g(x(t)) + J_s g(x(t - \tau_s(t)))\Big]. \tag{3}$$

It is a global nonlinear fuzzy system and its local delayed systems are

$$\dot{x}(t) = -x(t) + W_s g(x(t)) + J_s g(x(t - \tau_s(t))). \tag{4}$$

Lemma 4 [23] *Fuzzy system* (3)*is globally exponentially stable, if there exist constants* $\epsilon > 0$*and* $\Pi \ge 1$*such that* $\|x(t)\| \le \Pi\|\phi\|e^{-\epsilon t}$ *for all* $t \ge 0$.

Theorem 1 *For any* $s = 1, \ldots, r$, *if there exists a diagonal matrix* $C > 0$*and some constants* $k_s > 0$*such that*

$$-C + CW_s + \frac{k_s}{2} CJ_s C^{-1} J_s^T C + \frac{1}{k_s} C < 0, \tag{5}$$

then the free fuzzy system (3) *is globally exponentially stable.*

Proof Since $0 \le \tau_s(t) \le \tau$, by (5), there exists a sufficient small constant $\epsilon > 0$ such that

$$\epsilon C - C + CW_s + \frac{k_s}{2} CJ_s C^{-1} J_s^T C + \frac{e^{2\epsilon\tau}}{2k_s} C < 0.$$

For $C = diag(c_i) > 0 (i = 1, \ldots, n)$, we choose a differentiable function

$$V(t) = e^{2\epsilon t} \sum_{i=1}^n c_i \int_0^{x_i(t)} g(s)ds$$

The time derivative of $V(t)$ along the trajectories of (3) is given by

$$\begin{aligned}
\dot{V}(t) &= 2\epsilon e^{2\epsilon t} \sum_{i=1}^n c_i \int_0^{x_i(t)} g(s)ds + e^{2\epsilon t} \sum_{i=1}^n c_i g(x_i(t))\dot{x}_i(t) \\
&= 2\epsilon V(t) + e^{2\epsilon t} g^T(x(t)) C\dot{x}(t) \\
&= 2\epsilon V(t) + e^{2\epsilon t} \sum_{s=1}^r h_s(\alpha(t)) \Big[- g^T(x(t)) Cx(t) \\
&\quad + g^T(x(t)) CW_s g(x(t)) \\
&\quad + g^T(x(t)) CJ_s g\big(x(t - \tau_s(t))\big)\Big].
\end{aligned}$$

From *Lemma* 1 in last section, we know that

$$\begin{aligned}
\dot{V}(t) &\le 2\epsilon V(t) + e^{2\epsilon t} \sum_{s=1}^r h_s(\alpha(t)) \Big[- g^T(x(t)) Cx(t) \\
&\quad + g^T(x(t)) CW_s g(x(t)) + \frac{1}{2} k_s g^T(x(t)) CJ_s C^{-1} J_s^T C g(x(t)) \\
&\quad + \frac{1}{2k_s} g^T\big(x(t - \tau_s(t))\big) C g\big(x(t - \tau_s(t))\big)\Big].
\end{aligned}$$

Since $0 \le \tau_s(t) \le \tau$, we have

$$V(t - \tau_s(t)) \ge e^{-2\epsilon\tau} \frac{1}{2} e^{2\epsilon t} g^T\big(x(t - \tau_s(t))\big) C g\big(x(t - \tau_s(t))\big),$$

So using the *Lemma* 2 and *Lemma* 3 , it follows that

$$\begin{aligned}
\dot{V}(t) \le \sum_{s=1}^r h_s(\alpha(t)) \Big[& e^{2\epsilon t} g^T(x(t)) \Big(- C + \epsilon C + CW_s \\
& + \frac{k_s}{2} CJ_s C^{-1} J_s^T C\Big) g(x(t)) + \frac{e^{2\epsilon\tau}}{k_s} V(t - \tau_s(t))\Big],
\end{aligned}$$

From $-C+\epsilon C+CW_s+\frac{k_s}{2}CJ_sC^{-1}J_s^TC<\frac{e^{\epsilon\tau}}{2k_s}\left(-\frac{1}{2}C\right)$, we can get that

$$\begin{aligned}\dot V(t)&\le e^{2\epsilon\tau}\sum_{s=1}^{r}\frac{h_s(\alpha(t))}{k_s}\left[-\frac{1}{2}g^T(x(t))Cg(x(t))e^{2\epsilon t}+V(t-\tau_s(t))\right]\\&\le e^{2\epsilon\tau}\sum_{s=1}^{r}\frac{h_s(\alpha(t))}{k_s}\left[-V(t)+V(t-\tau_s(t))\right].\end{aligned}\tag{6}$$

Let $c_{\max}$ and $c_{\min}$ denote the largest and smallest ones of $c_i(i=1,\ldots,n)$, respectively. Obviously, $c_{\max}>c_{\min}>0$. For any $a>1$, by the *Lemma* 2 , we can see that for all $t\in[-\tau,0]$

$$\begin{aligned}V(t)&=\frac{1}{2}e^{2\epsilon t}\sum_{i=1}^{n}c_ig^2(x_i(t))<\frac{1}{2}ac_{\max}\sum_{i=1}^{n}x_i^2(t)\\&\le\frac{1}{2}ac_{\max}\parallel\phi\parallel^2.\end{aligned}\tag{7}$$

We will prove that $V(t)<\frac{1}{2}ac_{\max}\parallel\phi\parallel^2$ for all $t\ge 0$.

If this is not true, there must exist a $t_1>0$ such that $V(t_1)=\dfrac{1}{2}ac_{\max}\parallel\phi\parallel^2$ and $V(t)<\frac{1}{2}ac_{\max}\parallel\phi\parallel^2$, for all $t\in[-\tau,t_1)$. Hence, $\dot V(t_1)\ge 0$. However, from (7) we have

$$\dot V(t_1)\le e^{2\epsilon\tau}\sum_{s=1}^{r}\frac{h_s(\alpha(t_1))}{k_s}\left(\frac{1}{2}ac_{\max}\|\phi\|^2-\frac{1}{2}ac_{\max}\|\phi\|^2\right)=0.$$

This leads to a contradiction and it proves that $V(t)<\frac{1}{2}ac_{\max}\parallel\phi\parallel^2$.

By the definition of $V(t)$, we have

$$V(t)\ge\frac{1}{2}e^{2\epsilon t}\sum_{i=1}^{n}c_ig^2(x_i(t))\ge\frac{1}{2}c_{\min}e^{2\epsilon t}g^2(x_i(t)).$$

Hence, $|g(x_i(t))|\le\sqrt{\frac{2V(t)}{c_{\min}}}e^{-\epsilon t}<\sqrt{\frac{ac_{\max}}{c_{\min}}}\parallel\phi\parallel e^{-\epsilon t}$. Then

$$D^+|x_i(t)|\le|x_i(t)|+\sum_{j=1}^{n}\left(|W_{ij}^s|+|J_{ij}^s|e^{\epsilon\tau}\right)\cdot\sqrt{\frac{ac_{\max}}{c_{\min}}}\|\phi\|e^{-\epsilon t}.$$

$$\begin{aligned}|x_i(t)|\le\|\phi\|\Bigg[&\left(1-\frac{\sum_{j=1}^n\left(|W_{ij}^s|+|J_{ij}^s|e^{\epsilon\tau}\right)\sqrt{\frac{ac_{\max}}{c_{\min}}}}{1-\epsilon}\right)e^{-t}\\&+\frac{\sum_{j=1}^n\left(|W_{ij}^s|+|J_{ij}^s|e^{\epsilon\tau}\right)\sqrt{\frac{ac_{\max}}{c_{\min}}}}{1-\epsilon}e^{-\epsilon t}\Bigg].\end{aligned}$$

The proof is complete. □

From Theorem 1, we can get the condition to guarantee the exponential stability of the nonlinear time-delay fuzzy systems of (3). To check the inequalities of (5), it needs to find a common diagonal matrix $C>0$. Generally, it is not easy to solve inequalities of (5) to find such a common diagonal matrix $C>0$. However, we can rewrite the inequalities in (5) in the form of linear matrix inequalities (LMIs). LMIs can be numerically solved efficiently.

Corollary 1 *If there exists a common matrix $C>0$and constants $k_s>0(s=1,\ldots,r)$such that the following LMI's hold*

$$\begin{bmatrix}-C+CW_s+\dfrac{1}{2k_s}C & CJ_s\\ J_s^TC & -\dfrac{2}{k_s}C\end{bmatrix}<0,\quad(s=1,\ldots,r),$$

then the fuzzy system (3) is globally exponentially stable.

Corollary 2 *The free fuzzy system* (3) *is globally exponentially stable if*

$$\left(\sqrt{\lambda_{\max}\left(J_sJ_s^T\right)}-1\right)I+W_s<0,\quad(s=1,\ldots,r),$$

where I is the $n\times n$ identity matrix.

Proof Let $C=I$, and choose $k_s=\begin{cases}\dfrac{1}{\sqrt{\lambda_{\max}\left(J_sJ_s^T\right)}}, & \text{if }\lambda_{\max}\left(J_sJ_s^T\right)\ne 0\\ \to+\infty, & \text{otherwise}\end{cases}$. We can derive the above result from Theorem 1 directly. The proof is complete. □

It is hard to check the above matrix inequalities if the dimensions of the matrices are much high. In the following Theorem, we will derive some global exponential stability conditions which will be presented in some simple algebraic inequalities.

Theorem 2 *If* $-1+W_{ii}^s+\sum_{j=1}^n\left[|W_{ij}^s|(1-\delta_{ij})+|J_{ij}^s|\right]<0$, *where*

$$\delta_{ij}=\begin{cases}1, & i=j\\0, & i\ne j,\end{cases}\quad i=1,\ldots,n,\quad s=1,\ldots,r,$$

then, the free fuzzy system (3) *is globally exponentially stable.*

Proof For any delays $\tau_s(t)(s=1,\ldots,r)$, since $0\le\tau_s(t)\le\tau$, the free fuzzy system of (3) can be rewritten as

$$\begin{aligned}\dot x_i(t)=&-x_i(t)+\sum_{s=1}^{r}h_s(\alpha(t))\\&\times\left[\sum_{j=1}^{n}\left(W_{ij}^sg(x_j(t))+J_{ij}^sg(x_j(t-\tau_s(t)))\right)\right].\end{aligned}\tag{8}$$

Then, it follows that

$$D^+|x_i(t)| \leq -|x_i(t)| + \sum_{s=1}^{r} h_s(\alpha(t))\Big[W_{ii}^s|g(x_i(t))| + \sum_{j=1}^{n}\big(|W_{ij}^s|(1-\delta_{ij})|g(x_j(t))| + |J_{ij}^s||g(x_j(t-\tau_s(t)))|\big)\Big]. \tag{9}$$

Denote $\eta_{is} = -\Big[\epsilon - 1 + W_{ii}^s + \sum_{j=1}^{n}\Big[|W_{ij}^s|(1-\delta_{ij}) + e^{\epsilon\tau}|J_{ij}^s|\Big]\Big]$, and let $\sigma = \min_{1\leq i\leq n, 1\leq s\leq r}(\eta_{is})$. Obviously, $\sigma > 0$. Define $z_i(t) = |x_i(t)|e^{\epsilon t}$, for all $t \geq -\tau$; since $|g(x_i(t))| \leq |x_i(t)|$, it follows from (9) that

$$D^+z_i(t) \leq \sum_{s=1}^{r} h_s(\alpha(t))\Big[\big(-1 + W_{ii}^s + \epsilon\big)z_i(t) + \sum_{j=1}^{n}\big(|W_{ij}^s|(1-\delta_{ij})z_j(t) + e^{\epsilon\tau}|J_{ij}^s|z_j(t-\tau_s(t))\big)\Big]. \tag{10}$$

For any constant $a > 1$, it is easy to see that $z_i(t) = |\phi_i(t)|e^{\epsilon t} \leq \|\phi\| < a\|\phi\|$, for all $t \in [-\tau, 0]$. We will prove that $z_i(t) < a\|\phi\|$ $(i = 1, \ldots, n)$ for all $t \geq 0$. Otherwise, then there must exist some i and a time $t_1 > 0$ such that $z_i(t_1) = a\|\phi\|$ and

$$z_j(t)\begin{cases} < a\|\phi\|, & j = i, \quad \text{for } t \in [-\tau, t_1) \\ \leq a\|\phi\|, & j \neq i, \quad \text{for } t \in [-\tau, t_1]. \end{cases}$$

Then, we have $D^+z_i(t_1) \geq 0$. But on the other hand, it follows from (10) that

$$\begin{aligned} D^+z_i(t_1) &\leq \sum_{s=1}^{r} h_s(\alpha(t_1))\Bigg[\Big(-1 + W_{ii}^s + \epsilon\Big)a\|\phi\| + a\|\phi\|\sum_{j=1}^{n}\Big(|W_{ij}^s|(1-\delta_{ij}) + e^{\epsilon\tau}|J_{ij}^s|\Big)\Bigg] \\ &= -a\|\phi\|\sum_{s=1}^{r} h_s(\alpha(t_1))\cdot\eta_{is} \\ &\leq -\sigma a\|\phi\| \\ &< 0. \end{aligned}$$

This is a contradiction and then $z_i(t) < a\|\phi\|$ $(i = 1, \ldots, n)$ for all $t \geq 0$.

Letting $a \to 1$, we have $z_i(t) \leq \|\phi\|$ for all $t \geq 0$. Then, it follows that $|x_i(t)| \leq \|\phi\|e^{-\epsilon t}$ for all $t \geq 0$. The proof is complete. □

The above theorems provide some conditions to guarantee the exponential stability of the free fuzzy systems of (3) subject to any uncertain continuous bounded delays.

4 Fuzzy Feedback Controller Design

In this section, we will design a fuzzy state feedback controller for system (2) based on the results of the previous section. For each $l = 1, \ldots, r$, consider the following fuzzy control law:

Regulator Rule l: IF $\alpha_1(t)$ is M_{1l} AND $\cdots$ AND $\alpha_p(t)$ is M_{pl}, THEN
$u(t) = -K_l g(x(t))$

where each $K_l = \big(k_{ij}^l\big)$ is a $m \times n$ matrix.

The overall state feedback fuzzy controller can be inferred as

$$u(t) = -\sum_{l=1}^{r} h_l(\alpha(t))K_l g(x(t)). \tag{11}$$

Using the above fuzzy feedback controller, from (2), we get the closed loop delayed fuzzy system

$$\dot{x}(t) = \sum_{s,l=1}^{r} h_s(\alpha(t))h_l(\alpha(t))\Big[-x(t) + \Big(W_s - P_sK_l\Big)g(x(t)) + J_sg(x(t-\tau_s(t)))\Big]. \tag{12}$$

Similar to the analysis of the last section, we have the following theorems which will provide some criteria for the selection of the matrices of $K_l(l = 1, \ldots, r)$ such that the fuzzy system (12) is globally exponentially stable.

Theorem 3 *If there exists a diagonal matrix $C > 0$and some constants $\gamma_s > 0$such that*

$$-C + C(W_s - P_sK_l) + \frac{\gamma_s}{2}CJ_sC^{-1}J_s^TC + \frac{1}{2\gamma_s}C < 0 \tag{13}$$

or the LMI's

$$\begin{bmatrix} -C + C(W_s - P_sK_l) + \frac{1}{2\gamma_s}C & CJ_s \\ J_s^TC & -\frac{2}{\gamma_s}C \end{bmatrix} < 0$$

for $s, l = 1, \ldots, r$, then the fuzzy system (2)can be globally exponentially stabilized by the fuzzy controller (11).

Corollary 3 If $\Big(\sqrt{\lambda_{\max}\big(J_sJ_s^T\big)} - 1\Big)I + \Big(W_s - P_sK_l\Big) < 0$, for $s, l = 1, \ldots, r$, where I is the $n \times n$ identity matrix. Then the fuzzy system (2) can be globally exponentially stabilized by the fuzzy controller (11).

Theorem 4 *Suppose that*

$$-1 + W_{ii}^s - \sum_{p=1}^{m} P_{ip}^s k_{pi}^l + \sum_{j=1}^{n} \times\Bigg[\Bigg|W_{ij}^s - \sum_{p=1}^{m} P_{ip}^s k_{pj}^l\Bigg|\cdot(1-\delta_{ij}) + \Big|J_{ij}^s\Big|\Bigg] < 0$$

for all $i = 1, \ldots, n$ *and* $s, l = 1, \ldots, r$, *where* $\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$ *then, the fuzzy system* (2) *can be globally exponentially stabilized by the fuzzy controller* (11).

Since $\sum_{s,l=1}^{r} h_s(\alpha(t))h_l(\alpha(t)) = \sum_{s=1}^{r}\left(h_s(\alpha(t))\sum_{l=1}^{r} h_l(\alpha(t))\right) = 1$, the proofs of the above theorems can be derived by some slight modifications to the proofs of the theorems in last section. The details are omitted.

By solving the inequalities in the above theorems, the controllers can be obtained directly .

5 Simulations

In this section, we will give an example to illustrate the above theory. Consider the following T–S fuzzy time-delay system:

$$\begin{cases} \dot{x}_1(t) = -x_1(t) - g(x_1(t)) \cdot (1 + \sin^2 x_2(t)) - g(x_2(t)) \\ \qquad \cdot \sin^2 x_2(t) + g(x_1(t - \tau(t))) \cdot \cos^2 x_2(t) \\ \qquad + g(x_2(t - \tau(t))) \cdot (3\sin^2 x_2(t) - 1) - 3u(t) \\ \dot{x}_2(t) = -x_2(t) - g(x_1(t)) \cdot (1 - 6\sin^2 x_2(t)) - g(x_2(t)) \, .. \\ \qquad \cdot (7 + \sin^2 x_2(t)) + g(x_1(t - \tau(t))) \\ \qquad \cdot (-1 + 2\cos^2 x_2(t)) + g(x_2(t - \tau(t))) \\ \qquad \cdot 2\cos^2 x_2(t) - 4u(t) \end{cases} \tag{14}$$

The delay $\tau(t) = 1/(1 + |t|)$ is bounded, continuous but not differentiable.

Define some matrices

$$W_1 = \begin{pmatrix} -5 & -5 \\ 1 & -8 \end{pmatrix}, \quad J_1 = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

$$W_2 = \begin{pmatrix} -4 & 0 \\ -5 & -7 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}, \quad P_2 = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$$

and some functions $M_{11}(x_2(t)) = \sin^2 x_2(t)$, $M_{22}(x_2(t)) = \cos^2 x_2(t)$. We can interpret $M_{11}(x_2(t))$ and $M_{22}(x_2(t))$ as membership functions of some fuzzy sets M_{11} and M_{22}, respectively. Using these fuzzy sets, the above nonlinear system (14) can be presented by the following T–S fuzzy model

Plant Rule 1: IF $x_2(t)$ is M_{11}, THEN

$$\dot{x}(t) = -x(t) + W_1 g(x(t)) + J_1 g(x(t - \tau(t))) + P_1 u(t). \tag{15}$$

Plant Rule 2: IF $x_2(t)$ is M_{22}, THEN

$$\dot{x}(t) = -x(t) + W_2 g(x(t)) + J_2 g(x(t - \tau(t))) + P_2 u(t). \tag{16}$$

According to the controller designing method of Theorem 4, let $K_1 = (1, 3)$, $K_2 = (2, 1)$, it is easy to check that the

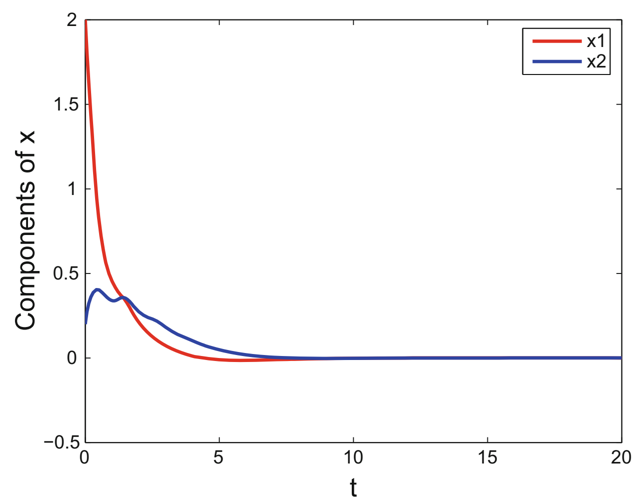

Fig. 2 Global exponential stability of (14)

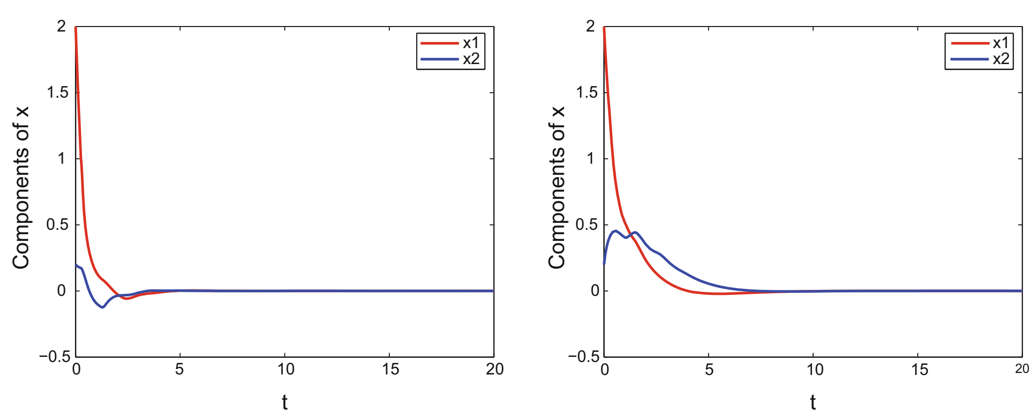

Fig. 1 Global exponential stability of (15) (*left*) and (16) (*right*)

two local systems (15) and (16) are global exponential stable in Fig. 1. Moreover, the T–S fuzzy time-delay system (14) is also globally exponentially stable. Fig. 2 shows the global exponential stability of the nonlinear system (14).

To further show the superiority of our results with some existing works such as [8]. Consider the following simple one-dimensional nonlinear T–S system

$$\dot{x}(t) = -x(t) + g(x(t)) + g\left(x\left(t - \frac{4\cos^2(t)}{5}\right)\right)$$

for all $t \geq 0$. Using Theorem 2, this system is globally exponentially stable. While, it is easy to see that the stability of this system cannot be checked by the results of the model with local linear systems [8].

6 Conclusions

In this paper, the global exponential stability analysis for a class of fuzzy systems with uncertain time delays has been studied. First, some global exponential stability conditions for free delayed fuzzy systems have been proposed. Then we have given some criteria for feedback fuzzy controller design. Finally, an example has been used to illustrate the results. We believe that all of the results obtained in this paper can be extended to the fuzzy systems with multiple time delays or with time-varying delay only by changing another Lyapunov function.

Acknowledgments This work was supported by the National Natural Science Foundation of China under Grant 61572112, 61432012, 61103041, and 11071178, and the Fundamental Research Funds for the Central Universities under Grant ZYGX2013Z005.

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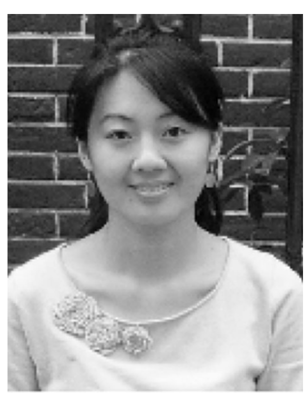

Jiali Yu received her Ph.D. in computer science from the University of Electronic Science and Technology of China, Chengdu, China, in 2009. She is currently an Associate Professor at the School of Mathematics Science, University of Electronic Science and Technology of China, Chengdu, China. From 2010 to 2013, she was a Research Scientist with the Institute for Infocomm Research, Singapore. Her current research interests include Fuzzy Control, Nonlinear System, and Neural Networks.

Zhang Yi received his Ph.D. in mathematics from the Institute of Mathematics, The Chinese Academy of Science, Beijing, China, in 1994. Currently, he is a Professor at the Machine Intelligence Laboratory, College of Computer Science, Sichuan University, Chengdu, China. He is the co-author of three books: Convergence Analysis of Recurrent Neural Networks (Kluwer Academic Publishers, 2004), Neural Networks: Computational Models and Applications (Springer, 2007), and Subspace Learning of Neural Networks (CRC Press, 2010). He was an Associate Editor of IEEE Transactions on Neural Networks and Learning Systems (2009–2012) and is an Associate Editor of IEEE Transactions on Cybernetics (2014). His current research interests include Neural Networks and Big Data. He is a fellow of IEEE.