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Membership-Function-Dependent Stability Analysis and Control Synthesis of Guaranteed Cost Fuzzy-Model-Based Control Systems

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Abstract This paper focuses on the guaranteed cost stability analysis of fuzzy-model-based (FMB) control systems. Representing the nonlinear plant using a Takagi–Sugeno (T–S) fuzzy model, a fuzzy controller is employed to close the feedback loop. A weighted linear quadratic cost function is considered as the cost index to measure the performance of the closed-loop fuzzy system in terms of the system states, system outputs, and control signals. The stability of the FMB control system is investigated by the Lyapunov stability theory subject to the minimization of cost index for performance realization. A membership-function-dependent approach using the piecewise-linear membership functions is employed to include the information of membership functions into the stability analysis. Membership-function-dependent stability conditions in terms of linear matrix inequalities are obtained to determine the system stability and feedback gains with the consideration of the system performance measured by the cost function. A simulation example is provided to illustrate the effectiveness and merits of the proposed approach.

Keywords Fuzzy controller · Guaranteed cost · Fuzzy-model-based control · Linear matrix inequalities (LMIs) · Membership-function dependent · Stability analysis

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1 Introduction

Takagi–Sugeno (T–S) fuzzy model was first developed by Takagi and Sugeno in 1985 [1], which provided an effective model to represent nonlinear plants which facilitates the system analysis and control synthesis. It is proved that any smooth nonlinear control systems can be approximated by T–S fuzzy models with linear rule consequence[2]. The inverted pendulum system can be one of these systems and other systems represented by T–S fuzzy models can be found in [2–5]. With the T–S fuzzy model, the system dynamics of the nonlinear systems can be represented as an average weighted sum of some local linear subsystems, where the weights are determined by membership functions [2] which embed the system nonlinearities. Based on the T–S fuzzy model, a fuzzy controller is proposed to close the feedback loop which forms a fuzzy-model-based (FMB) control system for feedback control [6]. Since then, the T–S FMB control systems have drawn the attention of fuzzy control researchers for more than 20 years due to its effectiveness on handling nonlinear control systems [7, 8]. In particular, the issues of stability analysis and control synthesis have been investigated extensively and fruitful results can be found in [2, 9–19] and the references therein.

The Lyapunov-based approach is a popular method used to investigate the stability of T–S FMB control systems. Through the Lyapunov stability theory, basic stability conditions of T–S FMB control systems can be achieved in terms of LMIs. If there exists a common solution of a group of Lyapunov inequalities in terms of LMIs which can be solved effectively by convex optimization methods such as interior point method [2], the FMB control system is guaranteed to be asymptotically stable [9]. With the parallel distribution compensation (PDC) [9] design approach, the stability conditions can be relaxed and some further related works can be found in [2, 9–16]. The work in [10] used the symmetry property of the membership functions of the T–S fuzzy model and fuzzy controller in the analysis and then managed to relax the LMI-based stability conditions. Inspired by the work in [10], various techniques have been proposed to gather the membership functions in the stability analysis [2, 11–15]. The work in [11] combined all the LMIs used in [10] to form a large symmetric matrix resulting in further reducing the conservativeness of stability conditions. The work in [16] generalized the stability conditions with the consideration of the permutations of membership functions using the Pólya theorem.

Under the PDC design technique [2, 9–16], both the T–S fuzzy model and fuzzy controller are required to share the same set of premise rules (the same premise variables, number of rules, and membership functions), which limits the flexibility of the controller design and as well as unnecessarily increase the complexities of the controller in some cases. However, if the premise rules of the fuzzy controller are different from those of the T–S fuzzy model, the stability analysis results will be very conservative as the permutations of the membership functions used in the PDC design cannot be applied due to the mismatched premised membership functions.

Furthermore, in most of the existing works, the membership functions have not been considered in the stability analysis which means that the stability conditions are valid for arbitrary membership functions. Given that only the specific membership functions used in T–S fuzzy model and fuzzy controller are needed to be considered in the control problem, the stability conditions are relatively conservative if the FMB control systems are unnecessarily guaranteed stable under all kinds of membership functions. Taking the membership functions and their information into account for stability analysis is a method to come up with membership-function-dependent stability conditions alleviating the conservativeness resulting from difficulty on handling the permutations of the mismatched premised membership functions.

One of the main difficulties to bring the information of membership functions into the analysis is the continuity property of the membership functions. When we consider continuous membership functions, the number of LMIs will reach infinity so it is impractical to apply numerical techniques to solve the solution to the stability conditions. In order to include the information of membership functions into the analysis, methods trying to add some constrains on the membership functions can be found in [20, 21]. Besides, approximation of membership functions is also one of the methods to circumvent this difficulty by approximating the infinite number of stability conditions with finite ones. Staircase membership functions were proposed in [22] to approximate the original membership functions of the FMB control system in the stability analysis. With the consideration of the approximation error, the stability of the FMB control system is implied by the stability of the FMB control systems having the membership grades at the flat regions of the staircase membership functions. Along this line, piecewise-linear membership function (PLMFs) [23] and Taylor-series membership functions (TSMFs) [24] were proposed to facilitate the stability analysis.

The performance of FMB control systems is another important issue to be considered during the controller design, and the index of performance can be the transient response and constrains on system variables (input, output, and control) [2]. The guaranteed performance control aims at not only stabilizing the system, but also guaranteeing the specific cost of the system through pre-defined cost function [25, 26]. Also there is a guaranteed cost approach introduced by works in [27], which is able to provide an upper bound on a given performance index and the performance of the system is guaranteed to be less than the boundary. Guan and Chen applied this method on T–S fuzzy systems with time delay in [28], Chen and Liu adopted the method in nonlinear systems with time-varying delay in [29], the problem of interval time-varying delay in T–S fuzzy systems is considered in [30], both state and input delays in the guaranteed cost T–S fuzzy systems are considered in work [31] and further related works can be found in [32–35], also some industrial applications of guaranteed cost T–S fuzzy systems can be found in [36–39]. This approach has also been extended from T–S fuzzy systems to polynomial fuzzy systems in works in [40]. In this paper, we have defined a weighted cost function as the performance criteria in the controller design. Through the guaranteed cost approach, we manage to stabilize the control system meanwhile maintain a constrained input, output, control cost, which depends on the weighted cost function we choose.

In this paper, we consider an FMB control system where the T–S fuzzy model and fuzzy controller do not share the same premise rules. Consequently, the fuzzy controller demonstrates a greater design flexibility by choosing its own number of rules and shapes of membership functions. PLMFs are adopted to approximate the original membership functions in a favorable form to facilitate the stability analysis. The PLMFs carrying the information of the original membership functions can be brought into the stability conditions so that the stability conditions become membership-function dependent. It implies that the stability conditions are dedicated to the FMB control system with specific membership functions to be handled and thus more relaxed stability results can be obtained compared with membership-function-independent analysis results [2, 9–16]. Furthermore, we consider a cost function to describe the system performance on top of the stability analysis. By taking the cost function on board along with the PLMFs, membership-function-dependent guaranteed cost stability conditions are obtained for the design of stable FMB control system.

This paper is organized as follows. In Sect. 2, the T–S fuzzy model and fuzzy controller are presented. In Sect. 3, the membership-function-dependent stability conditions in terms of LMIs are obtained through PLMFs with the consideration of the cost function describing the system performance. In Sect. 4, a simulation example is presented to verify the analysis results. A conclusion is drawn in Sect. 5.

2 Preliminaries

A nonlinear plant is described by the T–S fuzzy model [41, 42] with p rules of the following IF-THEN format.

$$\begin{aligned}&\text{Rule i: IF } f_1(\mathbf{x}(t)) \text{ is } M_1^i \text{ AND} \ldots \text{AND } f_\Psi(\mathbf{x}(t)) \text{ is } M_\Psi^i \\ &\qquad \text{THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t), \mathbf{y}(t) = \mathbf{C}_i\mathbf{x}(t),\end{aligned} \tag{1}$$

where M_α^i is a fuzzy term of rule i corresponding to the function $f_\alpha(\mathbf{x}(t))$, $\alpha = 1, 2, \ldots, \Psi$; $i = 1, 2, \ldots, p$; Ψ is a positive integer; $\mathbf{x}(t) \in \Re^n$ is the system state vector; $\mathbf{y}(t) \in \Re^l$ is the system output vector; $\mathbf{A}_i \in \Re^{n\times n}$, $\mathbf{B}_i \in \Re^{n\times m}$ and $\mathbf{C}_i \in \Re^{l\times n}$ are known system, input and output matrices, respectively; $\mathbf{u}(t) \in \Re^m$ is the input vector. The system dynamics and output are defined as follows,

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p} w_i(\mathbf{x}(t))(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t)), \tag{2}$$

$$\mathbf{y}(t) = \sum_{i=1}^{p} w_i(\mathbf{x}(t))\mathbf{C}_i\mathbf{x}(t), \tag{3}$$

where

$$w_i(\mathbf{x}(t)) \geq 0 \; \forall \, i, \sum_{i=1}^{p} w_i(\mathbf{x}(t)) = 1, \tag{4}$$

$$w_i(\mathbf{x}(t)) = \frac{\prod_{l=1}^{\Psi} \mu_{M_l^i}(f_l(\mathbf{x}(t)))}{\sum_{k=1}^{p}\prod_{l=1}^{\Psi} \mu_{M_l^k}(f_l(\mathbf{x}(t)))} \; \forall \, i, \tag{5}$$

$w_i(\mathbf{x}(t))$, $i = 1, 2, \ldots, p$, are the normalized grades of membership, $\mu_{M_\alpha^i}(f_\alpha(\mathbf{x}(t)))$, $\alpha = 1, 2, \ldots, \Psi$, are the grades of membership corresponding to the fuzzy term M_α^i.

A fuzzy controller with c rules of the following format is employed to control the nonlinear plant represented by the T–S fuzzy model (2).

$$\begin{aligned}&\text{Rule } j\text{: IF } g_1(\mathbf{x}(t)) \text{ is } N_1^j \text{ AND} \ldots \text{AND } g_\Omega(\mathbf{x}(t)) \text{ is } N_\Omega^j \\ &\qquad \text{THEN } \mathbf{u}(t) = \mathbf{G}_j\mathbf{x}(t)\end{aligned} \tag{6}$$

where N_β^j is a fuzzy term of rule j corresponding to the function $g_\beta(\mathbf{x}(t))$, $\beta = 1, 2, \ldots, \Omega$; $j = 1, 2, \ldots, c$; Ω is a positive integer; $\mathbf{G}_j \in \Re^{m\times n}$, $j = 1, 2, \ldots, c$, are constant feedback gains to be determined. The fuzzy controller is defined as follows,

$$\mathbf{u}(t) = \sum_{j=1}^{c} m_j(\mathbf{x}(t))\mathbf{G}_j\mathbf{x}(t), \tag{7}$$

where

$$m_j(\mathbf{x}(t)) \geq 0 \; \forall \, j, \sum_{j=1}^{c} m_j(\mathbf{x}(t)) = 1, \tag{8}$$

$$m_j(\mathbf{x}(t)) = \frac{\prod_{l=1}^{\Omega} \mu_{N_l^j}(g_l(\mathbf{x}(t)))}{\sum_{k=1}^{c}\prod_{l=1}^{\Omega} \mu_{N_l^k}(g_l(\mathbf{x}(t)))} \; \forall \, j, \tag{9}$$

$m_j(\mathbf{x}(t))$, $j = 1, 2, \ldots, c$, are the normalized grades of membership, $\mu_{N_\alpha^j}(g_\alpha(\mathbf{x}(t)))$, $\beta = 1, 2, \ldots, \Omega$, are the grades of membership corresponding to the fuzzy term N_β^j.

Considering the T–S fuzzy model (2) and the fuzzy controller (7) connected in a closed loop, with the property of the membership functions that $\sum_{i=1}^{p} w_i(\mathbf{x}(t)) = \sum_{j=1} cm_j(\mathbf{x}(t)) = \sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x}(t))m_j(\mathbf{x}(t)) = 1$, the FMB control system is obtained as follows,

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \sum_{i=1}^{p} w_i(\mathbf{x}(t))\Big(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\sum_{j=1}^{c} m_j(\mathbf{x}(t))\mathbf{G}_j\mathbf{x}(t)\Big) \\ &= \sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x}(t))m_j(\mathbf{x}(t))(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j)\mathbf{x}(t).\end{aligned} \tag{10}$$

The control objective is to drive the system state vector $\mathbf{x}(t)$ to the origin by determining the feedback gains $\mathbf{G}_j$. As the premise membership functions of the T–S fuzzy model and fuzzy controller are not the same, the analysis results with the PDC design [16, 43–48] cannot be applied to check for the stability of the FMB control system (10).

3 Stability Analysis

In this section, we will investigate the system stability of the FMB control system considering a guaranteed cost fuzzy controller in the form of (7) through a cost measuring the system performance. For brevity, the time t for variables is dropped for the situation without ambiguity, e.g., $\mathbf{x}(t)$ is denoted as $\mathbf{x}$.

The following quadratic Lyapunov function candidate is employed for the stability analysis of the FMB control system (10).

$$V = \mathbf{x}^T\mathbf{P}\mathbf{x}, \tag{11}$$

where $0 < \mathbf{P} = \mathbf{P}^T \in \Re^{n\times n}$. Denote $\mathbf{z} = \mathbf{P}^{-1}\mathbf{x}$ and $\mathbf{X} = \mathbf{P}^{-1}$. Define the feedback gains $\mathbf{G}_j = \mathbf{N}_j\mathbf{X}^{-1}$, where $\mathbf{N}_j \in \Re^{m\times n}$, $j = 1, 2, \ldots, c$, are matrices to be determined. From (10) and (11), we have,

$$\begin{aligned}\dot{V} &= \dot{\mathbf{x}}^T\mathbf{P}\mathbf{x} + \mathbf{x}^T\mathbf{P}\dot{\mathbf{x}} \\ &= \sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x})m_j(\mathbf{x})\mathbf{x}^T\big((\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j)^T\mathbf{P} \\ &\quad + \mathbf{P}(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j)\big)\mathbf{x} \\ &= \sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x})m_j(\mathbf{x})\mathbf{x}^T\mathbf{Q}_{ij}\mathbf{x}.\end{aligned} \tag{12}$$

$$J = \int_t^{\infty} \begin{bmatrix}\mathbf{x}\\ \mathbf{y}\\ \mathbf{u}\end{bmatrix}^T \mathbf{W} \begin{bmatrix}\mathbf{x}\\ \mathbf{y}\\ \mathbf{u}\end{bmatrix} dt, \tag{13}$$

where $0 \le \mathbf{W} \in \Re^{(n+l+m)\times(n+l+m)}$ is a pre-defined weighting matrix.

Remark 1 The cost $J > 0$ (except for $\mathbf{x} = \mathbf{0}$) is employed to measure the system performance. It can be considered as the energy consumed by the system state $\mathbf{x}$, the system output $\mathbf{y}$, and the control signal $\mathbf{u}$. With regard to the same weighting matrix $\mathbf{W}$, a smaller value of J implies a better system performance in terms of less energy consumption contributed by the combination of $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{u}$, which will eventually affect the transient behavior of the FMB control system (10) such as rise time, settling time, overshoot, undershoot, etc. The performance object is to suppress the value of J as much as possible through the design of the feedback gains $\mathbf{G}_j$ subject to the system stability.

Remark 2 The weighting matrix $\mathbf{W}$ plays an important role to the system performance. A special case is to choose $\mathbf{W} = \begin{bmatrix}\mathbf{W}_x & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{W}_y & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{W}_u\end{bmatrix}$, where $0 \le \mathbf{W}_x \in \Re^{n\times n}$ is the weighting matrix controlling the energy consumed by the system state $\mathbf{x}$; $0 \le \mathbf{W}_y \in \Re^{l\times l}$ is the weighting matrix controlling the energy consumed by the system output $\mathbf{y}$; and $0 \le \mathbf{W}_u \in \Re^{m\times m}$ is the weighting matrix controlling the energy consumed by the control signal $\mathbf{u}$.

From (3), (7), and (13), we have

$$J = \int_t^{\infty} \mathbf{x}^T \begin{bmatrix}\mathbf{I}\\ \sum_i^p w_i\mathbf{C}_i\\ \sum_j^c m_j\mathbf{G}_j\end{bmatrix}^T \mathbf{W} \begin{bmatrix}\mathbf{I}\\ \sum_i^p w_i\mathbf{C}_i\\ \sum_j^c m_j\mathbf{G}_j\end{bmatrix} \mathbf{x}dt\,, \tag{14}$$

where $\mathbf{I}$ is the identify matrix of compatible dimensions.

From (14) and (12), we have

$$\begin{aligned}\dot{V} &\le \sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x})m_j(\mathbf{x})\mathbf{x}^T\big((\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j)^T\mathbf{P} \\ &\quad + \mathbf{P}(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j)\big)\mathbf{x} \\ &\quad + \mathbf{x}^T \begin{bmatrix}\mathbf{I}\\ \sum_i^p w_i\mathbf{C}_i\\ \sum_j^c m_j\mathbf{G}_j\end{bmatrix}^T \mathbf{W} \begin{bmatrix}\mathbf{I}\\ \sum_i^p w_i\mathbf{C}_i\\ \sum_j^c m_j\mathbf{G}_j\end{bmatrix} \mathbf{x} \\ &= \sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x})m_j(\mathbf{x})\mathbf{z}^T\mathbf{Q}_{ij}\mathbf{z} \\ &\quad + \mathbf{z}^T \begin{bmatrix}\mathbf{X}\\ \sum_i^p w_i\mathbf{C}_i\mathbf{X}\\ \sum_j^c m_j\mathbf{N}_j\end{bmatrix}^T \mathbf{W} \begin{bmatrix}\mathbf{X}\\ \sum_i^p w_i\mathbf{C}_i\mathbf{X}\\ \sum_j^c m_j\mathbf{N}_j\end{bmatrix} \mathbf{z}\end{aligned}\,, \tag{15}$$

where $\mathbf{X} = \mathbf{P}^{-1}$; $\mathbf{z} = \mathbf{X}^{-1}\mathbf{x}$, $\mathbf{Q}_{ij} = \mathbf{A}_i\mathbf{x} + \mathbf{X}\mathbf{A}_i^T + \mathbf{B}_i\mathbf{N}_j + \mathbf{N}_j^T\mathbf{B}_i^T$; $\mathbf{G}_j = \mathbf{N}_j\mathbf{X}^{-1}$; $\mathbf{N}_j \in \Re^{m\times n}$ is a matrix to be determined for all j.

It is required that $\dot{V} \le 0$ (equality holds when $\mathbf{x} = \mathbf{0}$) for system stability which can be achieved by

$$\begin{aligned}&\sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x})m_j(\mathbf{x})\mathbf{Q}_{ij} \\ &+ \begin{bmatrix}\mathbf{X}\\ \sum_i^p w_i\mathbf{C}_i\mathbf{X}\\ \sum_j^c m_j\mathbf{N}_j\end{bmatrix}^T \mathbf{W} \begin{bmatrix}\mathbf{X}\\ \sum_i^p w_i\mathbf{C}_i\mathbf{X}\\ \sum_j^c m_j\mathbf{N}_j\end{bmatrix} < 0\end{aligned} \tag{16}$$

The non-convex inequalities can be converted to LMIs form using Schur complement [49]. The lemma of Schur complement is as follows:

Lemma 1 *The LMI is given as*

$$\mathbf{M} = \begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix} > 0\,,$$

where $\mathbf{A}\in\Re^{p\times p}$, $\mathbf{B}\in\Re^{p\times q}$, $\mathbf{C}\in\Re^{q\times p}$, $\mathbf{D}\in\Re^{q\times q}$, *and* $\mathbf{M}\in\Re^{(p+q)\times(p+q)}$, *also* $\mathbf{D}$ *is invertible, the linear inequality* $\mathbf{M}<0$ *is equivalent to*

$$\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C}>0$$

Then, from Schur complement lemma, the inequality (16) is equivalent to

$$\sum_{i=1}^{p}\sum_{j=1}^{c}w_i(\mathbf{x})m_j(\mathbf{x})\mathbf{H}_{ij}<0, \tag{17}$$

where $\mathbf{H}_{ij}=\begin{bmatrix}\mathbf{Q}_{ij} & \mathbf{T}_{ij}^T\\ \mathbf{T}_{ij} & -\mathbf{W}^{-1}\end{bmatrix}$; $\mathbf{T}_{ij}=\begin{bmatrix}\mathbf{X}\\ \mathbf{C}_i\mathbf{X}\\ \mathbf{N}_j\end{bmatrix}$.

As a result, it can be proved by the Lyapunov stability theory that the system stability is implied by $V>0$ and $\dot V<0$ (excluding $\mathbf{x}=\mathbf{0}$). The cost (13) reflects the system performance. Following from the fact $J>0$ in (14) and assuming that the FMB control system (10) is stable, from (12) and (16), we have

$$\dot V<-\mathbf{x}^T\begin{bmatrix}\mathbf{I}\\ \sum_i^p w_i\mathbf{C}_i\\ \sum_j^c m_j\mathbf{G}_j\end{bmatrix}^T\mathbf{W}\begin{bmatrix}\mathbf{I}\\ \sum_i^p w_i\mathbf{C}_i\\ \sum_j^c m_j\mathbf{G}_j\end{bmatrix}\mathbf{x} \tag{18}$$

Taking integration on both sides of (18) from 0 to ∞ and using the fact that $\mathbf{x}(\infty)\to\mathbf{0}$, we have

$$\mathbf{x}(0)^T\mathbf{P}\mathbf{x}(0)>J \tag{19}$$

Remark 3 It can be seen from (19) that $\mathbf{x}(0)^T\mathbf{P}\mathbf{x}(0)$, where $\mathbf{x}(0)$ is the initial condition, is the upper bound of J. By suppressing $\mathbf{x}(0)^T\mathbf{P}\mathbf{x}(0)$, the upper bound of J can be reduced reflecting a better system performance.

Let $\mathbf{x}(0)^T\mathbf{P}\mathbf{x}(0)\le\alpha\mathbf{x}(0)^T\mathbf{x}(0)$ which gives

$$\mathbf{P}<\alpha\mathbf{I}. \tag{20}$$

By minimizing the value of α, the upper bound of J, i.e., $\mathbf{x}(0)^T\mathbf{P}\mathbf{x}(0)$, can be minimized. By Schur complement, the inequality (20) is equivalent to the following:

$$\begin{bmatrix}\alpha\mathbf{I} & \mathbf{I}\\ \mathbf{I} & \mathbf{X}\end{bmatrix}>0 \tag{21}$$

Theorem 1 *The FMB control system* (10) *formed by a nonlinear system represented by the fuzzy model* (2) *and the fuzzy controller* (7) *connected in a closed loop is asymptotically stable and the system performance satisfies the cost* (13) *which is bound by a pre-determined value of* $\alpha>0$ *if there exist decision matrix variables* $\mathbf{N}_j\in\Re^{m\times n}$ *and* $\mathbf{x}\in\Re^{n\times n}$, *and pre-defined weighting matrix* $0\le\mathbf{W}\in\Re^{(n+l+m)\times(n+l+m)}$ *such that the following LMIs are satisfied*:

$$\begin{bmatrix}\alpha\mathbf{I} & \mathbf{I}\\ \mathbf{I} & \mathbf{X}\end{bmatrix}>0;$$

$$\mathbf{H}_{ij}<0,\quad\forall\quad i,j,$$

where $\mathbf{Q}_{ij}=\mathbf{A}_i\mathbf{X}+\mathbf{X}\mathbf{A}_i^T+\mathbf{B}_i\mathbf{N}_j+\mathbf{N}_j^T\mathbf{B}_i^T$; $\mathbf{H}_{ij}=\begin{bmatrix}\mathbf{Q}_{ij} & \mathbf{T}_{ij}^T\\ \mathbf{T}_{ij} & -\mathbf{W}^{-1}\end{bmatrix}$; $\mathbf{T}_{ij}=\begin{bmatrix}\mathbf{X}\\ \mathbf{C}_i\mathbf{X}\\ \mathbf{N}_j\end{bmatrix}$; *and the feedback gain is given as* $\mathbf{G}_j=\mathbf{N}_j\mathbf{X}^{-1}$ *for all j.*

Remark 4 The conditions $\mathbf{x}>0$ is omitted in Theorem 1 which is implied by $\begin{bmatrix}\alpha\mathbf{I} & \mathbf{I}\\ \mathbf{I} & \mathbf{X}\end{bmatrix}>0$.

Remark 5 The stability conditions in Theorem 1 are membership-function-dependent which does not consider the information of membership functions w_i and m_j in the stability analysis resulting in conservative stability analysis result.

In the following, we attempt to include the information of membership functions into the stability conditions to relax the stability analysis result. We approximate the membership function $h_{ij}(\mathbf{x})\equiv w_i(\mathbf{x})m_j(\mathbf{x})$ using the PLMF [50]. The basic idea constructing the PLMF is first to sample the original membership functions. Linear interpolation is then employed to approximate the grades of the original membership functions based on the sample points. Details are given as follows. The state space of interest Φ is first divided into q connected sub-state spaces Φ_k, $k=1,2,\ldots,q$. Consequently, we have $\Phi=\bigcup_{k=1}^{q}\Phi_k$. Mathematically, the PLMF $\hat h_{ij}(\mathbf{x})$ approximating the original membership function $h_{ij}(\mathbf{x})$ can be expressed as follows:

$$\hat h_{ij}(\mathbf{x})=\sum_{k=1}^{q}\sum_{i_1=1}^{2}\cdots\sum_{i_n=1}^{2}\prod_{r=1}^{n}v_{ri_rk}(x_r)\delta_{iji_1i_2\ldots i_nk},\quad\forall\, i,j,k, \tag{22}$$

$$0\le\hat h_{ijl}(\mathbf{x})\le 1, \tag{23}$$

$$0\le\delta_{iji_1i_2\ldots i_nk}\le 1, \tag{24}$$

where $\delta_{iji_1i_2\ldots i_nk}$ is a constant scalar to be determined which is in general a sample point of the original membership function $h_{ij}(\mathbf{x})$ at a chosen point $\mathbf{x}$; $0\le v_{ri_sk}(x_r(t))\le 1$ and $v_{r1k}(x_r(t))+v_{r2k}(x_r(t))=1$ for $r,s=1,2,\ldots,n$; $i_r=1,2$; $\mathbf{x}(t)\in\Phi_k$; otherwise, $v_{ri_sk}(x_r(t))=0$. As a result of the above settings, we have the following property:

$$\sum_{k=1}^{q}\sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\cdots\sum_{i_n=1}^{2}\prod_{r=1}^{n}v_{ri_rk}(x_r(t))=1. \tag{25}$$

The approximation error satisfies

$$\Delta\underline{h}_{ij}\le h_{ij}(\mathbf{x})-\hat h_{ij}(\mathbf{x})\le\Delta\overline{h}_{ij}, \tag{26}$$

where $\Delta \underline{h}_{ij}$ and $\Delta \overline{h}_{ij}$ are constant scalars to be determined.

From (17) and (22), we have

$$\begin{aligned} &\sum_{i=1}^{p}\sum_{j=1}^{c} h_{ij}(\mathbf{x})\mathbf{H}_{ij} \\ &= \sum_{i=1}^{p}\sum_{j=1}^{c} \hat{h}_{ij}(\mathbf{x})\mathbf{H}_{ij} + \sum_{i=1}^{p}\sum_{j=1}^{c}(h_{ij}(\mathbf{x}) - \hat{h}_{ij}(\mathbf{x}))\mathbf{H}_{ij} \\ &\leq \sum_{i=1}^{p}\sum_{j=1}^{c} \hat{h}_{ij}(\mathbf{x})\mathbf{H}_{ij} + \sum_{i=1}^{p}\sum_{j=1}^{c}(\Delta\overline{h}_{ij} - \Delta\underline{h}_{ij})\mathbf{y}_{ij}, \end{aligned} \tag{27}$$

where $0 \leq \mathbf{y}_{ij} = \mathbf{y}_{ij}^T \in \Re^{(n+l+m)\times(n+l+m)}$ and $\mathbf{y}_{ij} \geq \mathbf{H}_{ij}$ for all i and j

Expanding $\hat{h}_{ij}(\mathbf{x})$ in (27), we have

$$\begin{aligned} &\sum_{i=1}^{p}\sum_{j=1}^{c}\sum_{k=1}^{q}\sum_{i_1=1}^{2}\cdots\sum_{i_n=1}^{2}\prod_{r=1}^{n} v_{ri_rk}(x_r)\delta_{iji_1i_2\cdots i_nk}\mathbf{H}_{ij} \\ &+ \sum_{i=1}^{p}\sum_{j=1}^{c}(\Delta\overline{h}_{ij} - \Delta\underline{h}_{ij})\mathbf{y}_{ij} \\ &= \sum_{k=1}^{q}\sum_{i_1=1}^{2}\cdots\sum_{i_n=1}^{2}\prod_{r=1}^{n} v_{ri_rk}(x_r) \\ &\times \sum_{i=1}^{p}\sum_{j=1}^{c}(\delta_{iji_1i_2\ldots i_nk}\mathbf{H}_{ij} + (\Delta\overline{h}_{ij} - \Delta\underline{h}_{ij})\mathbf{y}_{ij}) \end{aligned} \tag{28}$$

Given the property (25), the satisfaction of $\sum_{i=1}^{p}\sum_{j=1}^{c}(\delta_{iji_1i_2\ldots i_nk}\mathbf{H}_{ij} + (\Delta\overline{h}_{ij} - \Delta\underline{h}_{ij})\mathbf{y}_{ij}) < 0$ implies the sanctification of (17) which further implies $\dot{V} \leq 0$ except $\mathbf{x} = \mathbf{0}$. The stability analysis result obtained through PLMFs is summarized in the following Theorem.

Theorem 2 *The FMB control system* (10) *formed by a nonlinear system represented by the fuzzy model* (2) *and the fuzzy controller* (7) *connected in a closed loop is asymptotically stable and the system performance satisfies the cost* (13) *which is bound by a pre-determined value of* $\alpha > 0$ *if there exist decision matrix variables* $\mathbf{N}_j \in \Re^{m\times n}$, $\mathbf{X} \in \Re^{n\times n}$ *and* $\mathbf{y}_{ij} = \mathbf{y}_{ij}^T \in \Re^{(n+l+m)\times(n+l+m)}$, *and pre-defined weighting matrix* $0 \leq \mathbf{W} \in \Re^{(n+l+m)\times(n+l+m)}$ *such that the following LMIs are satisfied*:

$$\begin{bmatrix} \alpha\mathbf{I} & \mathbf{I} \\ \mathbf{I} & \mathbf{X} \end{bmatrix} > 0;$$

$$\mathbf{y}_{ij} > 0, \forall\, i,j;$$

$$\mathbf{y}_{ij} > \mathbf{H}_{ij}, \forall\, i,j;$$

$$\sum_{i=1}^{p}\sum_{j=1}^{c}(\delta_{iji_1i_2\ldots i_nk}\mathbf{H}_{ij} + (\Delta\overline{h}_{ij} - \Delta\underline{h}_{ij})\mathbf{y}_{ij}) < 0,$$

$$\forall\, i, j, k, i_1, i_2, \ldots, i_n,$$

where $\mathbf{Q}_{ij} = \mathbf{A}_i\mathbf{X} + \mathbf{X}\mathbf{A}_i^T + \mathbf{B}_i\mathbf{N}_j + \mathbf{N}_j^T\mathbf{B}_i^T$; $\mathbf{H}_{ij} = \begin{bmatrix} \mathbf{Q}_{ij} & \mathbf{T}_{ij}^T \\ \mathbf{T}_{ij} & \mathbf{W}^{-1} \end{bmatrix}$; $\mathbf{T}_{ij} = \begin{bmatrix} \mathbf{X} \\ \mathbf{C}_i\mathbf{X} \\ \mathbf{N}_j \end{bmatrix}$; $\delta_{iji_1i_2\ldots i_nk}$ *is a sample point of the original membership function* $h_{ij}(\mathbf{x})$ *at a chosen point* $\mathbf{x}$; $\Delta\underline{h}_{ij}$ *and* $\Delta\overline{h}_{ij}$ *are constant scalars satisfying* $\Delta\underline{h}_{ij} \leq h_{ij}(\mathbf{x}) - \hat{h}_{ij}(\mathbf{x}) \leq \Delta\overline{h}_{ij}$ *for all i and j; and the feedback gain is given as* $\mathbf{G}_j = \mathbf{N}_j\mathbf{X}^{-1}$ *for all j.*

Remark 6 The problem of minimizing the value of α subject to the stability conditions in Theorems 1 and 2 can be formulated as a generalized eigenvalue problem that the solution can be solved numerically, say, using existing scientific engineering software package such as Matlab.

4 Simulation Example

A simulation example is given to verify the analysis results in terms of stability and performance. A 3-rule T–S fuzzy model inspired from [48] in the form of (2) is considered where the system, input and output matrices are chosen as $\mathbf{A}_1 = \begin{bmatrix} 1.59 & -7.29 \\ 0.01 & 0 \end{bmatrix}$, $\mathbf{A}_2 = \begin{bmatrix} 0.02 & -4.64 \\ 0.35 & 0.21 \end{bmatrix}$, $\mathbf{A}_3 = \begin{bmatrix} -3.25 & -4.33 \\ 0 & -0.05 \end{bmatrix}$, $\mathbf{B}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{B}_2 = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$, $\mathbf{B}_3 = \begin{bmatrix} -4 \\ -1 \end{bmatrix}$, $\mathbf{C}_1 = [1.21\ \ -3.65]$, $\mathbf{C}_2 = [3.15\ \ 6.37]$, $\mathbf{C}_3 = [-2.25\ \ 1.66]$, $\mathbf{x} = [x_1\ \ x_2]^T$. The membership functions are chosen as follows.

$$w_1(x_1) = \mu_{M_1^1}(x_1) = \begin{cases} 1 & \text{for} \quad x_1 < -10 \\ \dfrac{-x_1 + 2}{12} & \text{for} \quad -10 \leq x_1 \leq 2 \\ 0 & \text{for} \quad x_1 > 2 \end{cases} \tag{29}$$

$$w_2(x_1) = \mu_{M_1^2}(x_1) = 1 - w_1(x_1) - w_3(x_1) \tag{30}$$

$$w_3(x_1) = \mu_{M_1^3}(x_1) = \begin{cases} 0 & \text{for } x_1 < -2 \\ \dfrac{x_1 + 2}{12} & \text{for } -2 \leq x_1 \leq 10 \\ 1 & \text{for } x_1 > 10 \end{cases} \tag{31}$$

The 3-rule T–S fuzzy model is obtained as follows:

$$\dot{\mathbf{x}} = \sum_{i=1}^{3} w_i(x_1)(\mathbf{A}_i\mathbf{x} + \mathbf{B}_i u) \tag{32}$$

and its output is obtained as

Table 1 Weighting matrices $\mathbf{W}_x$, $\mathbf{W}_y$, and $\mathbf{W}_u$ for the 9 cases

Case	$\mathbf{W}_x$	$\mathbf{W}_y$	$\mathbf{W}_u$
1	$\begin{bmatrix} 0.01 & 0 \\ 0 & 1 \end{bmatrix}$	1	1
2	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	1	1
3	$\begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix}$	1	1
4	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	0.01	1
5	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	1	1
6	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	100	1
7	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	1	0.01
8	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	1	1
9	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	1	100

$$y = \sum_{i=1}^{3} w_i(x_1)\mathbf{C}_i\mathbf{x}. \quad (33)$$

We consider a 2-rule fuzzy controller in the form of (7) is employed to close the feedback loop. The membership functions of the fuzzy controller are chosen as follows.

$$m_1(x_1) = \mu_{N_1^1}(x_1) = 1 - \frac{1}{e^{\frac{-x_1}{2}}} \quad (34)$$

$$m_2(x_1) = \mu_{N_1^2}(x_1) = 1 - m_1(x_1) \quad (35)$$

The 2-rule fuzzy control is obtained as follows:

$$u = \sum_{j=1}^{2} m_j(x_1)\mathbf{G}_j\mathbf{x}. \quad (36)$$

Unlike the fuzzy controller using PDC design, the fuzzy controller uses different number of rules and shape of membership functions different from those of the T–S fuzzy model.

In order to investigate the impact of the weighting matrix on different signals, namely the system states $\mathbf{x}$, the system outputs $\mathbf{y}$, and the control signals $\mathbf{u}$, the weighting matrix $\mathbf{W}$ is chosen as shown in Remark 2. As the off-diagonal block entries of $\mathbf{W}$ are all set as zero, so that the mutual influence between $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{u}$ are eliminated. The influence from the weighting matrices $\mathbf{W}_x$, $\mathbf{W}_y$ and $\mathbf{W}_u$ to the system states $\mathbf{x}$, the system outputs $\mathbf{y}$ and the control signals $\mathbf{u}$, respectively, is more significant.

In this simulation, the system is tested by applying different weighting matrices $\mathbf{W}_x$, $\mathbf{W}_y$, and $\mathbf{W}_u$ as given in Table 1 that we take 1 as the reference and 0.01/100 as small/large value for the weighting matrices resulting in 9 cases in total. For cases 1–3, we only change $\mathbf{W}_x$ but keep $\mathbf{W}_y$ and $\mathbf{W}_u$ unchanged to investigate how $\mathbf{W}_x$ influences the system states in particular x_1. Similarly, for cases 4–6, we only change $\mathbf{W}_y$ but keep $\mathbf{W}_x$ and $\mathbf{W}_u$ unchanged to investigate how $\mathbf{W}_y$ influences the system output y. For cases 7–9, we only change $\mathbf{W}_u$ but keep $\mathbf{W}_x$ and $\mathbf{W}_y$ unchanged to investigate how $\mathbf{W}_u$ influences the control signal u.

Table 2 Feedback gains $\mathbf{G}_j$ for the 9 cases

Case	$\mathbf{G}_j$	$\mathbf{X}$
1	$\mathbf{G}_1 = [-5.9428$ $\mathbf{G}_2 = [8.5994 \times 10^{-1}$	$\begin{bmatrix} 2.7671 \times 10^{-2} & -1.0801 \times 10^{-3} \\ -1.0801 \times 10^{-3} & 3.8639 \times 10^{-4} \end{bmatrix}$
2	$\mathbf{G}_1 = [-6.6695 \quad -7.6349]$ $\mathbf{G}_2 = [1.1626 \quad 3.8265]$	$\begin{bmatrix} 2.4056 \times 10^{-2} & -9.6318 \times 10^{-4} \\ -9.6318 \times 10^{-4} & 3.6427 \times 10^{-4} \end{bmatrix}$
3	$\mathbf{G}_1 = [-1.2996 \times 10^{1} \quad -1.2636 \times 10^{1}]$ $\mathbf{G}_2 = [3.4361 \quad 6.1895]$	$\begin{bmatrix} 2.9074 \times 10^{-3} & -1.3968 \times 10^{-4} \\ -1.3968 \times 10^{-4} & 8.5156 \times 10^{-5} \end{bmatrix}$
4	$\mathbf{G}_1 = [-5.7126 \quad -6.2878]$ $\mathbf{G}_2 = [7.5703 \times 10^{-1} \quad 3.1231]$	$\begin{bmatrix} 3.1525 \times 10^{-2} & -1.2250 \times 10^{-3} \\ -1.2250 \times 10^{-3} & 4.2259 \times 10^{-4} \end{bmatrix}$
5	$\mathbf{G}_1 = [-6.6695 \quad -7.6349]$ $\mathbf{G}_2 = [1.1626 \quad 3.8265]$	$\begin{bmatrix} 2.4056 \times 10^{-2} & -9.6318 \times 10^{-4} \\ -9.6318 \times 10^{-4} & 3.6427 \times 10^{-4} \end{bmatrix}$
6	$\mathbf{G}_1 = [-1.0067 \times 10^{1} \quad -1.1047 \times 10^{1}]$ $\mathbf{G}_2 = [2.4792 \quad 5.4934]$	$\begin{bmatrix} 1.3238 \times 10^{-3} & -5.8860 \times 10^{-5} \\ -5.8860 \times 10^{-5} & 2.9299 \times 10^{-5} \end{bmatrix}$
7	$\mathbf{G}_1 = [-1.0397 \times 10^{1} \quad -1.1034 \times 10^{1}]$ $\mathbf{G}_2 = [2.5856 \quad 5.4546]$	$\begin{bmatrix} 9.2651 \times 10^{-2} & -4.1984 \times 10^{-3} \\ -4.1984 \times 10^{-3} & 2.1657 \times 10^{-3} \end{bmatrix}$
8	$\mathbf{G}_1 = [-6.6695 \quad -7.6349]$ $\mathbf{G}_2 = [1.1626 \quad 3.8265]$	$\begin{bmatrix} 2.4056 \times 10^{-2} & -9.6318 \times 10^{-4} \\ -9.6318 \times 10^{-4} & 3.6427 \times 10^{-4} \end{bmatrix}$
9	$\mathbf{G}_1 = [-5.1315 \quad -5.6521]$ $\mathbf{G}_2 = [5.1337 \times 10^{-1} \quad 2.7991]$	$\begin{bmatrix} 3.8101 \times 10^{-4} & -1.4576 \times 10^{-5} \\ -1.4576 \times 10^{-5} & 4.6047 \times 10^{-6} \end{bmatrix}$

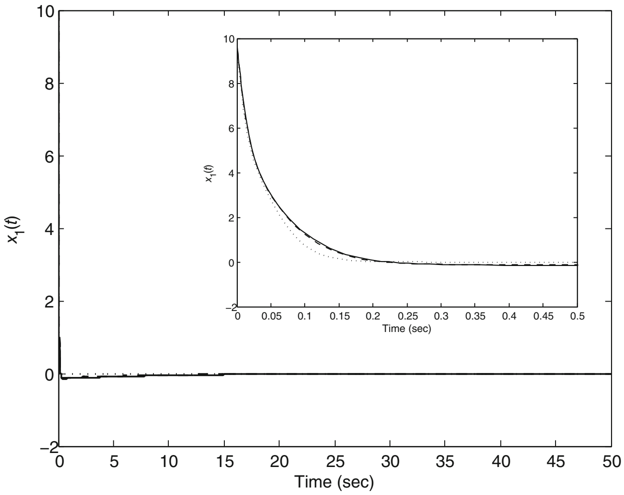

Fig. 1 Response of state $x_1(t)$ for Cases 1 (*solid line*), 2 (*dashed line*), and 3 (*dotted line*)

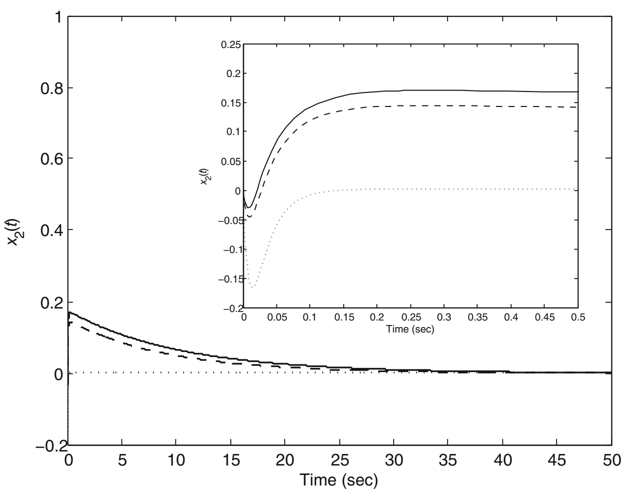

Fig. 2 Response of state $x_2(t)$ for Cases 1 (*solid line*), 2 (*dashed line*), and 3 (*dotted line*)

To apply Theorem 2, we need to define the PLMFs as in (22). As the membership functions of both T–S fuzzy model and fuzzy controller depends on x_1, the PLMFs can be constructed by considering only x_1. Considering $x_1 \in [-10, 10]$, δ_{iji_1k} is set as $h_{ij}(x_1)$ by considering the sample points of x_1 at $\{-10, -9.5, \ldots, 9.5, 10\}$, e.g., $\delta_{iji_11} = h_{ij}(-10)$, $\delta_{iji_12} = h_{ij}(-9.5)$ and so on. The function $v_{11k}(x_1) = \frac{x_1 - \underline{x}_{1k}}{\overline{x}_{1k} - \underline{x}_{1k}}$ and $v_{12k}(x_1) = 1 - v_{11k}(x_1)$, where $\overline{x}_{1k}$ and $\underline{x}_{1k}$ denote the lower and upper end points of x_1 at the k-th region, e.g., $\overline{x}_{1k} = -10$ and $\underline{x}_{1k} = -9.5$ when $k = 1$, $\overline{x}_{1k} = -9.5$ and $\underline{x}_{1k} = -9$ when $k = 2$ and so on. It should be noted that $v_{11k}(x_1) = 0$ and $v_{12k}(x_1) = 0$ when x_1 is outside the k-th region. According to the chosen original membership functions and PLMFs, it is found numerically that $\Delta\underline{h}_{11} = \Delta\underline{h}_{32} = -2.4426 \times 10^{-3}$, $\Delta\underline{h}_{12} = \Delta\underline{h}_{31} = -6.7708 \times 10^{-4}$, $\Delta\underline{h}_{21} = \Delta\underline{h}_{22} = -1.7826 \times 10^{-3}$, $\Delta\overline{h}_{11} =$

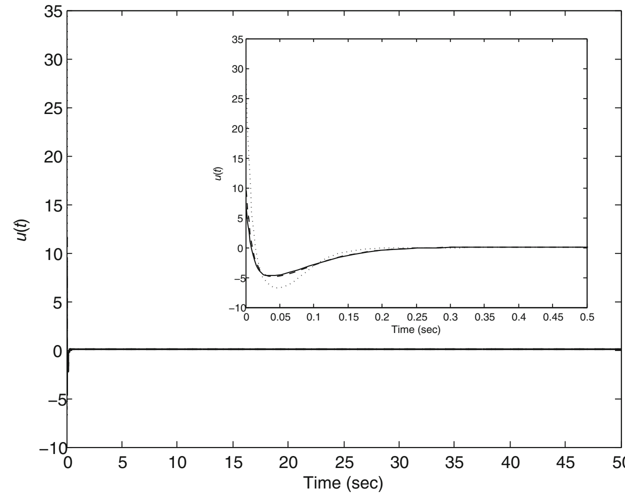

Fig. 3 Control signal $u(t)$ for Cases 1 (*solid line*), 2 (*dashed line*), and 3 (*dotted line*)

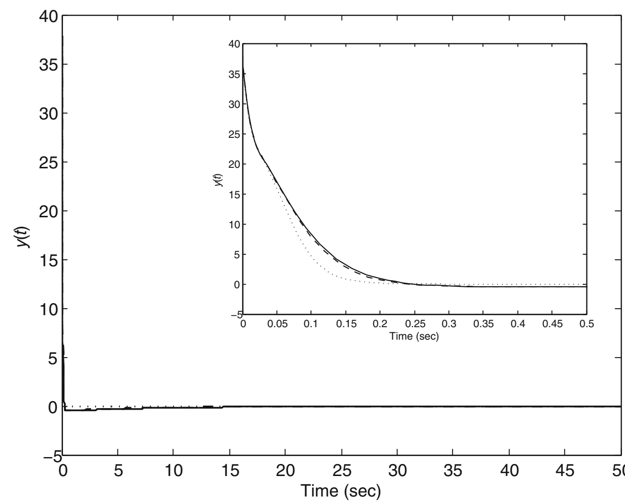

Fig. 4 Response of output $y(t)$ for Cases 1 (*solid line*), 2 (*dashed line*), and 3 (*dotted line*)

$\Delta\overline{h}_{32} = 1.7839 \times 10^{-3}$, $\Delta\overline{h}_{12} = \Delta\overline{h}_{31} = 1.3139 \times 10^{-3}$, $\Delta\overline{h}_{21} = \Delta\overline{h}_{22} = 2.4622 \times 10^{-3}$ satisfying the inequality. (26). For comparison purposes, we employ Theorem 1 to check the system stability. However, no feasible solution is found which indicates that the stability conditions in Theorem 2 are more relaxed thanks to the stability analysis using the PLMFs.

From the above settings, Theorem 2 is employed to check the system stability and determine the feedback gains. Table 2 tabulates the feedback gains $\mathbf{G}_j$ and $\mathbf{X}$ for the 9 cases. The 9 fuzzy controllers are employed to stabilize the T–S fuzzy model. The time responses of x_1, x_2, y, and u are shown in Figs. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12. It can be seen from the figures that all fuzzy controllers are

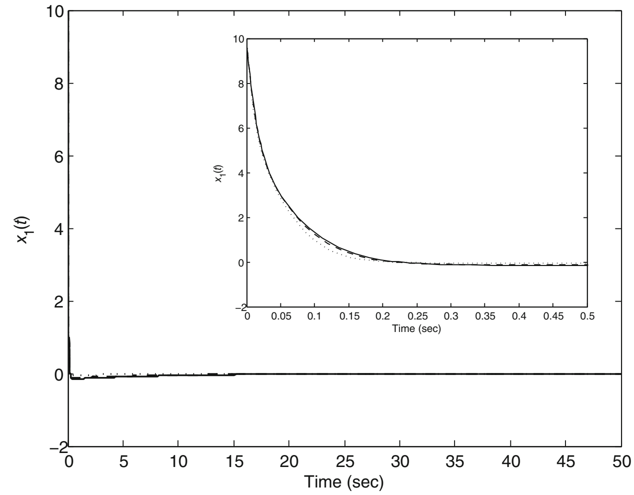

Fig. 5 Response of state $x_1(t)$ for Cases 4 (*solid line*), 5 (*dashed line*), and 6 (*dotted line*)

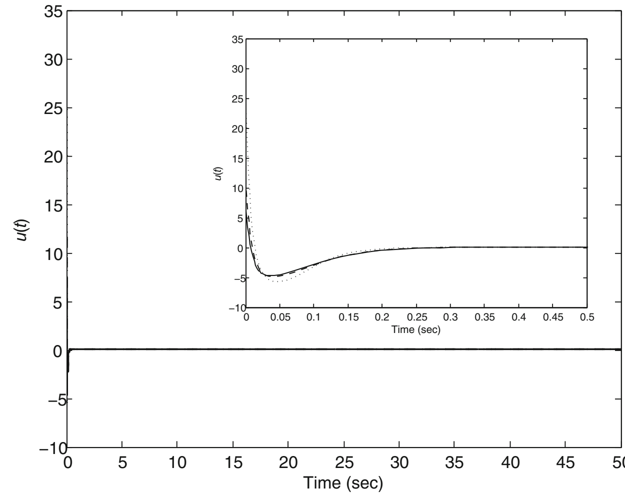

Fig. 7 Control signal $u(t)$ for Cases 4 (*solid line*), 5 (*dashed line*), and 6 (*dotted line*)

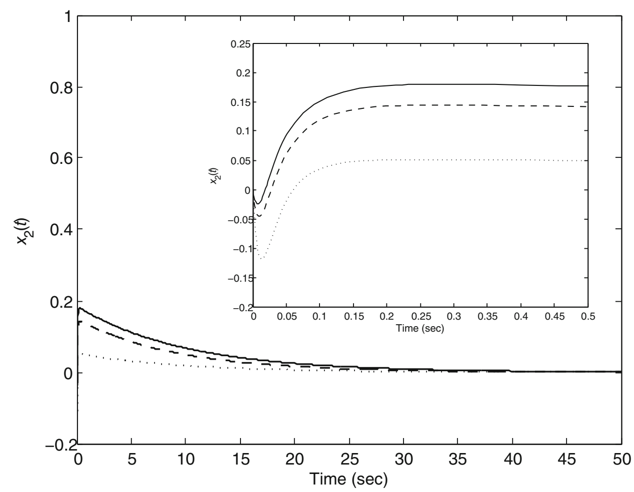

Fig. 6 Response of state $x_2(t)$ for Cases 4 (*solid line*), 5 (*dashed line*), and 6 (*dotted line*)

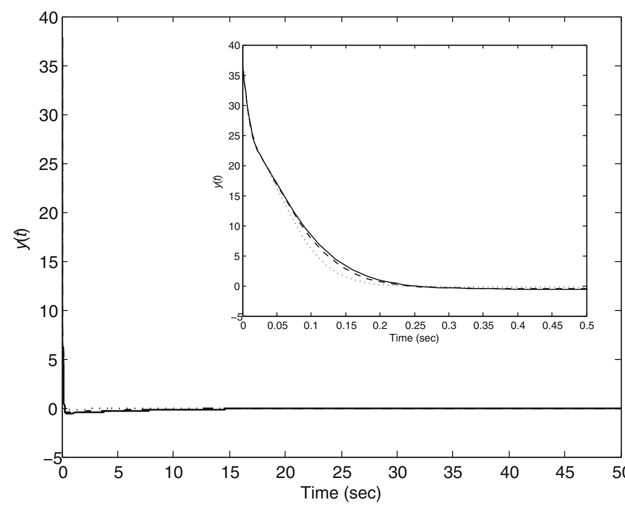

Fig. 8 Response of output $y(t)$ for Cases 4 (*solid line*), 5 (*dashed line*), and 6 (*dotted line*)

able to stabilize the T–S fuzzy model that the system states x_1 and x_2 approach the origin.

To facilitate comparison among cases, we define the following performance indexes J_{x_1}, J_y, and J_u which are the integral of squared signals.

$$J_{x_1} = \int_t^\infty {x_1}^T x_1 dt = \int_t^\infty {x_1}^2 \mathrm{d}t \tag{37}$$

$$J_y = \int_t^\infty y^T y dt = \int_t^\infty {y_1}^2 \mathrm{d}t \tag{38}$$

$$J_u = \int_t^\infty u^T u dt = \int_t^\infty u^2 \mathrm{d}t \tag{39}$$

A smaller value of performance index indicates a smaller consumption implying a better performance. Table 3 tabulates J_{x_1}, J_y, and J_u for the 9 cases in Table 1. In cases 1–3, the cost J_{x_1} decreases (increases) when placing heavier (lighter) weight on x_1. Referring to Fig. 1, the effect on different weights on x_1 can be seen that the response of state x_1 demonstrates a faster (slower) transient response with shorter (longer) settling time and smaller steady-state error with the increase (decrease) of weight on x_1. In cases 4 to 6, we place different weights on y. It can be seen from Table 1 that cost J_y decreases (increases) when placing

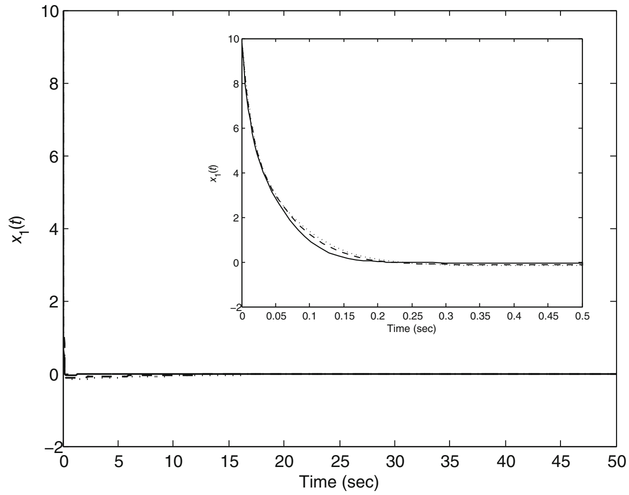

Fig. 9 Response of state $x_1(t)$ for Cases 7 (*solid line*), 8 (*dashed line*), and 9 (*dotted line*)

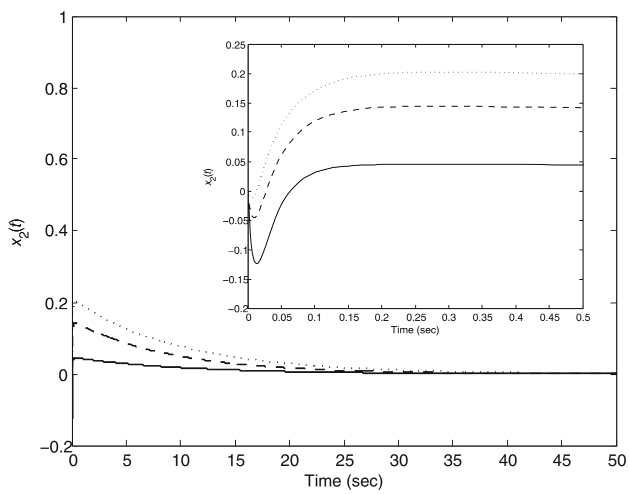

Fig. 10 Response of state $x_2(t)$ for Cases 7 (*solid line*), 8 (*dashed line*), and 9 (*dotted line*)

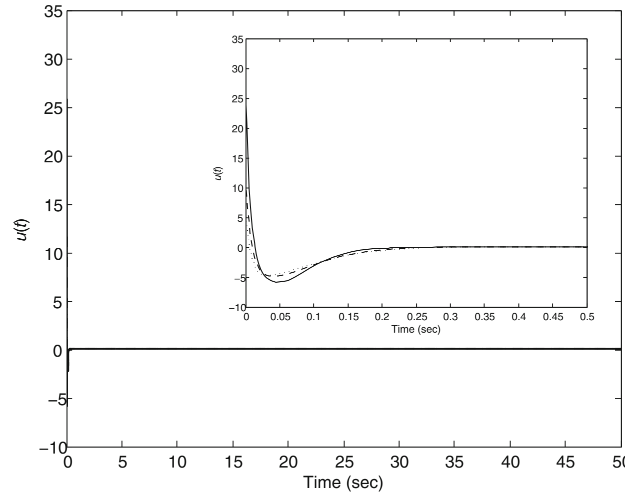

Fig. 11 Control signal $u(t)$ for Cases 7 (*solid line*), 8 (*dashed line*), and 9 (*dotted line*)

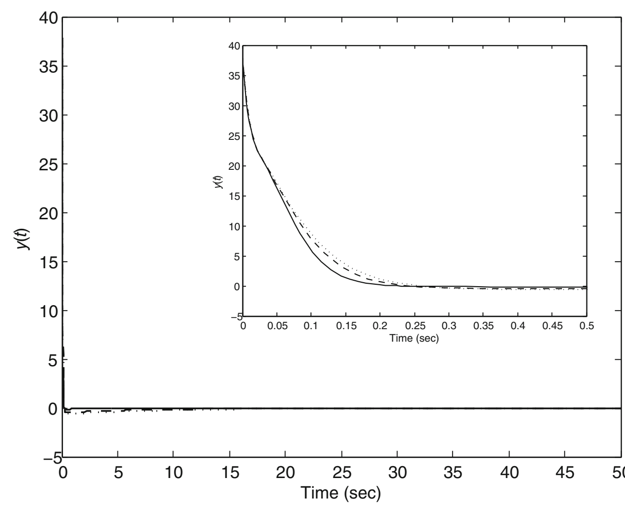

Fig. 12 Response of output $y(t)$ for Cases 7 (*solid line*), 8 (*dashed line*), and 9 (*dotted line*)

heavier (lighter) weight on $\mathbf{y}$. Referring to Fig. 7, it demonstrates that a faster (slower) transient response with shorter (longer) settling time and smaller steady-state error with the increase (decrease) of weight on y. Similarly, in cases 7–9, we place different weights on u to investigate how it is influenced. It is found that the cost J_u decreases (increases) when placing heavier (lighter) weight on u. Furthermore, Fig. 11 shows that a smaller (larger) control signal is required to stabilize the T–S fuzzy model corresponding to a heavier (lighter) weight on u.

Through this example, we can conclude that Theorem 2 offers relaxed stability conditions using the PLMFs in the stability analysis. Furthermore, with the consideration of

Table 3 Costs J, J_{x_1}, J_y, and J_u for the 9 cases

Case	J	J_{x_1}	J_y	J_u
1	4.2128×10^1	1.9715	4.0047×10^1	1.9053
2	4.3146×10^1	1.9053	3.9031×10^1	2.1110
3	2.0168×10^2	1.6208	3.3549×10^1	6.0541
4	4.4273	1.9931	4.0410×10^1	1.8627
5	4.3146×10^1	1.9053	3.9031×10^1	2.1110
6	3.5546×10^3	1.7130	3.5489×10^1	3.9376
7	3.6968×10^1	1.7006	3.5215×10^1	4.1588
8	4.3146×10^1	1.9053	3.9031×10^1	2.1110
9	2.2190×10^2	2.0534	4.1377×10^1	1.7826

cost function in the stability analysis, it offers an effective way to realize the system performance

5 Conclusion

In this paper, the T–S FMB control system equipped with different fuzzy rules of model and controller is investigated in terms of both stability and performance based on Lyapunov theory. In addition, unlike the membership-independent methods, the information of membership function of T–S FMB control systems has been included into the analysis through a PLMF approach to further relax the stability conditions. Furthermore, the weighted cost function is introduced into the analysis to improve the performance and suppress the cost. Different requirements on suppressing the cost can be satisfied through adjusting the weight matrix. The stability conditions are derived in terms of LMIs and solved in the simulation examples to show the effectiveness of the proposed approach.

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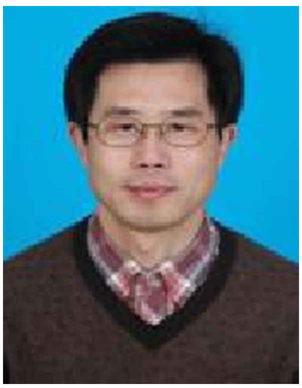

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