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A Note on L-fuzzy Closure Systems

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Abstract In this paper, a new notion of (v-consistent) $\mathbf{L}^*$-closure $\mathbf{L}$-system is proposed where $\mathbf{L}$ is a complete residuated lattice and $*$ is a truth stresser on $\mathbf{L}$. The one-to-one correspondence between (v-consistent) $\mathbf{L}^*$-closure $\mathbf{L}$-systems and (v-consistent) $\mathbf{L}^*$-closure operators is established. Furthermore, the notion of v-consistent $\mathbf{L}^*$-closure system is introduced. It is shown that the notion of (v-consistent) $\mathbf{L}^*$-closure $\mathbf{L}$-system provides an alternative way to characterize (v-consistent) $\mathbf{L}^*$-closure systems. Finally, the category of (v-consistent) $\mathbf{L}^*$-closure system spaces is introduced in virtue of the notion of continuous mapping. It is shown that the categories of $\mathbf{L}^*$-closure $\mathbf{L}$-system spaces, $\mathbf{L}^*$-closure spaces and $\mathbf{L}^*$-closure system spaces are isomorphic with each other.

Keywords Complete residuated lattice · Truth stresser · Closure operator · Closure system

1 Introduction

Closure operators and closure systems play an important role in many mathematics areas such as analysis, topology, logic, and geometry. In the classical setting, the close relationship between closure operators and closure systems has been investigated [8, 9]. In the framework of fuzzy set theory, the investigation of closure operators and closure systems may be traced back to the study of several special cases such as fuzzy subalgebra and fuzzy topology [2, 7, 18, 19]. Afterwards, Biacino and Gerla [3, 13] studied fuzzy closure operators and fuzzy closure systems themselves where the truth value structure is fixed to the unit interval [0, 1].

Fuzzy closure operators and closure systems have been studied in more general settings. For instance, Bělohlávek [4–6] generalized the notions of closure operator and closure system using complete residuated lattices as the truth value structures. Moreover, the one-to-one correspondence between fuzzy closure operators and closure systems has been established. In addition, Georgescu and Popescu [11, 12] presented analogous results in the non-commutative fuzzy framework where the truth value structure is a generalized residuated lattice. In another direction, there also have been a lot of works on generalizing fuzzy closure operators and closure systems onto the fuzzy partially ordered sets [14, 15, 21].

Recently, Fang and Yue [10] proposed a more general notion of (v-consistent) $\mathbf{L}$-fuzzy closure system which can be viewed as an extension of Bělohlávek's fuzzy closure system. It provides another way to think about the notion of fuzzy closure operator. However, as shown by Fang and Yue, one weakness is that the correspondence between $\mathbf{L}$-fuzzy closure systems and fuzzy closure operators is not one-to-one. This suggests that there may be more satisfactory notions of fuzzy closure system in order to characterize fuzzy closure operators.

In this paper, we propose a new notion of (v-consistent) $\mathbf{L}^*$-closure $\mathbf{L}$-system which can be viewed as an alternative of fuzzy closure systems introduced by Bělohlávek [6]. To

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demonstrate its capability of characterizing (v-consistent) fuzzy closure operators, we verify the bijective correspondence between (v-consistent) $\mathbf{L}^*$-closure **L**-systems and (v-consistent) $\mathbf{L}^*$-closure operators. Moreover, we introduce the notions of v-consistent $\mathbf{L}^*$-closure system and continuous mapping between $\mathbf{L}^*$-closure systems. It is shown that the categories of $\mathbf{L}^*$-closure **L**-system spaces, $\mathbf{L}^*$-closure spaces and $\mathbf{L}^*$-closure system spaces are isomorphic with each other from the categorical viewpoint.

This paper is organized as follows. In Sect. 2, we recall some preliminary notations and the notion of (v-consistent) $\mathbf{L}^*$-closure operator. In Sect. 3, we propose the new concept of (v-consistent) $\mathbf{L}^*$-closure **L**-system and establish a one-to-one correspondence between (v-consistent) $\mathbf{L}^*$-closure **L**-systems and (v-consistent) $\mathbf{L}^*$-closure operators. In Sect. 4, we introduce the notion of v-consistent $\mathbf{L}^*$-closure system and show that there exists a one-to-one correspondence between (v-consistent) $\mathbf{L}^*$-closure **L**-systems and (v-consistent) $\mathbf{L}^*$-closure systems. In Sect. 5, we introduce the concept of continuous mapping between $\mathbf{L}^*$-closure system spaces and prove that the categories of (v-consistent) $\mathbf{L}^*$-closure **L**-system spaces, (v-consistent) $\mathbf{L}^*$-closure system spaces and (v-consistent) $\mathbf{L}^*$-closure spaces are isomorphic with each other.

2 Preliminary

Truth value structures play an important role in fuzzy logic [17]. Throughout this paper, we use complete residuated lattices as truth value structures. Formally, a *complete residuated lattice* is a structure $\mathbf{L} = (L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ such that: $(L, \wedge, \vee, 0, 1)$ is a complete lattice where 0 is the least element and 1 is the greatest element; $(L, \otimes, 1)$ is a commutative monoid, i.e., $\otimes$ is commutative associative operator on L, and $x \otimes 1 = x$ holds for any $x \in L$; $x \le y \rightarrow z \Leftrightarrow x \otimes y \le z$ holds for any $x, y, z \in L$.

Let **L** be a complete residuated lattice. Given $x, y, z \in L$ and $\{x_i\}_{i\in I} \subseteq L$, where I is an index set, the following properties will be needed in the sequel:

(1) $0 \rightarrow x = 1$	(2) $1 \rightarrow x = x$
(3) $x \le y \Rightarrow x \otimes z \le y \otimes z$	(4) $x \le y \Rightarrow z \rightarrow x \le z \rightarrow y$
(5) $x \le y \Rightarrow x \rightarrow z \ge y \rightarrow z$	(6) $x \le y$ iff $x \rightarrow y = 1$
(7) $x \otimes (x \rightarrow y) \le y$	(8) $x \le (x \rightarrow y) \rightarrow y$
(9) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z) = (x \otimes y) \rightarrow z$	(10) $x \rightarrow (\bigwedge_{i\in I} x_i) = \bigwedge_{i\in I}(x \rightarrow x_i)$
(11) $x \otimes (\bigwedge_{i\in I} x_i) \le \bigwedge_{i\in I}(x \otimes x_i)$	

For more properties of complete residuated lattices one can refer to [20].

Recall that in [6], the notion of truth stresser was used to express the meaning "(very) true" in the study of fuzzy closure operators. Formally, a *truth stresser* on a complete residuated lattice **L** is a unary function $* : L \longrightarrow L$ which sends any element $a \in L$ to $a^* \in L$. In this paper, we only need the following properties of truth stresser: for any $a, b \in L$,

(1) $1^* = 1$; (2) $a^* \le a$; (3) $(a \rightarrow b)^* \le a^* \rightarrow b^*$.

It is easy to verify that the following consequences follows immediately from (1)–(3):

(4) $a \le b \Rightarrow a^* \le b^*$; (5) $a^* \otimes b^* \le (a \otimes b)^*$.

Particularly, if $*$ satisfies $a^* = a$ for any $a \in L$, then $*$ is called the *identity* on L. In the sequel of this paper, we will always use the notation $*$ to denote a general truth stresser unless otherwise specified. More properties of truth stresser can be referred to [16].

Now we recall some common notations in the framework of fuzzy set theory. Let X be a universe. The notation L^X denotes the family of all **L**-sets on X. Given $a \in L$, **L**-set χ_a is defined by $\chi_a(x) = a$ for any $x \in X$. For $a \in L$ and $A \in L^X$, **L**-set $a \otimes A$ is defined by $(a \otimes A)(x) = a \otimes A(x)$ for any $x \in X$, and $a \rightarrow A$ by $(a \rightarrow A)(x) = a \rightarrow A(x)$ for any $x \in X$. Given $A, B \in L^X$, $A \subseteq B$ means $A(x) \le B(x)$ for any $x \in X$. The *subsethood degree* $S(A, B)$ is defined by $S(A, B) = \bigwedge_{x\in X} A(x) \rightarrow B(x)$. It is obvious that $A \subseteq B$ is equivalent to $S(A, B) = 1$. For any family $\{A_i\}_{i\in I} \subseteq L^X$, **L**-sets $\bigcup_{i\in I} A_i$ and $\bigcap_{i\in I} A_i$ are given pointwisely, i.e., for any $x \in X$, $\bigcup_{i\in I} A_i(x) = \bigvee_{i\in I} A_i(x)$ and $\bigcap_{i\in I} A_i(x) = \bigwedge_{i\in I} A_i(x)$.

We recall the notion of $\mathbf{L}^*$-closure operator which may be initially developed in [6].

Definition 2.1 Let $C : L^X \longrightarrow L^X$ be a mapping, $v \in L$ and $A, A_1, A_2 \in L^X$. Consider the following conditions:

(LC1) $A \subseteq C(A)$;
(LC2) $S(A_1, A_2)^* \le S(C(A_1), C(A_2))$;
(LC3) $C(A) = C(C(A))$;
(LC4) $v \otimes C(A) \subseteq C(v \otimes A)$.

Then C is called an $\mathbf{L}^*$-closure operator on X if it satisfies (LC1)-(LC3). An $\mathbf{L}^*$-closure operator satisfying (LC4) is said to be v-consistent.

In the sequel, if C is an (v-consistent) $\mathbf{L}^*$-closure operator on X, then the pair (X, C) is called an (v-consistent) $\mathbf{L}^*$-closure space on X.

3 $\mathbf{L}^*$-closure L-systems and $\mathbf{L}^*$-closure Operators

In this section, we propose a new notion of **L**-fuzzy closure system, namely, $\mathbf{L}^*$-closure **L**-system, and investigate the fundamental properties. We show that there is a one-to-one correspondence between (v-consistent) $\mathbf{L}^*$-closure **L**-systems and (v-consistent) $\mathbf{L}^*$-closure operators.

Definition 3.1 Let $\lambda : L^X \longrightarrow L$ be an **L**-set on L^X and $v \in L$. Then λ is called an $\mathbf{L}^*$-*closure* **L**-*system* on X if for any $A \in L^X$, there exists $\tilde{A} \in L^X$ such that

(LLS1) $\lambda(A) = S(\tilde{A}, A)$;
(LLS2)$\lambda(\tilde{A}) = S(A, \tilde{A})^* = 1$;
(LLS3)for any $B \in L^X$, if $\lambda(B) = 1$, then $S(A, B)^* \leq S(\tilde{A}, B)$.

If, in addition, for any $A \in L^X$,
(LLS4) $\lambda(A) \leq \lambda(v \rightarrow A)$,
then λ is called a v-*consistent* $\mathbf{L}^*$-*closure* **L**-*system* on X.

For any $\mathbf{L}^*$-closure **L**-system λ on X, the **L**-set $\tilde{A}$ in the sense of Definition 3.1 is called the *closure of A with respect to* λ.

In the sequel, if λ is an (v-consistent) $\mathbf{L}^*$-closure **L**-system on X, then the pair (X, λ) is called an (v-consistent) $\mathbf{L}^*$-*closure* **L**-*system space* on X.

Remark 3.1 It is not difficult to verify that the closure of A with respect to an $\mathbf{L}^*$-closure **L**-system λ is unique. Indeed, suppose that both $\tilde{A}$ and $\tilde{A}'$ are $\mathbf{L}^*$-closures of $A \in L^X$ with respect to λ. Since $\lambda(\tilde{A}') = 1$, it holds that $S(A, \tilde{A}')^* \leq S(\tilde{A}, \tilde{A}')$ by Definition 3.1(LLS3). As $S(A, \tilde{A}')^* = 1$, it follows that $S(\tilde{A}, \tilde{A}') = 1$. On the other hand, $S(\tilde{A}', \tilde{A}) = 1$ can be proved similarly. Thus, we have $\tilde{A} = \tilde{A}'$.

Example 3.1 Consider the singleton $X = \{x\}$ and the real interval $L = [0, 1]$ with the Łukasiewicz t-norm $\otimes$ defined by $s \otimes t = \max\{s + t - 1, 0\}$ and the associated implication $\rightarrow$ defined by $s \rightarrow t = \min\{1 - s + t, 1\}$. It is easy to check that $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ is a complete residuated lattice. In this case, the family of all **L**-sets on X are exactly $\{\chi_a \mid a \in L\}$. Let $*$ be the identity on L, i.e., for any $a \in L$, $a^* = a$. Define a mapping $\lambda : L^X \longrightarrow L$ by letting $\lambda(\chi_a) = \frac{1}{2} + a$ for any $a \in [0, \frac{1}{2})$ and $\lambda(\chi_a) = 1$ for any $a \in [\frac{1}{2}, 1]$. It is trivial to check that λ is an $\mathbf{L}^*$-closure **L**-system on X. Particularly, the closure of χ_a for any $a \in [0, \frac{1}{2})$ with respect to λ is $\chi_{a\frac{1}{2}}$; and the closure of χ_a for any $a \in [\frac{1}{2}, 1]$ with respect to λ is χ_1. Moreover, if we choose $v = \frac{1}{2}$, then we can readily check that λ is v-consistent.

Proposition 3.1 Let λ be an $\mathbf{L}^*$-closure **L**-system on X. Then for any $A, B \in L^X$, we have $S(A, B)^* \leq S(\tilde{A}, \tilde{B})$.

Proof Suppose $A, B \in L^X$. As $S(B, \tilde{B})^* = 1$, we have $S(A, B)^* = S(A, B)^* \otimes S(B, \tilde{B})^* \leq (S(A, B) \otimes \quad S(B, \tilde{B}))^* \leq S(A, \tilde{B})^*$. Since $\lambda(\tilde{B}) = 1$, it follows that $S(A, \tilde{B})^* \leq S(\tilde{A}, \tilde{B})$ from Definition 3.1(LLS3). Thus, we have $S(A, B)^* \leq S(\tilde{A}, \tilde{B})$. □

Proposition 3.2 If λ is an $\mathbf{L}^*$-closure **L**-system on X, then $\lambda(\chi_1) = 1$ and $\bigwedge_{i \in I} \lambda(A_i) \leq \lambda(\bigcap_{i \in I} A_i)$ holds for any family $\{A_i\}_{i \in I} \subseteq L^X$.

Proof By Definition 3.1(LLS2), we have $S(\chi_1, \widetilde{\chi_1}) = 1$ which implies $\widetilde{\chi_1} = \chi_1$. By Definition 3.1(LLS1), it holds that $\lambda(\chi_1) = S(\widetilde{\chi_1}, \chi_1) = 1$.

For any family $\{A_i\}_{i \in I} \subseteq L^X$, we have

$$\begin{aligned}
\bigwedge_{i \in I} S(\widetilde{A_i}, A_i) &= \bigwedge_{i \in I} \bigwedge_{x \in X} \widetilde{A_i}(x) \rightarrow A_i(x) \\
&\leq \bigwedge_{i \in I} \bigwedge_{x \in X} (\bigwedge_{j \in I} \widetilde{A_j}(x) \rightarrow A_i(x)) \\
&= \bigwedge_{x \in X} \bigwedge_{i \in I} (\bigwedge_{j \in I} \widetilde{A_j}(x) \rightarrow A_i(x)) \\
&\leq \bigwedge_{x \in X} (\bigwedge_{i \in I} \widetilde{A_i}(x) \rightarrow \bigwedge_{i \in I} A_i(x)) \\
&\leq \bigwedge_{x \in X} (\widetilde{\bigcap_{i \in I} A_i}(x) \rightarrow (\bigcap_{i \in I} A_i)(x)) \\
&= S(\widetilde{\bigcap_{i \in I} A_i}, \bigcap_{i \in I} A_i)
\end{aligned}$$

This implies that $\bigwedge_{i \in I} \lambda(A_i) \leq \lambda(\bigcap_{i \in I} A_i)$. □

From Proposition 3.2, we can easily see that our proposed notion of $\mathbf{L}^*$-closure **L**-system is a special cases of **L**-fuzzy closure systems which were introduced in [10]. The following example shows that the inverse of Proposition 3.2 does not hold necessarily.

Example 3.2 Let $X = \{x\}$ be a single-point set, L the real interval [0, 1] with the Łukasiewicz t-norm $\otimes$ and the implication $\rightarrow$ as given in Example 3.1 and $*$ the identity on L. Define a mapping $\lambda : L^X \longrightarrow L$ by $\lambda(A) = 1$ if $A = \chi_{1/2}, \chi_1$ and $\lambda(A) = 0$ otherwise. It is trivial to check that λ satisfies $\lambda(\chi_1) = 1$ and $\bigwedge_{i \in I} \lambda(A_i) \leq \lambda(\bigcap_{i \in I} A_i)$ for any family $\{A_i\}_{i \in I} \subseteq L^X$. However, λ is not an $\mathbf{L}^*$-closure **L**-system. Indeed, for $A = \chi_{1/3}$, there does not exist any **L**-set $\tilde{A} \in L^X$ such that $\lambda(\chi_{1/3}) = S(\tilde{A}, \chi_{1/3})$ and $\lambda(\tilde{A}) = 1$.

Theorem 3.1 *Let λ be an* **L**-*set on L^X. Then the following are equivalent*:

(1) *λ is an $\mathbf{L}^*$-closure* **L**-*system on X.*
(2) *For any $A \in L^X$, it holds that $\lambda(A_0) = 1$ and $\lambda(A) = S(A_0, A)$, where*

$$A_0 = \bigcap_{A_i \in L^X, \lambda(A_i) = 1} S(A, A_i)^* \rightarrow A_i.$$

Proof $(1) \Rightarrow (2)$: Let λ be an $\mathbf{L}^*$-closure **L**-system on X. We want to prove A_0 is exactly the closure of A with respect to λ, i.e., $A_0 = \tilde{A}$. On the one hand, suppose $A_i \in L^X$ and $\lambda(A_i) = 1$. By Definition 3.1(LLS3), we have $S(A, A_i)^* \leq S(\tilde{A}, A_i)$ which implies that $S(A, A_i)^* \leq \tilde{A}(x) \rightarrow A_i(x)$ for any $x \in X$. This implies that $\tilde{A}(x) \leq S(A, A_i)^* \rightarrow A_i(x)$ for any $x \in X$, which means $\tilde{A} \subseteq S(A, A_i)^* \rightarrow A_i$. Hence, $\tilde{A} \subseteq A_0$. On the other hand, since

$\lambda(\tilde{A}) = S(A,\tilde{A})^* = 1$, we have $A_0(x) \leq S(A,\tilde{A})^* \rightarrow \tilde{A}(x) = 1 \rightarrow \tilde{A}(x) = \tilde{A}(x)$ for any $x \in X$, which implies $A_0 \subseteq \tilde{A}$. Therefore, $A_0 = \tilde{A}$. By Definition 3.1(LLS1) and (LLS2), we obtain $\lambda(A_0) = 1$ and $\lambda(A) = S(A_0, A)$.

(2) $\Rightarrow$ (1) : To prove λ is an $\mathbf{L}^*$-closure $\mathbf{L}$-system on X, we only need to verify that A_0 satisfies (LLS1)–(LLS3). First, it is clear that $\lambda(A) = S(A_0, A)$ and $\lambda(A_0) = 1$ by hypothesis. Second, to prove $S(A, A_0)^* = 1$, we have the following equivalences: $S(A, A_0) = 1$ iff for any $x \in X$, $A(x) \leq \bigwedge_{A_i \in L^X, \lambda(A_i)=1} S(A, A_i)^* \rightarrow A_i(x)$ iff for any $A_i \in L^X$ with $\lambda(A_i) = 1$, $A(x) \leq S(A, A_i)^* \rightarrow A_i(x)$, i.e., $S(A, A_i)^* \leq A(x) \rightarrow A_i(x)$ which immediately follows from $S(A, A_i)^* \leq S(A, A_i)$ and $S(A, A_i) \leq A(x) \rightarrow A_i(x)$. Finally, suppose $B \in L^X$ and $\lambda(B) = 1$. We have: $S(A,B)^* \leq S(A_0, B)$ iff for any $x \in X$, $S(A,B)^* \leq A_0(x) \rightarrow B(x)$ iff $S(A,B)^* \otimes A_0(x) \leq B(x)$, i.e., $S(A,B)^* \otimes \bigwedge_{A_i \in L^X, \lambda(A_i)=1} S(A, A_i)^* \rightarrow A_i(x) \leq B(x)$ which is true. Indeed, since $\lambda(B) = 1$, $S(A,B)^* \otimes \bigwedge_{A_i \in L^X, \lambda(A_i)=1} S(A, A_i)^* \rightarrow A_i(x) \leq S(A,B)^* \otimes (S(A,B)^* \rightarrow B(x)) \leq B(x)$. Therefore, λ is an $\mathbf{L}^*$-closure $\mathbf{L}$-system on X. □

Proposition 3.3 Let λ be an $\mathbf{L}^*$-closure $\mathbf{L}$-system on X. Then for any $A \in L^X$, we have

$$\bigcap_{A_i \in L^X, \lambda(A_i)=1} S(A, A_i)^* \rightarrow A_i = \bigcap_{B \in L^X} \lambda(B) \otimes S(A,B)^* \rightarrow B = \bigcap_{A \subseteq A_i, \lambda(A_i)=1} A_i.$$

Proof We first prove $\bigcap_{A_i \in L^X, \lambda(A_i)=1} S(A, A_i)^* \rightarrow A_i = \bigcap_{A \subseteq A_i, \lambda(A_i)=1} A_i$. On the one hand, it is easy to see that $A \subseteq A_i$ is equivalent to $S(A, A_i)^* = 1$. It thus follows that $\bigcap_{A_i \in L^X, \lambda(A_i)=1} S(A, A_i)^* \rightarrow A_i \subseteq \bigcap_{A \subseteq A_i, \lambda(A_i)=1} A_i$. On the other hand, since $S(A, \tilde{A})^* = 1$, we have $A \subseteq \tilde{A}$, which implies that $\bigcap_{A \subseteq A_i, \lambda(A_i)=1} A_i \subseteq \tilde{A}$. Since $\tilde{A} = \bigcap_{A_i \in L^X, \lambda(A_i)=1} S(A, A_i)^* \rightarrow A_i$ by the proof of Theorem 3.1, we have $\bigcap_{A \subseteq A_i, \lambda(A_i)=1} A_i \subseteq \bigcap_{A_i \in L^X, \lambda(A_i)=1} S(A, A_i)^* \rightarrow A_i$. Therefore, $\bigcap_{A_i \in L^X, \lambda(A_i)=1} S(A, A_i)^* \rightarrow A_i = \bigcap_{A \subseteq A_i, \lambda(A_i)=1} A_i$.

Now we prove $\bigcap_{A_i \in L^X, \lambda(A_i)=1} S(A, A_i)^* \rightarrow A_i = \bigcap_{B \in L^X} \lambda(B) \otimes S(A,B)^* \rightarrow B$. On the one hand, as $\{S(A, A_i)^* \rightarrow A_i \mid A_i \in L^X, \lambda(A_i) = 1\} \subseteq \{\lambda(B) \otimes S(A,B)^* \rightarrow B \mid B \in L^X\}$, we have $\bigcap_{B \in L^X} \lambda(B) \otimes S(A,B)^* \rightarrow B \subseteq \bigcap_{A_i \in L^X, \lambda(A_i)=1} S(A, A_i)^* \rightarrow A_i$. On the other hand, it suffices to prove $\tilde{A} \subseteq \bigcap_{B \in L^X} \lambda(B) \otimes S(A,B)^* \rightarrow B$ by Theorem 3.1. In this end, for any $B \in L^X$,

$$\begin{aligned} \tilde{A} \subseteq \lambda(B) \otimes S(A,B)^* \rightarrow B \quad & \Leftrightarrow \lambda(B) \otimes S(A,B)^* \otimes \tilde{A} \subseteq B \\ & \Leftrightarrow S(\tilde{B}, B) \otimes S(A,B)^* \otimes \tilde{A} \subseteq B \\ & \quad (\text{because } \lambda(B) = S(\tilde{B}, B) \\ & \quad \text{by Definition 3.1}(LLS1)) \\ & \Leftrightarrow S(\tilde{B}, B) \otimes S(A,B)^* \leq S(\tilde{A}, B) \end{aligned}$$

The last inequality holds because $S(A,B)^* \leq S(\tilde{A}, \tilde{B})$ by Proposition 3.1. □

Given an $\mathbf{L}^*$-closure $\mathbf{L}$-system λ, we can naturally define an operator $C_\lambda : L^X \longrightarrow L^X$ by

$$C_\lambda(A) = \tilde{A}$$

where $\tilde{A}$ is the closure of A with respect to λ. By Proposition 3.3, C_λ can be defined equivalently by

$$C_\lambda(A) = \bigcap_{A_i \in L^X, \lambda(A_i)=1} S(A, A_i)^* \rightarrow A_i.$$

Proposition 3.4 If λ is an (v-consistent) $\mathbf{L}^*$-closure $\mathbf{L}$-system on X, then C_λ is an (v-consistent) $\mathbf{L}^*$-closure operator.

Proof Suppose λ is an $\mathbf{L}^*$-closure $\mathbf{L}$-system on X. For (LC1), suppose $A \in L^X$, we have $S(A, C_\lambda(A)) = S(A, \tilde{A}) = 1$ which immediately follows from $S(A, \tilde{A})^* = 1$ and $S(A, \tilde{A})^* \leq S(A, \tilde{A})$.

For (LC2), suppose $A_1, A_2 \in L^X$. Because $S(A_2, \widetilde{A_2})^* = 1$, we have $S(A_1, A_2)^* = S(A_1, A_2)^* \otimes S(A_2, \widetilde{A_2})^* \leq (S(A_1, A_2) \otimes S(A_2, \widetilde{A_2}))^* \leq S(A_1, \widetilde{A_2})^*$. Since $\lambda(\widetilde{A_2}) = 1$, it follows from Definition 3.1(LLS3) that $S(A_1, \widetilde{A_2})^* \leq S(\widetilde{A_1}, \widetilde{A_2})$. Therefore, $S(A_1, A_2)^* \leq S(\widetilde{A_1}, \widetilde{A_2})$, i.e., $S(A_1, A_2)^* \leq S(C_\lambda(A_1), C_\lambda(A_2))$.

For (LC3), suppose $A \in L^X$. By Definition 3.1(LLS2), it is obvious that $S(\tilde{A}, \tilde{\tilde{A}}) = 1$. Moreover, since $\lambda(\tilde{A}) = 1$, by Definition 3.1(LLS3), $1 = S(\tilde{A}, \tilde{A})^* \leq S(\tilde{\tilde{A}}, \tilde{A})$. This implies $S(\tilde{\tilde{A}}, \tilde{A}) = 1$. We thus have $\tilde{A} = \tilde{\tilde{A}}$, i.e., $C_\lambda(A) = C_\lambda C_\lambda(A)$.

For (LC4), suppose λ is a v-consistent $\mathbf{L}^*$-closure $\mathbf{L}$-system on X. For any $A \in L^X$ and $x \in X$, we have

$$\begin{aligned} C_\lambda(A)(x) \quad & = \bigwedge_{A_i \in L^X, \lambda(A_i)=1} S(A, A_i)^* \rightarrow A_i(x) \\ & \leq \bigwedge_{A_i \in L^X, \lambda(A_i)=1} S(A, v \rightarrow A_i)^* \rightarrow (v \rightarrow A_i)(x) \\ & = \bigwedge_{A_i \in L^X, \lambda(A_i)=1} S(v \otimes A, A_i)^* \rightarrow (v \rightarrow A_i(x)) \\ & = \bigwedge_{A_i \in L^X, \lambda(A_i)=1} v \rightarrow (S(v \otimes A, A_i)^* \rightarrow A_i(x)) \\ & = v \rightarrow \bigwedge_{A_i \in L^X, \lambda(A_i)=1} S(v \otimes A, A_i)^* \rightarrow A_i(x) \\ & = v \rightarrow C_\lambda(v \otimes A)(x). \end{aligned}$$

This implies that $C_\lambda(A) \subseteq v \rightarrow C_\lambda(v \otimes A)$, i.e., $v \otimes C_\lambda(A) \subseteq C_\lambda(v \otimes A)$. □

Given an $\mathbf{L}^*$-closure operator C on X, define an $\mathbf{L}$-set λ_C on L^X by

$$\lambda_C(A) = S(C(A), A).$$

Proposition 3.5 If C is an (v-consistent) $\mathbf{L}^*$-closure operator, then λ_C is an (v-consistent) $\mathbf{L}^*$-closure $\mathbf{L}$-system.

Proof Suppose C is an $\mathbf{L}^*$-closure operator. For any $A \in L^X$, we prove that $C(A)$ satisfies conditions (LLS1)-(LLS3) in Definition 3.1. First, (LLS1) is clear from the definition of λ_C. For (LLS2), by Definition 2.1(LC3), $\lambda_C(C(A)) = S(CC(A), C(A)) = 1$. In addition, since $A \subseteq C(A)$, we have $S(A, C(A))^* = 1$. Finally, for (LLS3), suppose $B \in L^X$ and $\lambda_C(B) = 1$, then $S(C(B), B) = 1$. As $S(B, C(B)) = 1$, it follows that $B = C(B)$. By Definition 2.1(LC2), we have $S(A, B)^* \leq S(C(A), C(B)) = S(C(A), B)$.

Suppose C is a v-consistent $\mathbf{L}$ *-closure operator. From the proof above, we only need to prove $\lambda_C(A) \leq \lambda_C(v \rightarrow A)$ for any $A \in L^X$. By the definition of λ_C, it is equivalent to prove that for any $A \in L^X$, $S(C(A), A) \leq S(C(v \rightarrow A), v \rightarrow A)$ iff for any $x \in X$, $S(C(A), A) \leq C(v \rightarrow A)(x) \rightarrow (v \rightarrow A)(x)$ iff $S(C(A), A) \otimes C(v \rightarrow A)(x) \leq (v \rightarrow A)(x)$ iff $v \otimes S(C(A), A) \otimes C(v \rightarrow A)(x) \leq A(x)$ which is true. In fact,

$$\begin{aligned} v \otimes S(C(A),A) \otimes C(v \rightarrow A)(x) &\leq S(C(A),A) \otimes C(v \otimes (v \rightarrow A))(x) \\ &\leq S(C(A),A) \otimes C(A)(x) \\ &\leq A(x). \end{aligned}$$

□

From Proposition 3.4 and Proposition 3.5, we have the following theorem.

Theorem 3.2 *Let λ be an (v-consistent) $\mathbf{L}^*$-closure $\mathbf{L}$-system on X and C an (v-consistent) $\mathbf{L}^*$-closure operator on X. Then $\lambda = \lambda_{C_\lambda}$ and $C = C_{\lambda_C}$, i.e., mappings $\lambda \mapsto C_\lambda$ and $C \mapsto \lambda_C$ form a one-to-one correspondence between (v-consistent) $\mathbf{L}^*$-closure $\mathbf{L}$-systems on X and (v-consistent) $\mathbf{L}^*$-closure operators on X.*

Example 3.3 The $\mathbf{L}^*$-closure operator associated with the $\mathbf{L}$ *-closure $\mathbf{L}$-system given in Example 3.1 is precisely the mapping on L^X which sends χ_a to χ_a for any $a \in [\frac{1}{2}, 1]$ and χ_a to $\chi_{\frac{1}{2}}$ for any $a \in [0, \frac{1}{2})$. Moreover, it is routine to check that this $\mathbf{L}^*$-closure operator is v-consistent for $v = \frac{1}{2}$.

From Proposition 3.2 and Theorem 3.2, we immediately have the following result.

Corollary 3.1 *Let C be an $\mathbf{L}^*$-closure operator on X. Then $C(\chi_1) = \chi_1$ and for any family $\{A_i\}_{i \in I} \subseteq L^X$,*

$$\bigwedge_{i \in I} S(C(A_i), A_i) \leq S\left(C\left(\bigcap_{i \in I} A_i\right), \bigcap_{i \in I} A_i\right).$$

4 $\mathbf{L}^*$-closure Systems

In this section, we first introduce the notion of v-consistent $\mathbf{L}^*$-closure system based on the notion of $\mathbf{L}^*$-closure system introduced by Bělohlávek in [6]. Then we study the connection between (v-consistent) $\mathbf{L}^*$-closure $\mathbf{L}$-systems and (v-consistent) $\mathbf{L}^*$-closure systems. It is shown that a one-to-one correspondence can be established between them.

Definition 4.1 Let $\wp \subseteq L^X$ and $v \in L$. Then $\wp$ is called an $\mathbf{L}^*$-*closure system* on X if for any $A \in L^X$,

$$\bigcap_{A_i \in \wp} S(A, A_i)^* \rightarrow A_i \in \wp.$$

If, in addition, for any $A \in L^X$,

$$v \otimes \bigcap_{A_i \in \wp} S(A, A_i)^* \rightarrow A_i \subseteq \bigcap_{A_i \in \wp} S(v \otimes A, A_i)^* \rightarrow A_i,$$

then $\wp$ is called a v-*consistent* $\mathbf{L}^*$-*closure system* on X.

For any $\mathbf{L}^*$-closure system $\wp$ on X, the $\mathbf{L}$-set $\bigcap_{A_i \in \wp} S(A, A_i)^* \rightarrow A_i$ is called the *closure of A with respect to* $\wp$.

In the sequel, if $\wp$ is an (v-consistent) $\mathbf{L}^*$-closure system on X, then the pair $(X, \wp)$ is called an (v-*consistent*) $\mathbf{L}^*$-*closure system space* on X.

The following proposition gives an equivalent characterization of the notion of $\mathbf{L}^*$-closure system.

Proposition 4.1 [14] Let $\wp = \{A_i\}_{i \in I}$ be a subset of L^X. Then $\wp$ is an $\mathbf{L}^*$-closure system on X if and only if for any $A \in L^X$, there exists $A_0 \in \wp$ such that $S(A, A_0) = 1$ and $S(A, A_i)^* \leq S(A_0, A_i)$ for any $i \in I$.

Now we discuss the relationship between $\mathbf{L}^*$-closure systems and $\mathbf{L}$ *-closure $\mathbf{L}$-systems. Given an $\mathbf{L}^*$-closure $\mathbf{L}$-system λ on X, define a system $\wp_\lambda$ by

$$\wp_\lambda = \{A_i \in L^X \mid \lambda(A_i) = 1\}.$$

By Theorem 3.1, it is easy to see that $\wp_\lambda$ is an $\mathbf{L}^*$-closure system.

Conversely, given an $\mathbf{L}^*$-closure system $\wp$ on X, define a mapping $\lambda_\wp : L^X \longrightarrow L$ by

$$\lambda_\wp(A) = S\left(\bigcap_{A_i \in \wp} S(A, A_i)^* \rightarrow A_i, A\right).$$

Proposition 4.2 If $\wp$ is an (v-consistent) $\mathbf{L}^*$-closure system, then $\lambda_\wp$ is an (v-consistent) $\mathbf{L}^*$-closure $\mathbf{L}$-system.

Proof Suppose $\wp$ is an $\mathbf{L}^*$-closure system. Given $A \in L^X$, denote $A_0 = \bigcap_{A_i \in \wp} S(A, A_i)^* \to A_i$. In order to prove $\lambda_\wp$ is an $\mathbf{L}^*$-closure $\mathbf{L}$-system, we need to check that A_0 satisfies conditions (LLS1)-(LLS3) in Definition 3.1.

From the definition of $\lambda_\wp$, (LLS1) is clear.

For (LLS2), we observe that $\bigcap_{A_i \in \wp} S(A, A_i)^* \to A_i = A$ for any $A \in \wp$. Therefore, we have $\lambda_\wp(A_0) = S(\bigcap_{A_i \in \wp} S(A_0, A_i)^* \to A_i, A_0) = S(A_0, A_0) = 1$.

In addition, we have

$$\begin{aligned} S(A,A_0)^* = 1 \quad &\Leftrightarrow S(A,A_0) = 1 \\ &\Leftrightarrow A \subseteq A_0 \\ &\Leftrightarrow (\forall x \in X) A(x) \le \textstyle\bigwedge_{A_i \in \wp} S(A,A_i)^* \to A_i(x) \\ &\Leftrightarrow (\forall x \in X) A(x) \le S(A,A_i)^* \to A_i(x) \\ &\Leftrightarrow (\forall x \in X) S(A,A_i)^* \le A(x) \to A_i(x). \end{aligned}$$

Since the last statement is obvious, we have $S(A, A_0)^* = 1$.

Suppose $B \in L^X$ and $\lambda_\wp(B) = 1$. Denote $B_0 = \bigcap_{A_i \in \wp} S(B, A_i)^* \to A_i$. It is easy to verify that $S(B, B_0) = 1$. Since $S(B_0, B) = \lambda_\wp(B) = 1$, we have $B = B_0 \in \wp$. Therefore, for (LLS3), we only need to prove $S(A, B)^* \le S(A_0, B_0)$, i.e., $S(A, B)^* \le S(\bigcap_{A_i \in \wp} S(A, A_i)^* \to A_i, \bigcap_{A_i \in \wp} S(B, A_i)^* \to A_i)$ which is equivalent to $S(A, B)^* \otimes \bigwedge_{A_i \in \wp} S(A, A_i)^* \to A_i(x) \le \bigwedge_{A_i \in \wp} S(B, A_i)^* \to A_i(x)$ for any $x \in X$.

For the last inequality, it suffices to check that for any $j \in I$ with $A_j \in \wp$, $\bigwedge_{A_i \in \wp} S(A, B)^* \otimes (S(A, A_i)^* \to A_i(x)) \le S(B, A_j)^* \to A_j(x)$ which is equivalent to $S(B, A_j)^* \otimes \bigwedge_{A_i \in \wp} S(A, B)^* \otimes (S(A, A_i)^* \to A_i(x)) \le A_j(x)$ which is true because

$$\begin{aligned} &S(B,A_j)^* \otimes \textstyle\bigwedge_{A_i \in \wp} S(A,B)^* \otimes (S(A,A_i)^* \to A_i(x)) \\ &\le \textstyle\bigwedge_{A_i \in \wp} S(B,A_j)^* \otimes S(A,B)^* \otimes (S(A,A_i)^* \to A_i(x)) \\ &\le \textstyle\bigwedge_{A_i \in \wp} S(A,A_j)^* \otimes (S(A,A_i)^* \to A_i(x)) \\ &\le S(A,A_j)^* \otimes (S(A,A_j)^* \to A_j(x)) \\ &\le A_j(x). \end{aligned}$$

Suppose $\wp$ is a v-consistent $\mathbf{L}^*$-closure system. From the proof above, we only need to prove $\lambda_\wp(A) \le \lambda_\wp(v \to A)$ for any $A \in L^X$. By the definition of $\lambda_\wp$, we need to prove $S(\bigcap_{A_i \in \wp} S(A, A_i)^* \to A_i, A) \le S(\bigcap_{A_i \in \wp} S(v \to A, A_i)^* \to A_i, v \to A)$. Since $S(\bigcap_{A_i \in \wp} S(v \to A, A_i)^* \to A_i, v \to A) = S(v \otimes \bigcap_{A_i \in \wp} S(v \to A, A_i)^* \to A_i, A)$, it suffices to prove $v \otimes \bigcap_{A_i \in \wp} S(v \to A, A_i)^* \to A_i \subseteq \bigcap_{A_i \in \wp} S(A, A_i)^* \to A_i$ which is true. Indeed, for any $x \in X$,

$$\begin{aligned} (v \otimes \textstyle\bigcap_{A_i \in \wp} S(v \to A, A_i)^* \to A_i)(x) &= v \otimes \textstyle\bigwedge_{A_i \in \wp} S(v \to A, A_i)^* \to A_i(x) \\ &\le \textstyle\bigwedge_{A_i \in \wp} S(v \otimes (v \to A), A_i)^* \to A_i(x) \\ &\le \textstyle\bigwedge_{A_i \in \wp} S(A, A_i)^* \to A_i(x) \\ &= (\textstyle\bigcap_{A_i \in \wp} S(A, A_i)^* \to A_i)(x). \end{aligned}$$

□

Proposition 4.3 If λ is an (v-consistent) $\mathbf{L}^*$-closure $\mathbf{L}$-system, then $\wp_\lambda$ is an (v-consistent) $\mathbf{L}^*$-closure system.

Proof Suppose λ is an $\mathbf{L}^*$-closure $\mathbf{L}$-system. It is easy to see that $\wp_\lambda$ is an $\mathbf{L}^*$-closure system by Theorem 3.1.

Now suppose λ is a v-consistent $\mathbf{L}^*$-closure $\mathbf{L}$-system. We need to check that for any $A \in L^X$, $v \otimes \bigcap_{A_i \in L^X, \lambda(A_i)=1} S(A, A_i)^* \to A_i \subseteq \bigcap_{A_i \in L^X, \lambda} (A_i) = 1 S(v \otimes A, A_i)^* \to A_i$ which is equivalent to $\bigcap_{A_i \in L^X, \lambda(A_i)=1} S(A, A_i)^* \to A_i \subseteq v \to \bigcap_{A_i \in L^X, \lambda(A_i)=1} S(v \otimes A, A_i)^* \to A_i$ which is true. Indeed,

$$\begin{aligned} \textstyle\bigcap_{A_i \in L^X, \lambda(A_i)=1} S(A, A_i)^* \to A_i &\subseteq \textstyle\bigcap_{A_i \in L^X, \lambda(A_i)=1} S(A, v \to A_i)^* \to (v \to A_i) \\ &= \textstyle\bigcap_{A_i \in L^X, \lambda(A_i)=1} S(v \otimes A, A_i)^* \to (v \to A_i) \\ &= v \to \textstyle\bigcap_{A_i \in L^X, \lambda(A_i)=1} S(v \otimes A, A_i)^* \to A_i. \end{aligned}$$

□

Based on Proposition 4.2 and Proposition 4.3, it is easy to get the following theorem.

Theorem 4.1 *Let λ be an (v-consistent) $\mathbf{L}^*$-closure $\mathbf{L}$-system on X and $\wp$ an (v-consistent) $\mathbf{L}^*$-closure system on X. Then $\lambda = \lambda_{\wp_\lambda}$ and $\wp = \wp_{\lambda_\wp}$, i.e., mappings $\lambda \mapsto \wp_\lambda$ and $\wp \mapsto \lambda_\wp$ form a one-to-one correspondence between (v-consistent) $\mathbf{L}^*$-closure $\mathbf{L}$-systems on X and (v-consistent) $\mathbf{L}^*$-closure systems on X.*

Example 4.1 The $\mathbf{L}^*$-closure system associated with the $\mathbf{L}^*$-closure $\mathbf{L}$-system given in Example 3.1 is exactly the family $\wp = \{\chi_a \mid a \in [\frac{1}{2}, 1]\}$. It is routine to check that this $\mathbf{L}^*$-closure system is v-consistent for $v = \frac{1}{2}$. Moreover, we can readily check that: for any $a \in [0, \frac{1}{2})$, the closure of $A = \chi_a$ is $\chi_{\frac{1}{2}}$; and for any $a \in [\frac{1}{2}, 1]$, the closure of $A = \chi_a$ is χ_1.

5 The Category of $\mathbf{L}^*$-closure L-system Spaces

Let λ_X and λ_Y be $\mathbf{L}^*$-closure **L**-systems on X and Y respectively. A mapping $f : X \longrightarrow Y$ is said to be continuous from (X, λ_X) to (Y, λ_Y) if $\lambda_Y(W) \leq \lambda_X(f^{\leftarrow}(W))$ holds for any $W \in L^Y$, where $f^{\leftarrow}(W) = W \circ f$. The category of $\mathbf{L}^*$-closure **L**-system spaces with continuous mappings being morphisms is denoted as **CLSS**. Given a fixed element $v \in L$, the category of v-consistent $\mathbf{L}^*$-closure **L**-system spaces with continuous mappings being morphisms is denoted as **cCLSS**. Suppose C_X and C_Y are $\mathbf{L}^*$-closure operators on X and Y respectively. A mapping $f : X \longrightarrow Y$ is said to be continuous from (X, C_X) to (Y, C_Y) if $C_X(f^{\leftarrow}(W)) \subseteq f^{\leftarrow}(C_Y(W))$ holds for any $W \in L^Y$. The category of $\mathbf{L}^*$-closure spaces with continuous mappings being morphisms is denoted as **CS**. The category of v-consistent $\mathbf{L}^*$-closure spaces with continuous mappings being morphisms is denoted as **cCS**.

Based on the relationship between $\mathbf{L}^*$-closure systems, $\mathbf{L}^*$-closure **L**-systems and $\mathbf{L}^*$-closure operators obtained in Sects. 3 and 4, a natural question arises that whether we can develop continuous mappings between $\mathbf{L}^*$-closure system spaces. In this respect, we give the following definition.

Definition 5.1 Let $(X, \wp_X)$ and $(Y, \wp_Y)$ be $\mathbf{L}^*$-closure system spaces. A mapping $f : X \longrightarrow Y$ is said to be *continuous* if for any $W \in L^Y$,

$$\bigcap_{A_i \in \wp_X} S(f^{\leftarrow}(W), A_i)^* \to A_i \subseteq \bigcap_{B_i \in \wp_Y} S(W, B_i)^* \to f^{\leftarrow}(B_i).$$

In the following, the category of $\mathbf{L}^*$-closure system spaces with continuous mappings being morphisms is denoted as **CSS**. The category of v-consistent $\mathbf{L}^*$-closure system spaces with continuous mappings being morphisms is denoted as **cCSS**.

It is easy to see that all categories mentioned above are concrete categories. Moreover, categories **cCLSS**, **cCS**, and **cCSS** are subcategories of **CLSS**, **CS**, and **CSS**, respectively. In the sequel of this section, we will only focus on the interrelations of these categories instead of categorical properties of themselves. One can refer to [1] for more content about concrete categories.

Proposition 5.1 Let (X, C_X) and (Y, C_Y) be $\mathbf{L}^*$-closure spaces. If f is a continuous mapping from (X, C_X) to (Y, C_Y), then f is continuous from (X, λ_{C_X}) to (Y, λ_{C_Y}).

Proof Suppose $W \in L^Y$, we need to prove $\lambda_{C_Y}(W) \leq \lambda_{C_X}(f^{\leftarrow}(W))$, i.e., $S(C_Y(W), W) \leq S(C_X(f^{\leftarrow}(W)), f^{\leftarrow}(W))$ which is true. Indeed,

$$\begin{aligned} S(C_Y(W), W) &= \textstyle\bigwedge_{y \in Y} C_Y(W)(y) \to W(y) \\ &\leq \textstyle\bigwedge_{x \in X} C_Y(W)(f(x)) \to W(f(x)) \\ &\leq \textstyle\bigwedge_{x \in X} C_X(f^{\leftarrow}(W))(x) \to f^{\leftarrow}(W)(x) \\ &= S(C_X(f^{\leftarrow}(W)), f^{\leftarrow}(W)). \end{aligned}$$

□

Proposition 5.2 Let (X, λ_X) and (Y, λ_Y) be $\mathbf{L}^*$-closure **L**-system spaces. If f is a continuous mapping from (X, λ_X) to (Y, λ_Y), then f is continuous from (X, C_{λ_X}) to (Y, C_{λ_Y}).

Proof Suppose $W \in L^Y$, we need to prove $C_{\lambda_X}(f^{\leftarrow}(W)) \subseteq f^{\leftarrow}(C_{\lambda_Y}(W))$, which is true. In fact, we have

$$\begin{aligned} C_{\lambda_X}(f^{\leftarrow}(W)) &= \textstyle\bigcap_{A_i \in L^X, \lambda_X(A_i)=1} S(W \circ f, A_i)^* \to A_i \\ &\subseteq \textstyle\bigcap_{B_i \in L^Y, \lambda_Y(B_i)=1} S(W \circ f, B_i \circ f)^* \to (B_i \circ f) \\ &\subseteq \textstyle\bigcap_{B_i \in L^Y, \lambda_Y(B_i)=1} S(W, B_i)^* \to (B_i \circ f) \\ &= (\textstyle\bigcap_{B_i \in L^Y, \lambda_Y(B_i)=1} S(W, B_i)^* \to B_i) \circ f \\ &= f^{\leftarrow}(C_{\lambda_Y}(W)). \end{aligned}$$

□

Proposition 5.3 Let (X, λ_X) and (Y, λ_Y) be $\mathbf{L}^*$-closure **L**-system spaces. If f is a continuous mapping from (X, λ_X) to (Y, λ_Y), then f is continuous from $(X, \wp_{\lambda_X})$ to $(Y, \wp_{\lambda_Y})$.

Proof Suppose $W \in L^Y$, we have

$$\begin{aligned} \textstyle\bigcap_{A_i \in \wp_{\lambda_X}} S(f^{\leftarrow}(W), A_i)^* \to A_i &= \textstyle\bigcap_{A_i \in \wp_{\lambda_X}} S(W \circ f, A_i)^* \to A_i \\ &\subseteq \textstyle\bigcap_{B_i \in \wp_{\lambda_Y}} S(W \circ f, B_i \circ f)^* \\ &\quad \to (B_i \circ f) \\ &\subseteq \textstyle\bigcap_{B_i \in \wp_{\lambda_Y}} S(W, B_i)^* \to (B_i \circ f) \\ &= \textstyle\bigcap_{B_i \in \wp_{\lambda_Y}} S(W, B_i)^* \to f^{\leftarrow}(B_i). \end{aligned}$$

This means that f is continuous from $(X, \wp_{\lambda_X})$ to $(Y, \wp_{\lambda_Y})$. □

Proposition 5.4 Let $(X, \wp_X)$ and $(Y, \wp_Y)$ be $\mathbf{L}^*$-closure system spaces. If f is a continuous mapping from $(X, \wp_X)$ to $(Y, \wp_Y)$, then f is continuous from $(X, \lambda_{\wp_X})$ to $(Y, \lambda_{\wp_Y})$.

Proof Suppose $W \in L^Y$, we need to prove $\lambda_{\wp_Y}(W) \leq \lambda_{\wp_X}(f^{\leftarrow}(W))$. Indeed, we have

$$\begin{aligned} \lambda_{\wp_X}(f^{\leftarrow}(W)) &= S(\textstyle\bigcap_{A_i \in \wp_X} S(W \circ f, A_i)^* \to A_i, W \circ f) \\ &\geq S(\textstyle\bigcap_{B_i \in \wp_Y} S(W, B_i)^* \to B_i \circ f, W \circ f) \\ &= S((\textstyle\bigcap_{B_i \in \wp_Y} S(W, B_i)^* \to B_i) \circ f, W \circ f) \\ &\geq S(\textstyle\bigcap_{B_i \in \wp_Y} S(W, B_i)^* \to B_i, W) \\ &= \lambda_{\wp_Y}(W). \end{aligned}$$

□

From Theorem 3.2, Theorem 4.1 and Propositions 5.1–5.4, we have the following result.

Theorem 5.1 (1) *The categories* **CS**, **CSS** *and* **CLSS** *are isomorphic with each other;* (2) *The categories* **cCS**, **cCSS** *and* **cCLSS** *are isomorphic with each other.*

6 Conclusions

In this paper, we first proposed the notion of (v-consistent) $\mathbf{L}^*$-closure **L**-system where **L** is a complete residuated lattice and * is a truth stresser on **L**. We investigated the relationship between (v-consistent) $\mathbf{L}^*$-closure **L**-systems and (v-consistent) $\mathbf{L}^*$-closure operators. Our results show that a one-to-one correspondence can be established between these two structures. Furthermore, we proposed the notion of v-consistent $\mathbf{L}^*$-closure system and showed that the notion of (v-consistent) $\mathbf{L}^*$-closure **L**-system provides an alternative way to characterize (v-consistent) $\mathbf{L}^*$-closure systems. Finally, we put (v-consistent) $\mathbf{L}^*$-closure system spaces into categories in virtue of the notion of continuous mapping and proved that the categories of (v-consistent) $\mathbf{L}^*$-closure **L**-system spaces, (v-consistent) $\mathbf{L}^*$-closure spaces and (v-consistent) $\mathbf{L}^*$-closure system spaces are isomorphic with each other. Our results verify the capability of our proposed notion of (v-consistent) $\mathbf{L}^*$-closure **L**-system in characterizing fuzzy closure systems existing in the literature.

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