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L-E-Fuzzy Lattices

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Abstract A new definition of a fuzzy lattice (*L-E*-fuzzy lattice) as a particular fuzzy algebraic structure is introduced in the framework of fuzzy equalities and fuzzy identities. The membership values structure is a complete lattice. An *L-E*-fuzzy lattice is defined on a bi-groupoid M, as its fuzzy sub-bi-groupoid μ equipped with a fuzzy equality E, fulfilling fuzzy lattice identities. It is proved that the new notion is a generalization of known lattice-valued structures. Basic properties of the introduced new fuzzy lattices are presented. In particular, it is proved that the quotients of cuts of μ over the corresponding cuts of E are classical lattices. By a suitable example, it is shown how the new introduced structures can be applied.

Keywords Fuzzy lattice · Fuzzy identity · Fuzzy congruence · Fuzzy equality · Complete lattice

Mathematics Subject Classification 08A72 · 06D72 · 03E72

1 Introduction

The topic of this research is fuzzy lattice investigated in the framework of (*a*) lattice-valued structures, (*b*) fuzzy equality and (*c*) fuzzy identities.

(a) Lattice-valued structures in which the unit interval is replaced by a complete lattice date from Goguen's paper [1]. This approach is widely used for dealing with fuzzy algebraic topics (see e.g. [2], then also [3, 4]), and with the fuzzy topology (starting with [5] and many others). In the recent decades, a complete lattice is often replaced by a complete residuated lattice [6]. Our approach uses complete lattice, since it allows cutworthy [7] investigations of main algebraic properties.

(b) A fuzzy equality generalizing the crisp one has been introduced in fuzzy mathematics by Höhle in [8], and then it was used in investigations of fuzzy functions and fuzzy algebraic structures by many authors, in particular by Demirci [9], Bělohlávek and V. Vychodil [10] and others.

(c) Fuzzy identities were introduced in [11], and then developed in [12, 13]. These are lattice-theoretic formulas fuzzifying classical identities.

From the above mentioned aspects, we investigate fuzzy lattices. In [14], an *L*-valued fuzzy lattice was introduced both, as a lattice-valued fuzzy set on a lattice, and as a special *L*-valued fuzzy poset. Two definitions were proved to be equivalent, as in the classical case. Later, there were several further approaches to fuzzy lattices. Some of them were connected to fuzzy formal concept analysis (see [6]); there were also some recent investigations of fuzzy (complete) lattices [15, 16]. In particular, in a series of papers [9, 17, 18], Demirci investigates fuzzy equality, and in this framework fuzzy functions and fuzzy lattices as ordering structures. Our approach proposed in this paper is also based on a fuzzy equality instead of a crisp one; however, we use fuzzy identities, hence introducing a fuzzy lattice as a fuzzy algebra. In this way, our structure is more general than the classical fuzzy

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lattices [14, 19], since it covers the previous definitions. In our case, a fuzzy lattice is defined on a crisp bi-groupoid, and it is equipped with a fuzzy equality which is a fuzzy relation on this fuzzy structure, hence fulfilling particular reflexivity property. In this framework, we introduce fuzzy lattice identities. Namely, an *L*-*E*-fuzzy lattice on a bi-groupoid *M*, is a fuzzy sub-bi-groupoid μ of *M* equipped with a fuzzy equality *E*, fulfilling fuzzy lattice identities. Basic properties of these new fuzzy lattices are presented. In particular, it is proved that the quotient structures obtained by cuts of μ over the corresponding cuts of *E* are classical lattices.

1.1 Organization of the Paper

In the preliminary section, we review algebraic and lattice-theoretic notions, and we recall a fuzzy framework in which we define a fuzzy lattice.

The subsequent section contains the main results of the paper. We start with an algebra *M* with two binary operations (a bi-groupoid) and a fuzzy subalgebra μ of *M*, where the membership values structure is a complete lattice *L*. We also equip μ with an *L*-valued equality *E*, which is compatible with the operations in *M*. Then, using *E* instead of the classical equality, we introduce *L*-valued identities which are fuzzified identities analogous to the ones defining the lattice in the classical algebra. In this way, we obtain the structure (*M*, μ, *E*), which we call an *L*-*E*-fuzzy lattice. We prove that idempotency holds for both operations in the fuzzy framework, and also that the bi-groupoid *M* has to fulfil the same (crisp) property.

We also present a simple suitable example, and in concluding remarks we indicate some topics for future investigations.

2 Preliminaries

2.1 Lattices, Fuzzy Sets

Here, we fuzzify a **lattice** as an algebraic structure, which is known to be an algebra $(M, \wedge, \vee)$ with two binary operations, both commutative, associative and fulfilling absorbtion laws: $x \wedge (x \vee y) = x$, and $x \vee (x \wedge y) = x$. An ordering relation can be defined on *M* by $x \leqslant y$ if $x \wedge y = x$. Equivalently, a lattice is an ordered set $(M, \leqslant)$ in which the greatest lower bound (infimum) and the least upper bound (supremum) exist for every two-element subset $\{x, y\}$. As it is known, infimum and supremum can be considered as binary operations in $(M, \leqslant)$, fulfilling the above mentioned properties of $\wedge$ and $\vee$; conversely $x \wedge y$ and $x \vee y$ are respectively infimum and supremum of x and y in $(M, \wedge, \vee)$, with respect to the corresponding order.

Next, we introduce the membership values structure for fuzzy notions. This is a **complete lattice**, i.e. a structure $(L, \wedge, \vee, \leqslant)$, where $\leqslant$ is an ordering relation on a nonempty set *L*, such that for every subset there is an infimum (meet, greatest lower bound—glb) and a supremum (join, least upper bound—lub). Infimum and supremum of an arbitrary family $\{p_i \mid i \in I\}$ of elements from *L* are denoted by $\bigwedge_{i \in I} p_i$ and $\bigvee_{i \in I} p_i$, respectively. Every complete lattice possesses the smallest element, the bottom, 0, and the greatest element, the top, 1.

Remark 1 In this research, both, the domain and the co-domain of fuzzy structures are algebraic structures with two binary operations, since we introduce fuzzy i.e. lattice-valued lattices. The operations in these two lattices are denoted in the same way. A confusion could not arise, since the context determines the lattice in which the operations are used.

A **fuzzy set** μ on a nonempty set *A* (or a **fuzzy subset** of *A*) is a function $\mu : A \to L$, where *L* is a complete lattice. In the present framework the adjective *fuzzy* has the same meaning as *lattice valued*.

For $p \in L$, a **cut set** or a *p*-**cut** of a fuzzy set $\mu : A \to L$ is a subset μ_p of *A* which is the inverse image of the principal filter in *L*, generated by *p*:

$$\mu_p = \{x \in X \mid \mu(x) \geqslant p\}.$$

A **fuzzy** (binary) **relation** ρ **on** *A* is a fuzzy set on A^2, i.e. it is a mapping $\rho : A^2 \to L$.

Let $\mu : A \to L$ be a fuzzy set on *A* and let $\rho : A^2 \to L$ be a fuzzy relation on *A*. If for all $x, y \in A$, ρ satisfies

$$\rho(x, y) \leqslant \mu(x) \wedge \mu(y), \tag{1}$$

then we say that ρ is a **fuzzy relation on** μ (see e.g. [20]).

Let ρ be a fuzzy relation on a fuzzy set μ of *A*.

$$\rho \text{ is } \mathbf{reflexive\ if}\ \rho(\mathbf{x}, \mathbf{x}) = \mu(\mathbf{x}) \quad \mathbf{for\ every} \quad \mathbf{x} \in \mathbf{A}. \tag{2}$$

Observe that by (1 and 2), for a reflexive relation ρ on μ we have that for all $x, y \in A$

$$\rho(x, y) \leqslant \rho(x, x) \text{ and } \rho(y, x) \leqslant \rho(x, x).$$

$$\rho \text{ is } \textbf{symmetric} \text{ if } \rho(x, y) = \rho(y, x) \text{ for all } x, y \in A; \tag{3}$$

$$\rho \text{ is } \textbf{transitive} \text{ if } \rho(x, z) \wedge \rho(z, y) \leqslant \rho(x, y) \text{ for all } x, y, z \in A. \tag{4}$$

A reflexive, symmetric and transitive relation ρ on μ is a **fuzzy equivalence** on μ.

A fuzzy equivalence relation ρ on μ, fulfilling for all $x, y \in A$, $x \neq y$:

$$\text{if } \rho(x, x) \neq 0, \text{ then } \rho(x, y) < \rho(x, x), \tag{5}$$

is called a **fuzzy equality** relation on a fuzzy set μ.

2.2 Fuzzy Subalgebras, Fuzzy Identities

If $\mathcal{A} = (A, F)$ is an algebra, i.e. a nonempty set equipped with operations, then as it is known, a **fuzzy subalgebra** of $\mathcal{A}$ is any mapping $\mu : A \to L$ which is not constantly equal to 0, and which fulfils the following: For any operation f from F with arity greater than $0, f : A^n \to A, n \in \mathbb{N}$, and for all $a_1, \ldots, a_n \in A$, we have that

$$\bigwedge_{i=1}^{n} \mu(a_i) \leqslant \mu(f(a_1, \ldots, a_n)), \tag{6}$$

and for a nullary operation (constant) $c \in F$, $\mu(c) = 1$.

Next, we formulate a fuzzy version of a known property of term operations in universal algebra (the proof follows by induction on the complexity of terms).

Proposition 1 *Let $\mu : A \to L$ be a fuzzy subalgebra of an algebra $\mathcal{A}$ and let $t(x_1, \ldots, x_n)$ be a term in the language of $\mathcal{A}$. If t^A is the corresponding term-operation on $\mathcal{A}$ and $a_1, \ldots, a_n \in A$, then the following holds*:

$$\bigwedge_{i=1}^{n} \mu(a_i) \leqslant \mu(t^A(a_1, \ldots, a_n)).$$

Let $\mathcal{A} = (A, F)$ be an algebra. A fuzzy relation $\rho : A^2 \to L$ is **compatible** with the operations in F if the following holds: for every n-ary operation $f \in F$ and for all $a_1, \ldots, a_n, \quad b_1, \ldots, b_n \in A$

$$\bigwedge_{i=1}^{n} \rho(a_i, b_i) \leqslant \rho(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)), \tag{7}$$

and

$$\rho(c, c) = 1 \quad \text{for every constant (nullary operation) } c \in F. \tag{8}$$

If ρ is a fuzzy relation on fuzzy subalgebra μ of $\mathcal{A}$, then we say that it is **compatible** on μ if it is compatible with the operations in F, i.e. if (7) and (8) hold.

The following topics are introduced in [21] (see also [11, 13]). A compatible fuzzy equivalence on μ is a **fuzzy congruence** on this fuzzy subalgebra. A **fuzzy equality** on a fuzzy subalgebra μ is a fuzzy congruence on μ, fulfilling (5).

If $u(x_1, \ldots, x_n)$ and $v(x_1, \ldots, x_n)$ are terms in the language of an algebra $\mathcal{A}$, where variables appearing in these terms are among $x_1, \ldots, x_n$, and E is a binary relational symbol, we say that the expression

$$E(u(x_1, \ldots, x_n),\ v(x_1, \ldots, x_n))$$

is a **fuzzy identity**. Then, a fuzzy subalgebra μ of $\mathcal{A}$ **satisfies a fuzzy identity** $E(u, v)$ (or, a fuzzy identity is **valid** on a fuzzy subalgebra μ) **with respect to fuzzy equality** E^{μ} on μ, if the following condition is fulfilled for all $a_1, \ldots, a_n \in A$ and the term operations u^A and v^A on $\mathcal{A}$ corresponding to terms u and v, respectively:

$$\bigwedge_{i=1}^{n} \mu(a_i) \leqslant E^{\mu}(u^A(a_1, \ldots, a_n), v^A(a_1, \ldots, a_n)). \tag{9}$$

In other words, a fuzzy identity $E(u, v)$ is valid on a fuzzy subalgebra μ of $\mathcal{A}$ with respect to E^{μ}, if for any valuation replacing variables with elements from A, inequality (9) holds in lattice L.

The fact that a fuzzy subalgebra μ of an algebra $\mathcal{A}$ fulfils a fuzzy identity $E(u, v)$, does not imply that the crisp identity $u = v$ holds on $\mathcal{A}$. However, the converse does hold and we prove it in the sequel.

Proposition 2 *Let $u = v$ be an identity which holds on an algebra $\mathcal{A}$. If $\mu : A \to L$ is a fuzzy subalgebra on $\mathcal{A}$, and E^{μ} a fuzzy equality on μ, then the fuzzy identity $E(u, v)$ is satisfied on μ with respect to E^{μ}.*

Proof Suppose that $x_1, x_2, \ldots, x_n$ are variables appearing in terms u, v. If $u = v$ holds on $\mathcal{A}$, then by (2) and by Proposition 1, for any $a_1, \ldots, a_n \in A$,

$$E^{\mu}(u^A(a_1, \ldots, a_n), v^A(a_1, \ldots, a_n)) = \mu(u^A(a_1, \ldots, a_n)) \geqslant \mu(a_1) \wedge \ldots \wedge \mu(a_n).$$

3 Results

In order to define a new concept of a fuzzy lattice, we start from an algebra with two binary operations $(M, \wedge, \vee)$ without any additional conditions. This algebra is called a **bi-groupoid**. In this context and in connection with previous concepts, a fuzzy bi-groupoid is a fuzzy set $\mu : M \to L$ satisfying for all $x, y \in M$,

$$\mu(x) \wedge \mu(y) \leqslant \mu(x \wedge y) \quad \text{and} \quad \mu(x) \wedge \mu(y) \leqslant \mu(x \vee y).$$

Let $E : M^2 \to L$ be a fuzzy equality on μ which is also compatible with operations $\wedge$ and $\vee$ in the following sense:

$$E(x, y) \wedge E(z, t) \leqslant E(x \wedge z, y \wedge t) \text{ and } E(x, y) \wedge E(z, t) \leqslant E(x \vee z, y \vee t).$$

Remark 2 In order to simplify notation, we omit the upscript (as in the previous section), i.e. instead of E^{μ}, we write E, for the fuzzy equality on μ.

Then we say that $\mathcal{M} = (M, \mu, E)$ is an ***L-E*-fuzzy lattice** if the following six formulas are satisfied:

$$\mu(x) \wedge \mu(y) \leqslant E(x \wedge y, y \wedge x) \text{ commutative law;} \tag{10}$$

$$\mu(x) \wedge \mu(y) \leqslant E(x \vee y, y \vee x) \text{ commutative law;} \tag{11}$$

$$\mu(x) \wedge \mu(y) \wedge \mu(z) \leqslant E((x \wedge y) \wedge z, x \wedge (y \wedge z))$$
associative law; (12)

$$\mu(x) \wedge \mu(y) \wedge \mu(z) \leqslant E((x \vee y) \vee z, x \vee (y \vee z))$$
associative law; (13)

$\mu(x) \wedge \mu(y) \leqslant E((x \wedge y) \vee x, x)$ absorptive law; (14)

$\mu(x) \wedge \mu(y) \leqslant E((x \vee y) \wedge x, x)$ absorptive law (15)

Clearly, the above formulas are fuzzy identities, as they are defined in Section 2.2.

Lemma 1 If (M, μ, E) is an L-E-**fuzzy lattice**, then the following is satisfied:

$$\mu(x) \wedge \mu(y) \leqslant E((y \wedge x) \vee x, x); \tag{16}$$

$$\mu(x) \wedge \mu(y) \leqslant E((y \vee x) \wedge x, x). \tag{17}$$

Proof By compatibility and (10), we have that

$$\mu(x) \wedge \mu(y) \leqslant E(x \wedge y, y \wedge x) \wedge E(x, x) \leqslant E((x \wedge y) \vee x, (y \wedge x) \vee x).$$

By (14), symmetry and transitivity of E,

$$\mu(x) \wedge \mu(y) \leqslant E((x \wedge y) \vee x, (y \wedge x) \vee x) \wedge E((x \wedge y) \vee x, x) \leqslant E((y \wedge x) \vee x, x).$$

(17) is proved dually.

Proposition 3 *Let* (M, μ, E) *be an* L-E-**fuzzy lattice**, *Then, the idempotent laws* $\mu(x) \leqslant E(x \wedge x, x)$ *and* $\mu(x) \leqslant E(x \vee x, x)$ *are valid.*

Proof By the absorptive law (15) taking $y = x$, and using the fact that E is a reflexive relation on μ, we have that

$$\mu(x) \leqslant E((x \vee x) \wedge x, x) \wedge E(x, x).$$

Now, by the fact that E is a compatible relation, we obtain

$$E(x, x) \wedge E((x \vee x) \wedge x, x) \leqslant E(((x \vee x) \wedge x) \vee x, x \vee x).$$

Hence, we have that

$$\mu(x) \leqslant E(((x \vee x) \wedge x) \vee x, x \vee x).$$

By (16) from Lemma 1, taking $y = x \vee x$, it follows that

$$\mu(x) \leqslant E(((x \vee x) \wedge x) \vee x, x)$$

Further, by symmetry and transitivity of E, we have that

$$\mu(x) \leqslant E(((x \vee x) \wedge x) \vee x, x) \wedge E(((x \vee x) \wedge x) \vee x, x \vee x) \leqslant E(x \vee x, x).$$

Hence, $\mu(x) \leqslant E(x \vee x, x)$, which means that one of the idempotent laws is valid. The validity of the other law is proved analogously.

Next we prove that the idempotency in the fuzzy framework implies that the analogue crisp identity should be satisfied by the bi-groupoid on which an L-E-fuzzy lattice is defined.

Proposition 4 *The idempotent law* $\mu(x) \leqslant E(x \wedge x, x)$ *is valid in an* L-E-*fuzzy lattice* (M, μ, E) *if and only if operation* $\wedge$ *is idempotent in bi-groupoid* $(M, \wedge, \vee)$, *and analogously* $\mu(x) \leqslant E(x \vee x, x)$ *is valid if and only if* $\vee$ *is idempotent.*

Proof Suppose that the idempotent law $\mu(x) \leqslant E(x \wedge x, x)$ is valid. Since $E(x \wedge x, x) \leqslant E(x, x) = \mu(x)$, we have that $E(x \wedge x, x) = \mu(x)$. By the definition of the fuzzy equality, we have that $x \wedge x = x$. To prove the converse, if $x \wedge x = x$, then $E(x \wedge x, x) = \mu(x)$, and $\mu(x) \leqslant E(x \wedge x, x)$ is valid. Another part is proved analogously.

By the known definition (see [14]), a **fuzzy lattice** of a lattice $(M, \wedge, \vee)$ is a mapping $\mu : M \to L$, fulfilling

$$\mu(x) \wedge \mu(y) \leqslant \mu(x \wedge y) \text{ and } \mu(x) \wedge \mu(y) \leqslant \mu(x \vee y).$$

In the following, we prove that a fuzzy lattice of a lattice M, which is equipped with a fuzzy equality E is an L-E-fuzzy lattice as well. In other words, classical lattice properties imply fuzzy identities (10),..., (15).

Proposition 5 *If* $(M, \wedge, \vee)$ *is a lattice*, E *is a compatible fuzzy equality, and* $\mu : M \to L$ *is a fuzzy lattice, then* (M, μ, E) *is an* L-E-*fuzzy lattice.*

Proof Since $(M, \wedge, \vee)$ is a lattice, we have that all the lattice identities holds. Since, $x \wedge y = y \wedge x$, and since E is a fuzzy equality, we have that $E(x \wedge y, y \wedge x) = \mu(x \wedge y)$.

Since μ is a fuzzy lattice, we have that $\mu(x \wedge y) \geqslant \mu(x) \wedge \mu(y)$, hence, $\mu(x) \wedge \mu(y) \leqslant E(x \wedge y, y \wedge x)$. The proof for other five laws is analogous. □

The converse of Proposition 5 is not true, since not all L-E-fuzzy lattices are L-valued fuzzy lattices in the sense of [14]. This is illustrated by the following example presenting an L-E-fuzzy lattice, which is not a fuzzy lattice in the classical sense.

Example 1 Let $M = \{a, b, c\}$, let $(M, \wedge, \vee)$ be a three-element bi-groupoid given in Tables 1 and 2, and let L be a lattice in Figure 1.

Let $\mu : M \to L$ be a fuzzy bi-groupoid of M, given by:

$$\mu(x) = \begin{pmatrix} a & b & c \\ p & q & r \end{pmatrix}.$$

A fuzzy equality on μ is given in Table 3.

Table 1 Bi-groupoid M

$\wedge$	a	b	c
a	a	a	a
b	a	b	b
c	a	c	c

Table 2 Bi-groupoid M

$\vee$	a	b	c
a	a	a	a
b	a	b	c
c	a	b	c

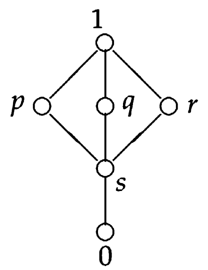

Fig. 1 Lattice L

Table 3 Fuzzy equality

E	a	b	c
a	p	s	s
b	s	q	s
c	s	s	r

Now, we can see that the commutative laws for both operations in the bi-groupoid are not satisfied, neither are the absorptive laws, since e.g. $b \wedge (a \vee b) = a \neq b$ Tables 1 and 2.

However, it is easy to check that (M, μ, E) is an L-E-fuzzy lattice.

Hence, (M, μ, E) is not a fuzzy lattice by the classical definition, but it is an L-E fuzzy lattice.

The following theorem presents a connection of an L-E-fuzzy lattice (M, μ, E) and classical lattices obtained by cuts of the bi-groupoid M.

Theorem 1 *Let $\mu : M \to L$ be a fuzzy set on a bi-groupoid $(M, \wedge, \vee)$ and let E be a fuzzy equality on μ. Then, (M, μ, E) is an L-E-fuzzy lattice if and only if for every $p \in L$, cut μ_p is a subalgebra (sub-bi-groupoid) of M, the cut relation E_p is a congruence on μ_p and a quotient structure μ_p/E_p is a lattice.*

Proof Let $\mu : M \to L$ be a fuzzy set on a bi-groupoid M and let E be a fuzzy equality on μ, such that (M, μ, E) is an L-E-fuzzy lattice. Since E is compatible with the operations on bi-groupoid μ, we have that $\mu(x) \wedge \mu(y) \leqslant \mu(x \wedge y)$ and $\mu(x) \wedge \mu(y) \leqslant \mu(x \vee y)$. Now, we prove that the cut μ_p is closed under operations $\vee$ and $\wedge$. Let $x, y \in \mu_p$, i.e. let $\mu(x) \geqslant p$ and $\mu(y) \geqslant p$. We have that $p \leqslant \mu(x) \wedge \mu(y) \leqslant \mu(x \wedge y)$ and $p \leqslant \mu(x) \wedge \mu(y) \leqslant \mu(x \vee y)$, hence $x \wedge y \in \mu_p$ and $x \vee y \in \mu_p$.

Therefore, μ_p is a subalgebra of $(M, \wedge, \vee)$.

Now, we prove that E_p is a congruence relation on μ_p. First, we prove that E_p is an equivalence relation on μ_p. Reflexivity: $(x, x) \in E_p$ if and only if $E(x, x) = \mu(x) \geqslant p$, if and only if $x \in \mu_p$. Symmetry and transitivity are proved straightforwardly. Finally, we prove that E_p is compatible with the operations in μ_p. We take $x, y, u, v \in \mu_p$, and suppose that $(x, y), (u, v) \in E_p$. Then $E(x, y) \geqslant p$ and $E(u, v) \geqslant p$. Now, we have that

$$E(x \wedge u, y \wedge v) \geqslant E(x, y) \wedge E(u, v) \geqslant p,$$

i.e. $(x \wedge u, y \wedge v) \in E_p$. Analogously we prove that E_p is compatible with operation $\vee$.

Now, we consider the quotient set of μ_p over E_p, μ_p/E_p. We denote operations on congruence classes in the same way as operations on $(M, \wedge, \vee)$, namely with $\wedge$ and $\vee$ (these operations can be always distinguished from the context). We define these operations in a natural way (by class representatives) and it is easy to prove that they are well defined. Now, $(\mu_p/E_p, \wedge, \vee)$ is a bi-groupoid. We prove that this bi-groupoid is a lattice. Let $[x]_{E_p}, [y]_{E_p}, [z]_{E_p}$ be three elements from μ_p/E_p. We have to prove six lattice axioms.

For $x, y \in \mu_p$, we have

$$E(x \wedge y, y \wedge x) \geqslant \mu(x) \wedge \mu(y) \geqslant p,$$

hence $(x \wedge y, y \wedge x) \in E_p$. Therefore, for $x, y \in \mu_p$, we have that

$$[x]_{E_p} \wedge [y]_{E_p} = [x \wedge y]_{E_p} = [y \wedge x]_{E_p} = [y]_{E_p} \wedge [x]_{E_p},$$

so we proved that the operation $\wedge$ in M_p/E_p is commutative.

Similarly, we prove all six lattice axioms, and M_p/E_p is a lattice.

Conversely, suppose that for every p from L, the cut μ_p is a sub-bi-groupoid of M, the cut relation E_p is a congruence on μ_p and a quotient structure μ_p/E_p is a lattice. Now, we would like to prove that (M, μ, E) is an L-E-fuzzy lattice, i.e. that six axioms for the fuzzy lattice hold. We will prove one of them, and others are proved analogously. We prove the absorptive law,

$$\mu(x) \wedge \mu(y) \leqslant E((x \vee y) \wedge x, x).$$

Let $\mu(x) \wedge \mu(y) = p$. Then $x, y \in \mu_p$. Since μ_p/E_p is a lattice by assumption, we have $([x]_{E_p} \vee [y]_{E_p}) \wedge [x]_{E_p} = [x]_{E_p}$, hence $[(x \vee y) \wedge x]_{E_p} = [x]_{E_p}$, and hence

$$E((x \vee y) \wedge x, x) \geqslant p = \mu(x) \wedge \mu(y).$$

Next, we deal with the relational aspect of L-E-fuzzy lattices.

As it is known, in a lattice as an algebra, it is possible to define the order in the following way: $x \leqslant y$ if and only if $x \wedge y = x$.

What is the analogue situation with L-E-fuzzy lattices? Since we use a fuzzy equality instead of the crisp one, in the sequel we introduce the notion of an L-E- fuzzy order. First, we deal with antisymmetry in the framework of a fuzzy equality.

Let $E : A^2 \to L$ be a fuzzy equality over a nonempty set A. We say that a fuzzy relation $\rho : A^2 \to L$ is E-**antisymmetric**, if the following holds:

$$\rho(x, y) \wedge \rho(y, x) \leqslant E(x, y), \quad \text{for all} \quad x, y \in A. \tag{18}$$

Let $\alpha : A \to L$ be a fuzzy set on a nonempty domain A, and $E : A^2 \to L$ a fuzzy equality over α. We say that (A, α, E) is an L-E-**fuzzy structure**. The set A in an L-E-fuzzy structure may be equipped with operations; an example of such a fuzzy structure is an L-E-fuzzy lattice.

We say that a fuzzy relation $\rho : A^2 \to L$ on $\alpha : A \to L$ is an **L-E-fuzzy order** on L-E-fuzzy structure $(A,\ \alpha,\ E)$, if it is reflexive in the sense of (2), E-antisymmetric, and transitive in the sense of (4).

For the main theorem of this part we need the following lemma.

Lemma 2 *Let (M, μ, E) be an L-E-fuzzy lattice, and $\rho : M^2 \to L$ a fuzzy relation on μ defined by $\rho(x, y) := \mu(x) \wedge \mu(y) \wedge E(x \wedge y, x)$. Let $p \in L$. Then, for $x, y \in \mu_p$ and $[x]_{E_p}, [y]_{E_p} \in \mu_p/E_p$,*

$[x]_{E_p} \leqslant [y]_{E_p}$ if and only if $\rho(x, y) \geqslant p$.

Proof By Theorem 1, μ_p/E_p is a lattice, hence $[x]_{E_p} \leqslant [y]_{E_p}$ if and only if $[x]_{E_p} \wedge [y]_{E_p} = [x]_{E_p}$

if and only if $[x \wedge y]_{E_p} = [x]_{E_p}$ if and only if $(x \wedge y, x) \in E_p$

if and only if $E(x \wedge y, x) \geqslant p$ if and only if $\rho(x, y) \geqslant p$, since

$\mu(x) \geqslant p$ and $\mu(y) \geqslant p$. □

Theorem 2 *If $\mathcal{M} = (M, \mu, E)$ is an L-E-fuzzy lattice, then a fuzzy relation $\rho : M^2 \to L$, such that $\rho(x, y) := \mu(x) \wedge \mu(y) \wedge E(x \wedge y, x)$ is an L-E-fuzzy order on $\mathcal{M}$.*

Proof By the definition, ρ is a fuzzy relation on μ.

To prove the reflexivity, we note that $\rho(x, x) = \mu(x) \wedge E(x \wedge x, x) = \mu(x) \wedge E(x, x) = \mu(x)$.

We prove that ρ is E-antisymmetric, i.e. that the formula $\rho(x, y) \wedge \rho(y, x) \leqslant E(x, y)$ holds for all $x, y \in M$. Indeed, suppose $\rho(x, y) \wedge \rho(y, x) = p \in L$. Then, by the definition of ρ, $\mu(x) \geqslant p$ and $\mu(y) \geqslant p$, hence $x, y \in \mu_p$. Further by $\rho(x, y) \geqslant p$ and $\rho(y, x) \geqslant p$,

by Lemma 2, we have also $[x]_{E_p} \leqslant [y]_{E_p}$, and $[y]_{E_p} \leqslant [x]_{E_p}$. Hence $[x]_{E_p} = [y]_{E_p}$, i.e. $(x, y) \in E_p$, or equivalently, $E(x, y) \geqslant p$. Therefore, $\rho(x, y) \wedge \rho(y, x) \leqslant E(x, y)$.

In order to prove transitivity, we prove that all the cuts of relation ρ are transitive as ordinary relations on the corresponding cuts of μ. Let $p \in L$. Now, suppose that $(x, y) \in \rho_p$ and $(y, z) \in \rho_p$. By the definition of ρ, this means that $x, y, z \in \mu_p$, $(x \wedge y, x) \in E_p$ and $(y \wedge z, y) \in E_p$. By $(z, z) \in E_p$ and $(x \wedge y, x) \in E_p$, we have that $((x \wedge y) \wedge z, x \wedge z) \in E_p$. On the other hand, from $(x, x) \in E_p$ and $(y \wedge z, y) \in E_p$, we have that $(x \wedge (y \wedge z), x \wedge y) \in E_p$. By transitivity of E_p and $(x \wedge (y \wedge z), (x \wedge y) \wedge z) \in E_p$ (which follows by Theorem 1), we have that $(x \wedge y, x \wedge z) \in E_p$. By $(x \wedge y, x) \in E_p$, and transitivity again, we have that $(x \wedge z, x) \in E_p$, hence $(x, z) \in \rho_p$. Therefore, all the cuts are transitive relations.

Now, we have to prove that from transitivity of all the cuts, it follows that also fuzzy relation ρ is transitive.

Let $\rho(x, y) \wedge \rho(y, z) = p$. By $\rho(x, y) \wedge \rho(y, z) = p$, it follows that $\rho(x, y) \geqslant p$ and $\rho(y, z) \geqslant p$., i.e. $(x, y) \in \rho_p$ and $(y, z) \in \rho_p$. Since ρ_p is a transitive relation, we have that $(x, z) \in \rho_p$, hence $\rho(x, z) \geqslant p = \rho(x, y) \wedge \rho(y, z)$. □

4 Application

As it is known, lattices appear in many mathematical branches (algebraic and ordering structures, vector spaces, topology, etc.) and consequently in applications of these fields. Let us mention the whole digital informatics which uses Boolean lattices, then Quantum Structures which are based on algebras derived from lattices [22], and also Formal Concept Analysis [23], having an impressive application in social sciences and related fields. In all mentioned disciplines also fuzzy logic has an important role [6] e.g., fuzzy relational equations are widely applied in connection to fuzzy control.

Fuzzy lattices have already been introduced and investigated [14, 16–19]. Why do we introduce this new concept? Our main reason is a fuzzy approach to identities. By this concept, our L-E-fuzzy lattice is defined on a weaker structure (idempotent bi-groupoid), hence it could be applied to many situations arising in real life problems. Moreover, by Proposition 5, the classical fuzzy lattice [14, 19] is a special case of our L-E-fuzzy lattice.

As an example of an application, consider a set $A = \{a, b, c, d, e, f, g, 1, h\}$ of, say, members of some unit in a company, or enterprize. By ten different tasks, $T_1, \ldots, T_{10}$ that are performed by some of people from A, each of members of A is connected to other members of the unit, as follows:

T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}
a, *h*	*a*, *b*, *h*	*e*, *h*	*c*, *e*, *h*	*d*, *e*, *h*	*c*, *d*, *e*, *f*, *h*	*c*, *d*, *e*, *g*, *h*	*c*, *d*, *e*, *f*, *g*, *h*	*A*	*h*

Clearly, for some task a single member of A is responsible, some are performed by several members; these groups form a collection of subsets of A, completed by the empty set; we denote them by $x_0, x_1, \ldots, x_{10}$, as follows: $x_0 = \emptyset$, $x_1 = \{a,h\}$, $x_2 = \{a,b,h\}$, $x_3 = \{e,h\}$, $x_4 = \{c,e,h\}$, $x_5 = \{d,e,h\}$, $x_6 = \{c,d,e,f,h\}, x_7 = \{c,d,e,g,1,h\}$, $x_8 = \{c,d,e,f,g,1,h\}, x_9 = \{a,b,c,d,e,f,g,1.h\}, x_{10} = \{h\}$.

For analysing inter-relations among members and groups, or among groups and several tasks, it is convenient to organize these groups structurally. Then, using set-intersection and set-union together with the lexicographic order where intersection and union are not closed within the sets of performers of tasks, we obtain the following tables of two binary operations $\wedge$ and $\vee$ over the set $\mathcal{F} = \{x_0, x_1, \ldots, x_{10}\}$:

$\wedge$	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
x_0	x_0	x_0	x_0	x_0	x_0	x_0	x_0	x_0	x_0	x_0	x_0
x_1	x_0	x_1	x_1	x_{10}	x_{10}	x_{10}	x_{10}	x_{10}	x_{10}	x_1	x_{10}
x_2	x_0	x_1	x_2	x_0	x_{10}	x_{10}	x_{10}	x_{10}	x_{10}	x_2	x_{10}
x_3	x_0	x_{10}	x_{10}	x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_{10}
x_4	x_0	x_{10}	x_{10}	x_3	x_4	x_3	x_4	x_4	x_4	x_4	x_{10}
x_5	x_0	x_{10}	x_{10}	x_3	x_3	x_5	x_5	x_5	x_5	x_5	x_{10}
x_6	x_0	x_{10}	x_{10}	x_3	x_4	x_5	x_6	x_4	x_6	x_6	x_{10}
x_7	x_0	x_{10}	x_{10}	x_3	x_4	x_5	x_4	x_7	x_7	x_7	x_{10}
x_8	x_0	x_0	x_0	x_3	x_4	x_5	x_6	x_7	x_8	x_8	x_{10}
x_9	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
x_{10}	x_0	x_{10}	x_{10}	x_{10}	x_{10}	x_{10}	x_{10}	x_{10}	x_{10}	x_{10}	x_{10}

$\vee$	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
x_0	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
x_1	x_1	x_1	x_2	x_9	x_9	x_9	x_9	x_9	x_9	x_9	x_1
x_2	x_2	x_2	x_2	x_9	x_9	x_9	x_9	x_9	x_9	x_9	x_2
x_3	x_3	x_9	x_9	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_3
x_4	x_4	x_9	x_9	x_4	x_4	x_6	x_6	x_7	x_8	x_9	x_4
x_5	x_5	x_9	x_9	x_5	x_6	x_5	x_6	x_7	x_8	x_9	x_5
x_6	x_6	x_9	x_9	x_6	x_6	x_6	x_6	x_8	x_8	x_9	x_6
x_7	x_7	x_9	x_9	x_7	x_7	x_7	x_8	x_7	x_8	x_9	x_7
x_8	x_8	x_9	x_9	x_8	x_8	x_8	x_8	x_8	x_8	x_9	x_8
x_9	x_9	x_9	x_9	x_9	x_9	x_9	x_9	x_9	x_9	x_9	x_9
x_{10}	x_{10}	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}

Specific various priorities concerning tasks (like e.g. relevance to the company, importance, etc.) are ordered as a lattice $L = \{1, p, q, r_1, r_2, r_3, r_4, r_5, r_6, s, t, v, w, z, 0\}$ in Figure 2 (where an element in the lattice is greater i.e. higher in the diagram, if it corresponds to a higher priority).

According to the assigned priorities, the collection $\mathcal{F}$ of groups can be considered as a fuzzy set:

$$\mu = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} \\ p & q & q & r_1 & r_2 & r_3 & r_4 & r_5 & r_6 & s & p \end{pmatrix}.$$

Finally, inter-relations among groups, according to the same priorities of assigned tasks, are presented as a fuzzy equality, given in the table.

E	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
x_0	p	0	0	0	0	0	0	0	0	0	z
x_1	0	q	t	0	0	0	0	0	0	0	0
x_2	0	t	q	0	0	0	0	0	0	0	0
x_3	0	0	0	r_1	w	w	w	z	z	0	0
x_4	0	0	0	w	r_2	w	w	z	z	0	0
x_5	0	0	0	w	w	r_3	w	z	z	0	0
x_6	0	0	0	w	w	w	r_4	z	z	0	0
x_7	0	0	0	z	z	z	z	r_5	v	0	0
x_8	0	0	0	z	z	z	z	v	r_6	0	0
x_9	0	0	0	0	0	0	0	0	0	s	0
x_{10}	z	0	0	0	0	0	0	0	0	0	p

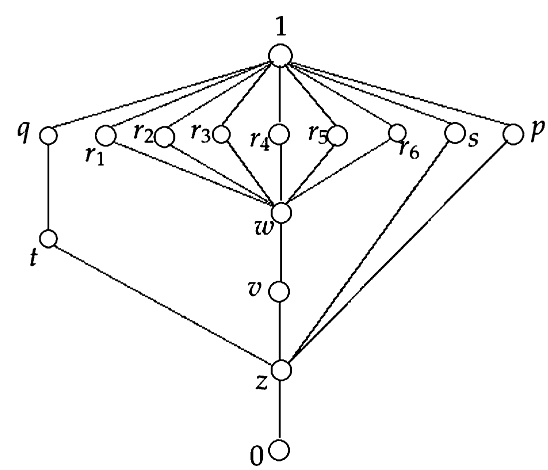

Fig. 2 Lattice L

Now, the collection $\mathcal{F}$ is under the operations $\wedge$ and $\vee$ an idempotent bi-groupoid, μ is a fuzzy sub(bi)groupoid of $\mathcal{F}$, and E is an L-fuzzy equality on $\mathcal{F}$. Moreover, formulas (axioms) (10)–(15) are satisfied. These can be checked straightforwardly. Therefore, $(\mathcal{F}, \mu, E)$ is an L-E-fuzzy lattice, where the co-domain lattice L is given in Figure 2.

Applying Theorem 1, it is possible to get answers to queries about groups and their connections with respect to criteria given in the lattice L. E.g. criterion z organize groups into a four-element Boolean lattice, consisting of sets $\{x_0, x_{10}\}$, $\{x_1, x_2\}$, $\{x_3, \ldots, x_8\}$ and $\{x_9\}$; this structure is obtained as a cut-structure μ_z/E_z. Similarly, μ_q/E_q is a two-element lattice $\{x_1\}, \{x_2\}$, etc.

5 Conclusion

Using fuzzy identities and a fuzzy equality instead of the corresponding classical notions, we have introduced *L-E*-fuzzy lattices and investigated their basic properties. There are many important lattice-theoretic properties that could still be investigated in this framework (e.g. distributive lattices, concept lattices). In addition, applications of this concept could be developed in several fields (economy, management, informatics in wider sense, etc.), as indicated by our example.

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