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Non-linear Programming Approach to Solve Bi-matrix Games with Payoffs Represented by I-fuzzy Numbers

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Abstract The aim of this paper is to develop a new methodology for solving bi-matrix games with payoffs of Atanassov's intuitionistic fuzzy (I-fuzzy) numbers. In this methodology, we define the concepts of I-fuzzy numbers, the value-index and ambiguity-index, and develop a difference-index based ranking method. Hereby the parameterized non-linear programming models are derived from a pair of auxiliary I-fuzzy mathematical programming models, which are used to determine solutions of bi-matrix games with payoffs represented by I-fuzzy numbers. Validity and applicability of the models and method proposed in this paper are illustrated with a practical example.

Keywords Atanassov's intuitionistic fuzzy (I-fuzzy) number · Fuzzy set · Fuzzy game theory · Mathematical programming · Fuzzy optimization

1 Introduction

Usually, the bi-matrix games assume that the payoffs are represented with crisp values, which indicate that the payoffs are exactly known by players. However, players often are not able to evaluate exactly the payoffs due to imprecision or lack of available information in real game situations [1, 2]. In order to make bi-matrix game theory more applicable to real competitive decision problems, the fuzzy set has been used to describe imprecise and uncertain information appearing in bi-matrix games [3, 4]. The common feature of these fuzzy games is that fuzziness is described by the fuzzy set with the membership degree. The non-membership degree is just automatically equal to the complement of the membership degree to 1. In reality, however, players often do not express the non-membership degree of a given element as the complement of the membership degree to 1. In other words, players may have some hesitation degree. The fuzzy set has no means to incorporate the hesitation degree.

The hesitation degree seems to be suitably expressed with the intuitionistic fuzzy (I-fuzzy) set [5]. The I-fuzzy set is of use in matrix game modelling due to the fact that in some situations players may describe their negative feelings, i.e. the degrees of dissatisfaction on the outcomes of the games. On the other hand, players could only approximately estimate their payoffs with some hesitation degrees. But it is possible that players are not so sure about them. Thus, the I-fuzzy set may provide players a natural tool for modelling such uncertain situations. As far as we know, however, there exists less investigation on matrix games using the I-fuzzy set. Dimitrov [6] used the I-fuzzy set to discuss some market structure problems. His work only involved the simple representation of game problems using the I-fuzzy set. Using the similar idea of the fuzzy goals, Nayak and Pal [7] and Li [8] constructed the linear programming models to solve bi-matrix games with goals expressed by I-fuzzy sets. Seikh and Pal [9] applied triangular I-fuzzy numbers to bi-matrix games. In this paper, we introduce the concepts of general I-fuzzy numbers and the value-index and ambiguity-index. Furthermore, we develop a difference-index-based ranking

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method and hereby derive a pair of parameterized non-linear programming models from the I-fuzzy mathematical programming models of bi-matrix games with payoffs represented by I-fuzzy numbers, which are called I-fuzzy number bi-matrix games for short in the sequent.

The rest of this paper is organized as follows. Section 2 gives the concepts of I-fuzzy numbers and arithmetic operations of their cut sets. Section 3 defines the value-index and ambiguity-index of an I-fuzzy number, and proposes a difference-index-based ranking method and discusses its properties. Section 4 formulates I-fuzzy number bi-matrix games and develops a parameterized non-linear programming method. An example of the strategy choice problem is given in Sect. 5. Section 6 concludes this paper.

2 Concepts of I-fuzzy numbers and cut sets

2.1 Concepts of I-fuzzy numbers

I-fuzzy numbers play an important role in optimization and decision-making problems [10, 11]. Inspired by the concept of the fuzzy number [12], an I-fuzzy number $\tilde{A}$ is defined as a special I-fuzzy set on the real number set R [13, 14], whose membership function $\mu_{\tilde{A}}: \mathrm{R} \to [0,1]$ and non-membership function $\upsilon_{\tilde{A}}: \mathrm{R} \to [0,1]$ should satisfy the four conditions (1)–(4) as follows: (1) there exist at least two real numbers $x'_0 \in \mathrm{R}$ and $x''_0 \in \mathrm{R}$ such that $\mu_{\tilde{A}}(x'_0) = 1$ and $\upsilon_{\tilde{A}}(x''_0) = 0$; (2) $\mu_{\tilde{A}}$ is quasi concave and upper semi-continuous on R; (3) $\upsilon_{\tilde{A}}$ is quasi convex and lower semi-continuous on R; and (4) the support sets $\{x|\mu_{\tilde{A}}(x) > 0, x \in \mathrm{R}\}$ and $\{x|\upsilon_{\tilde{A}}(x) < 1, x \in \mathrm{R}\}$ are bounded.

From the above definition of an I-fuzzy number, we can easily construct an I-fuzzy number $\tilde{A} = \langle(\underline{a}_1, a_{1l}, a_{1r}, \bar{a}_1), f_l, f_r; (\underline{a}_2, a_{2l}, a_{2r}, \bar{a}_2), g_l, g_r\rangle$, whose membership and non-membership functions are given as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & (x < \underline{a}_1) \\ f_l(x) & (\underline{a}_1 \le x < a_{1l}) \\ 1 & (a_{1l} \le x \le a_{1r}) \\ f_r(x) & (a_{1r} < x \le \bar{a}_1) \\ 0 & (x > \bar{a}_1) \end{cases} \quad (1)$$

and

$$\upsilon_{\tilde{A}}(x) = \begin{cases} 1 & (x < \underline{a}_2) \\ g_l(x) & (\underline{a}_2 \le x < a_{2l}) \\ 0 & (a_{2l} \le x \le a_{2r}) \\ g_r(x) & (a_{2r} < x \le \bar{a}_2) \\ 1 & (x > \bar{a}_2) \end{cases}, \quad (2)$$

respectively, depicted as in Fig. 1, where $\underline{a}_2 \le \underline{a}_1 \le a_{2l} \le a_{1l} \le a_{1r} \le a_{2r} \le \bar{a}_1 \le \bar{a}_2$; $f_l: [\underline{a}_1, a_{1l}) \to [0,1]$ and $g_r: (a_{2r}, \bar{a}_2] \to [0,1]$ are non-decreasing and piecewise upper semi-continuous functions, which satisfy the conditions: $f_l(\underline{a}_1) = 0$, $f_l(a_{1l}) = 1$, $g_r(a_{2r}) = 0$ and $g_r(\bar{a}_2) = 1$; $f_r: (a_{1r}, \bar{a}_1] \to [0,1]$ and $g_l: [\underline{a}_2, a_{2l}) \to [0,1]$ are non-increasing and piecewise lower semi-continuous functions, which fulfil the conditions: $f_r(a_{1r}) = 1$, $f_r(\bar{a}_1) = 0$, $g_l(\underline{a}_2) = 1$ and $g_l(a_{2l}) = 0$. $[a_{1l}, a_{1r}]$, $\underline{a}_1$ and $\bar{a}_1$ are called the mean interval and the lower and upper limits of the I-fuzzy number $\tilde{A}$ for the membership function, respectively. $[a_{2l}, a_{2r}]$, $\underline{a}_2$ and $\bar{a}_2$ are called the mean interval and the lower and upper limits of the I-fuzzy number $\tilde{A}$ for the non-membership function, respectively.

Let $\pi_{\tilde{A}}(x) = 1 - \mu_{\tilde{A}}(x) - \upsilon_{\tilde{A}}(x)$, which is called the I-fuzzy index of an element x in the I-fuzzy number $\tilde{A}$. It is the degree of indeterminacy membership of the element x to $\tilde{A}$.

If $\underline{a}_2 \ge 0$, then the I-fuzzy number $\tilde{A}$ is called non-negative, denoted by $\tilde{A} \ge 0$. Conversely, if $\bar{a}_2 \le 0$, then $\tilde{A}$ is called non-positive, denoted by $\tilde{A} \le 0$. Further, $\tilde{A}$ is called positive if $\underline{a}_2 \ge 0$ and $\bar{a}_2 > 0$, denoted by $\tilde{A} > 0$. Likewise, $\tilde{A}$ is called negative if $\underline{a}_2 < 0$ and $\bar{a}_2 \le 0$, denoted by $\tilde{A} < 0$. Particularly, if

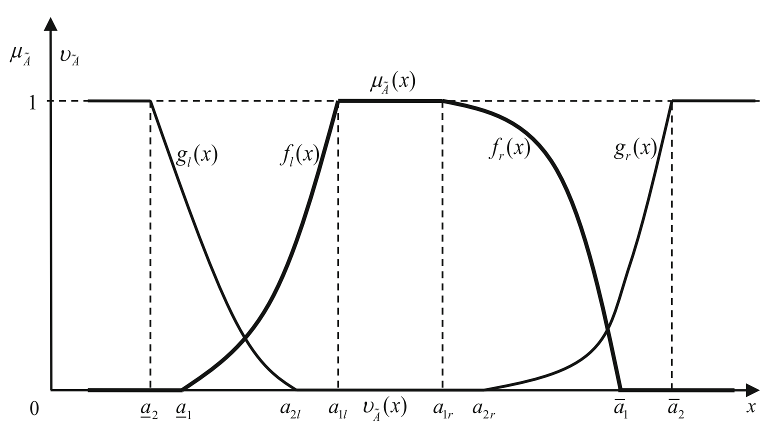

Fig. 1 An I-fuzzy number $\tilde{A}$

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & (x < \underline{a}_1) \\ (x - \underline{a}_1)/(a_{1l} - \underline{a}_1) & (\underline{a}_1 \leq x < a_{1l}) \\ 1 & (a_{1l} \leq x \leq a_{1r}) \\ (\bar{a}_1 - x)/(\bar{a}_1 - a_{1r}) & (a_{1r} < x \leq \bar{a}_1) \\ 0 & (x > \bar{a}_1) \end{cases},$$

and

$$\upsilon_{\tilde{A}}(x) = \begin{cases} 1 & (x < \underline{a}_2) \\ (a_{2l} - x)/(a_{2l} - \underline{a}_2) & (\underline{a}_2 \leq x < a_{2l}) \\ 0 & (a_{2l} \leq x \leq a_{2r}) \\ (x - a_{2r})/(\bar{a}_2 - a_{2r}) & (a_{2r} < x \leq \bar{a}_2) \\ 1 & (x > \bar{a}_2), \end{cases}$$

then the I-fuzzy number $\tilde{A}$ is reduced to a trapezoidal I-fuzzy number, denoted by $\tilde{A} = \langle(\underline{a}_1, a_{1l}, a_{1r}, \bar{a}_1); (\underline{a}_2, a_{2l}, a_{2r}, \bar{a}_2)\rangle$. Further, if $a_{2l} = a_{2r}$ (hereby $a_{1l} = a_{1r}$), i.e. $\underline{a}_2 \leq \underline{a}_1 \leq a \leq \bar{a}_1 \leq \bar{a}_2$, where $a = a_{2l} = a_{2r} = a_{1l} = a_{1r}$, then the trapezoidal I-fuzzy number $\tilde{A}$ is reduced to the triangular I-fuzzy number, denoted by $\tilde{A} = <(\underline{a}_1, a, \bar{a}_1); (\underline{a}_2, a, \bar{a}_2)>$.

Obviously, if $\underline{a}_2 = \underline{a}_1$, $a_{2l} = a_{1l}$, $a_{2r} = a_{1r}$ and $\bar{a}_2 = \bar{a}_1$, then $\mu_{\tilde{A}}(x) + \upsilon_{\tilde{A}}(x) = 1$ for all $x \in \mathrm{R}$. In this case, the trapezoidal I-fuzzy number $\tilde{A}$ degenerates to $\tilde{A} = (\underline{a}_1, a_{1l}, a_{1r}, \bar{a}_1)$, which is just the trapezoidal fuzzy number. Therefore, the trapezoidal I-fuzzy numbers are a generalization of the trapezoidal fuzzy numbers. Thus, the I-fuzzy numbers are also a generalization of the fuzzy numbers [15].

Generally, arithmetic operations of I-fuzzy numbers can be derived from the extension principle of I-fuzzy sets [14]. In the following, we discuss the addition and scalar multiplication of I-fuzzy numbers based on the concept of cut sets.

2.2 Cut sets of I-fuzzy Numbers and Arithmetic Operations

For any $\alpha \in [0, 1]$, a α-cut set of an I-fuzzy number $\tilde{A}$ can be expressed as a crisp subset of R, denoted by $\tilde{A}_\alpha = \{x | \mu_{\tilde{A}}(x) \geq \alpha, x \in \mathrm{R}\}$. It easily follows from the definition of the I-fuzzy number that $\tilde{A}_\alpha$ is a closed interval, denoted by $\tilde{A}_\alpha = [L_\alpha(\tilde{A}), R_\alpha(\tilde{A})]$. It is directly derived from Eq. (1) that

$$[L_\alpha(\tilde{A}), R_\alpha(\tilde{A})] = [f_l^{-1}(\alpha), f_r^{-1}(\alpha)], \quad (3)$$

where f_l^{-1} and f_r^{-1} are the inverse functions of f_l and f_r, respectively.

Likewise, for any $\beta \in [0, 1]$, a β-cut set of an I-fuzzy number $\tilde{A}$ can be expressed as a crisp subset of R, denoted by $\tilde{A}_\beta = \{x | \upsilon_{\tilde{A}}(x) \leq \beta, x \in \mathrm{R}\}$. Obviously, $\tilde{A}_\beta$ is a closed interval, denoted by $\tilde{A}_\beta = [L_\beta(\tilde{A}), R_\beta(\tilde{A})]$. It is directly derived from Eq. (2) that

$$[L_\beta(\tilde{A}), R_\beta(\tilde{A})] = [g_l^{-1}(\beta), g_r^{-1}(\beta)], \quad (4)$$

where g_l^{-1} and g_r^{-1} are the inverse functions of g_l and g_r, respectively.

According to the arithmetic operations of intervals [16] and the above concept of cut sets of I-fuzzy numbers, we can define the addition and scalar multiplication of I-fuzzy numbers.

Specifically, for any I-fuzzy number $\tilde{A} = \langle(\underline{a}_1, a_{1l}, a_{1r}, \bar{a}_1), f_l, f_r; (\underline{a}_2, a_{2l}, a_{2r}, \bar{a}_2), g_l, g_r\rangle$ and $\tilde{A}' = \langle(\underline{a}'_1, a'_{1l}, a'_{1r}, \bar{a}'_1), f'_l, f'_r; (\underline{a}'_2, a'_{2l}, a'_{2r}, \bar{a}'_2), g'_l, g'_r\rangle$, the sum of $\tilde{A}$ and $\tilde{A}'$ is defined as an I-fuzzy number $\tilde{A} + \tilde{A}'$, whose α-cut set and β-cut set are, respectively, given as follows:

$$\begin{aligned}(\tilde{A} + \tilde{A}')_\alpha &= \tilde{A}_\alpha + \tilde{A}'_\alpha = \left[L_\alpha(\tilde{A}) + L_\alpha(\tilde{A}'), R_\alpha(\tilde{A}) + R_\alpha(\tilde{A}')\right] \\ &= \left[f_l^{-1}(\alpha) + f_l'^{-1}(\alpha), f_r^{-1}(\alpha) + f_l'^{-1}(\alpha)\right],\end{aligned} \quad (5)$$

and

$$\begin{aligned}(\tilde{A} + \tilde{A}')_\beta &= \tilde{A}_\beta + \tilde{A}'_\beta = \left[L_\beta(\tilde{A}) + L_\beta(\tilde{A}'), R_\beta(\tilde{A}) + R_\beta(\tilde{A}')\right] \\ &= \left[g_l^{-1}(\beta) + g_l'^{-1}(\beta), g_r^{-1}(\beta) + g_r'^{-1}(\beta)\right].\end{aligned} \quad (6)$$

The scalar multiplication of $\tilde{A}$ and any real number ρ is defined as an I-fuzzy number $\rho\tilde{A}$, whose α-cut set and β-cut set are respectively given as follows:

$$\begin{aligned}(\rho\tilde{A})_\alpha = \rho\tilde{A}_\alpha &= \begin{cases} [\rho L_\alpha(\tilde{A}), \rho R_\alpha(\tilde{A})] \\ [\rho R_\alpha(\tilde{A}), \rho L_\alpha(\tilde{A})] \end{cases} \\ &= \begin{cases} [\rho f_l^{-1}(\alpha), \rho f_r^{-1}(\alpha)] & (\rho \geq 0) \\ [\rho f_r^{-1}(\alpha), \rho f_l^{-1}(\alpha)] & (\rho < 0) \end{cases},\end{aligned} \quad (7)$$

and

$$\begin{aligned}(\rho\tilde{A})_\beta &= \rho\tilde{A}_\beta \\ &= \begin{cases} [\rho L_\beta(\tilde{A}), \rho R_\beta(\tilde{A})] \\ [\rho R_\beta(\tilde{A}), \rho L_\beta(\tilde{A})] \end{cases} = \begin{cases} [\rho g_l^{-1}(\beta), \rho g_r^{-1}(\beta)] & (\rho \geq 0) \\ [\rho g_r^{-1}(\beta), \rho g_l^{-1}(\beta)] & (\rho < 0) \end{cases}.\end{aligned} \quad (8)$$

3 The Difference-Index-Based Ranking Method of I-fuzzy Numbers and Properties

3.1 The Value-Index and Ambiguity-Index of an I-fuzzy Number

For any I-fuzzy number $\tilde{A}$, its values of the membership and non-membership functions are defined as follows:

$$V_\mu(\tilde{A}) = \int_0^1 \left[(L_\alpha(\tilde{A}) + R_\alpha(\tilde{A}))/2\right] f(\alpha) \mathrm{d}\alpha, \quad (9)$$

and

$$V_\upsilon(\tilde{A}) = \int_0^1 \left[(L_\beta(\tilde{A}) + R_\beta(\tilde{A}))/2\right] g(\beta) \mathrm{d}\beta, \quad (10)$$

respectively, where $f(\alpha)$ is a non-decreasing function on the interval [0,1], which should satisfy the conditions: $f(0) = 0$ and $f(1) = 1$; $g(\beta)$ is a non-increasing function on the interval [0,1], which should fulfil the conditions: $g(0) = 0$ and $g(1) = 0$.

$f(\alpha)$ and $g(\beta)$ may reflect the attitude of players (or decision makers) towards uncertainty, which can be considered as weighting functions. $f(\alpha)$ gives different weights to elements at the α-cut sets of the I-fuzzy number $\tilde{A}$ so that the contribution of the lower α-cut sets can be lessened due to the fact that these cut sets arising from $\mu_{\tilde{A}}(x)$ have a considerable amount of uncertainty. Therefore, $V_\mu(\tilde{A})$ synthetically reflects the membership degrees of $\tilde{A}$. Likewise, $g(\beta)$ can lessen the contribution of the higher β-cut sets of $\tilde{A}$ since these cut sets arising from $\upsilon_{\tilde{A}}(x)$ have a considerable amount of uncertainty. $V_\upsilon(\tilde{A})$ synthetically reflects the non-membership degrees of $\tilde{A}$. And $f(\alpha)$ and $g(\beta)$ are specifically chosen according to need in real situations. Jafarian and Rezvani [17] gave more explanations and specific forms of the functions $f(\alpha)$ and $g(\beta)$, respectively. For example, $f(\alpha) = \alpha$ and $g(\beta) = 1 - \beta$ are simpler forms of such functions.

It is easy to see from Eqs. (9) and (10) that $V_\mu(\tilde{A}) \geq 0$ and $V_\upsilon(\tilde{A}) \geq 0$ for any I-fuzzy number $\tilde{A} \geq 0$.

Likewise, the ambiguities of the membership and non-membership functions for any I-fuzzy number $\tilde{A}$ are defined as follows:

$$W_\mu(\tilde{A}) = \int_0^1 \left(R_\alpha(\tilde{A}) - L_\alpha(\tilde{A})\right) f(\alpha)\mathrm{d}\alpha, \tag{11}$$

and

$$W_\upsilon(\tilde{A}) = \int_0^1 \left(R_\beta(\tilde{A}) - L_\beta(\tilde{A})\right) g(\beta)\mathrm{d}\beta, \tag{12}$$

respectively. Clearly, $R_\alpha(\tilde{A}) - L_\alpha(\tilde{A})$ and $R_\beta(\tilde{A}) - L_\beta(\tilde{A})$ are just the lengths of the intervals $\tilde{A}_\alpha$ and $\tilde{A}_\beta$. $W_\mu(\tilde{A})$ and $W_\upsilon(\tilde{A})$ basically measure how much there is uncertainty in $\tilde{A}$.

It is easy to see from Eqs. (11) and (12) that $W_\mu(\tilde{A}) \geq 0$ and $W_\upsilon(\tilde{A}) \geq 0$ for any I-fuzzy number $\tilde{A}$. Further, we can draw the following conclusion, which is summarized as in Theorem 1.

Theorem 1 *Assume that $\tilde{A}$ and $\tilde{A}'$ are any I-fuzzy numbers. Then, for any real number $\rho \in \mathrm{R}$, the following equalities are always valid*:

$$V_\mu(\rho\tilde{A} + \tilde{A}') = \rho V_\mu(\tilde{A}) + V_\mu(\tilde{A}'),$$

$$V_\upsilon(\rho\tilde{A} + \tilde{A}') = \rho V_\upsilon(\tilde{A}) + V_\upsilon(\tilde{A}'),$$

$$W_\mu(\rho\tilde{A} + \tilde{A}') = \rho W_\mu(\tilde{A}) + W_\mu(\tilde{A}'),$$

and

$$W_\upsilon(\rho\tilde{A} + \tilde{A}') = \rho W_\upsilon(\tilde{A}) + W_\upsilon(\tilde{A}').$$

Proof See Appendix 1. □

Theorem 1 shows that the values and ambiguities of any I-fuzzy number are linear.

The value-index and ambiguity-index of any I-fuzzy number $\tilde{A}$ are defined as follows:

$$V_\lambda(\tilde{A}) = \lambda V_\upsilon(\tilde{A}) + (1 - \lambda)V_\mu(\tilde{A}), \tag{13}$$

and

$$W_\lambda(\tilde{A}) = \lambda W_\mu(\tilde{A}) + (1 - \lambda)W_\upsilon(\tilde{A}), \tag{14}$$

respectively, where $\lambda \in [0, 1]$ is the weight which represents the attitude or preference information of players (or decision makers). $\lambda \in [0, 1/2)$ shows that the player prefers to uncertainty or negative feeling; $\lambda \in (1/2, 1]$ shows that the player prefers to certainty or positive feeling; $\lambda = 1/2$ shows that the player is indifferent between positive feeling and negative feeling. Therefore, the value-index and ambiguity-index may reflect players' attitude or preference to the I-fuzzy number.

Theorem 2 *Assume that $\tilde{A}$ and $\tilde{A}'$ are any I-fuzzy number. Then, for any real number $\rho \in \mathrm{R}$, the following equalities are always valid*:

$$V_\lambda(\rho\tilde{A} + \tilde{A}') = \rho V_\lambda(\tilde{A}) + V_\lambda(\tilde{A}'),$$

and

$$W_\lambda(\rho\tilde{A} + \tilde{A}') = \rho W_\lambda(\tilde{A}) + W_\lambda(\tilde{A}').$$

Proof See Appendix 2. □

Theorem 2 shows that the value-index and ambiguity-index of any I-fuzzy number are linear.

3.2 The Difference-Index of an I-fuzzy Number and the Ranking Method

From Eqs. (13) and (14), obviously, the larger the value-index and the smaller the ambiguity-index hereby the bigger the I-fuzzy number. Therefore, a ranking-index of any I-fuzzy number is defined as follows:

$$D_\lambda(\tilde{a}) = V_\lambda(\tilde{a}) - W_\lambda(\tilde{a}), \tag{15}$$

which is usually called the difference-index of the I-fuzzy number $\tilde{A}$ for short.

Theorem 3 Assume that $\tilde{A}$ and $\tilde{A}'$ are any I-fuzzy number. Then, for any real number $\rho \in \mathrm{R}$, the following equality is always valid:

$D_\lambda(\rho\tilde{A} + \tilde{A}') = \rho D_\lambda(\tilde{A}) + D_\lambda(\tilde{A}')$.

Proof See Appendix 2. □

Theorem 3 shows that the difference-index of any I-fuzzy number is linear. Further, it can be easily seen from Eq. (15) that the larger the difference-index the bigger the I-fuzzy number. Thus, the difference-index-based ranking method is proposed as follows.

Definition 1 Assume that $\lambda \in [0, 1]$ is any real number. For any I-fuzzy numbers $\tilde{A}$ and $\tilde{A}'$, we stipulate as follows:

(1) $D_\lambda(\tilde{A}) > D_\lambda(\tilde{A}')$ if and only if $\tilde{A}$ is larger than $\tilde{A}'$, denoted by $\tilde{A} >_{\text{IF}} \tilde{A}'$;

(2) $D_\lambda(\tilde{A}) > D_\lambda(\tilde{A}')$ if and only if $\tilde{A}$ is equal to $\tilde{A}'$, denoted by $\tilde{A} =_{\text{IF}} \tilde{A}'$; and

(3) $\tilde{A} \geq_{\text{IF}} \tilde{A}'$ if and only if $\tilde{A} >_{\text{IF}} \tilde{A}'$ or $\tilde{A} =_{\text{IF}} \tilde{A}'$.

The symbol "$>_{\text{IF}}$" is an I-fuzzy version of the order relation "$>$" on the real line and has the linguistic interpretation "essentially larger than". Similarly, "$=_{\text{IF}}$" and "$\geq_{\text{IF}}$" are I-fuzzy versions of "$=$" and "$\geq$" on the real line and have the linguistic interpretations "essentially being equal to" and "essentially larger than or being equal to", respectively. Analogously, we can define the order relations "$<_{\text{IF}}$" and "$\leq_{\text{IF}}$".

The above ranking method has some useful properties, which are summarized as in Theorem 4.

Theorem 4 *The difference-index-based ranking method of I-fuzzy numbers has the five properties as follows*:

(P1) For any I-fuzzy number $\tilde{A}$, then $\tilde{A} \geq_{\text{IF}} \tilde{A}$ is always valid;

(P2) For any I-fuzzy numbers $\tilde{A}$ and $\tilde{A}'$, if $\tilde{A} \geq_{\text{IF}} \tilde{A}'$ and $\tilde{A}' \geq_{\text{IF}} \tilde{A}$, then $\tilde{A} =_{\text{IF}} \tilde{A}'$;

(P3) For I-fuzzy numbers $\tilde{A}$, $\tilde{A}'$ and $\tilde{A}''$, if $\tilde{A} \geq_{\text{IF}} \tilde{A}'$ and $\tilde{A}' \geq_{\text{IF}} \tilde{A}''$, then $\tilde{A} \geq_{\text{IF}} \tilde{A}''$;

(P4) Assume that F_1 and F_2 are arbitrary finite subsets of I-fuzzy numbers. For any I-fuzzy number $\tilde{A} \in F_1 \cap F_2$ and $\tilde{A}' \in F_1 \cap F_2$, then $\tilde{A} >_{\text{IF}} \tilde{A}'$ on F_1 if and only if $\tilde{A} >_{\text{IF}} \tilde{A}'$ on F_2;

(P5) For any I-fuzzy numbers $\tilde{A}$ and $\tilde{A}'$, if $\tilde{A} \geq_{\text{IF}} \tilde{A}'$, then $\tilde{A} + \tilde{A}'' \geq_{\text{IF}} \tilde{A}' + \tilde{A}''$ for any I-fuzzy number $\tilde{A}''$;

(P5′) For any I-fuzzy numbers $\tilde{A}$ and $\tilde{A}'$, if $\tilde{A} >_{\text{IF}} \tilde{A}'$, then $\tilde{A} + \tilde{A}'' >_{\text{IF}} \tilde{A}' + \tilde{A}''$ for any I-fuzzy number $\tilde{A}''$.

Proof See Appendix 3. □

Remark 1 Wang and Kerre [18] proposed seven axioms A_1–A_7, which serve as the reasonable properties to figure out the rationality of a ranking method for the ordering of fuzzy quantities. The above properties (P1)–(P6) (or (P6′)) correspond to the axioms A1–A6 (or A6′), respectively. Unfortunately, it is very difficult to prove whether the last properties corresponding to the axioms A7 are valid.

It can be easily seen from the properties (P1)–(P3) of Theorem 4 that the difference-index-based ranking method of I-fuzzy numbers is a total order. Therefore, the above ranking method is different from those [19, 20].

4 Parameterized Non-linear Programming Models for I-fuzzy Number Bi-matrix Games

4.1 Bi-matrix Games and Non-linear Programming Models

The sets of pure strategies for players I and II are denoted by $S_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ and $S_2 = \{\beta_1, \beta_2, \ldots, \beta_n\}$; their payoff matrices are expressed with $\mathbf{A} = (a_{ij})_{m\times n}$ and $\boldsymbol{B} = (b_{ij})_{m\times n}$; their mixed strategy vectors are denoted by $\mathbf{y} = (y_1, y_2, \ldots, y_m)^T$ and $\mathbf{z} = (z_1, z_2, \ldots, z_n)^T$, respectively, where y_i $(i = 1, 2, \ldots, m)$ and z_j $(j = 1, 2, \ldots, n)$ are probabilities in which I and II choose their pure strategies $\alpha_i \in S_1$ $(i = 1, 2, \ldots, m)$ and $\beta_j \in S_2$ $(j = 1, 2, \ldots, n)$, respectively; the symbol "T" is the transpose of a vector/matrix. Their sets of mixed strategies are denoted by $Y = \{\mathbf{y} | \sum_{i=1}^m y_i = 1, y_i \geq 0\ (i = 1, 2, \ldots, m)\}$ and $Z = \{\mathbf{z} | \sum_{j=1}^n z_j = 1, z_j \geq 0\ (j = 1, 2, \ldots, n)\}$. Thus, a two-person non-zero-sum finite game is simply called the bi-matrix game $(\boldsymbol{A}, \boldsymbol{B})$ in which both players want to maximize his/her own payoffs. When I chooses any mixed strategy $\mathbf{y} \in Y$ and II chooses any mixed strategy $\mathbf{z} \in Z$, the expected payoffs of I and II can be computed as $E_1(\boldsymbol{y}, \boldsymbol{z}) = \boldsymbol{y}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{z} = \sum_{i=1}^m \sum_{j=1}^n y_i a_{ij} z_j$ and $E_2(\boldsymbol{y}, \boldsymbol{z}) = \boldsymbol{y}^{\mathrm{T}}\boldsymbol{B}\boldsymbol{z} = \sum_{i=1}^m \sum_{j=1}^n y_i b_{ij} z_j$, respectively.

Definition 2 If there is a pair $(\boldsymbol{y}^*, \boldsymbol{z}^*) \in Y \times Z$ so that $\boldsymbol{y}^T\boldsymbol{A}\boldsymbol{z}^* \leq \boldsymbol{y}^{*T}\boldsymbol{A}\boldsymbol{z}^*$ for any $\mathbf{y} \in Y$ and $\boldsymbol{y}^{*T}\boldsymbol{B}\boldsymbol{z} \leq \boldsymbol{y}^{*T}\boldsymbol{B}\boldsymbol{z}^*$ for any $z \in Z$, then $(\boldsymbol{y}^*, \boldsymbol{z}^*)$ is called a Nash equilibrium point of the bi-matrix game $(\boldsymbol{A}, \boldsymbol{B})$, $\boldsymbol{y}^*$ and $\boldsymbol{z}^*$ are called Nash equilibrium strategies of players I and II, $u^* = \boldsymbol{y}^{*T}\boldsymbol{A}\boldsymbol{z}^*$ and $v^* = \boldsymbol{y}^{*T}\boldsymbol{B}\boldsymbol{z}^*$ are called Nash equilibrium values of I and II, respectively. And $(\boldsymbol{y}^{*T}, \boldsymbol{z}^{*T}, u^*, v^*)$ is called a Nash equilibrium solution of $(\boldsymbol{A}, \boldsymbol{B})$.

The following theorem guarantees the existence of Nash equilibrium solutions of any bi-matrix game.

Theorem 5 *Any bi-matrix game* $(\boldsymbol{A}, \boldsymbol{B})$ *has at least one Nash equilibrium solution.*

A Nash equilibrium solution of any bi-matrix game $(\boldsymbol{A}, \boldsymbol{B})$ can be obtained by solving the non-linear programming model stated as the following Theorem 6 [21].

Theorem 6 *Let* $(\boldsymbol{A},\boldsymbol{B})$ *be any bi-matrix game.* $(\boldsymbol{y}^{*T},\boldsymbol{z}^{*T},u^*,v^*)$ *is a Nash equilibrium solution of the bi-matrix game* $(\boldsymbol{A},\boldsymbol{B})$ *if and only if it is a solution of the mathematical programming model, which is shown as follows*:

$$\max\{\boldsymbol{y}^T(\boldsymbol{A}+\boldsymbol{B})z-u-v\}$$
$$s.t.\begin{cases}\boldsymbol{Az}\le u\boldsymbol{e}^m\\ B^T\boldsymbol{y}\le v\boldsymbol{e}^n\\ \boldsymbol{y}^T\boldsymbol{e}^m=1\\ \boldsymbol{z}^T\boldsymbol{e}^n=1\\ \boldsymbol{y}\ge 0,\boldsymbol{z}\ge 0\end{cases}. \tag{16}$$

Furthermore, if $(\boldsymbol{y}^{*T},\boldsymbol{z}^{*T},u^*,v^*)$ is a solution of the above mathematical programming model, then $u^*=\boldsymbol{y}^{*T}\boldsymbol{Az}^*$, $v^*=\boldsymbol{y}^{*T}\boldsymbol{Bz}^*$ and $\boldsymbol{y}^{*T}(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{z}^*-u^*-v^*=0$.

4.2 Models and Method for I-fuzzy Number Bi-matrix Games

Let us consider an I-fuzzy number bi-matrix game, where sets of pure strategies S_1 and S_2 and sets of mixed strategies Y and Z for players I and II are defined as the above sections. If player I chooses any pure strategy $\alpha_i\in S_1\quad(i=1,2,\ldots,m)$ and player II chooses any pure strategy $\beta_j\in S_2\quad(j=1,2,\ldots,n)$, then at the situation (α_i,β_j) players I and II gain payoffs, which are expressed as I-fuzzy numbers

$$\tilde{A}_{ij}(\alpha_i,\beta_j)=\{\langle(\alpha_i,\beta_j);(\underline{a}_{1ij},a_{1lij},a_{1rij},\bar{a}_{1ij}),\ f_{ijl},f_{ijr};(\underline{a}_{2ij},a_{2lij},a_{2rij},\bar{a}_{2ij}),g_{ijl},g_{ijr}\rangle\}\quad(i=1,2,\ldots,m;\ j=1,2,\ldots,n),$$

where

$$\underline{a}_{2ij}\le\underline{a}_{1ij}\le a_{2ijl}\le a_{1ijl}\le a_{1ijr}\le a_{2ijr}\le\bar{a}_{1ij}\le\bar{a}_{2ij}$$

and

$$\tilde{B}_{ij}(\alpha_i,\beta_j)=\{\langle(\alpha_i,\beta_j);(\underline{b}_{1ij},b_{1lij},b_{1rij},\bar{b}_{1ij}),\ f_{ijl},f_{ijr};(\underline{b}_{2ij},b_{2lij},b_{2rij},\bar{b}_{2ij}),g_{ijl},g_{ijr}\rangle\}\quad(i=1,2,\cdots,m;\quad j=1,2,\cdots,n),$$

where $\underline{b}_{2ij}\le\underline{b}_{1ij}\le b_{2ijl}\le b_{1ijl}\le b_{1ijr}\le b_{2ijr}\le\bar{b}_{1ij}\le\bar{b}_{2ij}$. Thus, the payoff matrices of players I and II are expressed as $\tilde{\boldsymbol{A}}=(\tilde{A}_{ij}(\alpha_i,\beta_j))_{m\times n}$ and $\tilde{\boldsymbol{B}}=(\tilde{B}_{ij}(\alpha_i,\beta_j))_{m\times n}$, respectively. In the sequel, the above I-fuzzy number bi-matrix game is denoted by $(\tilde{\boldsymbol{A}},\tilde{\boldsymbol{B}})$ for short.

If players I and II, respectively, choose mixed strategies $\mathbf{y}\in Y$ and $\mathbf{z}\in Z$, then the expected payoff of player I is $\tilde{E}_1(\mathbf{y},z)=\boldsymbol{y}^{\mathrm{T}}\tilde{\mathbf{A}}\boldsymbol{z}=\sum_{i=1}^m\sum_{j=1}^n y_i\tilde{A}_{ij}z_j$, whose α-cut set and β-cut set can be, respectively, computed as follows:

$$(\tilde{E}(\boldsymbol{y},z))_\alpha=\left[\sum_{i=1}^m\sum_{j=1}^n y_iL_\alpha(\tilde{A}_{ij})z_j,\sum_{i=1}^m\sum_{j=1}^n y_iR_\alpha(\tilde{A}_{ij})z_j\right]=\left[\sum_{i=1}^m\sum_{j=1}^n f_{ijl}^{-1}(\alpha)y_iz_j,\sum_{i=1}^m\sum_{j=1}^n y_if_{ijr}^{-1}(\alpha)z_j\right],$$

and

$$(\tilde{E}(\boldsymbol{y},z))_\beta=\left[\sum_{i=1}^m\sum_{j=1}^n y_iL_\beta(\tilde{A}_{ij})z_j,\sum_{i=1}^m\sum_{j=1}^n y_iR_\beta(\tilde{A}_{ij})z_j\right]=\left[\sum_{i=1}^m\sum_{j=1}^n y_ig_{ijl}^{-1}(\beta)z_j,\sum_{i=1}^m\sum_{j=1}^n y_ig_{ijr}^{-1}(\beta)z_j\right],$$

where $\alpha\in[0,1]$ and $\beta\in[0,1]$.

According to the operations of I-fuzzy numbers, the expected payoff $\tilde{E}_1(\mathbf{y},z)$ of player I is an I-fuzzy number, which can be calculated as follows:

$$\tilde{E}_1(\mathbf{y},z)=\left\{\langle\sum_{i=1}^n\sum_{j=1}^n y_i(\underline{a}_{1ij},a_{1lij},a_{1rij},\bar{a}_{1ij})z_j,\min\{f_{ijl},f_{ijr}\};\ \sum_{i=1}^n\sum_{j=1}^n y_i(\underline{a}_{2ij},a_{2lij},a_{2rij},\bar{a}_{2ij})z_j,\max\{g_{ijl},g_{ijr}\}\rangle\right\}.$$

Similarly, the expected payoff of player II is $\tilde{E}_2(\mathbf{y},z)=\boldsymbol{y}^{\mathrm{T}}\tilde{\boldsymbol{B}}\boldsymbol{z}$, which can be calculated as follows:

$$\tilde{E}_2(\mathbf{y},z)=\left\{\langle\sum_{i=1}^n\sum_{j=1}^n y_i(\underline{b}_{1ij},b_{1lij},b_{1rij},\bar{b}_{1ij})z_j,\min\{f_{ijl},f_{ijr}\};\ \sum_{i=1}^n\sum_{j=1}^n y_i(\underline{b}_{2ij},b_{2lij},b_{2rij},\bar{b}_{2ij})z_j,\max\{g_{ijl},g_{ijr}\}\rangle\right\}.$$

Definition 3 Assume that there is a pair $(\boldsymbol{y}^*,\boldsymbol{z}^*)\in Y\times Z$. If any $\mathbf{y}\in Y$ and $z\in Z$ satisfy $\boldsymbol{y}^T\tilde{\boldsymbol{A}}\boldsymbol{z}^*\le_{IF}\boldsymbol{y}^{*T}\tilde{\boldsymbol{A}}\boldsymbol{z}^*$ and $\boldsymbol{y}^{*T}\tilde{\boldsymbol{B}}\boldsymbol{z}\le_{IF}\boldsymbol{y}^{*T}\tilde{\boldsymbol{B}}\boldsymbol{z}^*$, then $(\boldsymbol{y}^*,\boldsymbol{z}^*)$ is called a Nash equilibrium point of the I-fuzzy number bi-matrix game $(\tilde{\boldsymbol{A}},\tilde{\boldsymbol{B}})$, $\boldsymbol{y}^*$ and $\boldsymbol{z}^*$ are called Nash equilibrium strategies of players I and II, $\tilde{u}^*=\boldsymbol{y}^{*\mathrm{T}}\tilde{\boldsymbol{A}}\boldsymbol{z}^*$ and $\tilde{v}^*=\boldsymbol{y}^{*\mathrm{T}}\tilde{\boldsymbol{B}}\boldsymbol{z}^*$ are called Nash equilibrium values of I and II, respectively. $(\boldsymbol{y}^*,\boldsymbol{z}^*,\tilde{u}^*,\tilde{v}^*)$ is called a Nash equilibrium solution of $(\tilde{\boldsymbol{A}},\tilde{\boldsymbol{B}})$.

Stated as earlier, however, player I's expected payoff $\boldsymbol{y}^T\tilde{\mathbf{A}}\boldsymbol{z}$ and player II's expected payoff $\boldsymbol{y}^T\tilde{\boldsymbol{B}}\boldsymbol{z}$ are I-fuzzy numbers. Therefore, there are no commonly used concepts of solutions of the bi-matrix games. Furthermore, it is not easy to compute the membership degrees and the non-membership degrees of players' expected payoffs. As a result, solving Nash equilibrium solutions of I-fuzzy number bi-matrix games are very difficult. In the sequel, we use the ranking function D_λ to develop a new method for solving $(\tilde{\boldsymbol{A}},\tilde{\boldsymbol{B}})$.

Using Eq. (15), we can transform $\tilde{A}$ and $\tilde{B}$ into the payoff matrices as follows:

$$\tilde{A}_{\lambda_1} = D_{\lambda 1}((\tilde{A}_{ij}(\alpha_i, \beta_j))_{m\times n}) = (D_{\lambda 1}(\tilde{A}_{ij}(\alpha_i, \beta_j)))_{m\times n} \quad (17)$$

$$\tilde{B}_{\lambda_2} = D_{\lambda_2}((\tilde{B}_{ij}(\alpha_i, \beta_j))_{m\times n}) = (D_{\lambda_2}(\tilde{B}_{ij}(\alpha_i, \beta_j)))_{m\times n}, \quad (18)$$

where $\lambda_1 \in [0,1]$, $\lambda_2 \in [0,1]$, $D_{\lambda_1}(\tilde{A}_{ij}(\alpha_i, \beta_j)) = V_{\tilde{A}_{ij}(\alpha_i,\beta_j)} - A_{\tilde{A}_{ij}(\alpha_i,\beta_j)}$ and $D_{\lambda_2}(\tilde{B}_{ij}(\alpha_i, \beta_j)) = V_{\tilde{B}_{ij}(\alpha_i, \beta_j)} - A_{\tilde{B}_{ij}(\alpha_i,\beta_j)}$ $(i = 1,2,\ldots,m; \quad j = 1,2,\ldots,n)$.

According to the above usage and notations, the above parametric bi-matrix game can be simply denoted by $(\tilde{A}_{\lambda_1}, \tilde{B}_{\lambda_2})$, where the pure (or mixed) strategy sets of players I and II are S_1 and S_2 (or Y and Z) defined as the above. Then, the I-fuzzy number bi-matrix game $(\tilde{A}, \tilde{B})$ is transformed into the parametric bi-matrix game $(\tilde{A}_{\lambda_1}, \tilde{B}_{\lambda_2})$. Hereby, according to Definitions 1–3 and Theorem 3, we can give the definition of satisfying Nash equilibrium solutions of $(\tilde{A}_{\lambda_1}, \tilde{B}_{\lambda_2})$ as follows.

Definition 4 For given parameters $\lambda_1 \in [0,1]$ and $\lambda_2 \in [0,1]$, if there is a pair $(\mathbf{y}^*, \mathbf{z}^*) \in Y \times Z$ so that any $\mathbf{y} \in Y$ and $\mathbf{z} \in Z$ satisfy the following conditions: $\mathbf{y}^T\tilde{A}_{\lambda_1}\mathbf{z}^* \le \mathbf{y}^{*T}\tilde{A}_{\lambda_1}\mathbf{z}^*$ and $\mathbf{y}^{*T}\tilde{B}_{\lambda_2}\mathbf{z} \le \mathbf{y}^{*T}\tilde{B}_{\lambda_2}\mathbf{z}^*$, then $(\mathbf{y}^*, \mathbf{z}^*)$ is called a satisfying Nash equilibrium point of the I-fuzzy number bi-matrix game $(\tilde{\mathbf{A}}_{\lambda_1}, \tilde{B}_{\lambda_2})$, $\mathbf{y}^*$ and $\mathbf{z}^*$ are called satisfying Nash equilibrium strategies of players I and II, $u^*(\lambda_1) = \mathbf{y}^{*T}\tilde{A}_{\lambda_1}\mathbf{z}^*$ and $v^*(\lambda_2) = \mathbf{y}^{*T}\tilde{B}_{\lambda_2}\mathbf{z}^*$ are called satisfying equilibrium values of I and II, respectively. $(\mathbf{y}^*, \mathbf{z}^*, u^*(\lambda_1), v^*(\lambda_2))$ is called a satisfying Nash equilibrium solution of the I-fuzzy number bi-matrix game $(\tilde{A}_{\lambda_1}, \tilde{B}_{\lambda_2})$

Thus, for given parameters $\lambda_1 \in [0,1]$ and $\lambda_2 \in [0,1]$, according to Theorem 4, the parametric bi-matrix game $(\tilde{A}_{\lambda_1}, \tilde{B}_{\lambda_2})$ has at least one Nash equilibrium solution. Namely, the I-fuzzy number bi-matrix game $(\tilde{A}_{\lambda_1}, \tilde{B}_{\lambda_2})$ has at least one satisfying Nash equilibrium solution, which can be obtained through solving the following parametric non-linear programming model according to Theorem 6:

$$\max\left\{\sum_{j=1}^{n}\sum_{i=1}^{m} y_i\left[D_{\lambda_1}(\tilde{A}_{ij}(\alpha_i, \beta_j)) + D_{\lambda_2}(\tilde{B}_{ij}(\alpha_i, \beta_j))\right]z_j - u(\lambda_1) - v(\lambda_2)\right\}$$
$$s.t.\begin{cases} \sum_{j=1}^{n}[D_{\lambda_1}(\tilde{A}_{ij}(\alpha_i, \beta_j))]z_j \le u(\lambda_1)\ (i = 1,2,\ldots,m) \\ \sum_{i=1}^{m}[D_{\lambda_2}(\tilde{B}_{ij}(\alpha_i, \beta_j))]y_i \le v(\lambda_2)\ (j = 1,2,\ldots,n) \\ y_1 + y_2 + \cdots + y_m = 1 \\ z_1 + z_2 + \cdots + z_n = 1 \\ v(\lambda_2) \ge 0, u(\lambda_1) \ge 0 \\ y_i \ge 0\ (i = 1,2,\ldots,m),\ z_j \ge 0\ (j = 1,2,\ldots,n), \end{cases} \quad (19)$$

where $y_i (i = 1,2,\ldots,m)$, $z_j (j = 1,2,\ldots,n)$, $u(\lambda_1)$ and $v(\lambda_2)$ are decision variables.

According to Theorem 5, if $(\mathbf{y}^*, \mathbf{z}^*, u^*(\lambda_1), v^*(\lambda_2))$ is a solution of Eq. (19), then we have

$$u^*(\lambda_1) = \mathbf{y}^{*\mathrm{T}}\tilde{A}_{\lambda_1}\mathbf{z}^* = \sum_{j=1}^{n}\sum_{i=1}^{m}\left[V_{\lambda 1}(\tilde{A}_{ij}(\alpha_i, \beta_j)) - W_{\lambda 1}(\tilde{A}_{ij}(\alpha_i, \beta_j))\right]y_i^* z_j^*,$$

$$v^*(\lambda_2) = y^{*T}\tilde{B}_{\lambda_2}z^* = \sum_{j=1}^{n}\sum_{i=1}^{m}\left[V_{\lambda_2}(\tilde{B}_{ij}(\alpha_i, \beta_j)) - W_{\lambda_2}(\tilde{B}_{ij}(\alpha_i, \beta_j))\right]y_i^* z_j^*$$

and

$$\mathbf{y}^{*T}(D_{\lambda_1}(\widetilde{A}) + D_{\lambda_2}(\tilde{B}))z^* - u^*(\lambda_1) - v^*(\lambda_2) = 0$$

Noticing that $y_i^* \ge 0$, $z_j^* \ge 0$, and $V_\lambda(\tilde{a})$ and $A_\lambda(\tilde{a})$ are, respectively, continuous non-decreasing and non-increasing functions of the parameter $\lambda \in [0,1]$ if $\tilde{a}$ is a non-negative I-fuzzy number. Then, $u^*(\lambda_1)$ and $v^*(\lambda_2)$ are monotonic and non-decreasing functions of the parameters $\lambda_1 \in [0,1]$ and $\lambda_2 \in [0,1]$, respectively. Thus, the satisfying Nash equilibrium values of players I and II are obtained as $[u^*(0), u^*(1)]$ and $[v^*(0), v^*(1)]$, respectively, and can be written as the I-fuzzy numbers $\{\langle(\bar{y}^*, \bar{z}^*), u^*(0), 1 - u^*(1)\rangle\}$ and $\{\langle(\bar{y}^*, \bar{z}^*), v^*(0), 1 - v^*(1)\rangle\}$, where $(\bar{y}^*, \bar{z}^*)$ represents a mixed situation. Thus, $\tilde{u}^*(\bar{y}^*, \bar{z}^*)$ and $\tilde{v}^*(\bar{y}^*, \bar{z}^*)$ is Nash equilibrium values of players I and II, respectively.

In particular, for the parameters $\lambda_1 = 0$ and $\lambda_2 = 0$, Eq. (19) becomes the non-linear programming model as follows:

$$\max\left\{\sum_{j=1}^{n}\sum_{i=1}^{m} y_i[V_\mu(\tilde{A}_{ij}(\alpha_i, \beta_j)) - W_\upsilon(\tilde{A}_{ij}(\alpha_i, \beta_j)) + V_\mu(\tilde{B}_{ij}(\alpha_i, \beta_j)) - W_\upsilon(\tilde{B}_{ij}(\alpha_i, \beta_j))]z_j - u(0) - v(0)\right\}$$
$$\text{s.t.}\begin{cases} \sum_{j=1}^{n}[V_\mu(\tilde{A}_{ij}(\alpha_i, \beta_j)) - W_\upsilon(\tilde{A}_{ij}(\alpha_i, \beta_j))]z_j \le u(0) \quad (i = 1,2,\ldots,m) \\ \sum_{i=1}^{m}[V_\mu(\tilde{B}_{ij}(\alpha_i, \beta_j)) - W_\upsilon(\tilde{B}_{ij}(\alpha_i, \beta_j))]y_i \le v(0) \quad (j = 1,2,\ldots,n) \\ y_1 + y_2 + \cdots + y_m = 1 \\ z_1 + z_2 + \cdots + z_n = 1 \\ z_j \ge 0, y_i \ge 0 \quad (i = 1,2,\ldots,m; j = 1,2,\ldots,n) \\ u(0) \ge 0, v(0) \ge 0 \end{cases}, \quad (20)$$

where $u(0) = V_\mu(\tilde{u}(\bar{y}, \bar{z})) - W_\upsilon(\tilde{u}(\bar{y}, \bar{z}))$, $v(0) = V_\mu(\tilde{v}(\bar{y}, \bar{z})) - W_\upsilon(\tilde{v}(\bar{y}, \bar{z}))$, $y_i (i = 1,2,\ldots,m)$, $z_j (j = 1,2,\ldots,n)$, $u(0)$, and $v(0)$ are decision variables. The solution of Eq. (20) can be obtained by $(\mathbf{y}^{*T}, \mathbf{z}^{*T}, V_\mu(\tilde{u}^*(\bar{y}^*, \bar{z}^*)) - W_\upsilon(\tilde{u}^*(\bar{y}^*, \bar{z}^*)), V_\mu(\tilde{v}^*(\bar{y}^*, \bar{z}^*)) - W_\upsilon(\tilde{v}^*(\bar{y}^*, \bar{z}^*)))$.

Similarly, for the parameters $\lambda_1 = 1$ and $\lambda_2 = 1$, Eq. (19) becomes the non-linear programming model as follows:

$$\max\left\{\sum_{j=1}^{n}\sum_{i=1}^{m} y_i[V_\upsilon(\tilde{A}_{ij}(\alpha_i,\beta_j)) - W_\mu(\tilde{A}_{ij}(\alpha_i,\beta_j)) + V_\upsilon(\tilde{B}_{ij}(\alpha_i,\beta_j)) - W_\mu(\tilde{B}_{ij}(\alpha_i,\beta_j))]z_j - u(1) - v(1)\right\}$$

$$s.t.\begin{cases}\sum_{j=1}^{n}[V_\mu(\tilde{A}_{ij}(\alpha_i,\beta_j)) - W_\upsilon(\tilde{A}_{ij}(\alpha_i,\beta_j))]z_j \le u(1) \quad (i=1,2,\ldots,m)\\ \sum_{i=1}^{m}[V_\mu(\tilde{B}_{ij}(\alpha_i,\beta_j)) - W_\upsilon(\tilde{B}_{ij}(\alpha_i,\beta_j))]y_i \le v(1) \quad (j=1,2,\ldots,n)\\ y_1+y_2+\cdots+y_m=1\\ z_1+z_2+\cdots+z_n=1\\ z_j\ge 0, y_i\ge 0\,(i=1,2,\cdots,m; j=1,2,\ldots,m)\\ u(1)\ge 0, v(1)\ge 0\end{cases}, \tag{21}$$

where $u(1) = V_\upsilon(\tilde{u}^*(\bar{y}^*,\bar{z}^*)) - W_\mu(\tilde{u}^*(\bar{y}^*,\bar{z}^*))$, $v(1) = V_\upsilon(\tilde{v}^*(\bar{y}^*,\bar{z}^*)) - W_\mu(\tilde{v}^*(\bar{y}^*,\bar{z}^*))$, $y_i(i=1,2,\ldots,m)$, $z_j(j=1,2,\ldots,n)$, $u(1)$ and $v(1)$ are decision variables. Likewise, the solution of Eq. (21) can be obtained by $(\boldsymbol{y}'^{*T},\boldsymbol{z}'^{*T}$, $V_\upsilon(\tilde{u}^*(\bar{y}^*,\bar{z}^*)) - W_\mu(\tilde{u}^*(\bar{y}^*,\bar{z}^*))$, $V_\upsilon(\tilde{v}^*(\bar{y}^*,\bar{z}^*)) - W_\mu(\tilde{v}^*(\bar{y}^*,\bar{z}^*)))$.

Thus, we can explicitly obtain the satisfying Nash equilibrium values and corresponding satisfying Nash equilibrium strategies of players I and II through solving Eqs. (20) and (21). Furthermore, according to Eq. (19), any satisfying Nash equilibrium values and corresponding satisfying Nash equilibrium strategies of players I and II can be obtained through choosing different parameters $\lambda_1 \in [0,1]$ and $\lambda_2 \in [0,1]$.

5 An Application to the Strategy Choice Problem

There are lots of competitive decision problems which may be solved by using the game theory. In this section, we consider a manufacturers' production plan (or strategy) choice problem, which is used as a demonstration of the possible applications of the proposed methodology in realistic scenario.

Let us consider the case of two manufacturers P_1 and P_2 making a decision aiming to enhance the satisfaction degrees of customers. Assume that manufacturers P_1 and P_2 are rational, i.e. they will choose optimal strategies to maximize their own profits without cooperation. Suppose that manufacturer P_1 has two pure strategies: establishing a scientific and rational service system α_1 and providing customers with satisfying product α_2. Manufacturer P_2 has the same pure strategies as manufacturer P_1, i.e. establishing a scientific and rational service system β_1 and providing customers with satisfaction products β_2. Due to lack of information or imprecision of the available information, the customers' preference and satisfaction degrees are often vague, and the players' estimation often by their intuitive experience. Thus, the sales amount is not able to forecast exactly. In order to deal with the uncertainty, I-fuzzy numbers are used to express the sales amount of the product. The payoff matrices of manufacturers P_1 and P_2 are, respectively, expressed as follows:

$$\tilde{A} = \begin{matrix} & \alpha_1 & \alpha_2 \\ \beta_1 & <(160,170,180);f_{11l},f_{11r};(150,170,180,190),g_{11l},g_{11r}> & <(140,150,165,190);(130,155,180,190)> \\ \beta_2 & <(150,160,170,180);f_{12l},f_{12r};(140,160,180,190),g_{12l},g_{12r}> & <(140,150,160),(130,145,170)> \end{matrix}$$

and

$$\tilde{\boldsymbol{B}} = \begin{matrix} & \alpha_1 & \alpha_2 \\ \beta_1 & <(160,170,180,190),f_{11l},f_{11r};(155,165,180,200),g_{11l},g_{11r}> & <(140,150,170,190),(130,145,180,190)> \\ \beta_2 & <(130,150,160),f_{12l},f_{12r};(120,145,155,165),g_{12l},g_{12r}> & <(150,170,180),(140,170,190)> \end{matrix}$$

where $\tilde{A}_{11}=\langle(160,170,180);f_{11l},f_{11r};(150,170,180,190),g_{11l},g_{11r}\rangle$ is an I-fuzzy number with the membership (or satisfaction) and non-membership (or dissatisfaction) functions as follows:

$$\mu_{\tilde{A}_{11}}(x) = \begin{cases} 0 & (x<160)\\ (x-160)^2/100 & (160\le x<170)\\ 1 & (x=170)\\ (180-x)/10 & (170<x\le 180)\\ 0 & (x>180)\end{cases},$$

$$\upsilon_{\tilde{A}_{11}}(x) = \begin{cases} 1 & (x<150)\\ (170-x)^2/400 & (150\le x<170)\\ 0 & (170\le x\le 180)\\ (x-180)/10 & (180<x\le 190)\\ 1 & (x>200),\end{cases}$$

$\tilde{A}_{12}=\langle(150, 160, 170, 180); f_{12l}, f_{12r}; (140, 160, 180, 190), g_{12l}, g_{12r}\rangle$ is an I-fuzzy number with the membership and non-membership functions as follows:

$$\mu_{\tilde{A}_{12}}(x) = \begin{cases} 0 & (x<150)\\ (x-150)/10 & (150\le x<160)\\ 1 & (160\le x\le 170)\\ (180-x)^2/100 & (170<x\le 180)\\ 0 & (x>180)\end{cases}$$

$$\upsilon_{\tilde{A}_{12}}(x) = \begin{cases} 1 & (x<140)\\ (160-x)/20 & (140\le x<160)\\ 0 & (160\le x\le 180)\\ (x-180)^2/100 & (180<x\le 190)\\ 1 & (x>190)\end{cases},$$

$\tilde{A}_{21}=\langle(140,150,170,190);(130,145,180,190)\rangle$ is a trapezoidal I-fuzzy number with the membership and non-membership functions as follows:

$$\mu_{\tilde{A}_{21}}(x) = \begin{cases} 0 & (x < 140) \\ (x-140)/10 & (140 \le x < 150) \\ 1 & (150 \le x \le 170) \\ (190-x)/20 & (170 < x \le 190) \\ 0 & (x > 190) \end{cases},$$

$$\upsilon_{\tilde{A}_{21}}(x) = \begin{cases} 1 & (x < 130) \\ (145-x)/15 & (130 \le x < 145) \\ 0 & (145 \le x \le 180) \\ (x-180)/10 & (180 < x \le 190) \\ 1 & (x > 190), \end{cases}$$

Others in payoff matrices $\tilde{\boldsymbol{A}}$ and $\tilde{\boldsymbol{B}}$ can be similarly explained.

Taking $f(\alpha) = \alpha \quad (\alpha \in [0,1])$ and $g(\beta) = 1 - \beta$ $(\beta \in [0,1])$, according to Eqs. (9)–(15) and (19), the parametric non-linear programming model is constructed as follows:

$$\max\{(148.3+10.6\lambda_1+4.5\lambda_2)y_1z_1 + (127.9+9\lambda_1+8\lambda_2)y_1z_2 + (120.8+7.5\lambda_1+6.7\lambda_2)y_2z_1 + (144.2+1.7\lambda_1+3.3\lambda_2)y_2z_2 - u(\lambda_1) - v(\lambda_2)\}$$

$$s.t. \begin{cases} (73.9+10.6\lambda_1)z_1+(65.9+9\lambda_1)z_2 \le u(\lambda_1) \\ (61.7+7.5\lambda_1)z_1+(68.3+1.7\lambda_1)z_2 \le u(\lambda_1) \\ (74.4+4.5\lambda_2)y_1+(62+8\lambda_2)y_2 \le v(\lambda_2) \\ (59.1+6.7_2)y_1+(75.9+3.3\lambda_2)y_2 \le v(\lambda_2) \\ y_1+y_2=1 \\ z_1+z_2=1 \\ u(\lambda_1) \ge 0, v(\lambda_2) \ge 0 \\ y_i \ge 0, z_j \ge 0\,(i=1,2; j=1,2). \end{cases} \tag{22}$$

For the parameters $\lambda_1 \in [0,1]$ and $\lambda_2 \in [0,1]$, solving Eq. (22), we can obtain the satisfying Nash equilibrium values and corresponding satisfying Nash equilibrium strategies of manufacturers P_1 and P_2, respectively, depicted as in Tables 1, 2 and 3.

It is easy to see from Tables 1, 2 and 3 that the satisfying Nash equilibrium value of a manufacturer P_1 (or P_2) only depends on his/her own preference/parameter regardless of other player's preference/parameter. However, strategy choice of a player is only affected by other player' preference/parameter.

Table 1 Satisfying Nash equilibrium values and corresponding strategies

Parameters		P_1		P_2	
λ_1	λ_2	$\boldsymbol{y}^{*T}$	$u^*(\lambda_1)$	$\boldsymbol{z}^{*T}$	$v^*(\lambda_2)$
0	0	(0.476, 0.524)	67.22	(0.164, 0.836)	67.90
0.3	0.3	(0.460, 0.540)	68.73	(0.016, 0.984)	69.63
0.5	0.5	(0.449, 0.551)	70.40	(0, 1)	70.78
0.8	0.8	(0.428, 0.571)	73.10	(0, 1)	72.50
1	1	(0.413, 0.587)	74.90	(0, 1)	73.67

Table 2 Satisfying Nash equilibrium values and corresponding strategies

Parameters		P_1		P_2	
λ_1	λ_2	$\boldsymbol{y}^{*T}$	$u^*(\lambda_1)$	$\boldsymbol{z}^{*T}$	$v^*(\lambda_2)$
0	1	(0.413, 0.587)	67.22	(0.164, 0.836)	73.67
0.3	0.8	(0.428, 0.571)	68.73	(0.016, 0.984)	72.50
0.5	0.5	(0.449, 0.551)	70.40	(0, 1)	70.78
0.8	0.3	(0.460, 0.540)	73.10	(0, 1)	69.63
1	0	(0.476, 0.524)	74.90	(0, 1)	67.90

Table 3 Satisfying Nash equilibrium values and corresponding strategies

Parameters		P_1		P_2	
λ_1	λ_2	$\boldsymbol{y}^{*T}$	$u^*(\lambda_1)$	$\boldsymbol{z}^{*T}$	$v^*(\lambda_2)$
1	0	(0.476, 0.524)	74.90	(0, 1)	67.90
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0.5	0.5	(0.449, 0.551)	70.40	(0, 1)	70.78
0.3	0.8	(0.428, 0.571)	68.73	(0.016, 0.984)	72.50
0	1	(0.413, 0.587)	67.22	(0.164, 0.836)	73.67

6 Conclusion

Determining payoffs of bi-matrix games absolutely depends on players' judgments and intuition, which are often vague and not easy to be represented with crisp values and fuzzy numbers. This paper formulates bi-matrix games with payoffs expressed by I-fuzzy numbers and propose corresponding parameterized non-linear programming method. The main contributions include (1) giving the concepts of general I-fuzzy numbers and the value-index and ambiguity-index; (2) proposing the new ranking method based on the difference-index, which is proven to be a total order and has some useful properties; and (3) establishing parameterized non-linear programming models and method for any bi-matrix game with payoffs represented by I-fuzzy numbers.

Obviously, for any given parameter $\lambda \in [0,1]$, the parameterized non-linear programming models become a pair of primal–dual linear programming models, which are easily solved by using the simplex method of linear programming. Our work is remarkably different from those [6–9, 11, 13, 17, 19, 20], in that players' payoffs and/or goals were expressed with I-fuzzy sets [6–9, 11, 13] or fuzzy numbers [17, 20], and Nehi [19] established multi-objective programming models based on the average indices of the membership and non-membership functions of triangular I-fuzzy numbers, which are only a special form of I-fuzzy numbers.

Furthermore, it is easy to see that the derived parameterized non-linear programming models for bi-matrix games with payoffs represented by I-fuzzy numbers are an extension of the linear programming models for fuzzy matrix games. Therefore, effective and efficient methods for explicitly determining values of matrix games with payoffs of I-fuzzy numbers will be investigated in the near future.

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Appendix 1: Proof of Theorem 1

According to Eqs. (5) and (7), for any $\alpha \in [0,1]$, we have $(\rho\tilde{A}+\tilde{A}')^\alpha = \rho\tilde{A}^\alpha + \tilde{A}'^\alpha$. Hence, we have

$$\begin{aligned}&L^\alpha(\rho\tilde{A}+\tilde{A}') + R^\alpha(\rho\tilde{A}+\tilde{A}')\\&= L^\alpha(\rho\tilde{A}) + L^\alpha(\tilde{A}') + R^\alpha(\rho\tilde{A}) + R^\alpha(\tilde{A}')\\&= \rho(L^\alpha(\tilde{A}) + R^\alpha(\tilde{A})) + L^\alpha(\tilde{A}') + R^\alpha(\tilde{A}').\end{aligned}$$

Combining with Eq. (9), we have

$$\begin{aligned}V_\mu(\rho\tilde{A}+\tilde{A}') &= \rho\int_0^1 [(L^\alpha(\tilde{A}) + R^\alpha(\tilde{A}))/2]f(\alpha)\mathrm{d}\alpha\\&\quad + \int_0^1 [(L^\alpha(\tilde{A}') + R^\alpha(\tilde{A}'))/2]f(\alpha)\mathrm{d}\alpha\\&= \rho V_\mu(w,\tilde{A}) + V_\mu(w,\tilde{A}'),\end{aligned}$$

i.e. $V_\mu(\rho\tilde{A}+\tilde{A}') = \rho V_\mu(\tilde{A}) + V_\mu(\tilde{A}')$.

For any $\beta \in [0,1]$, it easily follows from Eqs. (6) and (8) that $(\rho\tilde{A}+\tilde{A}')_\beta = \rho\tilde{A}_\beta + \tilde{A}'_\beta$. According to Eq. (10), we can similarly prove that $V_\upsilon(\rho\tilde{A}+\tilde{A}') = \rho V_\upsilon(\tilde{A}) + V_\upsilon(\tilde{A}')$.

For any $\alpha \in [0,1]$, if $\rho \ge 0$, it is easily derived from Eqs. (5) and (7) that

$$\begin{aligned}&R^\alpha(\rho\tilde{A}+\tilde{A}') - L^\alpha(\rho\tilde{A}+\tilde{A}')\\&= R^\alpha(\rho\tilde{A}) - L^\alpha(\rho\tilde{A}) + R^\alpha(\tilde{A}') - L^\alpha(\tilde{A}')\\&= \rho(R^\alpha(\tilde{A}) - L^\alpha(\tilde{A})) + R^\alpha(\tilde{A}') - L^\alpha(\tilde{A}').\end{aligned}$$

Then, combining with Eq. (11), we have

$$\begin{aligned}W_\mu(\rho\tilde{A}+\tilde{A}') &= \rho\int_0^1 (R^\alpha(\tilde{A}) - L^\alpha(\tilde{A}))f(\alpha)\mathrm{d}\alpha\\&\quad + \int_0^1 (R^\alpha(\tilde{A}') - L^\alpha(\tilde{A}'))f(\alpha)\mathrm{d}\alpha\\&= \rho W_\mu(\tilde{A}) + W_\mu(\tilde{A}').\end{aligned}$$

Likewise, if $\rho < 0$, then $R^\alpha(\rho\tilde{A}+\tilde{A}') - L^\alpha(\rho\tilde{A}+\tilde{A}') = \rho(L^\alpha(\tilde{A}) - R^\alpha(\tilde{A})) + R^\alpha(\tilde{A}') - L^\alpha(\tilde{A}')$. Hereby, we have

$$\begin{aligned}W_\mu(\rho\tilde{A}+\tilde{A}') &= \rho\int_0^1 (L^\alpha(\tilde{A}) - R^\alpha(\tilde{A}))f(\alpha)\mathrm{d}\alpha\\&\quad + \int_0^1 (R^\alpha(\tilde{A}') - L^\alpha(\tilde{A}'))f(\alpha)\mathrm{d}\alpha\\&= \rho W_\mu(\tilde{A}) + W_\mu(\tilde{A}').\end{aligned}$$

Therefore, we have proven that $W_\mu(\rho\tilde{A}+\tilde{A}') = \rho W_\mu(\tilde{A}) + W_\mu(\tilde{A}')$ for any $\rho \in \mathrm{R}$.

Similarly, according to Eqs. (6), (8) and (12), we can prove that $W_\upsilon(u,\rho\tilde{A}+\tilde{A}') = \rho W_\upsilon(u,\tilde{A}) + W_\upsilon(u,\tilde{A}')$.

Appendix 2: Proof of Theorem 2

According to Theorem 1, it is derived from Eq. (13) that

$$\begin{aligned}&V_\lambda(\rho\tilde{A}+\tilde{A}') = \lambda V_\upsilon(\rho\tilde{A}+\tilde{A}') + (1-\lambda)V_\mu(\rho\tilde{A}+\tilde{A}')\\&=\rho[\lambda V_\upsilon(\tilde{A}) + (1-\lambda)V_\mu(\tilde{A})] + \lambda V_\upsilon(\tilde{A}') + (1-\lambda)V_\mu(\tilde{A}')\\&= \rho V_\lambda(\tilde{A}) + V_\lambda(\tilde{A}')\end{aligned}$$

i.e. $V_\lambda(\rho\tilde{A}+\tilde{A}') = \rho V_\lambda(\tilde{A}) + V_\lambda(\tilde{A}')$.

Likewise, according to Theorem 1 and Eq. (14), we can prove that $W_\lambda(\rho\tilde{A}+\tilde{A}') = \rho W_\lambda(\tilde{A}) + W_\lambda(\tilde{A}')$.

Proof of Theorem 3

According to Theorem 2, it is derived from Eq. (15) that

$$\begin{aligned}&D_\lambda(\rho\tilde{A}+\tilde{A}') = V_\lambda(\rho\tilde{A}+\tilde{A}') - W_\lambda(\rho\tilde{A}+\tilde{A}')\\&= (\rho V_\lambda(\tilde{A}) + V_\lambda(\tilde{A}')) - (\rho W_\lambda(\tilde{A}) + W_\lambda(\tilde{A}'))\\&= \rho(V_\lambda(\tilde{A}) - W_\lambda(\tilde{A})) + (V_\lambda(\tilde{A}') - W_\lambda(\tilde{A}'))\\&= \rho D_\lambda(\tilde{A}) + D_\lambda(\tilde{A}').\end{aligned}$$

Thus, we have completed the proof of Theorem 3.

Appendix 3: Proof of Theorem 4

(P1) For any I-fuzzy number $\tilde{A}$, it directly follows from Eq. (15) that $D_\lambda(\tilde{A}) \ge D_\lambda(\tilde{A})$ for any $\lambda \in [0,1]$. Hereby, according to Definition 1, we have $\tilde{A} \ge_{\mathrm{IF}} \tilde{A}$.

(P2) For any I-fuzzy numbers $\tilde{A}$ and $\tilde{A}'$, according to Definition 1, we have $D_\lambda(\tilde{A}) \ge D_\lambda(\tilde{A}')$ and $D_\lambda(\tilde{A}') \ge D_\lambda(\tilde{A})$ for any $\lambda \in [0,1]$. Thus, $D_\lambda(\tilde{A}) = D_\lambda(\tilde{A}')$. Hereby, we have proven that $\tilde{A} =_{\mathrm{IF}} \tilde{A}'$.

(P3) For any I-fuzzy numbers $\tilde{A}$, $\tilde{A}'$ and $\tilde{A}''$, according to Definition 1, we have $D_\lambda(\tilde{A}) \ge D_\lambda(\tilde{A}')$ and $D_\lambda(\tilde{A}') \ge D_\lambda(\tilde{A}'')$

for any $\lambda \in [0,1]$. Hence, $D_\lambda(\tilde{A}) \geq D_\lambda(\tilde{A}'')$. Therefore, we have proven that $\tilde{A} \geq_{\text{IF}} \tilde{A}''$.

(P4) It can be easily seen from Eqs. (9)–(15) that the difference-indices of I-fuzzy numbers $\tilde{A}$ and $\tilde{A}'$ are completely determined by themselves. Thus, the ranking order of $\tilde{A}$ and $\tilde{A}'$ completely depends on $D_\lambda(\tilde{A})$ and $D_\lambda(\tilde{A}')$, which have nothing to do with the other I-fuzzy numbers under comparison. Therefore, we have proven that $\tilde{A} >_{\text{IF}} \tilde{A}'$ on F_1 if and only if $\tilde{A} >_{\text{IF}} \tilde{A}'$ on F_2.

(P5) It is derived from Eqs. (9)–(10), we can obtain

$$V_\mu(\tilde{A}) = \int_0^1 [(L^\alpha(\tilde{A}) + R^\alpha(\tilde{A}))/2] f(\alpha)\mathrm{d}\alpha \geq \int_0^1 2\underline{a}_2\, \alpha \mathrm{d}\alpha = \underline{a}$$

and

$V_\mu(\tilde{A}') = \int_0^1 [(L^\alpha(\tilde{B}) + R^\alpha(\tilde{B}))/2] f(\alpha)\mathrm{d}\alpha \leq \int_0^1 2\bar{a}'\, \alpha\mathrm{d}\alpha = \bar{a}'$

Combining with $\sup p(\tilde{A}) >$ sup $\sup p(\tilde{A}')$, it directly follows that $V_\mu(\tilde{A}) > V_\mu(\tilde{A}')$.

Similarly, it follows that $V_\upsilon(\tilde{A}) = \int_0^1 [(L_\beta(\tilde{A}) + R_\beta(\tilde{A}))/2] g(\beta)\mathrm{d}\beta \geq \int_0^1 2\underline{a}\, \alpha\mathrm{d}\alpha = \underline{a}$ and $V_\upsilon(\tilde{A}') = \int_0^1 [(L_\beta(\tilde{B}) + R_\beta(\tilde{B}))/2] g(\beta)\mathrm{d}\beta \leq \int_0^1 2\bar{a}'\, \alpha\mathrm{d}\alpha = \bar{a}'$. Combining with $\sup p(\tilde{A}) >$ sup $\sup p(\tilde{A}')$, it directly follows that $V_\upsilon(\tilde{A}) > V_\upsilon(\tilde{A}^{'})$. Therefore, $\lambda V_\upsilon(\tilde{A}) + (1-\lambda)V_\mu(\tilde{A}) > \lambda V_\upsilon(\tilde{A}') + (1-\lambda)V_\mu(\tilde{A}')$, i.e. $V_\lambda(\tilde{A}) > V_\lambda(\tilde{A}')$.

In a similar way, we have $\lambda W_\mu(\tilde{A}) + (1-\lambda)W_\upsilon(\tilde{A}) > \lambda W_\mu(\tilde{A}') + (1-\lambda)W_\upsilon(\tilde{A}')$, i.e. $W_\lambda(\tilde{A}) > W_\lambda(\tilde{A}')$.

According to Definition 1, for any $\lambda \in [0,1]$, we have $D_\lambda(\tilde{A}) > D_\lambda(\tilde{A}')$ if and only if $\tilde{A}$ is larger than $\tilde{A}'$, i.e. $V(\tilde{A},\lambda) - W(\tilde{A},\lambda) > V(\tilde{A}',\lambda) - W(\tilde{A}',\lambda)$. Hence, $\tilde{A} >_{\text{IF}} \tilde{A}'$.

For instance, taking $f(\alpha) = \alpha$ ($\alpha \in [0,1]$) and $g(\beta) = 1-\beta$ ($\beta \in [0,1]$), by using Eqs. (9)–(15), the difference-indexes of any I-fuzzy number $\tilde{A}$ can be obtained as follows:

$$\begin{aligned} D_\lambda(\tilde{A}) &= V(\tilde{A},\lambda) - W(\tilde{A},\lambda) = [\lambda V_\upsilon(\tilde{A}) + (1-\lambda)V_\mu(\tilde{A})] \\ &\quad - [\lambda W_\mu(\tilde{A}) + (1-\lambda)W_\upsilon(\tilde{A})] \\ &= [(3\lambda-2)\bar{a}_2 + (6\lambda-4)a_{2r} + (4-2\lambda)a_{2l} \\ &\quad + (2-\lambda)\underline{a}_2)]/12 \quad + [(1-3\lambda)\bar{a}_1 + (2-6\lambda)a_{1r} \\ &\quad + (2+2\lambda)a_{1l} + (1+\lambda)\underline{a}_1)]/12, \end{aligned} \tag{23}$$

where $\underline{a}_2 \leq \underline{a}_1 \leq a_{2l} \leq a_{1l} \leq a_{1r} \leq a_{2r} \leq \bar{a}_1 \leq \bar{a}_2$. If $\sup p(\tilde{A}) >$ sup $\sup p(\tilde{A}')$, i.e. $\underline{a}'_1 \leq a'_{2l} \leq a'_{1l} \leq a'_{1r} \leq a'_{2r} \leq \bar{a}'_1 \leq \bar{a}'_2 < \underline{a}_2 \leq \underline{a}_1 \leq a_{2l} \leq a_{1l} \leq a_{1r} \leq a_{2r} \leq \bar{a}_1 \leq \bar{a}_2$, then it follows from Eq. (23) that

$$\begin{aligned} D_\lambda(\tilde{A}) - D_\lambda(\tilde{A}') &> (3\lambda-2)(\bar{a}_2 - \underline{a}_2) \\ &\quad + (6\lambda-4)(a_{2r} - \underline{a}_2) + (4-2\lambda)(a_{2l} - \underline{a}_2)]/12 \\ &\quad + [(1-3\lambda)(\bar{a}_1 - \underline{a}_2) + (2-6\lambda)(a_{1r} - \underline{a}_2) \\ &\quad + (2+2\lambda)(a_{1l} - \underline{a}_2) + (1+\lambda)(\underline{a}_1 - \underline{a}_2)]/12 \\ &\geq [(7\lambda-2)(a_{2l} - \underline{a}_2) + (6-6\lambda)(\underline{a}_1 - \underline{a}_2)]/12 \\ &\geq (\lambda+4)(\underline{a}_1 - \underline{a}_2) \geq 0 \end{aligned}.$$

Therefore, we have proven that if $\sup p(\tilde{A}) >$ sup $\sup p(\tilde{A}')$, then $\tilde{A} >_{\text{IF}} \tilde{A}'$.

(P6) In the same way to that of (P3), for any $\lambda \in [0,1]$, it follows from Definition 1 that

$$D_\lambda(\tilde{A}) \geq D_\lambda(\tilde{A}'). \tag{24}$$

Combining with Theorem 1, we have $D_\lambda(\tilde{A} + \tilde{A}'') = D_\lambda(\tilde{A}) + D_\lambda(\tilde{A}'') \geq D_\lambda(\tilde{A}') + D_\lambda(\tilde{A}'') = D_\lambda(\tilde{A}' + \tilde{A}'')$, i.e.

$$D_\lambda(\tilde{A} + \tilde{A}'') \geq D_\lambda(\tilde{A}' + \tilde{A}''). \tag{25}$$

Hence, we have $\tilde{A} + \tilde{A}'' \geq_{\text{IF}} \tilde{A}' + \tilde{A}''$.

(P6′) Eq. (24) is a strictly inequality due to $\tilde{A} >_{\text{IF}} \tilde{A}'$. Thus, Eq. (25) is also a strictly inequality. According to Definition 1, we have proven that $\tilde{A} + \tilde{A}'' >_{\text{IF}} \tilde{A}' + \tilde{A}''$.

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