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Generalized Interval-Valued Fuzzy Rough Set and its Application in Decision Making

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Abstract This paper presents a general study of generalized interval-valued fuzzy rough sets integrating the rough set theory with the interval-valued fuzzy set theory by constructive and axiomatic approaches. In the constructive approach, by employing an interval-valued fuzzy residual implicator and its dual operator, generalized upper and lower interval-valued fuzzy rough approximation operators with respect to an arbitrary interval-valued fuzzy approximation space are first defined. Then properties of generalized interval-valued fuzzy rough approximation operators are discussed. Furthermore, connections between special types of interval-valued fuzzy relations and properties of generalized interval-valued fuzzy approximation operator are also established. In the axiomatic approach, generalized interval-valued fuzzy rough approximation operators are defined by axioms. We prove that different axiom sets can characterize the essential properties of generalized interval-valued fuzzy rough approximation operators. Also the composition of two approximation spaces is explored. Finally, a practical application is provided to illustrate the efficiency of the generalized interval-valued fuzzy rough set model.

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1 Introduction

Rough set theory, developed by Pawlak [19, 20] as a framework for the construction of approximations of concepts, is mathematical approach to handle imprecision, vagueness, and uncertainty in data analysis. Generally speaking, there are mainly two methods for the development of this theory [15, 43], namely the constructive and axiomatic approaches.

In the constructive approach, the lower and upper approximation operators are constructed from the primitive notions, such as binary relations on the universe of discourse, partition (or coverings) of the universe of discourse, neighborhood systems, and Boolean algebras [20, 40, 43, 45, 54]. Recently, rough set approximations have also been developed into the fuzzy environment in which the results are called rough fuzzy sets [8, 14, 30, 35] and fuzzy rough sets [8, 21, 34, 36, 38, 41] based on the constructive method. Moreover, by combining rough set theory with the other uncertainty theory, such as interval-valued fuzzy set theory, intuitionistic fuzzy set theory, hesitant fuzzy set theory and soft set theory, many authors proposed some new rough sets model [6, 7, 13, 18, 22–25, 27, 47–49, 52, 53, 55–58]. On the other hand, the axiomatic approach [2, 12, 17, 21, 29–31, 36, 37, 39] is mainly engaged in algebraic systems of rough set theory by treating a pair of abstract operators as primitive notions. In this approach, a set of axioms is used to characterize approximation operators that are the same as the ones produced by using the constructive approach. Many authors explored and developed the axiomatic approach in the study of crisp rough set theory [28, 43–45]. The research of the axiomatic approach has also been extended to approximation operators in fuzzy environment [16, 17, 21, 29, 30, 36, 38, 39]. For example, a set of axioms on fuzzy rough sets was investigated by Moris and Yakout [17]. In [29, 30] Thiele explored axiomatic characterizations of fuzzy rough approximation operators and rough fuzzy approximation operators within modal logic. Furthermore, Wu et al. [35, 38–40] studied various generalized fuzzy approximation operators which are characterized by different sets of axioms. Recently, the axiomatic approach to approximation operators has been investigated by many authors in IF environment [49, 52, 53, 55–57], hesitant fuzzy environment [42], and interval-valued hesitant fuzzy environment [48].

As two generalizations of Zadeh's fuzzy sets [50], interval-valued fuzzy (IVF, for short) sets [32, 51], and intuitionistic fuzzy (IF, for short) sets [1] were conceived independently to avoid some of defects of fuzzy sets. As a method handling vagueness and uncertainty precisely, both IVF set theory and IF set theory have the virtue of complementing fuzzy sets. And they have been used in different research fields, for example, Sambuc [26] in medical diagnosis in thyroidian pathology; Gorzalczany [9], and Bustince [3] in approximate reasoning; Turksen and Zhong [33] and Cornelis et al. [5] in interval-valued and intuitionistic logic, etc.

As we mentioned above, many authors have extended rough set theory into IVF sets and IF sets [6, 7, 10, 13, 22, 27, 52, 55–58]. For example, according to fuzzy rough sets in the sense of Nanda and Majumda [18], Jena and Ghosh [13], Chakrabarty et al. [7] and Samanta and Monda [27] presented the concept of IF rough sets which is not defined by an approximation space. Comparing with the above approaches, Rizvi et al. [22] proposed the concept of rough IF sets base on a Pawlak approximation space (U, R) in which the lower and upper approximations are not IF sets in the universe of discourse U, but IF sets in the family of equivalence classes derived by equivalence relation R. To remedy this difficulty, on the basic of an IF triangular norm $\mathcal{T}_L$ and IF implicator $\mathcal{I}_L$, Cornelis et al. [6] introduced the concept of $(\mathcal{T}_L, \mathcal{I}_L)$ IF rough sets in which the lower and upper approximation operators are both IF sets in the universe. However, they have not investigated the properties of the lower and upper approximation operators generated by other relations, such as reflexive relation, symmetric relation, and transitive relation. Therefore, in [52] various relation-based IF rough approximation operators were discussed by Zhou and Wu through using a special type of IF triangular norm *min*. Meanwhile, on the basic of IF implicator Zhou et al. [53] investigated IF rough approximations on one universe, but they have not studied properties of $(\mathcal{I}, \mathcal{T})-$ IVF rough sets on two different universes of discourse. Therefore, Zhang et al. constructed $(\mathcal{I}, \mathcal{T})-$ IVF rough approximation operators on two different universes of discourse by the constructive and axiomatic approaches. However, we note that IVF implicators constituting for IVF rough approximation operators don't satisfy axioms of Smets and Magrez on L^I in [49], unless the conditions are further restrained. To overcome this defect, He et al. [11] presented a residual implicator on L^I called interval-valued fuzzy residual implicator. Meanwhile, Mi et al. [16] presented a generalized fuzzy rough set and discussed its some interesting properties. In this paper, by integrating the rough set theory with the residual implicator, we shall extend the approximation concepts in [16] to generalized interval-valued fuzzy lower and upper approximation operators which satisfy axioms of Smets and Magrez on L^I. We further study the generalized IVF rough approximation operators in which both the constructive and axiomatic approaches are considered. The generalized lower and upper approximations of IVF sets with respect to an IVF approximation space is constructed by using a residual implicator Θ and its dual operator on L^I.

The rest of this paper is organized as follows. In Sect. 2, we review some basic notions related to the lattice on L^I, IVF logical operators, and IVF sets. In Sect. 3, we construct the interval-valued fuzzy residual implicator and its dual operator on L^I which satisfy axioms of Smets and Magrez on L^I, and discuss their some interesting properties. Then the concepts of generalized lower and upper approximations of IVF sets with respect to an IVF approximation space is presented in Sect. 4, and the properties of the lower and upper approximation operators are examined. In Sect. 5, we investigate an operator-oriented characterization of generalized IVF rough sets, and give different sets of axioms to characterize various types of IVF approximation operators. Section 6 is devoted to studying the composition of two IVF approximation spaces. In Sect. 7, a general approach to decision making based on generalized IVF rough sets is established under the background of application in medical diagnosis. Section 8 illustrates the principal steps of the proposed decision method by a numerical example. Some conclusions and outlooks for further research are given in Sect. 9.

2 Lattice, Interval-Valued Fuzzy Sets and Interval-Valued Fuzzy Logical Operators

In this section, we recall briey a special complete lattice on $[0,1]^2$ with its logical operations originated by Cornelis et al. [5, 6], which will be used to construct the structure of generalized interval-valued fuzzy rough sets in the present paper.

Definition 2.1 ([5]) Let $L^I = \{[\mu, v] \in [0,1] \times [0,1] | \mu \leq v\}$. Denote

$$[\mu_1, v_1] \leq_{L^I} [\mu_2, v_2] \Leftrightarrow \mu_1 \leq \mu_2, v_1 \leq v_2, \quad \forall [\mu_1, v_1], [\mu_2, v_2] \in L^I.$$

Then the pair $(L^I, \leq_{L^I})$ is called a complete, bounded lattice. The operators $\wedge$ and $\vee$ on $(L^I, \leq_{L^I})$ are defined as follows:

$$[\mu_1, v_1] \wedge [\mu_2, v_2] = [\min\{\mu_1, \mu_2\}, \min\{v_1, v_2\}],$$
$$[\mu_1, v_1] \vee [\mu_2, v_2] = [\max\{\mu_1, \mu_2\}, \max\{v_1, v_2\}],$$

for $[\mu_1, v_1], [\mu_2, v_2] \in L^I$.

Obviously, a complete lattice on L^I has the smallest element $0_{L^I} = [0, 0]$ and the greatest element $1_{L^I} = [1, 1]$. The definitions of fuzzy logical operators can be straightforwardly extended to the interval-valued fuzzy case. The strict partial order $<_{L^I}$ is defined by

$$[\mu_1, v_1] <_{L^I} [\mu_2, v_2] \Leftrightarrow [\mu_1, v_1] \leq_{L^I} [\mu_2, v_2],$$

and

$$[\mu_1, v_1] \neq [\mu_2, v_2].$$

Definition 2.2 ([49]) An IVF triangular norm (t-norm) $\mathcal{T}$ on L^I is a commutative, associative mapping $\mathcal{T} : L^I \times L^I \to L^I$ which is increasing in both arguments and satisfies $\mathcal{T}(1_{L^I}, \alpha) = \alpha$, for all $\alpha \in L^I$.

Definition 2.3 ([49]) An IVF triangular conorm (t-conorm) $\mathcal{S}$ on L^I is a commutative, associative mapping $\mathcal{S} : L^I \times L^I \to L^I$ which is increasing in both arguments and satisfies $\mathcal{S}(0_{L^I}, \alpha) = \alpha$, for all $\alpha \in L^I$.

Definition 2.4 ([49]) An IVF negator $\mathcal{N}$ on L^I is a decreasing mapping $\mathcal{N} : L^I \to L^I$ satisfying $\mathcal{N}(0_{L^I}) = 1_{L^I}$ and $\mathcal{N}(1_{L^I}) = 0_{L^I}$. An IVF negator is involutive if and only if $\mathcal{N}(\mathcal{N}([\mu, v])) = [\mu, v]$, where $[\mu, v] \in L^I$. For all $[\mu, v] \in L^I$, the IVF negator $\mathcal{N}_S([\mu, v]) = [1 - v, 1 - \mu]$ is usually referred to as the standard negator.

Given an IVF negator $\mathcal{N}$ an IVF t-norm $\mathcal{T}$ and IVF t-conorm $\mathcal{S}$ are called dual with respect to $\mathcal{N}$ iff they satisfy the following conditions:

$$\mathcal{S}(I_1, I_2) = \mathcal{N}(\mathcal{T}(\mathcal{N}(I_1), \mathcal{N}(I_2))), \quad \text{for all } I_1, I_2 \in L^I;$$
$$\mathcal{T}(I_1, I_2) = \mathcal{N}(\mathcal{S}(\mathcal{N}(I_1), \mathcal{N}(I_2))), \quad \text{for all } I_1, I_2 \in L^I.$$

The above definitions are the counterparts on L^I of parallel definitions on ([0,1],$\leq$).

Theorem 2.1 ([49]) *Let T be a continuous t-norm on [0,1] and S a continuous t-conorm on [0,1]. Then an IVF t-norm $\mathcal{T}$ and an IVF t-conorm $\mathcal{S}$ are constructed by the following equations for two intervals $I_1 = [\mu_1, v_1]$ and $I_2 = [\mu_2, v_2]$,*

$$\mathcal{T}[I_1, I_2] = [T(\mu_1, \mu_2), T(v_1, v_2)], \tag{1}$$
$$\mathcal{S}[I_1, I_2] = [S(\mu_1, \mu_2), S(v_1, v_2)]. \tag{2}$$

An IVF t-norm $\mathcal{T}$ (respectively, IVF t-conorm $\mathcal{S}$) is called t-representable (respectively, s-representable) if they can be represented in the form of above two equations, respectively.

Definition 2.5 ([32]) An IVF set in U is an expression A denoted by

$$A = \{\langle x, A(x) \rangle | x \in U\},$$

where $A : U \to L^I, x \to A(x) = [\mu_A(x), v_A(x)] \in L^I$.

For simplicity, we write $A = [\mu_A, v_A]$. We denote by IVF(U) the set of all IVF sets in U.

For $[\alpha_1, \alpha_2] \in L^I$, $\widehat{[\alpha_1, \alpha_2]}$ denotes a constant IVF set: $\widehat{[\alpha_1, \alpha_2]}(x) = [\alpha_1, \alpha_2]$ for any $x \in U$, where $\alpha_1 \leq \alpha_2$. For any $y \in U$ and $M \subseteq U$, IVF sets $[1,1]_y$, $[1,1]_{U\text{-}\{y\}}$ and $[1,1]_M$ are, respectively, defined as follows: for $x \in U$,

$$[1,1]_y(x) = \begin{cases} [1,1], & x = y, \\ [0,0], & x \neq y. \end{cases}$$
$$[1,1]_{U-\{y\}}(x) = \begin{cases} [0,0], & x = y, \\ [1,1], & x \neq y. \end{cases}$$
$$[1,1]_M(x) = \begin{cases} [1,1], & x \in M, \\ [0,0], & x \notin M. \end{cases}$$

The IVF universe set is $U = [1,1]_U = \widehat{[1,1]} = \widehat{1_{L^I}} = \{\langle x, 1, 1 \rangle | x \in U\}$, and the IVF empty set is $\emptyset = \widehat{[0,0]} = \widehat{0_{L^I}} = \{< x, 0, 0 > | x \in U\}$.

The basic operations on IVF(U) are defined as follows [32]: for all $A, B \in$ IVF(U)

(1) $A \subseteq B$ iff $A(x) \leq_{L^I} B(x)$, i.e., $\mu_A(x) \leq \mu_B(x)$ and $v_A(x) \leq v_B(x)$, for all $x \in U$;
(2) $A = B$ iff $A \subseteq B$ and $B \subseteq A$;
(3) $\sim A = [1 - v_A, 1 - \mu_A]$;
(4) $(A \cap B)(x) = [\min\{\mu_A(x), \mu_B(x)\}, \min\{v_A(x), v_B(x)\}]$;
(5) $(A \cup B)(x) = [\max\{\mu_A(x), \mu_B(x)\}, \max\{v_A(x), v_B(x)\}]$.

Definition 2.6 ([4]) An IVF relation from U to W is an IVF set on $U \times W$, i.e., R is given by $R = \{[\mu_R(x, y), v_R(x, y)] | (x, y) \in U \times W\}$, for simplicity, $R = [\mu_R, v_R]$, where μ_R and v_R are two fuzzy relations on $U \times W$ satisfying $\mu_R(x,y) \leq v_R(x,y)$, for all $(x, y) \in U \times W$.

An IVF relation R from U to W is a serial IVF relation if it satisfies $\bigvee_{y \in W} R(x, y) = 1_{L^I}$ for all $x \in U$. If $U = W$, R is called an IVF relation on U. R is a reflexive IVF relation if $R(x, x) = 1_{L^I}$ for all $x, y \in U$. R is a symmetric IVF relation if $R(x, y) = R(y, x)$ for all $x, y \in U$. R is a $\mathcal{T}-$ transitive IVF relation if $R(x,z) \geq_{L^I} \bigvee_{y \in U} \mathcal{T}(R(x,y), R(y,z))$ for all $x, y, z \in U$. R is a $\mathcal{T}-$ similarity IVF relation if it is reflexive, symmetric and $\mathcal{T}-$ transitive.

3 Interval-Valued Fuzzy Residual Implicator and its Dual Operator

In [11], an interval-valued fuzzy residual implicator defined by the authors satisfies axioms of Smets and Magrez on L^I. In this section, by employing the interval-valued fuzzy residual implicator, we consider its dual operator which will be used to construct generalized interval-valued fuzzy rough sets in the present paper.

Definition 3.1 ([11]) Let $\mathcal{T}$ be a continuous IVF t-norm on L^I. An interval-valued fuzzy residual implicator on L^I generated by $\mathcal{T}$ generated by Θ defined as follows:

$$\Theta(I_1, I_2) = \sup\{I_3 \in L^I | \mathcal{T}(I_1, I_3) \leq_{L^I} I_2\},$$

where $I_1 = [\mu_1, \nu_1], I_2 = [\mu_2, \nu_2] \in L^I$.

Theorem 3.1 ([11]) *Let T be a continuous t-norm on [0,1]. Then the interval-valued fuzzy residual implicator Θ is given by*

$$\Theta(I_1, I_2) = [\theta(\mu_1, \mu_2) \wedge \theta(\nu_1, \nu_2), \theta(\nu_1, \nu_2)], \tag{3}$$

for all $I_1, I_2 \in L^I$, where θ is the residual implicator of t-norm T on [0,1] given by $\theta(a, b) = \sup\{c \in I | T(a, c) \leq b.\}, \quad \forall a, b \in [0, 1]$.

Suppose that T is a continuous t-norm on [0,1]. Then for all $I_1 = [\mu_1, \nu_1]$, $I_2 = [\mu_2, \nu_2] \in L^I$, the mapping $\mathcal{S}$: $L^I \times L^I \to L^I$ defined by

$$\mathcal{S}[I_1, I_2] = [1 - T(1 - \mu_1, 1 - \mu_2), 1 - T(1 - \nu_1, 1 - \nu_2)], \tag{4}$$

is an IVF t-conorm on L^I.

Theorem 3.2 *Let $\mathcal{N}$ be an IVF standard negator. Then we have*

$$\mathcal{N}\mathcal{S}(\mathcal{N}(I_1), \mathcal{N}(I_2)) = \mathcal{T}(I_1, I_2). \tag{5}$$

Theorem 3.2 *Shows that the IVF t-norm $\mathcal{T}$ given by Eq. (1) and the IVF t-conorm $\mathcal{S}$ given by Eq. (4) are dual to each other with respect to the IVF standard negator.*

Now, we can define a binary of operation on L^I as follows:

$$\Psi(I_1, I_2) = \inf\{I_3 \in L^I | \mathcal{S}(I_1, I_3) \geq_{L^I} I_2, I_1, I_2 \in L^I\}.$$

Theorem 3.3 *Let T be a continuous t-norm on [0,1]. Then the following equation holds:*

$$\Psi(I_1, I_2) = [1 - \theta(1 - \mu_1, 1 - \mu_2), 1 - \theta(1 - \nu_1, 1 - \nu_2) \wedge \theta(1 - \mu_1, 1 - \mu_2)], \tag{6}$$

for all $I_1, I_2 \in L^I$.

Theorem 3.4 *Let $\mathcal{N}$ be an IVF standard negator. Then for all $I_1, I_2 \in L^I$*

$$\mathcal{N}\Theta(\mathcal{N}(I_1), \mathcal{N}(I_2)) = \Psi(I_1, I_2).$$

Theorem 3.4 shows that the interval-valued fuzzy residual implicator Θ given by Eq. (3) and Ψ given by Eq. (6) are dual to each other with respect to the IVF standard negator.

Theorem 3.5 *Let T be a continuous t-norm on [0,1]. Then the dual operator Ψ of the interval-valued fuzzy residual implicator Θ enjoys the following properties: for all $I_1, I_2, I_3 \in L^I$,*

(1) $\Psi(0_{L^I}, I_2) = I_2, \Psi(I_2, 0_{L^I}) = 0_{L^I}$, $\Psi(1_{L^I}, I_2) = 0_{L^I}, \Theta(I_2, I_2) = 0_{L^I}$.
(2) $I_1 \leq_{L^I} I_2 \Rightarrow \Psi(I_3, I_1) \leq_{L^I} \Psi(I_3, I_2)$, $\Psi(I_1, I_3) \geq_{L^I} \Psi(I_2, I_3)$.
(3) $I_1 \geq_{L^I} I_2 \Leftrightarrow \Psi(I_1, I_2) = 0_{L^I}$.
(4) $\Psi(I_1, \Psi(I_2, I_3)) = \Psi(I_2, \Psi(I_1, I_3))$.
(5) $\Psi(\bigwedge_{i \in \Pi} I_i, I_2) = \bigvee_{i \in \Pi} \Psi(I_i, I_2), \quad \Psi(I_1, \bigvee_{j \in \Pi} I_j) = \bigvee_{j \in \Pi} \Psi(I_1, I_j)$ where $I_i, I_j \in L^I, i, j \in \Pi$, Π is any index set.
(6) $\bigvee_{I_2 \in L^I} \Psi(\Psi(I_1, I_2), I_2) = I_1$, i.e. $\Psi(\Psi(I_1, I_2), I_2) \leq_{L^I} I_1$.
(7) $\Psi(\mathcal{S}(I_1, I_2), I_3) = \Psi(I_1, \Psi(I_2, I_3))$.
(8) $\mathcal{S}(I_1, \Psi(I_1, I_2)) \geq_{L^I} I_2$.
(9) $\mathcal{S}(I_1, I_2) \geq_{L^I} I_3 \Leftrightarrow I_2 \geq_{L^I} \Psi(I_1, I_3)$.
(10) $\mathcal{S}(\Psi(I_1, I_3), \Psi(I_3, I_2)) \geq_{L^I} \Psi(I_1, I_2)$.
(11) $\mathcal{S}(\Psi(I_1, I_2), I_3) \geq_{L^I} \Psi(I_1, \mathcal{S}(I_2, I_3))$.
(12) $\Psi(I_1, I_2) \geq_{L^I} \Psi(\mathcal{S}(I_1, I_3), \mathcal{S}(I_2, I_3))$.
(13) $\Psi(I_1, \bigvee_{i \in \Pi} I_i) \geq_{L^I} \bigvee_{i \in \Pi} \Psi(I_1, I_i)$, where $I_i \in L^I, i \in \Pi$, Π is any index set.
(14) $\Psi(I_2, \mathcal{S}(I_1, I_2)) \leq_{L^I} I_1$.
(15) $\Psi(I_1, I_3) \leq_{L^I} \Psi(I_2, I_3) \Rightarrow I_1 \geq_{L^I} I_2$.

Proof Straightforward.

For the sake of convenience, we will use the following labels.

For $A \in \text{IVF}(U)$ and $x \in U$, $(\sim_{\mathcal{N}} A)(x) = \mathcal{N}(A)(x)$.

For $A, B \in IVF(U), \Psi(A, B)(\Theta(A, B)$, respectively) is an interval-valued fuzzy set in IVF(U), and satisfies $\Psi(A, B)(x) = \Psi(A(x), B(x))$ for any $x \in U$ $(\Theta(A, B)(x) = \Theta(A(x), B(x))$, respectively).

4 Generalized Interval-Valued Fuzzy Rough Approximation Operators and theirs Properties

In this section, by employing the interval-valued fuzzy residual implicator Θ residual operator Ψ, we will define the upper and lower approximations of IVF sets with respect to an arbitrary IVF approximation space and investigate the properties of IVF rough approximation operators.

In the sequel, we will assume that $\mathcal{N}$ is an IVF standard negator on L^I given by $\mathcal{N}([\mu,v]) = [1-v, 1-\mu]$ for $[\mu, v] \in L^I$, and N is a standard negator on [0,1] given by $N(x) = 1 - x$, for $x \in [0,1]$.

Definition 4.1 Let $R \in \mathrm{IVF}(U \times W)$ be an IVF relation from U to W. Then the triple (U, W, R) is called an IVF approximation space. For any $A \in \mathrm{IVF}(W)$, the upper and lower IVF rough approximations of A with respect to the approximation space (U, W, R), denoted by $\overline{R}(A)$ and $\underline{R}(A)$, respectively, are two IVF sets whose membership functions are defined respectively by:

$$\begin{aligned} \overline{R}(A)(x) &= \bigvee_{y\in W} \Psi\big(\sim_{\mathcal{N}} R(x,y), A(y)\big),\ x \in U, \\ \underline{R}(A)(x) &= \bigwedge_{y \in W} \Theta(R(x,y), A(y)),\ x \in U. \end{aligned} \tag{7}$$

The operators $\overline{R}, \underline{R} : \mathrm{IVF}(W) \to \mathrm{IVF}(U)$ are, respectively, referred to as the generalized upper and lower IVF rough approximation operators of (U, W, R). The pair $(\underline{R}A, \overline{R}A)$ is called the generalized IVF rough set of A with respect to (U, W, R).

Remark 4.1 When Θ is a residual implicator on [0,1], Ψ is its dual of the residual of implicator on [0,1], R is a fuzzy relation from U to W, $\mathcal{N}$ is a standard negator on [0,1] and A is a fuzzy set of W it can be observed that the IVF rough set defined by us degenerates to the fuzzy rough set introduced by Mi and Zhang in [16].

Example 4.1 Let

$$\begin{aligned} &U = W = \{x_1, x_2\}, \\ &A = \{\langle x_1, [0.1, 0.7]\rangle, \langle x_2, [0.6, 0.8]\rangle\} \in \mathrm{IVF}(U), \\ &R = \{\langle (x_1, x_1), [0.7, 0.8]\rangle, \langle (x_1, x_2), [0.3, 0.5]\rangle, \\ &\quad \langle (x_2, x_1), [0.4, 0.6]\rangle, \langle (x_2, x_2), [0.1, 1]\rangle\} \in \mathrm{IVF}(U \times U). \end{aligned}$$

$T = \min$. Then

$$\begin{aligned} \overline{R}(A)(x_1) &= \Psi\big(\sim_{\mathcal{N}} R(x_1,x_1), A(x_1)\big) \vee \Psi\big(\sim_{\mathcal{N}} R(x_1,x_2), A(x_2)\big) \\ &= \Psi([0.2, 0.3], [0.1, 0.7]) \vee \Psi([0.5, 0.7], [0.6, 0.8]) \\ &= [0, 0.7] \vee [0.6, 0.8] = [0.6, 0.8], \\ \overline{R}(A)(x_2) &= \Psi\big(\sim_{\mathcal{N}} R(x_2,x_1), A(x_1)\big) \vee \Psi\big(\sim_{\mathcal{N}} R(x_2,x_2), A(x_2)\big) \\ &= \Psi([0.4, 0.6], [0.1, 0.7]) \Psi([0, 0.9], [0.6, 0.8]) \\ &= [0, 0.7] \vee [0.6, 0.6] = [0.6, 0.7]. \end{aligned}$$

Hence, $\overline{R}(A) = \{\langle x_1, [0.6, 0.8]\rangle, \langle x_2, [0.6, 0.7]\rangle\}$.

Similarly, by Eq. (7), we have

$\underline{R}(A)(x_1) = [0.1, 0.7], \underline{R}(A)(x_2) = [0.1, 0.8]$.

Hence, $\underline{R}(A) = \{\langle x_1, [0.1, 0.7]\rangle, x_2\langle [0.1, 0.8]\rangle\}$.

Although IVF set theory has the virtue of complementing fuzzy sets to model vagueness and uncertainty precisely, it cannot solve some approximation problems of concepts in data analysis. To overcome this difficulty, it is natural for us to combine the interval-valued fuzzy set and rough set models. So the concept of generalized interval-valued fuzzy rough sets is presented by us. Because the new hybrid model includes both ingredients of IVF set and rough set, it is more flexible and effective to cope with imperfect and imprecise information than IVF set and rough set.

In what follows, by an example we will explain what kind of conditions make the method better than the traditional fuzzy rough set.

Example 4.2 Let (U, W, R) be a fuzzy approximation space, where $U = W = \{x_1, x_2\}$. Suppose that there is an expert who is invited to evaluate the possible membership degrees of the relationships between x_i and x_j with a crisp number. In that case, R is a fuzzy relation defined as follows:

$$R = \frac{0.7}{(x_1, x_1)} + \frac{0.4}{(x_1, x_2)} + \frac{0.5}{(x_2, x_1)} + \frac{0.8}{(x_2, x_2)}.$$

If a fuzzy set $A = \frac{0.6}{x_1} + \frac{0.7}{x_2}$, then by the definition of fuzzy approximation operators in [16], we obtain

$$\begin{aligned} &\underline{R}(A)(x_1) = 0.6,\ \underline{R}(A)(x_2) = 0.7; \\ &\overline{R}(A)(x_1) = 0.7,\ \overline{R}(A)(x_2) = 0.7. \end{aligned}$$

Hence, we can conclude that

$$\underline{R}(A) = \frac{0.6}{x_1} + \frac{0.7}{x_2},\ \overline{R}(A) = \frac{0.7}{x_1} + \frac{0.7}{x_2}.$$

By the above fuzzy rough approximations $\overline{R}(A)$ and $\underline{R}(A)$, we can cope with some decision-making problems.

However, in many real decision-making problems, due to the shortage of the expert's experience and insufficiency in available information, the decision-makers are easy to lose information and cannot supply correct policies by using traditional fuzzy rough set theory. So, it may be difficult for decision-makers to exactly quantify their opinions with a crisp number. Instead, the basic characteristics of the decision-making problems described by an interval number within [0,1] can overcome such a situation. For example, due to the shortage of an expert's experience and insufficiency in available information, we cannot present the precise membership degree of the relationship between x_2 and x_1 by a crisp number 0.5, but we can provide an interval number [0.4, 0.6] to depict the possible membership degree of the relationship between x_2 and x_1 (see Example 4.1). Considering the fact, it is necessary for us to extend a fuzzy relation (set) to an IVF relation (set). In this case, R is an IVF relation defined in Example 4.1 above. Meanwhile, A is an IVF set defined in Example 4.1.

Thus we have

$$\underline{R}(A) = \{\langle x_1, [0.1, 0.7]\rangle, \langle x_2, [0.1, 0.8]\rangle\};$$
$$\overline{R}(A) = \{\langle x_1, [0.6, 0.8]\rangle, \langle x_2, [0.6, 0.7]\rangle\}.$$

Comparing with the results of two type approximation operators, we can see that generalized IVF rough sets in Definition 4.1 can contain more information than the traditional fuzzy rough set in [16] due to insufficiency in available information. So in many real decision-making problems, the generalized IVF rough set is more comprehensive and objective method than the traditional fuzzy rough set.

Theorem 4.1 For any IVF approximation space (U, W, R) if Θ is an interval-valued fuzzy residual implicator on L^I and Ψ is dual to Θ with respect to the IVF standard negator $\mathcal{N}$, then

$$\overline{R}(A) = \sim_{\mathcal{N}} \underline{R}(\sim_{\mathcal{N}} A), \quad \forall A \in \mathrm{IVF}(W),$$
$$\underline{R}(A) = \sim_{\mathcal{N}} \overline{R}(\sim_{\mathcal{N}} A), \quad \forall A \in \mathrm{IVF}(W).$$

Proof By Definition 4.1 and Theorem 3.4, we can easily get the conclusion of the theorem.

Theorem 4.1 shows that the generalized IVF rough operators $\overline{R}$ and $\underline{R}$ are dual to each other.

Theorem 4.2 *Let (U, W, R) be an IVF approximation space. Then the upper and lower IVF rough approximation operators defined by Eq. (7) admit the following properties: for any* $A, B, A_i \in \mathrm{IVF}(W)$, $\forall i \in \Pi, \Pi$ *is an index set,* $M \subseteq W, [\alpha_1, \alpha_2] \in L^I, (x, y) \in U \times W$,

(IVFU1) $\overline{R}(\Psi(\widehat{[\alpha_1, \alpha_2]}, A)) = \Psi(\widehat{[\alpha_1, \alpha_2]}, \overline{R}(A))$,

(IVFL1) $\underline{R}(\Theta(\widehat{[\alpha_1, \alpha_2]}, A)) = \Theta(\widehat{[\alpha_1, \alpha_2]}, \underline{R}(A))$.

(IVFU2) $\overline{R}(\bigcup_{i\in\Pi} A_i) = \bigcup_{i\in\Pi} \overline{R}(A_i)$,

(IVFL2) $\underline{R}(\bigcap_{i\in\Pi} A_i) = \bigcap_{i\in\Pi} \underline{R}(A_i)$.

(IVFU3) $\overline{R}(\widehat{[\alpha_1, \alpha_2]}) \subseteq \widehat{[\alpha_1, \alpha_2]}$,

(IVFL3) $\underline{R}(\widehat{[\alpha_1, \alpha_2]}) \supseteq \widehat{[\alpha_1, \alpha_2]}$.

(IVFU4) $\overline{R}(\varnothing) = \varnothing$,

(IVFL4) $\underline{R}(W) = U$.

(IVFU5) $\overline{R}(\bigcap_{i\in\Pi} A_i) \subseteq \bigcap_{i\in\Pi} \overline{R}(A_i)$,

(IVFL5) $\underline{R}(\bigcup_{i\in\Pi} A_i) \supseteq \bigcup_{i\in\Pi} \underline{R}(A_i)$.

(IVFU6) $A \subseteq B \Rightarrow \overline{R}(A) \subseteq \overline{R}(B)$,

(IVFL6) $A \subseteq B \Rightarrow \underline{R}(A) \subseteq \underline{R}(B)$.

(IVFU7) $\overline{R}(\Psi([1,1]_{W-\{y\}}, \widehat{[\alpha_1, \alpha_2]}))(x) = \Psi(\sim_{\mathcal{N}} R(x,y), [\alpha_1, \alpha_2])$,

(IVFL7) $\underline{R}(\Theta([1,1]_y, \widehat{[\alpha_1, \alpha_2]}))(x) = \Theta(R(x,y), [\alpha_1, \alpha_2])$.

(IVFU8) $\overline{R}([1,1]_y)(x) = \Psi(\sim_{\mathcal{N}} R(x,y), [1,1])$,

(IVFL8) $\underline{R}([1,1]_{W-\{y\}})(x) = \Theta(R(x,y), [0,0])$.

(IVFU9) $\overline{R}([1,1]_M)(x) = \bigvee_{y\in M} \Psi(\sim_{\mathcal{N}} R(x,y), [1,1])$,,

(IVFL9) $\underline{R}([1,1]_M)(x) = \bigwedge_{y\notin M} \Theta(R(x,y), [0,0])$.

Proof Since the IVF rough operators $\overline{R}$ and $\underline{R}$ are dual to each other, we only investigate the case of $\overline{R}$.

(IVFU1). According to Eq. (7) and Theorem 3.5(4) and (5), for all $x \in U$, we derive

$$\begin{aligned}\overline{R}(\Psi(\widehat{[\alpha_1, \alpha_2]}, A))(x) &= \bigvee_{y\in W} \Psi(\sim_{\mathcal{N}} R(x,y), \Psi([\alpha_1, \alpha_2], A(y))) \\ &= \bigvee_{y\in W} \Psi([\alpha_1, \alpha_2], \Psi(\sim_{\mathcal{N}} R(x,y), A(y))) \\ &= \Psi\left([\alpha_1, \alpha_2], \bigvee_{y\in W} \Psi(\sim_{\mathcal{N}} R(x,y), A(y))\right) \\ &= \Psi\left(\widehat{[\alpha_1, \alpha_2]}, \overline{R}(A)\right)(x).\end{aligned}$$

Hence, (IVFU1) holds.

(IVFU2). Similar to (IVFU1), it can be easily verified.

(IVFU3). For all $x \in U$, by Theorem 3.5(5) and (2), we obtain

$$\begin{aligned}\overline{R}(\widehat{[\alpha_1, \alpha_2]})(x) &= \bigvee_{y\in W} \Psi(\sim_{\mathcal{N}} R(x,y), [\alpha_1, \alpha_2]) \\ &= \Psi\left(\bigwedge_{y\in W} (\sim_{\mathcal{N}} R(x,y)), [\alpha_1, \alpha_2]\right) \\ &\leq_{L^I} \Psi([0,0], [\alpha_1, \alpha_2]) \\ &= [\alpha_1, \alpha_2] = \widehat{[\alpha_1, \alpha_2]}(x).\end{aligned}$$

Thus, (IVFU3) holds.

(IVFU4). By taking $\alpha_1 = 0, \alpha_2 = 0$ instead of $\widehat{[\alpha_1, \alpha_2]}$ in (IVFU3).

(IVFU5) and (IVFU6). They follow immediately from Eq. (7) and Theorem 3.5(2).

(IVFU7). By the definitions of $[1,1]_{W-\{y\}}$ and $\overline{R}$, we obtain

$$\begin{aligned}&\overline{R}\left(\Psi\left([1,1]_{W-\{y\}}, \widehat{[\alpha_1, \alpha_2]}\right)\right)(x) \\ &= \bigvee_{z\in W} \Psi\left(\sim_{\mathcal{N}} R(x,z), \Psi\left([1,1]_{W-\{y\}}(z), [\alpha_1, \alpha_2]\right)\right) \\ &= \bigvee_{z\neq y} \Psi(\sim_{\mathcal{N}} R(x,z), \Psi([1,1], [\alpha_1, \alpha_2])) \vee \Psi(\sim_{\mathcal{N}} R(x,y), [\alpha_1, \alpha_2]) \\ &= \bigvee_{z\neq y} \Psi(\sim_{\mathcal{N}} R(x,z), [0,0]) \vee \Psi(\sim_{\mathcal{N}} R(x,y), [\alpha_1, \alpha_2]) \\ &= \Psi(\sim_{\mathcal{N}} R(x,y), [\alpha_1, \alpha_2]),\end{aligned}$$

which implies that (IVFU7) holds.

(IVFU8). From Eq. (7), we can see that

$$\overline{R}([1,1]_y)(x) = \bigvee_{z\in W} \Psi\Big(\sim_{\mathcal{N}} R(x,z), [1,1]_y(z)\Big)$$
$$= \bigvee_{z\neq y} \Psi(\sim_{\mathcal{N}} R(x,z), [0,0]) \vee \Psi(\sim_{\mathcal{N}} R(x,y), [1,1])$$
$$= \Psi(\sim_{\mathcal{N}} R(x,y), [1,1]).$$

Hence, (IVFU8) holds.

(IVFU9). By the definition of $[1,1]_M$ and $\overline{R}$, we get

$$\overline{R}([1,1]_M)(x) = \bigvee_{y\in W} \Psi\big(\sim_{\mathcal{N}} R(x,y), [1,1]_M(y)\big)$$
$$= \bigvee_{y\notin M} \Psi(\sim_{\mathcal{N}} R(x,y), [0,0])$$
$$\vee \left(\bigvee_{y\in M} \Psi(\sim_{\mathcal{N}} R(x,y), [1,1])\right)$$
$$= \bigvee_{y\in M} \Psi(\sim_{\mathcal{N}} R(x,y), [1,1]).$$

Thus, (IVFU9) holds

Now we discuss the relationships between the properties of special IVF relations and the properties of the generalized IVF rough approximation operators. We show that the properties of some special IVF relations can be characterized by IVF rough approximation operators.

Theorem 4.3 *Let (U, W, R) be an IVF approximation space.* $\overline{R}$ and $\underline{R}$ *are the generalized IVF approximation operators defined by Eq. (7). Then R is serial iff one of the following properties holds:*

(IVFU0) $\overline{R}(\widehat{[\alpha_1,\alpha_2]}) = \widehat{[\alpha_1,\alpha_2]}, \forall [\alpha_1,\alpha_2] \in L^I$;
(IVFU0)′ $\overline{R}(W) = U$;
(IVFL0) $\underline{R}(\widehat{[\alpha_1,\alpha_2]}) = \widehat{[\alpha_1,\alpha_2]}, \forall [\alpha_1,\alpha_2] \in L^I$;
(IVFL0)′ $R(\varnothing) = \varnothing$.

Proof First, we need to prove that (IVFU0)′ ⇔ R is serial ⇔ (IVFU)▹. If R is serial then $\bigvee_{y\in W} R(x,y) = [1,1]$ for all $x \in U$. By (IVFU3), we can obtain $\overline{R}(\widehat{[\alpha_1,\alpha_2]}) = \widehat{[\alpha_1,\alpha_2]}$, for any $[\alpha_1,\alpha_2] \in L^I$. So, (IVFU0) holds.

Conversely, by assuming that (IVFU0) holds and using (IVFU3), we have

$$\Psi\left(\sim_{\mathcal{N}}\left(\bigvee_{y\in W} R(x,y)\right), [\alpha_1,\alpha_2]\right) = \Psi([0,0],[\alpha_1,\alpha_2]).$$

According to Theorem 3.5(15), it follows that $\bigvee_{y\in W} R(x,y) = [1,1]$. So R is serial. On the other hand, if R is serial, then

$$\overline{R}(W)(x) = \overline{R}\left(\widehat{[1,1]})(x) = \bigvee_{y\in W} \Psi(\sim_{\mathcal{N}} R(x,y), [1,1]\right)$$
$$= \Psi\left(\sim_{\mathcal{N}}(\bigvee_{y\in W} R(x,y)), [1,1]\right)$$
$$= \Psi([0,0],[1,1]) = [1,1] = U(x).$$

Therefore, (IVFU0)′ holds.

Conversely, if (IVFU0)′ holds, then by Theorem 3.5(15) and the above equation, it can be directly obtained that R is serial.

Second, by the Theorem 4.1, we can observe that (IVFU0) ⇔ (IVFL0), (IVFU0)′ ⇔ (IVFL0)′, from which we conclude that R is a serial ⇔ (IVFU0) ⇔ (IVFU0)′ ⇔ (IVFL0) ⇔ (IVFL0)′.

Theorem 4.4 *Let (U, R) be an IVF approximation space. If R is an IVF relation on U,* $\overline{R}$ *and* $\underline{R}$ *are the generalized IVF approximation operators of (U, R), then*

(1) R is reflexive ⇔ (IVFUR)$A \subseteq \overline{R}(A)$
⇔ (IVFLR)$\underline{R}(A) \subseteq A$.

(2) R is symmetric ⇔ (IVFUS)
$$\overline{R}\Big(\Psi\Big([1,1]_{U-\{x\}}, \widehat{[\alpha_1,\alpha_2]}\Big)\Big)(y)$$
$$= \overline{R}\Big(\Psi\Big([1,1]_{U-\{y\}}, \widehat{[\alpha_1,\alpha_2]}\Big)\Big)(x)$$
$$\Leftrightarrow \text{(IVFLS)}\underline{R}\Big(\Theta([1,1]_{\{x\}}, \widehat{[\alpha_1,\alpha_2]})\Big)(y)$$
$$= \underline{R}\Big(\Theta\Big([1,1]_{\{y\}}, \widehat{[\alpha_1,\alpha_2]}\Big)\Big)(x).$$

(3) R is $\mathcal{T}-$ transitive ⇔ (IVFUT) $\overline{R}(\overline{R}(A)) \subseteq \overline{R}(A)$
⇔ (IVFLT) $\underline{R}(A) \subseteq \underline{R}(\underline{R}(A))$

Proof (1) If R is reflexive, then, for any $A \in \mathrm{IVF}(U)$ and $x \in U$, we have

$$\overline{R}(A)(x) = \bigvee_{y\in U} \Psi(\sim_{\mathcal{N}} R(x,y), A(y))$$
$$\geq_{L^I} \Psi(\sim_{\mathcal{N}} R(x,x), A(x))$$
$$= \Psi([0,0], A(x)) = A(x),$$

which implies that $A \subseteq \overline{R}(A)$

Conversely, if (IVFUR) holds, then by (IVFU7), we obtain

$$\Psi(\sim_{\mathcal{N}} R(x,x), [\alpha_1,\alpha_2]) = \overline{R}\Big(\Psi([1,1]_{U-\{x\}}, \widehat{[\alpha_1,\alpha_2]})\Big)(x)$$
$$\geq_{L^I} \Psi\Big([1,1]_{U-\{x\}}, \widehat{[\alpha_1,\alpha_2]}\Big)x)$$
$$= \Psi\Big([1,1]_{U-\{x\}}(x), [\alpha_1,\alpha_2]\Big)$$
$$= \Psi([0,0],[\alpha_1,\alpha_2]).$$

According to Theorem 3.5(15), we have $R(x,x) \geq_{L^I} [1,1]$

Hence, we conclude that R is reflexive. On the other hand, by Theorem 4.1 we can observe that (IVFUR) ⇔ (IVFLR). So R is reflexive ⇔ (IVFUR) ⇔ (IVFLR).

It follows immediately from (IVFU7) and (IVFL7). If R is $\mathcal{T}-$ transitive, then, for any $A \in \mathrm{IVF}(U)$ and $x \in U$ we have

$$\begin{aligned}\overline{R}\big(\overline{R}(A)\big)(x) &= \bigvee_{y\in U}\Psi\big(\sim_{\mathcal{N}}R(x,y),\overline{R}(A)(y)\big)\\ &= \bigvee_{y\in U}\Psi\Big(\sim_{\mathcal{N}}R(x,y),\bigvee_{z\in U}\Psi\big(\sim_{\mathcal{N}}R(y,z),A(z)\big)\Big)\\ &= \bigvee_{y\in U}\bigvee_{z\in U}\Psi\big(\sim_{\mathcal{N}}R(x,y),\Psi\big(\sim_{\mathcal{N}}R(y,z),A(z)\big)\big)\\ &= \bigvee_{y\in U}\bigvee_{z\in U}\Psi\big(\mathcal{S}\big(\sim_{\mathcal{N}}R(x,y),\sim_{\mathcal{N}}R(y,z)\big),A(z)\big)\\ &= \bigvee_{y\in U}\bigvee_{z\in U}\Psi\big(\sim_{\mathcal{N}}\mathcal{T}(R(x,y),R(y,z)),A(z)\big)\\ &\leq_{L^I}\bigvee_{y\in U}\bigvee_{z\in U}\Psi\big(\sim_{\mathcal{N}}R(x,z),A(z)\big)\\ &= (\overline{R}(A))(x).\end{aligned}$$

So, (IVFUT) holds.

Conversely, if (IVFUT) holds, then by (IVFU7), for any $x,y\in U$, one has

$$\begin{aligned}&\Psi(\sim_{\mathcal{N}}R(x,y),[\alpha_1,\alpha_2])\\ &\quad=\overline{R}\Big(\Psi([1,1]_{U-\{y\}},\widehat{[\alpha_1,\alpha_2]})\Big)(x)\\ &\quad\geq_{L^I}\overline{R}\Big(\overline{R}\Big(\Psi\Big([1,1]_{U-\{y\}},\widehat{[\alpha_1,\alpha_2]}\Big)\Big)\Big)(x)\\ &\quad=\bigvee_{z\in U}\Psi\Big(\sim_{\mathcal{N}}R(x,z),\overline{R}\Big(\Psi\Big([1,1]_{U-\{y\}},\widehat{[\alpha_1,\alpha_2]}\Big)\Big)\Big)(z)\\ &\quad=\bigvee_{z\in U}\Psi(\sim_{\mathcal{N}}R(x,z),\Psi(\sim_{\mathcal{N}}R(z,y),[\alpha_1,\alpha_2]))\\ &\quad=\bigvee_{z\in U}\Psi(\mathcal{S}(\sim_{\mathcal{N}}R(x,z),\sim_{\mathcal{N}}R(z,y)),[\alpha_1,\alpha_2])\\ &\quad=\bigvee_{z\in U}\Psi(\sim_{\mathcal{N}}\mathcal{T}R(x,z),R(z,y),[\alpha_1,\alpha_2])\\ &\quad=\Psi\Big(\sim_{\mathcal{N}}\Big(\bigvee_{z\in U}\mathcal{T}(R(x,z),R(z,y))\Big),[\alpha_1,\alpha_2]\Big).\end{aligned}$$

By virtue of Theorem 3.5(15), we have

$R(x,\ y)\geq_{L^I}\bigvee_{z\in U}\mathcal{T}(R(x,\ z),R(z,\ y)).$

So R is $\mathcal{T}-$ transitive.

On the other hand, by Theorem 4.1, we can observe that (IVFUT) ⇔ (IVFLT). Hence, R is $\mathcal{T}-$ transitive ⇔ (IVFUT) ⇔ (IVFLT).

5 Axiomatic Characterization of Generalized IVF Rough Approximation Operators

In this section, we will present an axiomatic characterization of generalized IVF rough sets by defining a pair of abstract IVF approximation operators.

Now we consider the abstract interval-valued fuzzy set-theoretic operators $L,H:\mathrm{IVF}(W)\to\mathrm{IVF}(U)$.

Definition 5.1 Let $L,H:\mathrm{IVF}(W)\to\mathrm{IVF}(U)$ be two operators. They are referred to as dual operators if for all $A\in\mathrm{IVF}(W)$ the following holds:

(1) $L(A)=\sim_{\mathcal{N}}H\big(\sim_{\mathcal{N}}A\big)$,
(2) $H(A)=\sim_{\mathcal{N}}L\big(\sim_{\mathcal{N}}A\big)$.

Definition 5.2 Suppose that $L,H:\mathrm{IVF}(W)\to\mathrm{IVF}(U)$ are two dual operators. Then L and H are referred to as IVF approximation operator iff H satisfies the axioms (H_1) and (H_2), or equivalently L satisfies the axioms (L_1) and (L_2), where

(H_1) $H(A\cup B)=H(A)\cup H(B)$,
(H_2) $H(\Psi(\widehat{[\alpha_1,\alpha_2]}),A))=\Psi(\widehat{[\alpha_1,\alpha_2]},HA)$;
(L_1) $L(A\cap B)=L(A)\cap L(B)$,
(L_2) $L(\Theta(\widehat{[\alpha_1,\alpha_2]}),A))=\Theta(\widehat{[\alpha_1,\alpha_2]},LA)$,

For any $A,B\in\mathrm{IVF}(W)$ and $[\alpha_1,\alpha_2]\in L^I$.

Lemma 5.1 *Suppose that* $E:\mathrm{IVF}(W)\to L^I$ *satisfies the following conditions:*

(1) $E(A\cap B)=E(A)\cap E(B)$,
(2) $E(\Theta(\widehat{a},A))=\Theta(a,E(A))$ *where* $A,B\in\mathrm{IVF}(W)$ *and* $a\in L^I$.

Then there exists $v\in\mathrm{IVF}(W)$, *such that* $E(A)=\bigwedge_{y\in W}\Theta(v(y),A(y))$, $\forall A\in\mathrm{IVF}(W)$

Proof For any $\forall A\in\mathrm{IVF}(W)$,we denote $\gamma=E(A)$. It follows from the item (2) that $E(\Theta(\widehat{\gamma},A))=\Theta(\gamma,E(A))=\Theta(\gamma,\gamma)=[1,1]$.

Define $v=\wedge\{A\in\mathrm{IVF}(W):E(A)=[1,1]\}$. Clearly, $\Theta(\widehat{\gamma},A)\supseteq v$. By virtue Theorem 4(2) in [11], we can see that $\Theta(\Theta(\gamma,A(y)),A(y))\leq_{L^I}\Theta(v(y),A(y))$. Thus, by Theorem 5(6) in [11], we further obtain $\gamma\leq_{L^I}\bigwedge_{y\in W}\Theta(v(y),A(y))$.

On the other hand, denote

$$\eta=\sup\big\{c\in L^I:E(\Theta(\hat{c},A))=[1,1]\big\}.$$

Then

$$\begin{aligned}\Theta(\eta,\gamma)&=\inf\{\Theta(c,\gamma):E(\Theta(\hat{c},A))=[1,1]\}\\ &=\inf\{\Theta(c,\gamma):\Theta(c,E(A))=[1,1]\}\\ &=[1,1].\end{aligned}$$

Hence, by Theorem 4(3) in [11], we get $\eta\leq_{L^I}\gamma$. For any $a>_{L^I}E(A)=\gamma$, we have $a>_{L^I}\eta$. It then follows that

$E(\Theta(\widehat{a}, A)) <_{L^I} [1,1]$. By the definition of v, $E(v) = [1,\ 1]$, we get $E(\Theta(\widehat{a}, A)) <_{L^I} E(v)$. It is easy to see that E is monotone. Therefore, $v \not\subset \Theta(\widehat{\gamma}, A)$. By Theorem 5(6) in [11], $a >_{L^I} \bigwedge_{y\in W} \Theta(v(y), A(y))$. Thus $\gamma \geq_{L^I} \bigwedge_{y\in W} \Theta(v(y), A(y))$. Hence, $\gamma = \bigwedge_{y\in W} \Theta(v(y), A(y))$.

Lemma 5.2 *Suppose that $L, H : \mathrm{IVF}(W) \to \mathrm{IVF}(U)$ are two dual IVF approximation operators. Then for each $x \in U$, there exist IVF sets v_x and $u_x \in \mathrm{IVF}(W)$ such that for any $A \in \mathrm{IVF}(W)$,*

$$H(A)(x) = \bigvee_{y\in W} \Psi(v_x(y), A(y)),$$

$$L(A)(x) = \bigwedge_{y\in W} \Theta(u_x(y), A(y))$$

Proof Let $E_x(A) = \sim_{\mathcal{N}} H(\sim_{\mathcal{N}} A)(x), \forall x \in U$. Then

$$\begin{aligned} E_x(A \cap B) &= \sim_{\mathcal{N}} H\big(\sim_{\mathcal{N}} (A \cap B)\big)(x) \\ &= \sim_{\mathcal{N}} H\big((\sim_{\mathcal{N}} A) \cup (\sim_{\mathcal{N}} B)\big)(x) \\ &= \sim_{\mathcal{N}} \big(H(\sim_{\mathcal{N}} A) \cup H(\sim_{\mathcal{N}} B)\big)(x) \\ &= \sim_{\mathcal{N}} \big(H(\sim_{\mathcal{N}} A)(x) \vee H(\sim_{\mathcal{N}} B)(x)\big) \\ &= \big(\sim_{\mathcal{N}} H(\sim_{\mathcal{N}} A)(x)\big) \\ &\quad \wedge \big(\sim_{\mathcal{N}} H(\sim_{\mathcal{N}} B)(x)\big) \\ &= E_x(A) \wedge E_x(B), \end{aligned}$$

$$\begin{aligned} E_x(\Theta(\widehat{[\alpha_1, \alpha_2]}), A)) &= \sim_{\mathcal{N}} H\Big(\sim_{\mathcal{N}} \Theta(\widehat{[\alpha_1, \alpha_2]}, A)\Big)(x) \\ &= \sim_{\mathcal{N}} H\Big(\Psi\Big(\sim_{\mathcal{N}} \widehat{[\alpha_1, \alpha_2]}, \sim_{\mathcal{N}} A\Big)\Big)(x) \\ &= \sim_{\mathcal{N}} \Psi\Big(\sim_{\mathcal{N}} \widehat{[\alpha_1, \alpha_2]}, H(\sim_{\mathcal{N}} A)\Big)(x) \\ &= \Theta\Big(\widehat{[\alpha_1, \alpha_2]}, \sim_{\mathcal{N}} H(\sim_{\mathcal{N}} A)\Big)(x) \\ &= \Theta([\alpha_1, \alpha_2], E_x(A)). \end{aligned}$$

By Lemma 5.1, there exists, there exists $u_x \in \mathrm{IVF}(W)$ such that $E_x(A) = \bigwedge_{y\in W} \Theta(u_x(y), A(y))$. Let $v_x = \sim_{\mathcal{N}} u_x$. Then we have

$$\begin{aligned} H(A)(x) &= \sim_{\mathcal{N}} E_x(\sim_{\mathcal{N}} A) \\ &= \sim_{\mathcal{N}} \Big(\bigwedge_{y\in W} \Theta\big(u_x(y), \sim_{\mathcal{N}} A(y)\big)\Big) \\ &= \bigvee_{y\in W} \big(\sim_{\mathcal{N}} \Theta\big(u_x(y), \sim_{\mathcal{N}} A(y)\big)\big) \\ &= \bigvee_{y\in W} \Psi\big(\sim_{\mathcal{N}} u_x(y), A(y)\big) \\ &= \bigvee_{y\in W} \Psi(v_x(y), A(y)). \end{aligned}$$

On the other hand, since $H(A)(x) = \bigvee_{y\in W} \Psi(v_x(y), A(y))$, we obtain $L(A)(x) = \bigwedge_{y\in W} \Theta(u_x(y), A(y))$.

Let H be an operator from IVF(W) to IVF(U). We define a special IVF relation Rel$H$ from U to W as follows: for all $(x, y) \in U \times W$

$$\mathrm{Rel}H(x,y) = \sim_{\mathcal{N}} \Big(\bigvee_{[\alpha_1,\alpha_2]\in L^I} \Psi\Big(H\Big(\Psi\Big([1,1]_{W-\{y\}}, \widehat{[\alpha_1, \alpha_2]}\Big)\Big)(x), [\alpha_1, \alpha_2]\Big)\Big).$$

Now we consider the relations between IVF approximation operators and the general IVF relations.

Theorem 5.1 *Let $R \in \mathrm{IVF}(U \times W)$. Then $\mathrm{Rel}\bar{R} = R$.*

Proof For any $(x, y) \in U \times W$, we have

$$\begin{aligned} &\mathrm{Rel}\bar{R}(x,y) \\ &= \sim_{\mathcal{N}} \Big(\bigvee_{[\alpha_1,\alpha_2]\in L^I} \Psi\Big(\bar{R}\Big(\Psi\Big([1,1]_{W-\{y\}}, \widehat{[\alpha_1, \alpha_2]}\Big)\Big)(x), [\alpha_1, \alpha_2]\Big)\Big) \\ &= \sim_{\mathcal{N}} \Big(\bigvee_{[\alpha_1,\alpha_2]\in L^I} \Psi(\Psi(\sim_{\mathcal{N}} R(x,y), [\alpha_1, \alpha_2]), [\alpha_1, \alpha_2])\Big) \\ &= \sim_{\mathcal{N}} (\sim_{\mathcal{N}} R(x,y)) = R(x,y). \end{aligned}$$

Theorem 5.2 *Suppose that $L, H : \mathrm{IVF}(W) \to \mathrm{IVF}(U)$ are two dual IVF approximation operators. Then*

$$\overline{\mathrm{Rel}H} = H, \underline{\mathrm{Rel}H} = L.$$

Proof For any $A \in \mathrm{IVF}(W)$ and $x \in U$, we get

$$\begin{aligned} &\overline{\mathrm{Rel}H}(A)(x) \\ &= \bigvee_{y\in W} \Psi\Big(\bigvee_{[\alpha_1,\alpha_2]\in L^I} \Psi\Big(H\Psi\Big([1,1]_{W-\{y\}}, \widehat{[\alpha_1, \alpha_2]}\Big)(x), [\alpha_1, \alpha_2]\Big), A(y)\Big) \\ &= \bigvee_{y\in W} \Psi\Big(\bigvee_{[\alpha_1,\alpha_2]\in L^I} \Psi\Big(\bigvee_{z\in W} \Psi\Big(v_x(z), \Psi\Big([1,1]_{W-\{y\}}, \widehat{[\alpha_1, \alpha_2]}\Big)(z)\Big), [\alpha_1, \alpha_2]\Big), A(y)\Big) \\ &= \bigvee_{y\in W} \Psi\Big(\bigvee_{[\alpha_1,\alpha_2]\in L^I} \Psi(\Psi(v_x(y), [\alpha_1, \alpha_2]), [\alpha_1, \alpha_2]), A(y)\Big) \\ &= \bigvee_{y\in W} \Psi(v_x(y), A(y)) = H(A)(x). \end{aligned}$$

It is easy to see that Rel$H = L$ holds due to the assumption and $\overline{\mathrm{Rel}H}(A)(x) = H(A)(x)$.

Theorem 5.3 *Let L,H be a pair of dual operators. Then there exists an IVF relation $R \in \mathrm{IVF}(U \times W)$ such that $L = \underline{R}$ and $H = \overline{R}$ iff L, H are IVF approximation operators.*

Proof (⇒) It follows immediately from Theorem 4.2 (⇐). Let $R = \mathrm{Rel}H$. Then $H = \overline{\mathrm{Rel}H} = \overline{R}$ and $L = \underline{\mathrm{Rel}H} = \underline{R}$. By Theorem 5.2, we can obtain the conclusion immediately.

Theorem 5.3 shows that IVF approximation operators defined in Sect. 4 can be characterized by the axioms L_1, L_2, H_1 and H_2.

Example 5.1 Let $U = W = \{x_1, x_2\}$. Define $H : \mathrm{IVF}(W) \to \mathrm{IVF}(U)$ as $H(A) = \{\langle x_1, \max\{A(x_1), A(x_2)\rangle\}, \langle x_2, \max\{A(x_1), A(x_2)\rangle\}$ for any $A \in \mathrm{IVF}(U)$. By Theorem 3.5(5), we can computer that for all $[\alpha_1, \alpha_2] \in L^I$ and $A \in \mathrm{IVF}(U)$,

$$\begin{aligned}H\Big(\Psi\Big(\widehat{[\alpha_1,\alpha_2]}\Big),A\Big)(x_i)\\ &= \max\Big\{\Psi\Big(\widehat{[\alpha_1,\alpha_2]},A\Big)(x_1),\Psi\Big(\widehat{[\alpha_1,\alpha_2]},A\Big)(x_2)\Big\}\\ &= \max\{\Psi([\alpha_1,\alpha_2],A(x_1)),\Psi([\alpha_1,\alpha_2],A(x_2))\}\\ &= \Psi([\alpha_1,\alpha_2],\max\{A(x_1),A(x_2))\\ &= \Psi(\widehat{[\alpha_1,\alpha_2]},H(A))(x_i),\end{aligned}$$

which implies that $H(\Psi(\widehat{[\alpha_1,\alpha_2]}),A)) = \Psi(\widehat{[\alpha_1,\alpha_2]},H(A))$.

Thus, (H_2) holds.

Let

$$A = \{\langle x_1,[0.1,0.2]\rangle,\langle x_2,[0,0.8]\rangle\},$$
$$B = \{\langle x_1,[0.1,1]\rangle,\langle x_2,[0,1]\rangle\}.$$

Then $H(A\cup B)(x_i)=[0.1,1],(H(A)\cup H(B))(x_i)=[0.1,0.8]$.

Thus (H_1) does not hold. Hence $H_2 \not\Rightarrow H_1$. Similarly, we can prove that $H_1 \not\Rightarrow H_2$.

Remark 5.1 From Example 5.1, we conclude that $\{H_1,H_2\}$, or equivalently $\{L_1,L_2\}$, is the minimal axiom set to characterize the generalized IVF rough approximation operators produced by an arbitrary IVF relation.

6 The Composition of IVF Approximation Spaces

In the section, we will investigate the composition of generalized IVF rough set models. First, the concept of the composition of IVF relations is introduced.

Definition 6.1 Let $G_1 = (U,V,R_1)$ and $G_2 = (V,W,R_2)$ be two generalized IVF approximation spaces. The composition of IVF relations R_1 and R_2 is an IVF relation R from U to W, denoted by $R = R_1 \circ R_2$., and is defined as follows: for all $(x,z) \in U\times W$

$$R(x,z) = \bigvee_{y\in V} \mathcal{T}(R_1(x,y),R_2(y,z)). \tag{8}$$

The generalized IVF approximation space $G = (U,W,R)$ is referred to as the composition of $G_1 = (U,V,R_1)$ and $G_2 = (V,W,R_2)$, denoted by $G = G_1 \otimes G_2$.

Now, it is natural to ask, "what is the relationship between generalized IVF rough approximation operators in the composition space G and in the original two IVF approximation spaces G_1 and G_2?" The following theorem answers the question.

Theorem 6.1 *Let $G_1 = (U,V,R_1)$ and $G_2 = (U,W,R_1)$ be two generalized IVF approximation spaces, and $G = G_1 \otimes G_2$ be the composition of G_1 and G_2. Then*

(1) $\overline{R} = \overline{R}_1 \circ \overline{R}_2$,
(2) $\underline{R} = \underline{R}_1 \circ \underline{R}_2$

Proof We only prove the conclusion in (1). The assertion in (2) can be easily obtained by Theorem 4.1 and the result in (1). For every $A \in \mathrm{IVF}(W)$ and $x\in U$, we have

$$\begin{aligned}\overline{R}_1(\overline{R}_2(A))(x) &= \bigvee_{y\in V}\Psi(\sim_{\mathcal{N}}R_1(x,y),\overline{R}_2(A)(y))\\ &= \bigvee_{y\in V}\Psi\Big(\sim_{\mathcal{N}}R_1(x,y),\bigvee_{z\in W}\Psi(\sim_{\mathcal{N}}R_2(y,z),A(z))\Big)\\ &= \bigvee_{y\in V}\bigvee_{z\in W}\Psi(\sim_{\mathcal{N}}R_1(x,y),\Psi(\sim_{\mathcal{N}}R_2(y,z),A(z)))\\ &= \bigvee_{y\in V}\bigvee_{z\in W}\Psi(\mathcal{S}(\sim_{\mathcal{N}}R_1(x,y),\sim_{\mathcal{N}}R_2(y,z)),A(z))\\ &= \bigvee_{z\in W}\Psi\Big(\sim_{\mathcal{N}}\Big(\bigvee_{y\in V}\mathcal{T}(R_1(x,y),R_2(y,z))\Big),A(z)\Big)\\ &= \bigvee_{z\in W}\Psi(\sim_{\mathcal{N}}R(x,z),A(z))\\ &= \overline{R}(A)(x).\end{aligned}$$

Example 6.1 Let $U = V = W = \{x_1,x_2\}$. Assume that

$$A = \{\langle x_1,[0.1,0.7]\rangle,\langle x_2,[0.6,0.8]\rangle\},$$
$$R_1 = R_2 = \{\langle(x_1,x_1),[0.7,0.8]\rangle,\langle(x_1,x_2),[0.3,0.5]\rangle,$$
$$\langle(x_2,x_1),[0.4,0.6]\rangle,\langle(x_2,x_2),[0.1,1]\rangle\}$$

and $T = \min$. Then $\overline{R}_2(A) = \{\langle x_1,[0.6,0.8]\rangle,\langle x_2,[0.6,0.7]\rangle\}$, $\overline{R}_1(\overline{R}_2)(A)(x_1) = [0.6,0.8]$, $\overline{R}_1(\overline{R}_2)(A)(x_2) = [0.6,0.8]$. Hence $\overline{R}_1(\overline{R}_2)(A) = \{\langle x_1,[0.6,0.8]\rangle,\langle x_2,[0.6,0.8]\rangle\}$.

On the other hand, from Eq. (8), we have $R = R_1\circ R_2 = \{\langle(x_1,x_1),[0.7,0.8]\rangle,\langle(x_1,x_2),[0.3,0.5]\rangle,\langle(x_2,x_1),[0.4,0.6]\rangle,\langle(x_2,x_2),[0.3,1]\rangle\}$.

Thus, $\overline{R}(A) = \{\langle x_1,[0.6,0.8]\rangle,\langle x_2,[0.6,0.8]\rangle\}$. Obviously, $\overline{R} = \overline{R}_1\circ\overline{R}_2$. Similarly, we can obtain $\underline{R} = \underline{R}_1\circ\underline{R}_2$.

7 Application of the Generalized Interval-Valued Fuzzy Rough Set Model in Medical Diagnosis

In this section, in order to illustrate the efficiency of generalized interval-valued fuzzy rough set, we present an approach to the decision making based on the generalized interval-valued fuzzy rough set.

In order to rank the interval values, Xu [46] gave the definition as follows.

Definition 7.1 ([46]) Let $a = [a^L,a^U]$ and $b = [b^L,b^U]$ then the degree of possibility of $a \geq b$ is defined as:

$$p(a\geq b) = \max\left\{1-\max\left(\frac{b^U-a^L}{a^U-a^L+b^U-b^L},0\right),0\right\}. \tag{9}$$

Similarly, the degree of possibility of $b \geq a$ is defined as:

$$p(b \geq a) = \max\left\{1 - \max\left(\frac{a^U - b^L}{a^U - a^L + b^U - b^L}, 0\right), 0\right\}. \quad (10)$$

Equations (9) and (10) are proposed in order to compare two interval values, and to rank all the input arguments. Further details could be found in [46].

In the following we will apply generalized interval-valued fuzzy rough set model to medical diagnosis problems.

Let (*U*, *W*, *R*) be an IVF approximation space. Suppose that the universe $U = \{x_1, x_2, \ldots, x_m\}$ denotes a symptom set, and the universe $W = \{y_1, y_2, \ldots y_n\}$ denotes a disease set. Let $R \in \mathrm{IVF}(U \times W)$ be an IVF relation from U to W. For any $(x_i, y_j) \in U \times W, R(x_i, y_j)$ represents interval membership degree of the relationships between the symptom $x_i(x_i \in U)$ and the disease $y_j(y_j \in W)$, which is evaluated by a doctor in advance. For any a patient set A who has some symptoms in universe U, patient set A is an IVF set on symptom set U. That is, $A = \{\langle x_i, A(x_i)\rangle | x_i \in U\}$, where $A(x_i) \in L^I$ represents the membership degree to the symptom $x_i \in U$ of A. Now, the problem is that a decision-maker needs to make a reasonable decision about how to judge what kind of the disease y_j patient A is suffering from.

In what follows, we present an approach to the decision making for this kind of problem by using the generalized interval-valued fuzzy rough set theory with three steps.

First, according to Definition 4.1, we calculate the lower and upper approximations $\underline{R}(A)$ and $\overline{R}(A)$ of IVF set A with respect to (*U*, *W*, *R*). Without loss of generality, for the lower and upper approximations of IVF set A we can take $T = \min$.

Second, we introduce two operations on two IVFs, shown as follows, for all $A, B \in \mathrm{IVF}(U)$.

- Ring sum operation:

$$A \oplus B \in = \{\langle x, [\mu_A(x) + \mu_B(x) - \mu_A(x)\,\mu_B(x), v_A(x) + v_B(x) - v_A(x)v_B(x)]\rangle | x \in U\},$$

- Ring product operation:

$$A \otimes B \in= \{\langle x, [\mu_A(x)\mu_B(x), v_A(x)v_B(x)]\rangle | x \in U\}.$$

So, by the ring sum operation, we can obtain

$$\underline{R}(A) \oplus \overline{R}(A) = \Big\{\langle y_j, [\mu_{\underline{R}(A)}(y_j) + \mu_{\overline{R}(A)}(y_j) - \mu_{\underline{R}(A)}(y_j)\mu_{\overline{R}(A)}(y_j), v_{\underline{R}(A)}(y_j) + v_{\overline{R}(A)}(y_j) - v_{\underline{R}(A)}(y_j)v_{\overline{R}(A)}(y_j)]\rangle | y_j \in W\Big\},$$

Denote $\lambda_j = \underline{R}(A) \oplus \overline{R}(A)(y_j)$.

Finally, by Eq. (9), we rank the interval values λ_j. Then the optimal decision is to select y_1 if $\lambda_l = \max_j \lambda_j, j = 1, 2, \ldots, |W|$, $j = 1, 2, \cdots, |W|$. In other words, if $\lambda_l = \max_j \lambda_j$, $j = 1, 2, \ldots, |W|$, we can conclude that patient A is suffering from the disease y_l. Note that if l has more than one value, then all the y_l may be chosen, which implies that patient A is suffering from the various diseases.

Therefore, we have established an approach to uncertainty decision making based on the generalized interval-valued fuzzy rough set theory. In the next section, the application of this method will be shown by using a medical diagnosis decision-making problem.

8 A Numerical Example

In this section, we will apply the decision approach proposed in Sect. 7 to a medical diagnosis problem.

Let $U = \{x_1, x_2, x_3, x_4, x_5\}$ be five symptoms in clinic, where x_i stand for "temperature", "headache", "stomach pain", "cough," and "chest-pain," respectively, and the universe $W = \{y_1, y_2, y_3, y_4, y_5\}$ be four diseases, where y_i stand for Viral fever", "Malaria", "Typhoid", "Stomach problem" and "Chest problem" respectively. Let $R \in \mathrm{IVF}(U \times W)$ be an IVF relation from U to W. And R is a medical knowledge statistic data of the relationship of the symptom $x_i(x_i \in U)$ and the disease $y_i(y_i \in W)$. The statistic data are given in Table 1.

In this example, we suppose that A represents a patient. And the symptoms of patient A are described by an IVF set on the universe U. Let

$$A = \{\langle x_1, [0.4, 0.5]\rangle, \langle x_2, [0.5, 0.6]\rangle, \langle x_3, [0.7, 0.9]\rangle, \langle x_4, [0.2, 0.3]\rangle, \langle x_5, [0.5, 0.7]\rangle\}.$$

For example, for $A(x_3) = [0.7{,}0.9]$, a doctor cannot present the precise membership degree of how pain the stomach of patient A is, but he (she) provides a certain interval value [0.7,0.9] to depict the membership degree of how pain the stomach of patient A is.

In what follows, we give the decision-making process by using the three steps given in Sect. 7 in detail.

First, let $T = \min$, then by Definition 4.1, we calculate the lower and upper approximations $\underline{R}(A)$ and $\overline{R}(A)$ of patient A as follows:

$$\underline{R}(A) = \{\langle y_1, [0.2, 0.3]\rangle, \langle y_2, [0.2, 0.3]\rangle, \langle y_3, [0.4, 0.5]\rangle, \langle y_4, [0.4, 0.5]\rangle, \langle y_5, [0.5, 0.6]\rangle\},$$

$$\overline{R}(A) = \{\langle y_1, [0.7, 0.9]\rangle, \langle y_2, [0.7, 0.9]\rangle, \langle y_3, [0.7, 0.9]\rangle, \langle y_4, [0.7, 0.9]\rangle, \langle y_5, [0.7, 0.9]\rangle\}.$$

Then, we have

Table 1 Symptoms characteristic for the considered diagnoses

R	y_1	y_2	y_3	y_4	y_5
x_1	[0.3,0.4]	[0.2,0.3]	[0.6,0.9]	[0.6,0.7]	[0.4,0.5]
x_2	[0.4,0.6]	[0.7,0.9]	[0.7,0.8]	[0.4,0.5]	[0.6,0.7]
x_3	[0.4,0.5]	[0.3,0.5]	[0.4,0.5]	[0.3,0.6]	[0.8,0.9]
x_4	[0.5,0.5]	[0.7,0.8]	[0.1,0.3]	[0.2,0.3]	[0.1,0.2]
x_5	[0.8,0.9]	[0.4,0.5]	[0.6,0.8]	[0.5,0.6]	[0.2,0.4]

$$\underline{R}(A) \oplus \overline{R}(A) = \{\langle y_1, [0.76, 0.93]\rangle, \langle y_2, [0.76, 0.93]\rangle, \langle y_3, [0.82, 0.95]\rangle, \langle y_4, [0.82, 0.95]\rangle, \langle y_5, [0.85, 0.96]\rangle\}.$$

So according to Eq. (9), it is clear that the maximum interval value is $\lambda_5 = [0.85, 0.96]$. Hence, the optimal decision is to select y_5. That is, we can conclude that patient A is suffering from the disease Chest problem (y_5).

On the other hand, if we adopt the ring product operation, then

$$\underline{R}(A) \otimes \overline{R}(A) = \{\langle y_1, [0.14, 0.27]\rangle, \langle y_2, [0.14, 0.27]\rangle, \langle y_3, [0.28, 0.45]\rangle, \langle y_4, [0.28, 0.45]\rangle, \langle y_5, [0.35, 0.54]\rangle\}.$$

We can note that the optimal decision is still to select y_5. In other word, patient A is still suffering from the disease Chest problem (y_5). In general, no matter we adopt the ring sum operation or ring product operation in decision making, the decision result is the same.

9 Conclusion

In this paper, we have developed a general framework for the study of generalized interval-valued fuzzy rough sets by using constructive and axiomatic approaches. This work may be viewed as the extension of Mi and Zhang [16]. Then composition of two approximation spaces was also studied. At last, by using the generalized IVF rough set theory, we have developed a general framework for dealing with uncertainty decision making. The approach will be helpful for making scientific and reasonable decision on fuzzy and uncertainty decision problems. Further, we use a medical diagnosis decision-making problem to demonstrate the principal steps of the decision methodology.

Knowledge reduction is one of the important contents in the research on rough set theory. So in the future we mainly focus on knowledge reduction based on generalized IVF rough set theory under complete information systems. Moreover, it is important and interesting to further investigate characterization and uncertain measures of generalized IVF rough sets.

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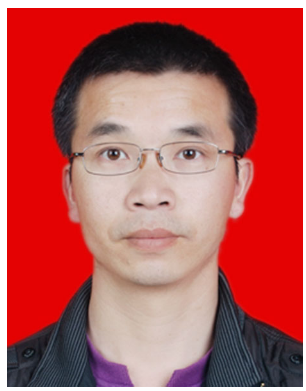

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