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SHORT COMMUNICATION

A comment on "Mathematical study of a Leslie-Gower type tritrophic population model in a polluted environment" [Modeling in Earth Systems and Environment 2 (2016) 1–11]

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Abstract In the current manuscript we comment on (Misra and Babu, Model Earth Syst Environ 2(1):1–11, 2016), where two novel five-species ODE models are proposed and analyzed, in order to investigate the population dynamics of a three-species food chain, in a polluted environment. It is shown in Misra and Babu (Model Earth Syst Environ 2(1):1–11, 2016) that under certain restrictions on the parameters, the models have bounded solutions for *all positive* initial conditions. Furthermore, a globally attracting set is explicitly constructed for initial conditions in $\mathbb{R}^5_+$. We prove these results are *not true*. To the contrary, solutions to these models can *blow-up in finite time*, even under the parametric restrictions derived in Misra and Babu (Model Earth Syst Environ 2(1):1–11, 2016), for sufficiently large initial conditions. We provide both analytical proofs and numerics to confirm our results.

Keywords Three-species food chain · Pollution · Finite time blow-up

Mathematics subject classification Primary: 34K12 · 34D20 · Secondary: 92D25 · 92D40

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Introduction

Pollution has many adverse effects on ecosystems. In order to better understand the pollution problem, many works in the literature couple the dynamics of populations in ecosystems, with those of the pollutants therein (Hallam and De Luna 1984; Misra and Babu 2014, 2016). This serves two purposes

- One can better understand the spread and subsequent control of these pollutants, so as to provide a cleaner living environment.
- One can better understand the effects of the pollutants on the species residing in the ecosystem, so as to protect them, and maintain biodiversity.

In this regard the work of the authors in Misra and Babu (2016) is very interesting, and such investigations are highly desireable. Therein, two five-species ordinary differential equation (ODE) models are proposed and analyzed in an attempt to understand the effect of toxicity/pollution on a three species food chain, modeled via a combination of a modified Leslie-Gower type scheme and Holling type II functional response (Upadhyay and Rai 1997; Aziz-Alaoui 2002). However, one must take *extreme care* in deriving global existence results for such systems, given recent results that show finite time blow-up in such models (Parshad et al. 2013; Parshad 2015; Parshad and Basheer 2016).

The purpose of this short communication is to remark on certain results in Misra and Babu (2016). The authors therein derive global boundedness results for solutions to the ODE model systems proposed, under certain restrictions on the parameters. We note, these results are *incorrect*. In particular there is *no bound* on the solutions to the proposed models, *even under* the parametric restrictions imposed in Misra and Babu (2016). Our primary contributions in the current manuscript are:

- We consider the second model proposed in Misra and Babu (2016), and show solutions to this model system are not bounded. In fact the solutions can blow-up in finite time, for sufficiently large initial data. This is shown via theorem 2.3.
- Similar results are proved via corollary 1, for the first model system proposed in Misra and Babu (2016).
- We demonstrate our results via numerical simulations. In particular, we choose parameters satisfying the parameter restrictions derived in Misra and Babu (2016), and show that solutions of the models considered, *still* blow-up in finite time, for sufficiently large initial condition.
- We provide concluding remarks and discuss interesting future directions.

Finite time blow-up

Given a system of ODE's, depending on the nonlinearities in the equations, one might not expect a solution to always exist globally in time. In particular, solutions of some ODE's may blow-up in finite time (Straughan 1998). Recall,

Definition 2.1 (*finite time blow-up*) Given a ODE, with suitable initial conditions, one says finite time blow-up occurs if

$$\lim_{t \to T^* < \infty} \|z\|_\infty \to \infty,$$

here the norm $\|\cdot\|$ is the supremum norm, z is the solution to the ODE in question, and $T^* < \infty$, is the blow-up time.

In the literature, finite time blow-up is also referred to as an *explosive instability* (Straughan 1998), and there are many interpretations of blow-up in physical phenomenon. The interested reader is referred to Quittner and Souplet (2007); Straughan (1998).

We now turn our attention back to Misra and Babu (2016). Of the two models proposed in Misra and Babu (2016), the non-dimensionalised form of the second one is,

$$\frac{dX}{dt} = a_0 X - b_0 X^2 - \left(\frac{v_0 XY}{d_0 + X}\right), \tag{1}$$

$$\frac{dY}{dt} = \left(\frac{v_1 XY}{d_1 + X}\right) - a_1 Y - i_2 Y C_0 - \left(\frac{v_2 YZ}{d_2 + Y}\right), \tag{2}$$

$$\frac{dZ}{dt} = c_3 Z^2 - v_3 \frac{Z^2}{d_3 + Y}, \tag{3}$$

$$\frac{dC_0}{dt} = c_1 X C_E - s_1 C_0 - s_2 X C_0, \tag{4}$$

$$\frac{dC_E}{dt} = q_0 - s_3 C_E - c_1 X C_E + s_2 X C_0. \tag{5}$$

Following the scaling from Misra and Babu (2016), one converts (1–5) into,

$$\frac{dx}{dt} = x(1-x) - \left(\frac{xy}{a+x}\right), \tag{6}$$

$$\frac{dy}{dt} = \left(\frac{cxy}{a+x}\right) - \beta_2 y c_0 - by - \left(\frac{yz}{d+y}\right), \tag{7}$$

$$\frac{dz}{dt} = pz^2 - q\frac{z^2}{r+y}, \tag{8}$$

$$\frac{dc_0}{dt} = t_2 x c_e - o_1 c_0 - t_2 x c_0, \tag{9}$$

$$\frac{dc_e}{dt} = p_1 - o_2 c_e - u_2 x c_e + u_2 x c_o. \tag{10}$$

Remark 1 (1–5) is used for the purpose of numerical simulations in Misra and Babu (2016), while (6–10) is used for analysis. In keeping with this, we do the very same in the current manuscript.

For complete details on the above model formulation the reader is refereed to Upadhyay and Rai (1997), Aziz-Alaoui (2002), Misra and Babu (2016) and Leslie (1948). Essentially z is a specialist top predator, depredating on a specialist middle predator y. The interaction between z and y is modeled via a modified Leslie-Gower scheme. Also the middle predator y depredates on a prey species x. The interactions between y and x are modeled via a standard Holling type II functional response. Also c_0 and c_e represent the pollutant concentration in the ecosystem, in both the prey x, and the environment respectively. There is an additional death term $-\beta_2 c_0 y$, in the middle predator, due to toxicant concentration. This is realistic, as there could easily be enhanced death in the middle predator y, due to ingestion of toxic prey perse.

We first recall the following result (Theorem 4.1 from Misra and Babu (2016))

Theorem 2.2 *Let us assume*

$$c + \frac{c}{4b} + r < \frac{q}{p}, \tag{11}$$

and let Ω_2*be the set defined by:*

$$\Omega_2 = \left\{ (x, y, z, c_0, c_e) \in \mathbb{R}^5_+ : 0 \le x \le 1, 0 \le x + \frac{y}{c} \le 1 + \frac{1}{4b}, \right.$$
$$\left. x + \frac{y}{c} + \alpha z \le 1 + \frac{1}{4b} + \frac{M}{b}, c_0 + c_e \le \frac{1 + p_0 t_1}{\theta_2} \right\} \tag{12}$$

where $\alpha = \frac{1}{b^2(c + \frac{c}{4b} + r)}$, $M = \frac{1}{4(q - (c + \frac{c}{4b} + r)p)}$*and* $\theta_2 = \min(o_1 u_1, t_1 o_2)$ *then*

- (i) Ω_2 *is positively invariant*
- (2i) *All non-negative solutions (i.e. solutions initiating in* $\mathbb{R}^5_+$) of (6)-(10) *are uniformly bounded forward in time, (thus they exist for all positive times), they eventually enter the attracting set* Ω_2.
- (3i) *System* (6–10) *is dissipative*

We next show that (Theorem 4.1 (2i), (3i) Misra and Babu (2016)) is incorrect. In particular we state the following theorem

Theorem 2.3 *Consider the three species food chain model* (6–10). *Even if*

$$c + \frac{c}{4b} + r < \frac{q}{p}, \tag{13}$$

z(t) blows up in finite time, that is

$$\lim_{t \to T^* < \infty} ||z(t)||_\infty \to \infty, \tag{14}$$

as long as the initial data (y_0, z_0)*are large enough.*

Remark 2 Note the above theorem shows, that contrary to what is claimed in Misra and Babu (2016), there is *no bound* for z.

Proof Here we will show that the z solving (8), blows-up in finite time. Consider (6–10), with positive initial conditions. By integrating (8) we obtain

$$-\frac{1}{z} + \frac{1}{z_0} = pt - q\int_0^t \frac{ds}{y+r},$$

which gives

$$z = \frac{1}{\frac{1}{z_0} - pt + q\int_0^t \frac{ds}{y+r}}.$$

Our goal is to show that the function: $t \to \psi(t) = \frac{1}{z_0} - pt + q\int_0^t \frac{dt}{y+r}$ vanishes at a time $T^{**} > 0$. Then the solution z will blow-up in finite time, at $t = T^{**}$.

Since the reaction terms in (6–10) are continuous functions, then the solutions (On their maximal interval of existence) are classical and continuous and

$$\lim_{t \to 0}\left(\frac{1}{t}\int_0^t \frac{ds}{y+r}\right) = \frac{1}{y_0 + r}.$$

For y_0 chosen sufficiently large, there exists a $\delta > 0$ such that

$$\frac{1}{t}\int_0^t \frac{ds}{y+r} < \frac{p}{2q}, \qquad \text{for all } t \in (0, \delta).$$

Then

$$\frac{1}{z_0} - pt + q\int_0^t \frac{ds}{y+r} = \frac{1}{z_0} + \left[-p + \frac{q}{t}\int_0^t \frac{ds}{y+r}\right]t < \frac{1}{z_0} - \frac{p}{2}t,$$

for all $t \in (0, \delta)$.

If now z_0 is chosen sufficiently large, then we can find $T^* \in (0, \delta)$ such that

$$\frac{1}{z_0} - \frac{p}{2}T^* = 0.$$

This entails

$$\psi(T^*) = \frac{1}{z_0} - pT^* + q\int_0^{T^*} \frac{ds}{y+r} < \frac{1}{z_0} - \frac{p}{2}T^* = 0. \tag{15}$$

Thus one has $\psi(T^*) < 0$, but $\psi(0) > 0$, and since $\psi(t)$ is continuous in time, by application of the mean value theorem, we obtain the existence of some $T^{**} \in (0, \delta)$, $T^{**} < T^*$, s.t $\psi(T^{**}) = 0$. This implies $z(t)$ the solution of (8), blows-up in finite time, at $t = T^{**}$. This proves the theorem. □

We next state the following Theorem

Theorem 2.4 *The model system* (6–10)*, even under condition* (13) *is not dissipative in all of* $\mathbb{R}^5_+$. *It is dissipative only for initial data starting within* Ω_2.

Proof Via Theorem 2.3, there exists initial data in $\mathbb{R}^5_+$, for which solutions blow-up in finite time, and thus do not enter the set Ω_2, which is claimed to be invariant and attracting. Thus system (6–10) is not dissipative. However for initial data initiated in Ω_2, the trajectories remain in Ω_2. □

Remark 3 Note, the set Ω_2 is thus *only* invariant, *not* globally attracting for $\mathbb{R}^5_+$.

We now consider the first model proposed in Misra and Babu (2016),

$$\frac{dx}{dt} = x(1-x) - \beta_1 x c_0 - \left(\frac{xy}{a+x}\right), \tag{16}$$

$$\frac{dy}{dt} = \left(\frac{cxy}{a+x}\right) - by - \left(\frac{yz}{d+y}\right), \tag{17}$$

$$\frac{dz}{dt} = pz^2 - q\frac{z^2}{r+y}, \tag{18}$$

$$\frac{dc_0}{dt} = t_1 x c_e - o_1 c_0 - t_1 x c_0, \tag{19}$$

$$\frac{dc_e}{dt} = p_1 - o_2 c_e - u_1 x c_e + u_1 x c_o. \tag{20}$$

The dynamics here are essentially the same as that of the second model, except that now the prey x has an additional non-linear death term $-\beta_1 x c_0$, due to pollution/toxicants.

We next show that (Theorem 3.1 (2i), (3i) Misra and Babu (2016)) which claims global boundedness of the above model (16–20) is incorrect. In particular we state the following corollary

Corollary 1 *Consider the three species food chain model* (16–20). *Even if*

$$c + \frac{c}{4b} + r < \frac{q}{p}, \tag{21}$$

z(t) blows up in finite time, that is

$$\lim_{t \to T^* < \infty} ||z(t)||_\infty \to \infty, \tag{22}$$

as long as the initial data (y_0, z_0)*are large enough.*

Proof The proof follows by a direct application of theorem 2.3. That is we mimic the proof of theorem 2.3, by integrating (18) in time, and proceeding as earlier. Hence the result follows as a corollary. □

Numerical simulation

In this section we perform simple numerical simulations that *disprove* theorem 4.1 (Misra and Babu 2016) (recapped in the current manuscript as theorem 2.2).

The following is the list of parameters given in Misra and Babu (2016), for numerical simulations around the interior equilibrium $(X^*, Y^*, Z^*, C_0^*, C_E^*)$ for (1–5): $a_0 = 0.97$, $b_0 = 0.06$, $v_0 = 1$, $d_0 = 10$, $a_1 = 1$, $v_1 = 2$, $d_1 = 10$, $v_2 = 0.405$, $d_2 = 10$, $c_3 = 0.0403$, $v_3 = 1$, $d_3 = 20$, $i_2 = 0.85$, $c_1 = 0.0183$, $s_1 = 3.8199$, $s_2 = 0.0270$, $q_0 = 1.6883$, $s_3 = 0.6984$.

However, for the above values using the scaling introduced in Misra and Babu (2016), we obtain

$$c + \frac{c}{4b} + r = 3.8372 > 1.5842 = \frac{q}{p} \tag{23}$$

Remark 4 Note that theorem 4.1 from Misra and Babu (2016) (recapped in the current manuscript as theorem 2.2) requires that we strictly maintain

$$c + \frac{c}{4b} + r < \frac{q}{p} \tag{24}$$

Hence showing blow-up numerically, for the above parameter set *does not* contradict the theorems in Misra and Babu (2016).

We proceed by changing v_1. We set $v_1 = .23$, and keep all other parameters the same as above. With $v_1 = .23$ one easily calculates,

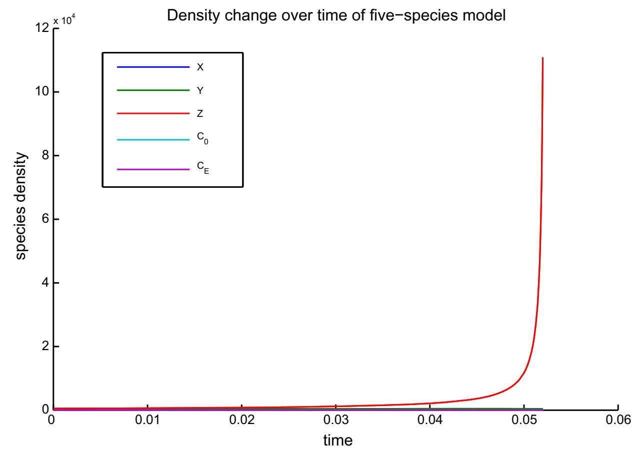

Fig. 1 In this simulation we consider the list of parameters mentioned earlier, but with $v_1 = .23$. We take large initial data for the middle predator Y, and top predator Z. The initial data tried is $(12.62, 400, 400, 0.103, 0.1741) \in \mathbb{R}_+^5$. For this set of parameters we have, $c + \frac{c}{4b} + r = 1.57 < 1.5842 = \frac{q}{p}$. However, we still observe that Z blows-up at approximately $t = 0.053$. We show the solution just before the blow-up time, at $t = 0.052$. Hence there is no bound on the solutions for sufficiently large initial data

$$c + \frac{c}{4b} + r = 1.57 < 1.5842 = \frac{q}{p} \tag{25}$$

Thus the parametric restrictions imposed via theorem 4.1 from Misra and Babu (2016) are satisfied, and so according to Misra and Babu (2016), we should have globally bounded solutions, for any initial condition in $\mathbb{R}_+^5$. The following numerical simulation, see Fig. 1, shows this is *not true*.

Conclusion

In the current manuscript we have shown that the five-species models introduced in Misra and Babu (2016), *do not* posess bounded solutions, *even under* the parametric restrictions imposed in Misra and Babu (2016). Note, solutions to these models can actually blow-up in finite time. In summary,

- We caution against deriving global boundedness results whence modeling ecosystem dynamics, via the modified Leslie-Gower scheme, introduced in Upadhyay and Rai (1997).
- Any global boundedness results, where the modified Leslie-Gower scheme is incorporated, *must* impose restrictions on initial conditions, and specify them explicitly, such as in Parshad et al. (2014).
- Possible future avenues of research should be aimed at introducing damping mechanisms, that can *impede* or *delay* the blow-up, such as Parshad et al. (2016). To

this end, the additional non-linear death term in the middle predator $-\beta_2 y c_0$, due to ingestion of toxic prey, is a right step.

One can investigate the effect on blow-up times, or blow-up impedence altogether (for certain initial conditions), with and without such a non-linear term. This would make an extremely interesting future work.

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