



# Student Interpretations of Eigenequations in Linear Algebra and Quantum Mechanics

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Accepted: 29 April 2024  
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## Abstract

This work investigates how students interpret various eigenequations in different contexts for  $2 \times 2$  matrices:  $A\vec{x} = \lambda\vec{x}$  in mathematics and either  $\hat{S}_x|+\rangle_x = \frac{\hbar}{2}|+\rangle_x$  or  $\hat{S}_z|+\rangle = \frac{\hbar}{2}|+\rangle$  in quantum mechanics. Data were collected from two sources in a senior-level quantum mechanics course; one is video, transcript and written work of individual, semi-structured interviews; the second is written work from the same course three years later. We found two principal ways in which students reasoned about the equal sign within the mathematics eigenequation and at times within the quantum mechanical eigenequations: with a functional interpretation and/or a relational interpretation. Second, we found three distinct ways in which students explained how they made sense of the physical meaning conveyed by the quantum mechanical eigenequations: via a measurement interpretation, potential measurement interpretation, or correspondence interpretation of the equation. Finally, we present two themes that emerged in the ways that students compared the different eigenequations: attention to form and attention to conceptual (in)compatibility. These findings are discussed in relation to relevant literature, and their instructional implications are also explored.

**Keywords** Eigentheory · Linear algebra · Quantum mechanics · Student reasoning

## Introduction

Students encounter many of the same advanced mathematical concepts across undergraduate mathematics classes and physics classes. For instance, quantum mechanics is a core course for an undergraduate physics degree; within quantum

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mechanics alone, students are expected to carry out computations and reason with linear algebra concepts such as matrix–vector operations, basis and change of basis, normalization, projection, inner products, and eigentheory. There is evidence that there may be a mismatch between how many of these mathematical concepts are emphasized and interpreted in math courses and in other disciplinary courses. For example, previous work in the context of definite integrals showed that certain conceptualizations may not be productive in physics contexts (Jones, 2015) and that the interpretations typically taught in a calculus class are not those that are typically used in physics contexts (Pina & Loverude, 2020).

The experience of learning quantum mechanics may present some unique challenges (Emigh et al., 2015; Gire & Manogue, 2008, 2012; Hoehn & Finkelstein, 2018; Passante et al., 2020; Singh, 2008; Singh & Marshman, 2015). For instance, its abstract nature may add extra layers of complexity when trying to understand eigenequations in a quantum mechanics context. Furthermore, many quantum mechanics courses also introduce a new notation, known as Dirac or bra-ket notation. Students need to learn what the various symbols represent, how to work with Dirac notation mathematically, and how to make sense of what it conveys physically. Research on student understanding of representations in quantum mechanics includes comparing notations' affordances and limitations more broadly (Gire & Price, 2015; Schermerhorn et al., 2019), students' preferences and metarepresentational competence (Wawro et al., 2020), expert interpretations of Dirac expressions (Dreyfus et al., 2017), and representations-focused instructional materials (Kohnle & Passante, 2017). Related literature is discussed in more detail in the [Literature Review](#) section.

In this paper, we investigate how physics students make sense of eigenequations that they encounter in a mathematics course and in a quantum mechanics course. Building upon our prior work (Wawro et al., 2020), our research project investigates students' meanings for eigentheory in quantum mechanics and how their language for eigentheory compares and contrasts across mathematics and quantum physics contexts. In particular, we present students' interpretations of various eigenequations for  $2 \times 2$  matrices:  $A\vec{x} = \lambda\vec{x}$  from mathematics and either  $\hat{S}_x|+\rangle_x = \frac{\hbar}{2}|+\rangle_x$  or  $\hat{S}_z|+\rangle = \frac{\hbar}{2}|+\rangle$  from quantum mechanics. We also highlight instances of synergistic and potentially incompatible interpretations, and we discuss ways in which those interpretations may be productive or unproductive in different contexts.

## Background

In this section, we first summarize some foundations of quantum mechanical theory, focusing on the origins and role of eigentheory. We conclude with a comparison of quantum mechanical and mathematical interpretations of eigenequations.

### Physics Background: Postulates and Spin- $\frac{1}{2}$ Systems

In many cases, classical phenomena can be predicted exactly due to their deterministic nature. Quantum phenomena, however, are inherently probabilistic: only the probability of a specific outcome can be predicted. Quantum mechanical

phenomena were discovered empirically rather than theorized; the initial theoretical work was motivated by phenomenological results that challenged contemporary theories. Thus, in contrast to the empirical laws that underpin classical physics, the foundations of quantum mechanics are encapsulated in six postulates that relate the physical system to mathematical formalism. The postulates, developed to provide the necessary framing for the mathematization of experimental observations, are connected to physical states of a system, outcomes of measurements of those systems, and mathematical formalism used to represent and work with those entities. We summarize the four postulates most relevant to this paper.

The first postulate of quantum mechanics establishes that quantum mechanical systems and all knowable information about them are represented mathematically by normalized state vectors, which are referred to as “kets” and symbolized as  $|\psi\rangle$ . This notation to describe systems, known as Dirac notation or bra-ket notation (Dirac, 1939), was developed to bridge the matrix mechanics developed by Heisenberg and the wave mechanics developed by Schrödinger (Schrödinger, 1926), and is used in quantum mechanics for its generalizability between discrete and continuous systems. Research has supported the notion that Dirac notation can productively support student reasoning about quantum mechanical systems (Gire & Price, 2015; Wawro et al., 2020).

Kets mathematically behave like vectors in a complex Hilbert space, with the condition of normalization. Any ket can be expressed as a linear combination (superposition) of basis kets for the Hilbert space. Normalization is a consequence of the probabilistic nature of quantum mechanics. For a general ket, the coefficient of each basis state is related (by complex square) to the probability of a measurement outcome associated with that state. This is the essence of the fourth postulate. The sum of the individual probabilities, and thus the sum of the squares of individual coefficients, has to be one.

The second postulate of quantum mechanics states that any physical observable, i.e., measurable quantity, can be represented by a Hermitian operator that acts on kets. Operators are typically labeled using the common variable for the relevant quantity with a “hat” over it; a general operator can be written as  $\hat{A}$ . The third postulate states that the only possible result of a measurement of an observable is one of the eigenvalues  $a_n$  of the corresponding operator  $\hat{A}$ ; thus, it is relevant that eigenvalues of Hermitian operators are necessarily real-valued. That eigenvalue is associated with an eigenket  $|a_n\rangle$  representing the state of the system corresponding to that particular measurement value.

The most common physical observables in a standard undergraduate quantum mechanics class are energy, position, momentum, and spin. The observable that is pertinent to this paper is that of spin. Spin is a measure of a particle’s “intrinsic angular momentum” and is related to the particle’s magnetic moment. Measurements can be made of a spin component along a specified axis; that quantity is represented by an operator such as  $\hat{S}_z$  (where the  $z$  indicates the component axis). The simplest spin system has two possible values,  $\pm\frac{\hbar}{2}$ , for the  $\hat{S}_z$  measurement;

these systems are known as spin- $\frac{1}{2}$  systems and are often used to introduce quantum mechanical phenomena due to their mathematical simplicity. The measurements correspond to the states typically labeled  $|+\rangle$  and  $|-\rangle$ , respectively, which comprise a set of orthonormal basis vectors called the  $S_z$  basis. The analogous information can be determined for other component axes, such as  $x$ . In that instance, the possible results of the  $\hat{S}_x$  measurement are also  $\pm\frac{\hbar}{2}$ , which correspond to the set of orthonormal basis vectors  $|\pm\rangle_x$ , called the  $S_x$  basis.

The introduction of eigentheory as a representation for quantum mechanical systems establishes explicitly that eigenvalues are the only possible outcomes of measurements, and implicitly that there are eigenvectors corresponding to those eigenvalues. Both of these ideas are consistent with the third postulate, and together they form the basis of an expert interpretation of an eigenvalue equation in quantum mechanics: it is a statement about the possible results of measurement of an observable and the relationships between those results and physical states. For example, the relationship of some of these objects and quantities in the spin- $\frac{1}{2}$  context can be expressed symbolically as the eigenvalue equation  $\hat{S}_x|+\rangle_x = \frac{\hbar}{2}|+\rangle_x$ .

### Interpretations of Eigenequations in Mathematics and Quantum Mechanics

Given the role of eigenequations in quantum mechanics, it makes sense to compare and contrast some possible interpretations of such equations by mathematicians and physicists. For a standard matrix eigenvalue equation,  $A\vec{x} = \lambda\vec{x}$ , a mathematician might give one of two common interpretations:

- M1. There are some nonzero vectors  $x$  such that when  $A$  acts on  $x$ , the result is a scalar multiple (or stretch) of the vector  $x$ .
- M2. There are some nonzero vectors  $x$  such that the result of  $A$  times  $x$  is the same as the result of  $\lambda$  times  $x$ .

Interpretation M1 can be particularly central in  $\mathbb{R}^2$  and  $\mathbb{R}^3$ , where graphical or embodied interpretations of vectors being stretched can be useful. However, for rotation matrices, which have no real-valued eigenvalues, the notion of "stretching" in M1 is less helpful, as there are no real-valued vectors that point in the same direction after being acted on by a rotation matrix. Thus, with complex-valued eigenvalues with non-zero imaginary components, the "scalar multiple" aspect of interpretation M1, or interpretation M2, may be more useful. A set-theoretic interpretation, such as the collection of all  $x$  in  $\mathbb{R}^n$  that satisfy  $A\vec{x} = \lambda\vec{x}$ , allows for the zero vector and is consistent with reasoning about eigenspaces as subspaces of  $\mathbb{R}^n$ .

Now consider a spin- $\frac{1}{2}$  eigenequation,  $\hat{S}_x|+\rangle_x = \frac{\hbar}{2}|+\rangle_x$ . A physicist might use one of two interpretations of this eigenequation:

- P1. The  $S_x$  operator acting on the up- $x$  eigenstate yields an eigenvalue of  $\frac{\hbar}{2}$ .

P2. If you were to measure the  $x$ -component of the spin of an up- $x$  state, you would get  $\frac{\hbar}{2}$  for that measurement.

Statement P1 identifies what each quantity in the equation - operator, eigenstate, and eigenvalue - represents. Statement P2 emphasizes the physical meaning of the empirical relationship between these three quantities and is more consistent with the third postulate. What is notable is that neither interpretation of the spin- $\frac{1}{2}$  eigenequation is a “readout” of the equation; in particular, the eigenstate on the right-hand side is not explicitly mentioned in either interpretation. Furthermore, a geometric interpretation of the spin- $\frac{1}{2}$  eigenequation could not convey the physical significance of this equation. First, the eigenstates must be normalized, so the idea of scaling or stretching an eigenvector by the eigenvalue, the M1 interpretation above, does not make sense. Second, the terms in the equation are tied to physical quantities, and the relationship between these quantities needs to be acknowledged.

Although terms in physics equations are generally tied to physical quantities, the ties manifest differently in quantum mechanical eigenequations than in most physics equations. Generally, physics equations provide symbolic representations of physical relationships between quantities (see Fig. 1). For example, Newton’s second law (Fig. 1a) relates the mass and acceleration of an object to the sum of all forces (the “net force”) acting on the object. The ideal gas law (Fig. 1b) relates the pressure, volume, temperature, and number of gas particles via Boltzmann’s constant,  $k_B$ . These equations are mathematical representations of physical operations, such as exerting force(s) on a system, measuring the acceleration of the system and determining the net force on that system, or determining the pressure of an ideal gas system if the other quantities are known. The allowable energies of a quantum mechanical harmonic oscillator - the eigenvalues of the total energy (Hamiltonian) operator - can be expressed in terms of the energy level index ( $n$ ), the angular oscillation frequency of the system ( $\omega$ ), and Planck’s constant divided by  $2\pi$  ( $\hbar$ ). The resulting equation (Fig. 1c) can be used to determine how one property of a system (here the energy eigenvalue) will change in response to a change in a different property (the angular frequency or the state index), effectively demonstrating covariation or multivariation between physical quantities (Carlson et al., 2002; Jones, 2022; Jones and Kuster, 2021). An interpretation of these equations typically consists of transliterating the symbols left-to-right into a sentence, effectively reversing the symbolization of the physical relationship established when mathematizing. A desired interpretation of a quantum mechanical eigenequation, however, is not a transliteration of the symbols, because it does not convey a direct mapping of the physical world. For example, statement P2 does not “read out” the vector on the right side of the equal sign.

(a) $\Sigma \vec{F} = m\vec{a}$	(b) $PV = Nk_B T$	(c) $E_n = \hbar\omega(n + \frac{1}{2})$
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**Fig. 1** Canonical physics equations: **a** Newton’s second law, **b** the ideal gas law, and **c** the energy spectrum of a quantum harmonic oscillator

These distinct interpretations between the mathematics and the physics for quantum mechanical eigenequations are what motivated this study. In order to frame the study further, we discuss prior work on student understanding of linear algebra focused on eigentheory, in the research on undergraduate mathematics education (RUME) and physics education research (PER) literatures, as well as some discussions of the equal sign in both domains.

## Literature Review

We first summarize pertinent findings related to student reasoning about matrix equations and eigentheory in mathematics and to student reasoning about eigentheory in physics contexts. As we carried out our analysis of how students made sense of the eigenequations in our data set, it became pertinent to build on literature related to the equal sign. Thus, we conclude this section with a review of relevant findings on student reasoning about the equal sign in mathematics and in physics.

### Matrix Equations and Eigentheory in Linear Algebra

In linear algebra, any linear transformation  $T$  between vector spaces  $\mathbb{R}^n$  and  $\mathbb{R}^m$  can be represented by a matrix  $A$  defined by  $T(x) = Ax$ . Thus, one could interpret the matrix equation  $A\vec{x} = \vec{b}$  as  $A$  transforming vectors  $\vec{x}$  in  $\mathbb{R}^n$  and producing vectors  $\vec{b}$  in  $\mathbb{R}^m$ . Larson and Zandieh (2013) call this a *transformation interpretation of the matrix equation*  $A\vec{x} = \vec{b}$ . They state that in this interpretation, “the vector  $\vec{x}$  is related to the vector  $\vec{b}$  through a transformation via multiplication by the matrix  $A$  (an interpretation related to developing an understanding of properties of linear transformations such as injectivity, surjectivity, and invertibility)” (p. 12). Larson and Zandieh offer two other interpretations of  $A\vec{x} = \vec{b}$ : one in which  $\vec{b}$  is a linear combination of the columns of  $A$ , and one in which  $\vec{x}$  is the solution to the system of equations corresponding to  $A\vec{x} = \vec{b}$ . Her and Loverude (2023) found that the transformation interpretation of matrix multiplication was the most consistent and stable interpretation among a small sample of interviewed physics majors on a set of tasks, some of which were designed to elicit other interpretations (e.g., linear combination); they also found that students who had the most consistent use of the transformation interpretation were those that had completed two quantum mechanics courses. With respect to eigenequations, a specific type of matrix equation, we see interpretation M1, “there are some nonzero vectors  $x$  such that when  $A$  acts on  $x$ , the result is a scalar multiple or a stretch of the vector  $x$ ,” as a transformation interpretation of the eigenequation. In a different study focused on comparing how linear algebra students reasoned about high school functions and about linear transformations, Zandieh et al. (2017) documented five clusters of metaphorical expressions that students leveraged for function and/or transformation; the three most relevant to our work are *input–output*, in which  $\vec{x}$  serves as the input vector and  $\vec{b}$  as the output vector; *morphing*, in which  $\vec{x}$  morphs into  $\vec{b}$ ; and *machine*, in which  $A$  acts on  $\vec{x}$  to produce  $\vec{b}$ . Zandieh et al. found that a focus on properties (such as linear independence

of  $A$ 's column vectors implying the transformation is one-to-one) “appeared to impede students’ development of a unified concept image of function while an ability to draw on metaphors facilitated such development” (p. 12).

Research on students’ understanding of eigentheory has grown over the past decade, providing several insights into the complexity of the topic, students’ sophisticated ways of reasoning, and pedagogical suggestions for supporting conceptual understanding (Bouhjar et al., 2018; Karakok, 2019; Salgado & Trigueros, 2015; Wawro et al., 2019). For example, Thomas and Stewart (2011) noted students’ need to understand how both matrix multiplication and scalar multiplication on the two sides of the eigenequation  $A\vec{x} = \lambda\vec{x}$  yield the same result, a vector. They also advocated for instructors to help their students develop a graphical conception of eigenvectors and eigenvalues, something they noted was weak in their study participants. Gol Tabaghi and Sinclair (2013) investigated students’ visual and kinesthetic understanding of eigenvector and eigenvalue. They found that working with an interactive sketch promoted students’ flexibility between synthetic-geometric and analytic-arithmetic modes of reasoning. Serbin et al. (2020) investigated physics students’ reasoning about the characteristic equation (CE) for a  $2 \times 2$  matrix  $A$ . They found that both conceptually and procedurally, most students’ knowledge of using the CE was deeper than their knowledge of deriving the CE. They also found that in deriving the CE, most students’ procedural knowledge was deeper than their conceptual knowledge, suggesting that students’ knowledge of principles underlying the relevant procedures might need further development to help them better understand how the CE is derived conceptually. Although this is most likely valued in a mathematics course when students are first learning the material, it may be less relevant in a physics course where students are using the CE to solve specific applied problems.

Eigenvalues and eigenvectors can take on various meanings, such as principal moments of inertia along principal axes for a rotating solid body. Beltrán-Meneu et al. (2017) found that a teaching intervention about moments of inertia had a positive impact on students’ geometric understanding of eigenvalues and eigenvectors. There can also be various emphases related to eigenvalues and eigenvectors. For example, Caglayan (2015) found that interacting with a dynamic geometry software helped some students to reason about the existence of infinitely many eigenvectors for a given eigenvalue of a  $2 \times 2$  matrix  $A$ ; in particular, some students evidenced analytical-structural thinking (Sierpinska, 2000) through using the applet sliders to drag a vector  $u$  until it was collinear with the vector  $Au$ , noticing a family of solution vectors that have such a property. Wawro et al. (2019) investigated linear algebra and quantum mechanics students’ understanding of linear combinations of eigenvectors. They found that students reasoned algebraically, structurally, and geometrically to successfully reach conclusions, but that some did not seem to leverage that there are infinitely many eigenvectors for a given eigenvalue. The authors suggest helping mathematics students understand eigenvectors as an infinite set of vectors satisfying the eigenequation before focusing on finding eigenbases for finite-dimensional vector spaces. In quantum mechanics, however, due to normalization conditions, only one eigenvector of length one associated with every eigenvalue of an operator is considered, rather than infinitely many. Therefore, because focusing on one eigenvector

for a given eigenvalue is acceptable and common in quantum mechanics, physics students may not attend to infinitely many eigenvectors in their reasoning.

Although these interpretations of eigenvectors and eigenvalues are important for understanding eigentheory in general, in this paper we demonstrate how eigenequations in quantum mechanics require students to think about eigenvectors and eigenvalues in different ways than they have learned in the past, and how there are some unique challenges that arise with interpreting eigenequations in quantum mechanics.

## Eigentheory in Physics

There are several physical systems for which linear algebra is the mathematical medium. Examples include finding moments of inertia for rotating rigid bodies and determining normal modes of oscillation of coupled systems, such as a double pendulum or a system of multiple masses connected by springs (e.g., Taylor, 2005). In rigid body rotation, the rotational inertia is represented as a tensor and is diagonalized to find the principal axes; the eigenvalues are the moments of inertia about each axis, known as the “principal moments.” For oscillating systems, the purpose of the eigenvalue equation is to solve for eigenvalues and eigenvectors, using the characteristic equation, to characterize the system by its normal modes (eigenvectors) and resonant frequencies (eigenvalues). However, individual eigenequations are not written out and interpreted as in quantum mechanics. Typically, the eigenvalues are fed into the general solution form to describe the motion in terms of rotations about principal axes or sinusoidal oscillations of the normal modes.

When an eigenequation is presented in classical physics textbooks, the main interpretation is geometric, with a transformation interpretation of the equation. For example:

“[Eigenvalue equations] always express the same idea, that some mathematical operation performed on a vector ( $\omega$  in our case) produces a second vector ( $I\omega$  here) that has the same direction as the first. A vector  $\omega$  that satisfies  $[I\omega = \lambda\omega]$  is called an eigenvector and the corresponding number  $\lambda$ , the corresponding eigenvalue.” (Taylor, 2005, p. 389)

In an intermediate-level mathematical methods for physics course, Her and Loverude (2020) investigated student performance on problems dealing with determining normal modes of oscillation of coupled systems. They found that students were fairly successful in technical operations of matrix mechanics, in particular working through the characteristic equation and taking the determinant, but were less successful in mathematizing the scenario - finding the general structure of the matrix and the correct eigenequations - and in interpreting the solutions as the oscillation frequencies of the system.

A common text for a Mathematical Methods for Physics course (Boas, 2006) discusses eigentheory in the context of diagonalization of matrices for a coordinate transformation; the author uses an elastic membrane as a metaphor for the coordinate system. In this metaphor, the eigenvectors have the property that “Along these [...] directions (and only these), the deformation of the elastic membrane was a pure stretch with no shear (rotation)” (p. 149). Much later in the text, eigenvalues are



discussed in the context of “separation constants” in partial differential equations, noting that:

“The resulting values of the separation constants are called eigenvalues and the solutions of the differential equation [...] corresponding to the eigenvalues are called eigenfunctions. [...] Having found the eigenfunctions, the next step is to expand the given function in terms of them. [...] the eigenfunctions are a set of basis functions for this expansion.” (Boas, 2006, p. 625)

Textbook emphasis on eigenequations as having centrality in physical interpretation is not as prevalent in classical contexts as in quantum mechanical contexts, perhaps because the classical systems are more tangible in their meaning.

In quantum mechanics in particular, eigenequations play a significant role. One common use of eigenequations is as reference for computation. For example, when determining an expectation value of a quantity (the predicted mean value of a set of measurements on an ensemble of identically prepared systems) for a given state, the state vector is typically expanded in the basis of the quantity’s associated operator, and each eigenstate is acted on by the operator, resulting in the corresponding eigenvalue “replacing” via algebraic substitution the operator in front of each eigenstate. If the state vector is an eigenstate of that operator, then the expectation value is the associated eigenvalue. For example, for a system prepared in the state  $|+\rangle$ , the expectation value of  $\hat{S}_z$  is  $\langle \hat{S}_z \rangle = \langle + | \hat{S}_z | + \rangle = \langle + | \frac{\hbar}{2} | + \rangle = \frac{\hbar}{2} \langle + | + \rangle = \frac{\hbar}{2}$ . This calculation explicitly uses the eigenequation  $\hat{S}_z | + \rangle = \frac{\hbar}{2} | + \rangle$  to algebraically substitute  $\frac{\hbar}{2} | + \rangle$  in for  $\hat{S}_z | + \rangle$ . It also leverages the inner product  $\langle + | + \rangle = 1$ , because of the normality of  $| + \rangle$ , to determine the expectation value to be  $\frac{\hbar}{2}$ . This calculation is sensible physically because  $\frac{\hbar}{2}$  is the only possible result of a measurement of  $S_z$  for the  $| + \rangle$  state.

Another common use of eigenequations in quantum mechanics is as a concise representation of postulates that relate the physical system, measurable quantities, and possible measured values of those quantities (e.g., as explained in the [Physics Background: Postulates and Spin- \$\frac{1}{2}\$  Systems](#) section for spin- $\frac{1}{2}$  systems). Physics education researchers have studied student understanding of the eigenequation, often focusing on the time-independent Schrödinger equation or the eigenequation for the Hamiltonian (total energy) operator,  $\hat{H}|\varphi_n\rangle = E_n|\varphi_n\rangle$ . This is a major focus of quantum mechanics courses because time evolution of a system is associated with the Hamiltonian operator. Singh (2008) reported several ideas students have about the Schrödinger equation, including that students have been found to incorrectly interpret it as representing a measurement of energy. In a study focused on resources (Hammer, 2000) students use to understand quantum mechanical spin operators, Gire and Manogue (2008) identified the resource “quantum measurement as an agent,” coming from the idea that taking a measurement changes the system. The students in their study also articulated that operating on a vector generally transforms the vector, which the researchers labeled “operating as agent.” The researchers argued that by accessing both of these resources in parallel, students arrive at the unproductive conclusion that an operator acting on a state vector represents a measurement of the quantity represented by that operator. Gire and Manogue (2012) followed this up, noting that navigating unfamiliar language in addition to eigentheory

may present additional challenges for students. One student in their study used the term “observable” when describing an eigenvalue. They also note that “It is common for textbooks to say that observables are represented by operators” (p. 4), enabling further conflation between a measurable quantity and a measurement.

There is evidence that students overgeneralize eigenequations of physical observables to apply to any state (even for a state that is an explicit superposition of the eigenstates of the operator), perhaps connecting to the emphasis on the energy eigenvalue equation in instruction (Singh, 2008; Styer, 1996), the relationship of the Hamiltonian operator with time dependence, and/or the connection mentioned above between operators acting on a state and measurements of the state (Gire & Manogue, 2008; Singh, 2008).

Dreyfus et al. (2017) engaged in a theory-building effort meant to use data to illustrate their conjectures and explored a method for analyzing student mathematical sensemaking in quantum mechanical problem solving. They proposed that a student view of an eigenequation as “a transformation that reproduces the original,” consistent with a geometric interpretation, might support an appropriate interpretation of the energy eigenvalue equation. The authors argue that the transformation in a quantum mechanics context is less directly associated with physical meaning than are typical equations in physics.

### Student Conceptions of the Equal Sign in Mathematics and Physics

Another relevant aspect of student interpretation of eigenequations relates to the equal sign. Generally, students’ conceptions of the equal sign in mathematics, initially studied among K-12 students, have been described as either operational or relational (e.g., Behr et al., 1980; Carpenter et al., 2003; Kieran, 1981; Knuth et al., 2005). An operational view of the equal sign is one in which a person interprets the equal sign as a signal to “do something,” typically computation; for example, performing operations on the left side produces the result on the right side. A relational view is typified by a person reasoning about the equal sign as representing a relationship of equivalence between the quantities or expressions on either side of the equals sign; for example, a person might compare the expressions on either side of the equal sign and confirm that they are indeed equivalent, i.e., that one side is “the same as” the other. In a study of undergraduate and graduate students, McNeil and Alibali (2005) found that student interpretations can be context dependent, but that with experience, the relational interpretation can supersede the operational one. Larson et al. (2009) asked linear algebra students, before instruction about eigentheory, to explain how they thought about  $A \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$ , where  $A$  is a  $2 \times 2$  matrix.

Some students concluded that because the two sides are the same, it must be that  $A$  somehow equals 2. The authors concluded these students leveraged a relational view of the equal sign because of their need for sameness on both sides of the equation. We note that although this student view has room for growth in linear algebra, it is understandable based on students’ prior experiences with equations in which all

entities are numbers. If, instead, a student understands that for  $A\vec{x} = \lambda\vec{x}$  in each side of the equation the final object is a vector (Thomas & Stewart, 2011) and relates those final objects as equivalent, we posit that would be demonstrating a rich relational view of the equal sign in that linear algebra context.

In the [Matrix Equations and Eigentheory in Linear Algebra](#) section, we summarized the transformational interpretation of the matrix equation  $A\vec{x} = \vec{b}$  from Larson and Zandieh (2013), as well as three metaphors for function from Zandieh et al. (2017): input–output, machine, and morphing. We note that all students in the 2017 study agreed that a linear transformation is a type of function, and almost all used one of these three metaphors for comparison between functions and linear transformations. In our analysis we leverage these prior works to create an additional category related to the equal sign: we classify students' interpretations of the equal sign in eigenequations consistent with these specific metaphors within the transformational category of Zandieh et al. (2017) as a *functional interpretation of the equal sign*.

Interpretations of the equal sign in physics equations are less studied. Alaei et al. (2022) explored equations in multiple chapters of representative physics texts at introductory, intermediate, and advanced levels, classifying the role of the equal sign in each. They found five categories - *definitional, causality, assignment, balancing, and calculate* - and showed the relative prevalence of each category in the different texts. Upper-division texts on electricity and magnetism and quantum mechanics showed a far smaller proportion of assignment equal signs, a greater proportion of definitional signs, and twice as many calculate signs. These differences are attributed to the increase in the number and complexity of derivations requiring more symbolic manipulations and quantities that are more carefully defined. The process of computing expectation values may reinforce an operational interpretation of the equal sign for quantum mechanical eigenequations, placing the equation in the Alaei et al. (2022) calculate or definition category for students. This may negatively impact the development of a nonstandard conceptual interpretation of the eigenequations related to the third postulate, as discussed in the [Physics Background: Postulates and Spin- \$\frac{1}{2}\$  Systems](#) section.

## Methods

In this paper, we investigate how physics students make sense of eigenequations that they encounter in a mathematics course and in a quantum mechanics course. To do so, we analyze data that were collected from two sources. The first set consists of video, transcript, and written work from individual, semi-structured interviews (Bernard, 1988), drawn on a voluntary basis, with 9 students from a senior-level quantum mechanics course at a medium-sized public research university in the northeast US. The second data set, collected three years later, consists of written work from 10 students enrolled in the same quantum mechanics course. The course was taught by the same instructor both times, who was part of the research team for both data collections.

The interviews took place in week 8 of the semester, after the completion of a unit on spin- $\frac{1}{2}$  systems with Dirac notation. All interview participants were recruited during

class, gave informed consent before being interviewed, and received compensation for their time and effort. For this paper, we focus on one particular interview question, which was used to investigate how students conceptualize and make sense of the eigenequations  $A\vec{x} = \lambda\vec{x}$  and  $\hat{S}_x|+\rangle_x = \frac{\hbar}{2}|+\rangle_x$  (see Fig. 2a). In Wawro et al. (2020), a subset of the author team used discourse analysis to characterize and examine the (in)compatibility of the students' interpretations of the two equations. That preliminary analysis highlighted an opportunity to support student understanding. The written question used in the second data collection was developed to assess the instructional goals of the course. In order to probe student thinking on this topic more directly within the course itself, the question was used on a homework assignment to investigate how students conceptualize and make sense of  $A\vec{x} = \lambda\vec{x}$  and  $\hat{S}_z|+\rangle = \frac{\hbar}{2}|+\rangle$ , which is the eigenequation for the "up"  $z$  component of spin rather than the "up"  $x$  component (see Fig. 2b). The  $\hat{S}_x$  eigenequation was used for the interview data collection to align with a subsequent interview question (regarding a "spin flip" non-eigenequation, see Wawro et al., 2020). Noting that the potentially less familiar  $x$  basis did not seem problematic nor relevant to student reasoning about eigentheory in the interview data, the standard  $\hat{S}_z$  eigenequation was used in the written assignment. Subscripts for the kets are not included in the  $\hat{S}_z$  eigenequation as a matter of convention; physicists assume that when no basis is given, states are written in the  $z$  basis.

The written question explicitly stated that both equations were eigenvalue equations that could be thought of in similar and different ways, and it asked students to discuss the equations' individual meanings as well as to compare them. It explicitly asked for

(a)

"I have a few questions prepared. For each one, I want you to explain what the equation means to you."

- $A\vec{x} = \lambda\vec{x}$ , where  $A$  is a  $2 \times 2$  matrix,  $\vec{x}$  is a  $2 \times 1$  vector, and  $\lambda$  is a scalar.
- $\hat{S}_x|+\rangle_x = \frac{\hbar}{2}|+\rangle_x$

"You mentioned both related to eigentheory. Please compare and contrast how you personally conceptualize eigentheory in the two situations."

(b)

In previous courses you have probably seen an eigenvalue equation, which looks something like this:

$$A\vec{x} = \lambda\vec{x}$$

There are a number of ways to interpret this equation, one of which is similar to the way we interpret

$$\hat{S}_z|+\rangle = \frac{\hbar}{2}|+\rangle$$

in quantum mechanics. In general, however, the most productive ways to think about these two equations are quite different. In no more than a page, discuss the various meanings & interpretations of these two equations, their similarities and differences, and why we identify them both as eigenvalue equations.

**Fig. 2** Questions asked in this study. **a** The interview question. **b** The written question

similarities and differences between the two equations in order to elicit ideas that may have required prompting in the interviews. Student responses varied in amount of detail given: some students primarily addressed the first portion of the prompt asking them to interpret the two equations, and others primarily addressed the portion asking them to make comparisons.

Students' interpretations of the equations in the two tasks were so similar that the authors analyzed them together for the purposes of this study. Students from interview data are pseudonymized with "C#", and students from written task are pseudonymized with "S#". Students were not asked for their pronouns, so we use the singular pronoun "they" for all participants.

In order to document students' various interpretations of the two eigenequations, we coded the interview data for nuances in student imagery and meanings for the various concepts that students brought to bear as they engaged with the question. We drew inferences from the language students used, which is type of discourse analysis (Gee, 2014). We leveraged a modified grounded theory approach (Glaser & Strauss, 1967; Zimmerman et al., 2019) to construct analytic codes primarily from the data and also our knowledge of existing literature on student understanding of eigentheory in both the RUME and PER communities. For instance, the notion of an eigenvector being a vector that is stretched by its associated transformation is documented in the literature as a productive conceptualization (e.g., Sinclair & Gol Tabaghi, 2010), so we were primed by our prior knowledge to notice such meanings in the data. Modified grounded theory also leaves space for us as researchers to be open to discovering unanticipated ways that students made sense of the two eigenequations, as well as how they reconciled the two in comparison with each other.

A similar approach was taken with the written data, with the exception that its coding was informed by the interview data as well. Some ideas presented by students in the written data still stood out as unique from the codes established during interview analysis. We used constant comparative methods to compare student ideas and determine if responses are indicative of the same knowledge (Strauss & Corbin, 1998). These ideas were added to the existing codes and, upon returning to the interview data, we found similar ideas to those presented in the written data and coded them accordingly. In this way the two data sets were found to be consistent with one another despite the differences between time of collection and different groups of individuals in the course.

## Results

Students from both the interview and written data sets interacted with  $A\vec{x} = \lambda\vec{x}$ , which we refer to as Equation 1 (E1). Students from the interview data interacted with  $\hat{S}_x|+\rangle_x = \frac{\hbar}{2}|+\rangle_x$ , and students from the written data set interacted with  $\hat{S}_z|+\rangle = \frac{\hbar}{2}|+\rangle$ . We refer to both of these spin eigenequations as Equation 2 (E2).

In this section, we summarize our findings regarding students' interpretations of E1 and E2. First, we found two main ways in which students explained E1: via a functional or a relational interpretation of the equal sign. We detail and exemplify both of these interpretations, as well as give instances of students leveraging both

interpretations. We then detail ways that the students' language in explaining E2 was consistent with functional or relational interpretations of the equal sign. Second, we present results regarding three main ways that students explained how they made sense of the physical meaning conveyed by E2: as measurement, potential measurement, or as correspondence. We acknowledge that our analysis is inferential and that alternative interpretations of student meanings may be plausible in some instances. Finally, we present two themes regarding student comparisons of E1 and E2: attention to form and attention to conceptual (in)compatibility.

## Student Reasoning About Equation 1

As discussed in the Introduction, two common ways that members of a mathematics classroom might interpret the eigenequation  $A\vec{x} = \lambda\vec{x}$  are (1) there are some nonzero vectors  $\vec{x}$  such that when  $A$  acts on  $\vec{x}$ , the result is just a scalar multiple or a stretch of the vector  $\vec{x}$ ; and (2) there are some nonzero vectors  $\vec{x}$  such that the result of  $A$  acting on  $\vec{x}$  is the same as the result of  $\lambda$  times  $\vec{x}$  or  $\lambda$  stretching  $\vec{x}$ . These interpretations of the eigenequation are consistent with existing literature and what we found in the data. In particular, the first is consistent with a functional interpretation of the equal sign, and the second is consistent with a relational interpretation of the equals sign (e.g., Knuth et al., 2005).

### Functional Interpretation of the Equal Sign in Equation 1

In the interview data with 9 students, we found five instances of students discussing Equation 1 in a way consistent with a functional interpretation of the equal sign. All but one of these instances occurred when students were asked to read Equation 1 out loud and explain how they thought about it. For example, C1 stated, "the solutions  $x$  to this, are the vectors such that when operated on by  $A$ , you get the same vector  $um$ , to within a scalar constant." We code this utterance as a functional interpretation because of the conveyance of a vector being "operated on by  $A$ " and the result of that action being a scalar multiple of the same vector. A different student, C7, stated, "So I see that, as a matrix  $A$  operates on a vector  $x$  and it has a special property such that it's just that vector again times a scalar." In this example we also see evidence of the operation of  $A$  on the vector  $\vec{x}$  producing a scalar multiple of the vector; C7's use of the word "again" helps convey that the action of  $A$  on  $\vec{x}$  produced the result of the scaled vector.

In the written data, 6 students presented views consistent with a functional interpretation of the equal sign. Some of these responses were very explicit, e.g., S3: " $A\vec{x} = \lambda\vec{x}$  is more generally interpreted as a linear transformation  $A$  acting on a vector  $\vec{x}$ , the end result of which is the vector  $\vec{x}$  being scaled by a constant  $\lambda$ ," and S4: "Mathematically, the eigenvalue equation states that the transformation represented by the matrix  $A$  only scales the vectors in its eigenspaces (i.e., vectors that are eigenvectors)." In both examples, the scaling of the vector is being framed as the result

of the operator acting on the vector, consistent with the functional interpretation of the equal sign. Other students were more reminiscent of student C1 in the interview data, such as S7: “Given  $A$  is a matrix,  $\lambda$  is the value that the matrix scales by... thus, when the eigenvector is operated on it is only scaled, never rotated.” In all of these cases, just as with the students in the interview data, the focus is on the scalar multiplication being the result of the operation of the matrix on its eigenvector.

### Relational Interpretation of the Equal Sign in Equation 1

The second common way that members of a mathematics classroom might interpret the eigenequation  $A\vec{x} = \lambda\vec{x}$ , (2) there are some nonzero vectors  $\vec{x}$  such that the result of  $A$  acting on  $\vec{x}$  is the same as the result of  $\lambda$  times  $\vec{x}$  or  $\lambda$  stretching  $\vec{x}$ , is consistent with existing K-16 literature on student understanding of the equal sign. In particular, in our data there is evidence of students interpreting Equation 1 in ways consistent with a relational interpretation of the equals sign (e.g., Knuth et al., 2005).

In the interview data, there are five instances of students discussing Equation 1 in this way. All of these instances occurred as students were responding to the interviewer’s prompt, “How do you think about what the equals sign means in that equation?” For example, C8 stated, “ $A$  acting on this vector is the exact same thing as this scalar quantity acting on the vector.” In C8’s response, they explain how the result of the action on one side of the equation (“ $A$  acting on this vector”) produces the same result as the action on the other side of the equation (“scalar acting on this vector”). Similarly, but with a bit more specificity about that result, C2 stated, “the two-by-one vector that is made by multiplying this matrix by this vector, um is the same as the two-by-one vector that is, um, this eigenvector—the same eigenvector [points to  $\vec{x}$  on both sides], scaled by the eigenvalue [points to  $\lambda$ ].”

### Simultaneous Views of Equation 1

One student, C6, explained the equation in ways consistent with both the functional interpretation (twice) and the relational interpretation (once). When asked to explain how they thought about the equation, C6 stated: “this looks like the uh, eigenvalue eigenvector equation...where you uh operate on vector  $x$  with  $A$ , and you get the same vector  $x$  times a scalar multiple,” which we coded as a functional interpretation. When responding to how they thought about the equals sign means, C6 stated:

Um, so they have to be describing the same, like they’re identical. So like, you do this [points to  $A\vec{x}$  on the left-hand side (LHS) of E1] to get that [points to  $\lambda\vec{x}$  on the right-hand side (RHS)] usually um, but like doing this [points to LHS] and doing this [points to RHS] same thing are—you’d get the same thing back as a scaled vector. [Int: Gotcha] But this one you’ve pulled the scale out [points to RHS]. And on this side it’s a matrix vector operation [points to LHS].

We coded C6’s statements of “you do this to get that” and associated gestures as a functional view. The rest of C6’s statement, paired with their gestures, is consistent with a relational view of the equal sign.

Similar to C6 in the interview data, S17 presented both views of the eigenvalue equation in different portions of their response. They first wrote, “The basic equation  $A\vec{x} = \lambda\vec{x}$  can be thought of similarly, with  $\lambda\vec{x}$  representing the corresponding values for the operation  $A\vec{x}$ .” This correspondence view of the eigenvectors, eigenvalues, and operator seems consistent with the relational view of the equal sign. The student then wrote, “More commonly however, the eigenvalue equation is generally used to find the vectors  $\vec{x}$  where, when  $A$  is applied, only change their magnitude by  $\lambda$ ”, which is consistent with a functional view of the equation.

## Student Reasoning About Equation 2

The physics eigenequation that students in our data set were asked to reason about was either  $\hat{S}_x|+\rangle_x = \frac{\hbar}{2}|+\rangle_x$  (the interview) or  $\hat{S}_z|+\rangle = \frac{\hbar}{2}|+\rangle$  (the written task). We first detail ways in which the students’ language in explaining Equation 2 was consistent with the functional or relational interpretations of the equal sign. We then present results regarding students’ interpretations of the physical meaning conveyed by Equation 2. In particular, our analysis revealed three main interpretations of Equation 2: measurement, potential measurement, and correspondence. Although measurement is not a meaningful/valid interpretation of the equation, potential measurement and correspondence are both meaningful and connected. We make a distinction between the two ideas, potential measurement and correspondence, because students from both data sets presented ideas consistent with the third postulate, but some of those students placed an emphasis on the explicit potential measurement portion of the postulate while others focused primarily on the implicit connections between the terms established in the postulate.

## Functional or Relational Interpretations of Equation 2

Some students in the interview data initially presented ideas consistent with either the functional or relational interpretations of the equal sign. Some examples come from C12, “I mean, this is the  $S_x$  operator, and it’s operating on just one eigenvector. You’re getting the, like one eigenvalue times the eigenvector,” and C1, “Um, so, if you take a vector plus  $x$  [points to the  $|+\rangle_x$  on left], and operate it on, with some operator,  $S_x$  [points to  $\hat{S}_x$ ], then [pause] you, you will, you would get solutions that are the form [points to right side of equation]  $\hbar$  over two times plus  $x$ .” In the written data, S3, S7, and S14 all presented ideas consistent with a functional interpretation of E2 as well.

C5 provides an explanation of E2 consistent with the relational interpretation of the equal sign: “So, um, for the up  $x$  state [pointing to both plus- $x$  kets], um, there is an eigenvalue [pointing to  $\frac{\hbar}{2}$ ] for  $h$ , er, the eigenvalue  $\hbar$  over two results in the same thing [points to  $\hat{S}_x$ ] as operating  $S_x$  on the up  $x$  state [points to  $|+\rangle_x$  on left].” Student C2 gives another example of relational interpretations of the equal sign in E2 saying, “two-by-one vector scaled by the measured observable eigenvalue is the same as this [points to left side], or is the same vector that is made by multiplying



this, um, matrix by this eigenvalue, er eigenvector.” We point out that C2’s explanation of E1 (see the [Relational Interpretation of the Equal Sign in Equation 1](#) section) was also consistent with a relational interpretation of the equation.

From the written data, two students presented views consistent with the relational view of the equal sign. For example, S14 wrote, “We identify equation 2 as an eigenvalue equation because the matrix  $\hat{S}_z$  yields the same result as the scalar  $\frac{\hbar}{2}$ . This is, more or less, how eigenvalue equations work.” Although S14 explicitly writes about Equation 2, they also generalize to all eigenvalue equations. Given that the prompt explicitly states that both are eigenvalue equations, it seems reasonable to assume S14’s view might also extend to E1, even though they only explicitly address E2.

### Measurement Interpretation of Equation 2

In our data set, five of the nine students from the interview data and no students from the written data conveyed an understanding that is consistent with what we call a measurement interpretation of Equation 2. By a *measurement interpretation* of the eigenequation, we mean that a student seems to be making sense of the eigenequation by thinking that it is conveying that an operator acting on a state is the same as taking a physical measurement of the quantity that the operator represents. This is consistent with findings related to students activating both “quantum measurement as agent” and “operation as agent” as a reason they might think that operating represents measuring (Gire & Manogue, 2008). For example, after reading the equation, C8 volunteered,

I guess everything I have said earlier [for E1] would apply to this, however, I don’t really think of it in terms of that way when I see it in this notation [E2]. This makes me more think of taking a measurement.

When asked how they describe what is going on in E2 physically, C6 stated,

you’re measuring  $S_x$  [...] not only are you mathematically operating, like it represents physically doing something to the system, which in this case is measuring the spin in the  $x$  direction. [...] you’re getting a measurement of  $\hbar$  over two in the plus  $x$  direction.

The main reasoning leveraged by students in these responses seems to describe operation as measurement by thinking that the equation is a mathematical representation of a physical operation. Although this is not a desired interpretation, one can see why it seems sensible to a student; the terms “operator,” “operation,” and “acting” may actually suggest measurement to these students.

### Potential Measurement Interpretation of Equation 2

In our data set, two of the nine students from the interview data and three of the ten students from the written data conveyed an understanding that is consistent with what we call a *potential measurement interpretation* of Equation 2. By a “potential measurement” interpretation of the eigenequation, we mean that a student seems to

be making sense of the eigenequation by thinking that it is conveying information about what value would be measured if a measurement of that quantity (in this case, spin) were to occur. Statements coded in this category tended to use some sort of conditional language to indicate that the equation provided information about possible measurements, rather than it representing some actual action of measurement. We see the potential measurement interpretation to be consistent with the third postulate of quantum mechanics (see the [Physics Background: Postulates and Spin- \$\frac{1}{2}\$  Systems](#) section) but with an emphasis on the explicit potential measurement portion of the postulate.

For example, when asked to explain what E2 meant physically, student C7 replied, “this is a representation that I think is meant to show what the measurement of this spin, of a spin-up-in-the-x particle would be,” later adding, “I know that this eigenvalue equation just tells you that if you take a particle that’s up x, the, you try to measure the observable... that’s the value you would get.” In both portions of C7’s response, they leverage conditional language to communicate that the eigenequation is indicating potential measurement values, if one were to measure the spin observable. In response to the written prompt (Fig. 1(b)), student S21 wrote, “ $\hat{S}_z|+\rangle = \frac{\hbar}{2}|+\rangle$  expresses that you will measure  $\frac{\hbar}{2}$  in the upward spin state. This will also be true when measuring  $|-\rangle$  state, you will receive a measurement of  $-\frac{\hbar}{2}$ .” We coded S21’s response as a potential measurement interpretation of the eigenequation because of the future tense; they do not indicate that the equation represents the action of measuring, but rather that it provides information about measured values when the related measurements do occur. To the same written prompt, student S25’s response included: “in quantum eigenvalues represent one of the possible measurement results... quantum eqn represents the possible measured values of the  $\hat{S}_z$  operator.” S25 used the word “possible” twice - once to indicate eigenvalues are possible measured values, and once to indicate that the (given) eigenequation conveys information about possible measurement results of the spin operator. Because of this, we coded S25 as demonstrating a potential measurement interpretation of E2.

### Correspondence Interpretation of Equation 2

In our data set, three of the nine students from the interview data and two of the ten students from the written data conveyed an understanding that is consistent with what we call a *correspondence interpretation* of Equation 2. By a “correspondence interpretation” of the eigenequation, we mean that a student seems to be making sense of the eigenequation by thinking that it is conveying information about the relationship between eigenstates and eigenvalues. We see the correspondence interpretation to be consistent with the third postulate of quantum mechanics (see the [Physics Background: Postulates and Spin- \$\frac{1}{2}\$  Systems](#) section) but with an emphasis on the implicit connections between the terms established in the postulate. This was first noted in the written data with S14’s response: “[E2] tells us that the spin in the z direction,  $S_z$ , of a spin up particle or beam,  $|+\rangle$ , has a value of  $\frac{\hbar}{2}$ .” They are not relating the equation to measurement but rather are providing an interpretation consistent with the idea that the eigenvalue equation conveys information about

the value of an observable for a given eigenstate. This prompted us to examine the interview data for similar interpretations.

While explaining the meaning of E2, C3 said, “I think of it as-if the particle is spinning in the  $x$ , has spin in the  $x$  direction, it, that spin will have a value of  $\hbar$  over two.” We coded C3’s statement as a correspondence interpretation because they are describing, without any reference to measurement, the physical information that the equation is communicating. When asked how they interpret the equation physically, C2 said, “this shows that the energy, or the, the measured um, value of, uh, the positive  $x$  spin is  $\hbar$  over two, I suppose.” Although the student said “measured,” their response does not include discussion of an operator, which would be indicative of measurement interpretation, nor the conditional language indicative of potential measurement interpretation. Thus, C2’s response most closely aligns with correspondence, though it may have some overlap with potential measurement given that both interpretations are consistent with the third postulate.

## Student Comparisons of Equations 1 and 2

All students noticed (in the interviews) or were told (in the written prompt) that both E1 and E2 are eigenequations. Thus, in this section our analysis is focused on how students compared and contrasted the two equations. In our analysis two themes emerged in the ways that students compared Equations 1 and 2: attention to form and attention to conceptual compatibility. Student work that paid attention to form manifested in two main ways: drawing parallels between individual terms in the two equations and/or between the two equations’ overall structure; and commenting on the literal symbols and notational systems used in the equations. The second theme, conceptual compatibility, focuses on aspects of student thinking in the data that highlight ways in which various interpretations make sense or are potentially not productive for either or both of the two eigenequations.

### Attention to Form

In the written data, just over half of students (5/9) noted some sort of morphological similarity between the two equations, where morphological means relating to the form or structure of things. For example, when comparing the two equations, S21 wrote, “ $\vec{x}$  and  $|+\rangle$  are both vector quantities.  $\lambda$  and  $\frac{\hbar}{2}$  are both scalars,” making element-by-element comparisons of the two equations. S4 wrote, “both equations... contain a vector operated on by a quantity... that results in the vector being rescaled but preserving its initial direction,” commenting on the similar structure of the two equations holistically. We note S4’s statement is consistent with a functional interpretation of the equal sign for both E1 and E2. These two students are representative of the group of students who noted morphological similarities between the two equations. Similarly, some students (5/9) in the interview data also made morphological comparisons between the two equations. For example, C3 stated, “The  $A$  [points to  $A$  in E1] matches the spin  $x$  [points to  $S_x$  in E2] and the lambda [points to  $\lambda$  in E1] is  $\hbar$  over two [points to  $\frac{\hbar}{2}$  in E2],” and C8 said, “Okay so, um, I can’t deny it, the parallel here. We have a matrix acting on a vector is equal to a scalar quantity times

the same vector,” when presented with E2. We note that C8’s statement about both E1 and E2 is consistent with a relational interpretation of the equal sign.

A few students (3/9) in the written data indicated that notation was a notable difference between the two equations. Students in the interview data did not make the same kinds of notational comparisons in response to these two equations. S14 wrote, “equation 1 uses vectors and matrices while equation 2 uses kets and can only be represented by matrices,” noting that the notations are different for the two equations and that due to the embedded meaning of E2 these are only representations. Unlike the interview prompt, the written assignment did not explicitly state what  $A$  is, but these students still interpreted  $A$  as being a matrix. In the same vein, S7 focused on the implications of working with the two different notations writing, “[ $|+\rangle$ ] requires inner products, while  $\vec{x}$  just require dot products.” We interpret this as comparison of working in Dirac notation and in matrix notation for real-valued vectors.

### Attention to Conceptual Compatibility

The second theme from our analysis focuses on conceptual compatibility, that is, the ways that students made sense of the interpretation(s) of the two equations, and in cases where they were different, reconciled them. In their comparison of the two equations, some students remarked on a conceptual tension between the two equations; in other cases, we as researchers noticed a potential conceptual tension between the two equations in how the students described the equations’ meanings. For the student-noticed tension, we focus on student C1 as a vignette.

When asked how they think about what is going on physically in what is represented by Equation 2, student C1 began an explanation but paused and conveyed some uncertainty with how to reconcile various conceptual interpretations that they were considering:

- 1 C1: So, physically, the  $S_x$  operator [points to  $\hat{S}_x$ ] is representative of a measurement, um,
- 2 of a vector, er, of a, of a state that’s represented by this vector [points to  $|+\rangle_x$  on left]. So
- 3 if you have, um a system which is prepared in this state, and you perform a measurement
- 4 to measure what the state is, um, then, then if, if this is a state such that it’s not changed
- 5 – hmm. [pauses]
- 6 Int: What’s giving you pause?
- 7 C1: I’m not exactly sure what this represents physically would be [ $S_x |+\rangle_x$ ].
- 8 This is a measurement on some, [ $\hat{S}_x |+\rangle_x$  again] measurement of the state of
- 9 some system. With respect to the  $x$  part of the state, this will only tell you what state it’s in
- 10 with respect to the  $x$ , the  $x$  component of the state I think. And since this is already entirely
- 11 in the plus  $x$  component [pointing to  $|+\rangle_x$  on left], and because that’s an eigenvector of this
- 12 [points to  $\hat{S}_x$ ], that means that [ $\frac{\hbar}{2} |+\rangle_x$ ] this state is unchanged. Although the
- 13 state isn’t represented by this then [ $\frac{\hbar}{2} |+\rangle_x$  again], because this isn’t normalized
- 14 for one.
- 15 Int: ... What kind of things are you thinking about that seem to be a struggle for you?
- 16 Or, like what makes you uncertain?
- 17 C1: I guess I’m uncertain about what this side represents [points to  $\frac{\hbar}{2} |+\rangle_x$ ]. I feel like I
- 18 know what the individual pieces mean okay. Like this, this eigenvalue [circles  $\frac{\hbar}{2}$ ] is the value
- 19 that you would measure when you operate this [circles  $\hat{S}_x$ ] in this state [circles  $|+\rangle_x$  on the
- 20 left]. I’m not sure what this [ $\frac{\hbar}{2} |+\rangle_x$  again] represents as a vector. Because to
- 21 say that the, the state is scaled by a vector seems wrong because the particles aren’t then
- 22 in this state [ $\frac{\hbar}{2} |+\rangle_x$ ], they’re just in this state [ $|+\rangle_x$  on the right],
- 23 just the plus  $x$  state.

Overall, C1 seems to be working to coordinate a conceptual interpretation of the various parts of the equation and the equation as a whole. Their initial explanation is consistent with a measurement interpretation of E2 ( $\hat{S}_x$  “is representative of a measurement” (line 1), “perform a measurement” (line 3),  $\hat{S}_x|+\rangle_x$  “is a measurement of the state of some system” (lines 8-9)). We note that C1 seems to be referring to both the operator  $\hat{S}_x$  and the operator times a ket  $\hat{S}_x|+\rangle_x$  within their measurement interpretation. In the middle of this initial explanation, though, C1 expresses some hesitation (lines 4-5, 7), saying they were not sure what  $\hat{S}_x|+\rangle_x$  would represent physically.

After reiterating their measurement interpretation of  $\hat{S}_x|+\rangle_x$  (lines 8-9), C1 articulates that the left-most ket  $|+\rangle_x$  indicates that the state is entirely in the “plus x component” (line 11). They then state that the same ket is an eigenvector of  $\hat{S}_x$  (line 12), which they had previously said is a “state such that it’s not changed” (line 4). From this, C1 concludes that “this state is unchanged” (lines 12-13) and underlines  $\frac{\hbar}{2}|+\rangle_x$  on the right side of E2. We note that C1 describes both the left ket  $|+\rangle_x$  and the right ket  $\frac{\hbar}{2}|+\rangle_x$  as unchanged states. However, C1 immediately explains something conceptually incompatible in E2, namely that because  $\frac{\hbar}{2}|+\rangle_x$  is not of magnitude 1 (“this isn’t normalized,” line 14), it cannot represent the state under consideration; this is consistent with the first postulate of quantum mechanics. Finally, C1 confirms that although they know what the individual pieces of the equation mean (lines 17-18), they are uncertain what  $\frac{\hbar}{2}|+\rangle_x$  represents (line 17). They again state  $\frac{\hbar}{2}$  is an eigenvalue of  $\hat{S}_x$  (line 18) and that it is a measurement value associated with  $\hat{S}_x$  and  $|+\rangle_x$  (line 19-20). Their wording here, however, is now consistent with a potential measurement interpretation of E2 (“this eigenvalue is the value that you would measure when you operate this in this state”). C1 revisits the right side of E2 and explains that interpreting  $\frac{\hbar}{2}|+\rangle_x$  as  $|+\rangle_x$  being scaled by  $\frac{\hbar}{2}$  seems wrong because particles are in the  $|+\rangle_x$  state not in the  $\frac{\hbar}{2}|+\rangle_x$  state (lines 22-23). It is unclear if C1 thought of “scale” as a geometric action (such as stretching a vector) or an algebraic action (multiplying a vector by a scalar). However, C1 explains the tension further by comparing it to how they would interpret E2 in a way compatible with ideas from a mathematics course. C1 begins by drawing axes in 2-space and a line through the origin:

- 24 *C1:* This vector [ $|+\rangle_x$  on the left] is represented by some line on, some line in the space,  
 25 and then this matrix [underlines  $\hat{S}_x$ ] means that if you start out somewhere on this line  
 26 and you operate with this matrix, you end up somewhere still on this line but scaled by  
 27 one of the eigenvalue scalars. However, that doesn’t make physical sense because the states  
 28 that we allow, maybe not that we allow, but the states that we say that particles are in are  
 29 represented by our normalized vectors. So if you scale this thing by anything other than  
 30 one, then you get something that’s not one. And, so yeah, that’s, that’s confusing.

In lines 24-30, C1 explains one last time the tension they experience between the mathematical and quantum mechanical interpretation of an eigenequation. Using a sketch, they give a functional interpretation of E2 (“if you start out somewhere on this line and you operate with this matrix, you end up somewhere still

on this line,” lines 25-26). This also leverages a geometric interpretation of scaling (“still on this line but scaled by one of the eigenvalue scalars,” lines 26-27). C1 concludes by reiterating that this is confusing because physical states must be represented by normalized kets (lines 29-30). Overall, C1 invokes the requirement of a state vector to be normalized to reject the idea that operation with the matrix scales the eigenstate. C1 recognizes but did not reconcile the disconnect they felt between geometric and quantum mechanical interpretations of the quantum mechanical eigenequation. We posit that this disconnect can be evidenced in C1 communicating both a measurement interpretation and a functional interpretation of E2; we unpack the tension between possible interpretations of various eigenequations in the [Discussion](#) section.

Other students also presented differences in their interpretations of the two equations, but they did not necessarily mark them as a point of tension for them conceptually. For example, student C8 who was categorized as demonstrating a “possible measurement” interpretation of E2, made unprompted comparisons between the two equations when asked to interpret E2. When C8 read E1, they said, “A acting on this vector is the exact same thing as this scalar quantity [ $\lambda$ ] acting on the vector.” When subsequently asked for their physical interpretation for E2, C8 said, “I guess everything I have said earlier [for E1] would apply to this, however, I don’t really think of it in terms of that way when I see it in this notation [E2]. This makes me more think of taking a measurement.” C8 demonstrated an explicit separation between interpreting eigenequations in a mathematical context and in a quantum mechanical context. C8 did not express any confusion or concern about this distinction.

Although thematically similar, some comparisons made by students were broader, saying that E2 was a special or specific case of the more general E1. A few examples include: “[E1 is the] generalized mathematic [sic] approach to finding an eigenvalue,” (S22), “[E1] is slightly more generalizable and easier to understand” (S7), and “[E1] is more general” (S21). C12 indicated that this was something they thought about in their interview, stating: “Um, it’s just like this yeah, this could be more generalizable. Like we don’t know what eigenvector  $x$  is, and it could be both, I think, and we don’t know what eigenvalue that is. Whereas here [pointing to E2] it’s specified.”

Finally, other students presented parallel interpretations of the two equations. For example, S7 wrote, “ $S_z$  is an operator,  $|+\rangle$  is an eigenvector that only scales along its axis, &  $\frac{\hbar}{2}$  is the eigenvalue  $A$  scales  $|+\rangle$  by.” Similarly, S4 wrote “both equations... contain a vector operated on by a quantity... that results in the vector being rescaled but preserving its initial direction.” We interpret both S7 and S4 as communicating that the vector on the right side of the equation is being “rescaled” or is a scalar multiple of the vector in the left side of the equation, with none of the tension C1 described. As mentioned earlier, it is not necessarily conceptually productive to interpret E2 as scaling an eigenvector because quantum mechanical eigenvectors must be normalized, but S7 and S4 did not notice or draw attention to this incompatibility.

## Discussion

In this paper, we presented results regarding students' interpretations of  $A\vec{x} = \lambda\vec{x}$  (E1) from mathematics and either  $\hat{S}_x|+\rangle_x = \frac{\hbar}{2}|+\rangle_x$  or  $\hat{S}_z|+\rangle = \frac{\hbar}{2}|+\rangle$  (E2) from quantum mechanics. We found that students reasoned with E1 and at times with E2 via a functional interpretation and/or a relational interpretation of the equal sign. Second, we found that students made sense of E2 via a measurement, potential measurement, and/or correspondence interpretation of the equation. Finally, two themes emerged in how students compared E1 and E2: attention to form and attention to conceptual (in)compatibility.

We conclude with a discussion the various interpretations of the equations and the limitations of the study, and we end with implications for instruction.

## Interpretations of the Equal Sign

We believe our introduction of *functional interpretation of the equal sign* to be a novel contribution for research. The existing operational and relational dichotomy of the equal sign (e.g., Knuth et al., 2005) was insufficient to capture student conceptualizations of the eigenequations in our data; this is likely because the foundational research on views of the equal sign originated in K-8 settings not focused on reasoning about functions. Thus, we drew inspiration from the notion of transformational interpretation of the matrix equation  $A\vec{x} = \vec{b}$  from Larson and Zandieh (2013) to introduce the functional interpretation of the equal sign. This contributes to the literature by connecting meanings of the equal sign to meanings of specific types of equations. In this paper we leveraged this categorization for student reasoning about eigenequations in linear algebra and in quantum mechanics, but it could be applied to other equations and in other domains as well.

Quantum mechanical eigenequations are conceptually distinct from standard mathematical eigenequations. A functional or relational interpretation of the equal sign, in which the right side of the equation is read out as a scale or stretch (if in a graphical setting) of the vector on the left side, either by transformation or by comparison, is completely acceptable and encouraged for E1. Several students in each data set reasoned in one of these ways about E1. However, as noted in the [Background](#) section, these are not physically appropriate for E2: scaling an eigenstate of a quantum mechanical system does not make sense because the states must be normalized, according to the first postulate of quantum mechanics. The subtlety of this distinction is reflected in our results: some students (e.g., S4 and S7) did not demonstrate that they understood that it is unproductive and perhaps inappropriate to think of QM eigenvectors in terms of scaling, while C1 was able to articulate this inconsistency very well (e.g., lines 10-14 in the [Attention to Conceptual Compatibility](#) section). Similarly, interpreting the action of the operator on the state vector as making a measurement of the associated physical quantity (spin) is not correct. However, this interpretation is understandable precisely because so many scientific equations, including some in quantum mechanics, are direct mappings of relationships in the

physical world, and are often functional and/or relational expressions as well. This is consistent with the discussion by Dreyfus et al. (2017) related to the indirectness of physical interpretation of the time-independent Schrödinger equation.

## Interpretations of Quantum Mechanical Eigenequations

One physically meaningful interpretation of a quantum mechanical eigenequation is as a potential measurement, that is, to connect the eigenvalues to the possible outcomes of hypothetical measurements of the quantity represented by the operator. This interpretation is more refined than a measurement interpretation, is often used by instructors, and is supported by the phrasing of Postulate 3 in the text: “The only possible result of a measurement of an observable is one of the eigenvalues  $a_n$  of the corresponding operator  $A$ ” (McIntyre, 2012, p. 35). The phrase here is “possible result of a measurement” rather than “result of a possible measurement,” but students may not discern that distinction. Gire and Manogue (2012) stated the role of eigenvalue equations in quantum mechanics is “to determine the eigenvalues (possible values of measurements) and eigenstates (possible states after a measurement is made)” (p. 195). A concern with the measurement interpretation of an operator times a state is that when a linear combination of (non-degenerate) eigenstates - typically referred to in quantum mechanics as a superposition state - is operated on, the result is not an eigenvalue multiplied by the initial state. On the other hand, measuring an observable for a particle in that superposition state does yield a measured value of the observable that corresponds to a single eigenstate; the eigenstate describes the state of the system/particle after the measurement (often referred to as the “collapse” of the system into an eigenstate upon measurement). In our data, we only asked students about their interpretations of specific eigenequations (thus necessarily involving eigenstates), so we cannot gauge the generalizability of statements consistent with a measurement interpretation to superposition states. However, Singh (2008) reported that students tend to overgeneralize the applicability of eigentheory in quantum mechanics to superposition states.

It is possible that the subtlety between “measurement” and “potential measurement” may not be reflected in some student responses. Moreover, student reasoning that suggests a measurement interpretation may be a lower bound for the student developing a potential measurement interpretation. Eigentheory in quantum mechanics is intrinsically linked to measurement, and students thinking an eigenequation as representative of an actual measurement rather than a potential measurement may be a point from which instructors can build.

## Limitations

Limitations in our study include the complexity of using linguistic cues to make inferences about student reasoning. For example, “would” is often used in English in a way not necessarily meant to convey conditionality (e.g., one might ask “how would you explain addition?” and expect an explanation of addition in response).



Thus, there is a danger that student explanations coded with the potential measurement interpretation of E2 may not be relying on a conditional use of “would” as strongly as we as researchers thought they were. A second and related limitation is that the interviewer from the first data set did not always thoroughly prompt students via follow-ups to explain their reasoning with respect to the physical meaning related to E2. A limitation of the second data set (written homework assignment) is that there was not a way to ask clarifying questions. Similarly, we did not have control over the environment the assignment was completed in or what resources the students may have accessed. Ways to improve the prompt could have been to ask students to list any resources accessed while completing the assignment and perhaps provide additional scaffolding questions to the task.

### Implications for Instruction

The research we presented is beneficial for the improvement of undergraduate mathematics education, as well as physics education. Our results demonstrate students’ rich ways of interpreting eigenequations in both math and physics. The results also highlight that different interpretations may be more productive or nonproductive in various contextual settings.

We conclude with some suggested implications for teaching. Linear algebra instructors could expose students to a variety of interpretations of eigenvectors and eigenvalues. For instance, in  $\mathbb{R}^2$  and  $\mathbb{R}^3$ , leveraging the graphical interpretations of eigenvectors of a matrix or linear transformation as vectors that are stretched or shrunk by a real-valued eigenvalue is common, can be visualized through dynamic geometry software, and has been shown to help students develop conceptual understanding of eigentheory (e.g., Beltrán-Meneu et al., 2017; Gol Tabaghi & Sinclair, 2013; Plaxco et al., 2018). From other contexts, eigenvectors associated with eigenvalue 1 can help determine the eventual or steady state of phenomena modeled by stochastic processes, such as long-term population in a predator–prey system or stable distribution of cars in a rental company franchise (e.g., Larson, 2010), as well as in network analyses that sort information such as in the PageRank algorithm for webpages (Brin & Page, 1998). Both math and physics instructors could discuss with their students ways that some underlying interpretations may not be fruitful or appropriate in certain contexts. The aforementioned examples all use eigentheory differently than the spin eigenequations discussed in this paper. We suggest that physics instructors could help their students recognize and appreciate that there are idiosyncratic interpretations of eigenequations in quantum mechanics that prior relevant mathematics instruction does not address. In addition, physics instructors could support students in identifying the correspondence between physical quantities and mathematical entities in the various terms of quantum mechanical equations, which is a skill students can build (Karakok, 2019). We posit that understanding such disciplinary distinctions is a worthwhile and rewarding experience, not only for ourselves but also for our future students, that can help us in our collective endeavor to improve undergraduate mathematics and physics education.

**Acknowledgements** We are grateful for the opportunity to collaborate with Kevin Watson, 1988–2020, who contributed towards study conceptualization, data collection, and early stages of analysis. This material is based upon work supported by the National Science Foundation under Grant Numbers DUE-1452889 and PHY-1912087. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the NSF.

**Data Availability** Due to privacy reasons of sensitive data, the dataset is not available.

## Declarations

**Ethics Approval** This study was approved by two overseeing Institutional Review Boards.

**Consent to Participate** All participants in this study gave informed consent to participate.

**Consent for Publication** All participants in this study gave informed consent for publication.

**Competing Interests** Not applicable.

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