International Journal of Research in Undergraduate Mathematics Education (2021) 7:235–260
https://doi.org/10.1007/s40753-021-00138-9

On the Analysis of Mathematical Practices in Signal Theory Courses

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Accepted: 16 February 2021/ Published online: 18 May 2021

Abstract
The contribution aims at subject-specific analyses of student solutions of an exercise from an electrical engineering signal theory course. The basis for the analyses is provided by praxeological studies (in the sense of the Anthropological Theory of Didactics) and the identification of two institutional mathematical discourses, one related to higher mathematics for engineers and one related to electrical engineering. Regarding the relationship between institutional observations and analyses of students' solutions, we refer, among others, to Weber's (1904) concept of ideal types. In the subject-specific analyses of student solutions we address in particular transitions and interrelations within single processing steps that refer to the two mathematical discourses and different forms of embedding of mathematics into the electrical engineering context. Finally, we present a few ideas for teaching.

Keywords Mathematical practices · Student solutions · Ideal typical discourses · Institutions · Anthropological theory of the didactic

Introduction

The use of mathematics in engineering courses and its adequate conceptualisation is a frequently addressed issue (e.g. Alpers, 2017; Alpers et al., 2013; Barquero et al., 2011, 2013; Czocher, 2013; Harris et al., 2015; Rooch et al., 2016). In previous research we have dealt with this in the context of signal theory (Hochmuth & Peters, in press; Hochmuth & Schreiber, 2015). In particular, we have referred to Castela and Romo Vázquez (2011) in which institutional references of mathematical practices (in the sense of the Anthropological Theory of Didactics (ATD) (Bosch & Gascón, 2014; Chevallard, 1992)) were reconstructed. In (Peters & Hochmuth, in press) we have also

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taken up their idea of an extended praxeological model, which they used to distinguish between practical and academic knowledge aspects in a vocational context. In contrast, we focused on investigating various elements of mathematical knowledge taught within a signal theory course in the context of an electrical engineering university program. We distinguished between a mathematical HM- and a mathematical ET-discourse[1] and showed, that the extended model and the identified discourses allow to figure out a complex interweaving of mathematical and engineering practices that cannot be conceptualised on the basis of a view that separates mathematics, the application of mathematics and engineering science. For characterising the mathematical discourses, we most notably used studies by Bissell and Dillon (Bissell & Dillon, 2000; Bissell, 2004, 2012).

In this study, too, the ATD serves as our theoretical framework. ATD is also used by González-Martín and Hernandes-Gomes (2018, 2019), where they address curricular differences between mathematics and engineering courses. In particular, they ask about the appropriateness of practices with regard to aspects of the integral concept and the use of integrals in calculus and mechanics courses. Similarly to Dammann (2016, p. 97), they notice that the mathematical requirements in statics lie primarily in the areas of basic arithmetic, the processing of linear equations and systems of equations, and that it is not necessary for students within statics to master the mathematical procedures of differential and integral calculus. By curricular differences we mean the phenomenon that there are mathematical topics or argumentation contexts that are dealt with in higher mathematics, but not in engineering courses, or that are substantially different in engineering courses and vice versa. This lack of fit is an important observation and certainly gives cause to think about appropriate adjustments and, if necessary, to incorporate them into the curriculum.

In contrast to these investigations, our research interest is directed towards refined analyses of mathematical practices in the engineering sciences and the interplay of different mathematical discourses within these practices. In (Peters & Hochmuth, in press) we analysed an exercise and could identify transitions and interrelations within single processing steps of a sample solution that goes beyond the vision of a pure application of mathematics in the engineering context, a vision which has been coined by Barquero, Bosch and Gascón (2011) as applicationism. Both the analysis of the exercise and the figured-out characteristics of the HM- and the ET-discourse referred to the institutional level in the sense of the ATD. If one now looks at solutions of the exercise by individual students, the question arise, whether the identified transitions and interrelations possibly appear as breaking points in solution processes, or, more generally, whether and how they can be found there. To answer this question is the main objective of this paper.

Since ATD distinguishes between institutional praxeologies and individual activities the question comes up how institutional analyses can be used to analyse student solutions. In our view, this question is not fully answered in ATD. For more details regarding the appraisal and argumentations in ATD studies we refer the reader to

[1] The acronyms HM and ET were introduced by Peters and Hochmuth (in press) to denote the two relevant contexts of “Höhere Mathematik” (HM, higher mathematics) and “Elektrotechnik” (ET, electrical engineering) and associated discourses. HM and ET are the standard German actronyms for these contexts. Although the English term electrical engineering requires the acronym EE, for reasons of consistency we stick here to the acronym ET.

section 3. To make progress on this issue, we need to clarify the meaning and analytical status of discourse more concretely than in previous contributions. Following our preceding work our use of discourse still focuses on subject-specific[2] aspects and is based on the concept of praxeology in ATD. Beyond that, however, it will prove fruitful in the following to link the notion of discourse with Weber's (1904) concept of ideal types. The ideal type concept turns out to be compatible with the ATD framework and from it methodical steps can be derived for analysing individual contributions on the basis of institutionally based models. From the point of view of the ATD Gascón and Nicolás (2017) have dealt with Weber: With reference to Weber's distinction between normative and scientific statements, the authors specified the position of the ATD with regard to responsibilities and objectives of didactics as a science. To our knowledge, the ideal type concept has not been discussed in the ATD so far.

We have structured our contribution as follows: In section 2 we characterise both the electrical engineering (ET) and the higher mathematics (HM) context representing important reference points for our analyses of the signal theory exercise and related student solutions. To illustrate the characterisations of two different mathematical discourses and to delimit the two contexts, we use the topic of complex numbers and their different subject-specific rationales in electrical engineering and higher mathematics. Finally, we give a short introduction to amplitude modulation, the specific subject of the exercise. In section 3, we introduce the ATD notions that we will subsequently use to grasp subject-specific aspects. Against the background of differences regarding the two course-contexts we introduce the two institutional mathematical discourses, the HM- and the ET-discourse and connect them to Weber's (1904) concept of ideal types. Additionally referring back to the theory of rational explanation (Schwemmer, 1976) and with the focus on subject-specific aspects we finally identify a methodical procedure with four steps for applying the institutional analysis to individual student solutions. Section 4 then starts by an institutional analysis of the exercise. The presented analysis is based on the analysis in (Peters & Hochmuth, in press), but develops it further with a view to the intended use in the current contribution. Applying the ideal typical mathematical discourses, we generate a praxeological model which we also present as a graphical scheme. The model is subsequently used to analyse the student solutions following the previously identified methodical steps. In particular we provide answers to the question of whether and how the institutionally identified transitions and interrelations regarding the mathematical discourses can be found there. In section 5 we finally discuss the obtained insights and present a few ideas for teaching based on them.

Context of the Study: Signal Theory and Amplitude Modulation

Focus of our analyses are student solutions of an exercise from a signal theory course. Signal theory courses are one of the first in-depth courses in electrical engineering studies at German universities. They are usually scheduled for the third or fourth

[2] In the institutional context, we usually skip the institutional and simply speak of subject-specific. This seems justified to us, since discipline-specificity is unthinkable without institutions. With regard to the individual level, we use the term individual subject-specific. The term subject-related, on the other hand, addresses aspects that consider the individual as subject, including societal and psychological moments.

semester, after students have attended courses on higher mathematics for engineering and introductory theory-orientated electrical engineering courses. Signal theory is considered to be very mathematical, while the extent to which formal mathematical concepts are elaborated varies. Sometimes it is offered as a Fourier analysis course, as found in mathematics studies, where electrical engineering terms or problems are hardly considered. In contrast our contribution deals with data from a signal theory course that is strongly oriented towards electrical engineering. The variation of mathematical formalism is also present in the definitions of electrical engineering concepts.[3] Engineering concepts are closely related to physical quantities. They are always related to measurement and to real phenomena.[4] These observations already indicate two characteristics of the electrical engineering context: on the one hand the reference to reality, and on the other hand a very different degree of explication of this reference to reality, which is accompanied by a different degree of mathematical formalisation. This is also mentioned by Fettweis (1996, p. i) and addressed as a dilemma: an increasing mathematical formalisation of concepts can make it increasingly difficult to understand their physical meaning and justification.

The notion of system refers to a third characteristic: Frey and Bossert (2009) generally understand a system to be "an abstracted arrangement that relates several signals to one another. This corresponds to the mapping of one or more input signals to one or more output signals". (p. 3). They first introduce a system type that is easy to handle mathematically and consider only one input and one output signal each, since "this makes the system thinking [Systemgedanke] easier to grasp." (p. 6). A system can be understood as a black box that responds to a specific input signal with a specific output signal. Studies by Bissell and Dillon (Bissell & Dillon, 2000; Bissell, 2004, 2012) show that system thinking and the electrical engineering way of doing and talking about mathematics, differs from the way of mathematicians. The authors argue that "this linguistic shift is more than just jargon, and more than just a handy way of coping with the mathematics" (Bissell & Dillon, 2000, p. 10).

To further illustrate the electrical engineering way of thinking and doing mathematics, we give a short overview on how complex numbers and relating concepts like phasors are treated in an introductory course on electrical engineering and in a course on higher mathematics for engineers respectively. The following observations are based on standard literature, lecture notes and students' notes for two consolidated standard courses which are held every year at the University of Kassel. Complex numbers play also an important role in the exercise (cf. Appendix) we examine in this paper. We will argue that in those courses the respective rationales of complex numbers and their justifications are different and that these variations constitute partly conflicting resp. complementary reference points for students in the signal theory course.

In Albach (2011), a standard textbook for introductory courses on electrical engineering, phasors are introduced with the purpose to graphically describe time-dependent sinusoidal[5] functions, see Fig. 1. The first introduction of phasors is without

[3] For example, compare definitions in the two books by Frey and Bossert (2009) and Fettweis (1996). Both books are recommended as standard literature for the signal theory course we are studying.

[4] For a more detailed discussion of such epistemological issues regarding the relationship of mathematics and empirical sciences we refer to Hochmuth und Peters (in press).

[5] Circuits are operated with sinusoidal current- and voltage forms in the power supply network as well as in many other important areas.

references to complex numbers: A phasor [Zeiger] is an arrow with a specific length and a specific angle with respect to a reference.

When analysing electrical components, the amplitude ratio of the input signal to the output signal and the phase shift caused by the component are of primary interest. Therefore, phasors are important graphic tools for interpretation and analysis of electrical engineering processes. Current- and voltage ratios in electrical networks can be displayed and analysed graphically in phasor diagrams without using complex numbers or differential equations. For the purpose of an algebraic description of phasors, the plane in which phasors are drawn, can be considered as the complex plane. The phasor can now be understood as a complex quantity that symbolically represents the time-dependent voltage (see Albach, 2011, p. 42). The compatibility of the rules for manipulating phasors and the calculation rules of complex numbers is justified via physical relations. Furthermore, for a sinusoidal quantity the following holds: $A\cos(\omega t+\varphi)\ =\Re\left(Ae^{j(\omega t+\varphi)}\right)\ =\Re(Ae^{j\omega t}e^{j\varphi})$, where A is the amplitude, ω is angular velocity, φ is the phase angle (each independent of time) and j denotes the complex unit in electrical engineering. The factor $A_{\_}=Ae^{j\varphi}$ then is the mathematical representation of the phasor, graphically represented by an arrow with length A and angle φ with respect to a reference zero angle. The function $A_{\_}e^{j(\omega t)}$ is a representation of a rotating phasor in the phasor- or Argand diagram.

In the course on higher mathematics for engineers, complex numbers are considered in the first semester in the context of Linear Algebra (Strampp, 2012). Their introduction is motivated by the solvability of the equation $x^2 + 1 = 0$. For this purpose, real numbers are extended by a number i with the property $i^2 = -1$. This approach is typical for the whole chapter: the rational is aimed at an elaboration of the solvability of equations. This results in considerations about the general solution of algebraic equations, the fundamental theorem of algebra and Vieta's formula. Calculation rules for complex numbers are derived without introducing and proving formal concepts, but by stating that all rules which are relevant for calculating with real numbers should continue to be applicable (p. 59): Also, in further contexts it is pointed out that various terms are an extension of already known concepts from real numbers. For example, the complex exponential function $e^{i\phi}$, which is introduced to serve as an abbreviation for $\cos(\phi)+\sin(\phi)i$. Although the chapter is clearly designed to develop a practical approach to the concepts and rules of calculation, it is subject to an orientation towards the inner-mathematical, generalisation-oriented rational of academic mathematics.

In addition to the algebraic view on complex numbers, the chapter contains another, geometric, orientation: An analogy to vectors is established, but the vector concept is also distinguished from complex numbers: "We speak of phasors[6] [Zeiger] and not of vectors, since complex numbers, unlike vectors, can also be multiplied. This

[6] We translated the German term Zeiger with the term phasor, which already refers to electrical engineering concepts. But electrical engineering aspects play no role in the course and Strampp (2012) does not refer to them either. Another possible translation of Zeiger, without the connection to engineering concepts would be pointer. But we decided to use phasor for the following reason: In German, the term Zeiger is used both in electrical engineering and in mathematics courses for engineers, but with different meanings (reference to electrical engineering concepts vs. geometrical object with no further references). By using the term Zeiger instead of vector Strampp (2012) can thus establish a connection to the electrical engineering courses without dropping the inner mathematical conception of complex numbers. This aspect of using the same term, that has different meanings in different course-contexts is in jeopardy of being lost through translation.

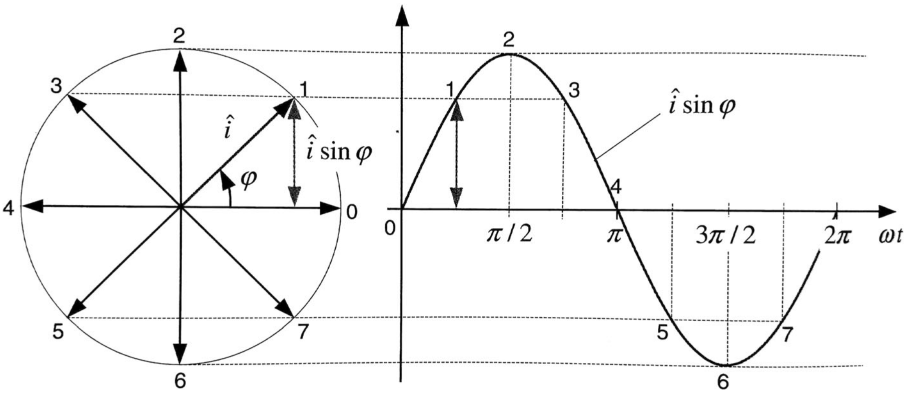


Fig. 1 Relationship between phasor and time-dependent function (Albach, 2011, p. 32)

multiplication extends the multiplication of real numbers." (p. 60). This phasor concept in the higher mathematics context differs from the phasor concept in the electrical engineering context: In HM the geometrical representation of complex numbers as arrows in the Argand diagram is used as a visualisation of calculation rules. In ET phasors are arrows that represent measurable, time-dependent quantities such as alternating voltages or currents. Complex numbers are then used for the convenient algebraic description of phasors.

In summary, whereas the electrical engineering context is notable for system thinking and for references to reality with different degrees of explication of this reference accompanied by different degrees of mathematical formalisation, the higher mathematics context is characterised by statements without references to reality and an inner-mathematical understanding and justification of concepts, in particular and, a generalisation-oriented rational following academic mathematics and a concentration on calculation rules.

The exercise we investigate in this paper belongs to *amplitude modulation* (AM), a central topic in signal theory. The principle of amplitude modulation is illustrated in Fig. 2:

The amplitude of a high-frequency carrier signal (Fig. 2, left) is varied corresponding to the course of the low-frequency message signal $s(t)$ (Fig. 2, middle). The AM signal (Fig. 2, right) can be represented as $x(t) = A[1 + m\ s(t)] \cos(2\pi f_0\ t)$ where $\cos(2\pi f_0)$ is the carrier signal. The modulation index m is the ratio between the amplitude of the carrier signal and the amplitude of the message signal, in addition the restrictions $\max\limits_{t \in R} |s(t)| = 1$ und $0 < m < 1$ apply. With amplitude modulation, several message signals (e.g. for different radio stations) with different carrier frequencies can be transmitted via antenna and received without crosstalk between signals at the receiver (radio set) depending on the chosen frequency.

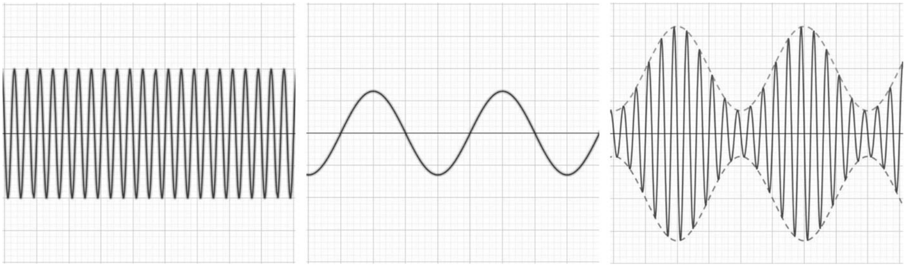


Fig. 2 Carrier signal (left), message signal s(t) (middle), and AM signal x(t) (right)

The Anthropological Theory of the Didactic and Ideal Typical Mathematical Discourses

ATD is a research program to study (mathematical) practices from an institutional perspective. The notion of *institution* within the ATD is related to the work by Mary Douglas (1986), who draws on ideas of Durkheim and Fleck. Her main point relevant for ATD is the elaboration of the idea that all knowledge depends on (social) institutions and conversely all institutions are based on shared knowledge (p. 45). In the following we will take a closer look of how observations regarding institutional practices and contexts can be referred to individual subject-specific analyses. In this respect, we will refer in particular to Weber's concept of the ideal type and, finally, propose a procedure in four steps. But first we will introduce some basic terms of the ATD. These constitute our starting point for linking the characterisations reported in section 2 with Weber's concept of ideal types.

The 4 T-Model and the Institutional Dependence of Knowledge

In ATD knowledge is related to human activities including not only aspects of know-why but also practical knowledge in the sense of know-how. This is subsumed under the term *praxeology*:

> What exactly is a praxeology? ... One can analyse any human doing into two main, interrelated components: *praxis*, i.e. the practical part, on the one hand, and

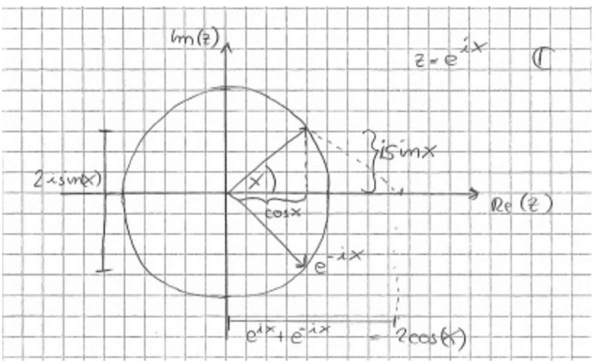


Fig. 3 Graphical representation of complex numbers. Students' lecture note

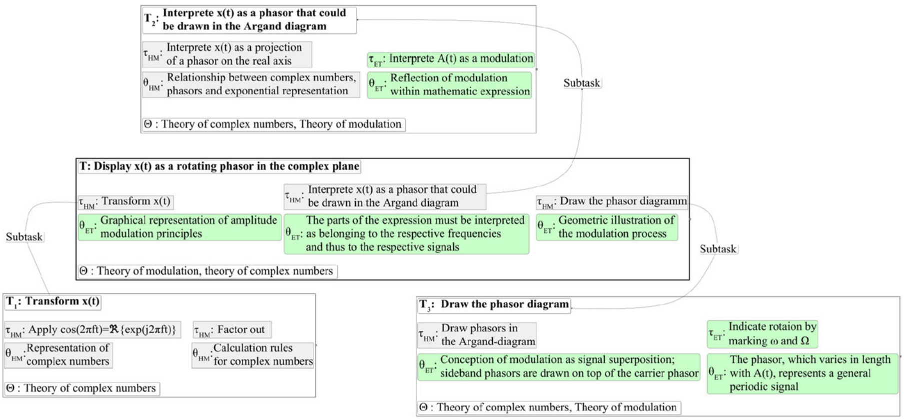


Fig. 4 Graphical representation of the institutional analysis

> *logos*, on the other hand. … How are *P* [Praxis] and *L* [Logos] interrelated within the praxeology [*P/L*], and how do they affect one another? The answer draws on one of the fundamental principles of ATD ... according to which no human action can exist without being, at least partially, 'explained', made 'intelligible', 'justified', 'accounted for', in whatever style of 'reasoning' such as an explanation or justification may be cast. *Praxis* thus entails *logos* which in turn backs up *praxis*. (Chevallard, 2006, p. 23)

A praxeology thus is a basic epistemological model to describe knowledge in the form of the two inseparable and interrelated blocks Praxis and Logos. Those two blocks can be differentiated further: the praxis block *P* (know-how) consists of problems or *tasks T* and a set of relevant *techniques* τ used to solve them. The logos block *L* (know-why) consists of a two-levelled reasoning discourse.[7] On the first level, the *technology* θ describes, justifies, explains etc. the techniques and on the second level the *theory* Θ organises, supports and explains the technology. Since praxis and logos are dialectically interrelated, every aspect of praxis (i.e. tasks or techniques) is related to the discourse. In short praxeologies are denoted by the *standard 4T-model* $[T, \tau, \theta, \Theta]$.

In the Chevallard quote, the part "in whatever style of 'reasoning' such as an explanation or justification may be cast" refers to the idea that institutional conditions constitute the technological-theoretical discourse and the practices available. Regarding in particular the relationship of institutions and techniques, Chevallard (1999) writes:

> Finally, in a given institution *I*, with regard to a given type of task *T*, there is usually *only one* technique, or at least *a small number* of *institutionally recognised* techniques, to the exclusion of possible alternative techniques - which may actually exist, but then *in other institutions*. (p. 225, our translation)

Accordingly, we use the notion *scope of the technique* to address the set of tasks, which can be solved with the institutionally recognised technique. In considering one specific

[7] Within the ATD the term discourse, e.g. in expressions like "reasoning discourse" or "a discourse on praxis" is used in the etymological sense (e.g. Bosch & Gascón, 2014, p. 68).

piece of knowledge in different institutions, different praxeologies emerge: different types of tasks are relevant, different solution techniques are adequate, and different reasoning discourses are acceptable and constitutive. These relationships are addressed by the term *institutional dependence of knowledge*. In view of section 2, we can accordingly say that in the institution of electrical engineering, the ET-context, and in the institution of higher mathematics, the HM-context, there are different praxeologies concerning complex numbers. Furthermore, the characterisations of the two contexts can be understood as descriptions of institutional aspects that shape the logos block and thus, due to the dialectic of praxis and logos, also the practical part of praxeologies. In the following we understand the characterisations of the two contexts as characterisations of two different mathematical *discourses*[8] and associate praxeologies or praxeological elements to the mathematical *ET-discourse* or the mathematical *HM-discourse*, if they can be characterised according to the institutional ET- or the institutional HM context, respectively.[9] In order to articulate the references to the different institutional mathematical discourses, we use the further notations τ_{HM} and τ_{ET} as well as θ_{HM} and θ_{ET}.

The Difference between Institutional Praxeologies and Individual Activities

The institutional dependency of knowledge implies that human activities are constituted and located in institutions: A praxeology does not present itself as something individual, but as something institutional and societal. To stress the difference between institutional praxeologies and actual individual activities, Chevallard uses the notion of *relation to objects* (of knowledge): Institutions are based on shared knowledge, every object of knowledge O is in relation to the institution I, noted as $R_I(O)$. A praxeology is a concept to study the subject-specific content of those relations. Similarly, every person X, that acts with an object of knowledge O is in an individual relation to it, noted as $R(X, O)$.

> A person X becomes a good subject of I relative to the institutional object O when his personal relation $R(X, O)$ is judged to be *consistent* with the institutional relation $R_I(O)$. This person may also prove to be a bad subject, … and may, in the end, be expelled from I. Here is where a development relating to *intra-institutional evaluation* comes into play, relating to the mechanisms according to which I is led to pronounce, through some of its agents, a verdict of conformity (or non-conformity) of $R(X, O)$ to $R_I(O)$. . … In particular, the institutional relation … is nobody's personal relation, … : conformity is not identity. (Chevallard, 1992, p. 146/7)

[8] This use of the term discourse goes beyond an etymological understanding (cf. footnote 6). We extend thereby a term which already exists within the ATD. Our extended understanding blends into the already existing concepts (e.g. institutional dependence of knowledge). We do not use the term discourse in the sense of discourse theory. Due to the limited word count, we refrain from further elaboration of possible connections and delimitations.

[9] The two mathematical discourses can also be connected to the work of Artaud (2020), where she describes two types of didactical transposition processes: An external didactical transposition process, originating in academic mathematics research institutions. Here one can locate the HM-discourse. And an endogenous didactical transposition process concerning processes within the engineering institution. Here one can locate the ET-discourse.

In summary, identified institutional praxeologies and discourses must be distinguished from individual actions and their products. But, although persons do not have to reproduce the specific institutional logos in a specific institution context, they provide *points of references* for their actual practices. Consequently, institutional praxeologies also provide important reference points for analyses of individual products, but cannot be directly and unmediatedly be related to them. Research in mathematics education sometimes neglect this difference[10] and rather identifies individual actions and institutional praxeologies. Regarding these arguments, it is important to stress once again that we are focusing on subject-specific aspects. For example, Hardy (2009) analyses students' interpretations of institutional praxeologies in view of political and educational issues. Thus, of course, differences between students' actions and institutional praxeologies are considered. But, focusing on the subject-specific, if students use the right symbols and write things down as they have been worked out in practice, both actions are identified with each other. Because of these implicit identifications the insightful explanations of the student's interpretation possess an hypothetical character also with respect to subject-specific aspects and not only with respect to the considered political and educational issues. To support the overall relevance of these latter issues Hardy proposed further studies (p. 357). However, it is not reflected in detail how this could actually contribute to clarify differences regarding subject-specific issues, particularly in view of general characteristics of discourses.

HM- and ET-Discourse and Weber's Notion of Ideal Type

To deal further with the issue of the difference between institutional praxeologies and individual products regarding the HM- and ET-discourse we will recur in the following to Max Weber's (1904) construct of ideal type. Its use in empirical research to explain individual actions has been investigated for many decades. We will refer to and adapt these diverse and in part in-depth investigations focusing on subject-specific aspects. This methodological considerations will enable us in the following to indicate a suitable approach (see the four steps at the end of this section) and to clarify its possibilities and limitations.

Weber (1904) introduces *ideal types* as a construction,

> which is obtained by *mental* enhancement of certain elements of reality. Its relationship to the empirically given facts of life consists merely in the fact that where connections of the kind represented abstractly in that construction ... are *determined* or *suspected* to be effective to some degree in reality, we can pragmatically *visualize* and understand the *peculiarity* of this connection on an *ideal type*. This possibility can be both heuristic and indispensable for the representation of value. … The ideal typical term … is not a 'hypothesis', but it wants to show the direction of hypothesis formation. It is not a *representation* of the real, but it wants to give the representation unambiguous means of expression. (p. 64/5, our translation)

[10] As expressed within ATD as difference between institutional and individual relations to objects of knowledge.

According to Weber's introduction, we interpret the two mathematical discourses as ideal types and the underlying characterisations of the discourses as a result of "mental enhancements" regarding aspects of mathematical practices within specified institutional contexts. Furthermore, we use the two ideal typical mathematical discourses as heuristics for our subject-specific analyses of the exercises and the student solutions as well as to formulate hypotheses. Thus, following Weber's formulation, we assume that the ideal typical discourses are to a certain extent effective and represent a means of expressing something real in sample solutions and students works on exercises. Thereby the relevance of this something in individual productions cannot be substantiated by the ideal typical discourses themselves, but has to be shown in the individual productions in a concrete and subject-specific way.

Weber's construct of the ideal type has been widely criticised and expanded in the social sciences (see Shubat, 2011, especially Chapter 1.4). There is no space here to elaborate on this in detail. Regarding our use, the following question adapted from Schwemmer (1984, p. 177) is particularly relevant to avoid circularity: In which way do the characterisations associated with the mathematical discourses enter into the empirical studies of individual productions and their results without withdrawing the characterisations from empirical criticism? Our standpoint on this is the following: Whether something is apparent in individual-related data whose concrete meaning can be demonstrated by means of one of the two mathematical discourses is empirically open with regard to the following two dimensions. (1) From a subject-specific point of view, the appearance must be shown in the specific context of exercises and related praxeologies. Thus, for example, it could be proven empirically that a subject-specific context cannot be reconstructed as a particular case of application of the mathematical discourses. (2) In subject-related respects, for example with regard to the question of whether a student experiences the identified aspects as such, corresponding claims are accessible to empirical criticism. Here, it could be proven empirically that the connections formulated by means of mathematical discourses do not contribute to an understanding of the students' thoughts about her actions. In this article we focus on (1), the subject-specific perspective. With regard to the rational explanation of concrete individual products, we adopt a further argument by Schwemmer (1976, p. 142) concerning the precondition of purpose-rationality as a methodological postulate: Our subject-specific analyses are based on the assumption that solutions and works on exercises follow the demand for a respective institutionally set subject-specific rationale. This is a methodological requirement, since otherwise (i.e., assuming that the students and their works do not follow any particular institutionalised disciplinary rationale) any relationship between institutional disciplinary analyses and the analysis of concrete exercises would be questioned from the outset.[11]

Methodological Consequences for Subject-Specific Analyses

Against the background of ATD concrete individual actions or their products are thus considered to be *explained* if their rational can be connected to praxeological analyses and

[11] This does not contradict the empirical openness discussed above with regard to the two mathematics discourses, since a concrete case could follow a different disciplinary rational, which might be completely independent of the ones we reconstructed.

the identified ideal typical discourses. In this respect, observations should be justified that certain mathematical actions can be substantiated on the basis of praxeologically reconstructed practices, including the consideration of ideal typical discourses. This implies formulating hypotheses in observation language [Beobachtungssprache][12] with a view to the respective subject-specific context. Specifically examined action situations are then to be validated with regard to the existence of corresponding observation correlates [Beobachtungskorrelate].[13] Hereby, the ideal typical discourses identified by us prove to be genetic concepts, i.e. they function as a guideline for hypothesis formation and allow the rational to be grasped in a concrete action situation with regard to its institutional meanings. The insights into practices gained through the discourses are of course not considered to be valid for all times. Thus, for example, the modification of institutionalised practices (for example, a modified treatment of complex numbers in higher mathematics courses) can change the role of the identified ideal typical discourses in the reconstruction of concrete practices.

In summary and with regard to our subject-specific focus, this results in the following sequence of methodical steps:

1. At the institutional level, praxeologies are to be identified and connected to ideal typical discourses.
2. Hypotheses in the specific context of signal theory or the exercises are to be formulated using observational language.
3. Concrete material (exercises, sample solutions and students' solutions) is to be validated with regard to corresponding observation correlates.
4. If these are available, concrete material can be explained on this basis.

In the next section we first present the institutional analysis of the lecturer's sample solution with a view to using it as a reference for the analyses of the students' solutions (step 1). This will also illustrate the ATD concepts introduced in this section. Thereby we will use the graphical method developed in (Peters & Hochmuth, in press) to present the result of the institutional analysis, see Fig. 4. In the next step we move on to the student work and formulate corresponding hypotheses (step 2) in observational language. With reference to these hypotheses, we will then identify observation correlates with respect to the rationales and, in particular, the ideal typical mathematical discourses (step 3). Finally, we explain the student solutions regarding their institutionalised subject-specific rationales (step 4). Here we use the graphical scheme, i.e. Figure 3 without text, to represent the analysis results of the students' solutions, see Figs. 5a, 6a, 7a, 8a, and 9a.

Analyses of the Student Solutions

The exercise and the sample solution are presented in the appendix: The complete exercise consists of three items. Results for items 1 and 2 and the complete sample

[12] Using observation language means to describe an observation without interpretations (Schwemmer, 1976, p. 165), whereas interpreting means to show actions as rational in purpose or sense (p. 168).
[13] The observation correlate of an action is the part of the action that is observed and described in observation language (Schwemmer, 1976, p. 168).

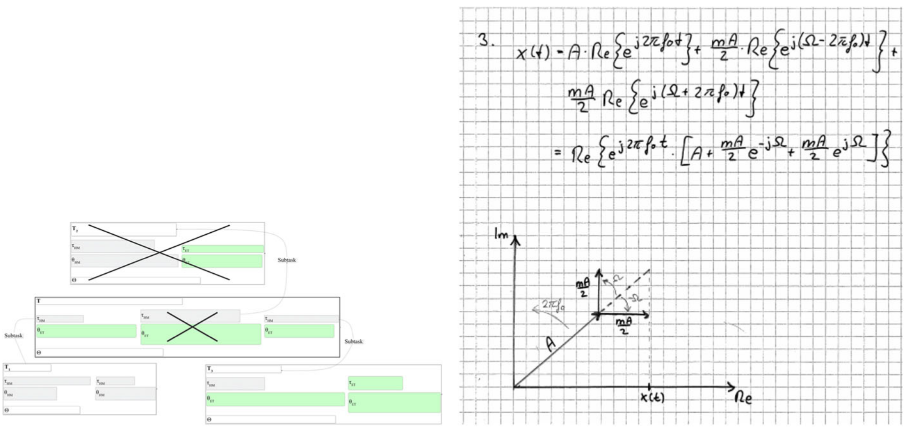
solution for item 3 is shown. In item 1, a given message signal should be amplitude modulated. For this purpose, the given term $s(t) = \cos(\Omega t)$ must be inserted in the formula for amplitude modulation: $x(t) = A[1 + m\ s(t)] \cos(2\pi f_0\ t)$. In item 2, this term should then be represented as an expression of three harmonic oscillations. And finally, in item 3, that is focussed in our analyses, $x(t)$ has to be graphically displayed in the complex plane as a rotating phasor with varying amplitude, using the relationship cos $(2\pi ft) = \Re\{\exp(j2\pi ft)\}$ and the result of item 2.

Step 1: Institutional Analysis

Item 3 is solved in three steps: [1] Transforming mathematical expressions, [2] Interpreting mathematical expression to draw a diagram, and [3] drawing the phasor diagram. The main part of the exercise, to display $x(t)$ as a rotating phasor in the complex plane, is a task (T) in the sense of the ATD. We then assign technique and technology to each of the three solution steps [1] to [3], and only roughly summarise theoretical aspects (see the bold framed rectangle in Fig. 4). This assignment of techniques and technologies is then differentiated and refined in a second analysis step, in which the three techniques assigned to the steps [1] to [3], are considered as subtasks T_1 to T_3 (see the corresponding light framed rectangles in Fig. 4). All in all, we also present the result of the analysis in Fig. 4 graphically. This graphic representation will afterwards serve as a scheme for the analysis of the student solutions.

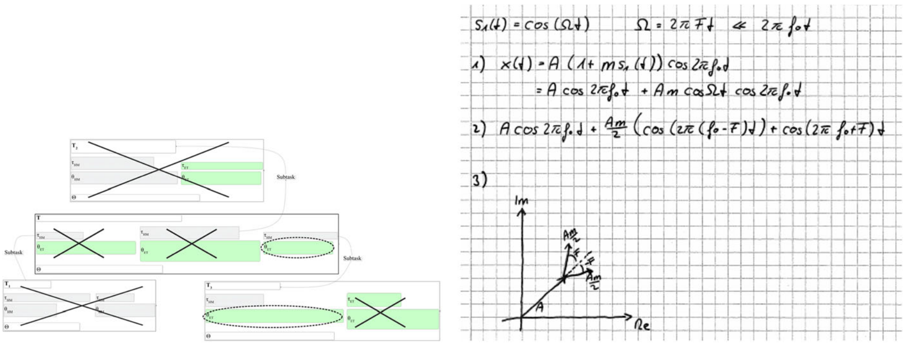
First we consider the three techniques that we assigned to the steps [1] to [3]: Each of the three techniques in T is located within the HM-discourse: [1] To transform mathematical expressions (τ_{HM}), [2] to interpret this expression for drawing a diagram (τ_{HM}), and finally [3] to draw a phasor diagram (τ_{HM}) are activities that are present in the corresponding higher mathematics course. For example, Fig. 3 shows a diagram from the first lecture of the signal theory course which dealt with repeating the properties of complex numbers according to the prior HM-course (definitions, calculation rules, phasors as visualisations of properties of complex numbers).

The mathematical expressions in the exercise and the diagram to be drawn are more complicated than corresponding content in the higher mathematics course, but the



a b

Fig. 5 **a** Analysis result for C1. **b** Student solution C1



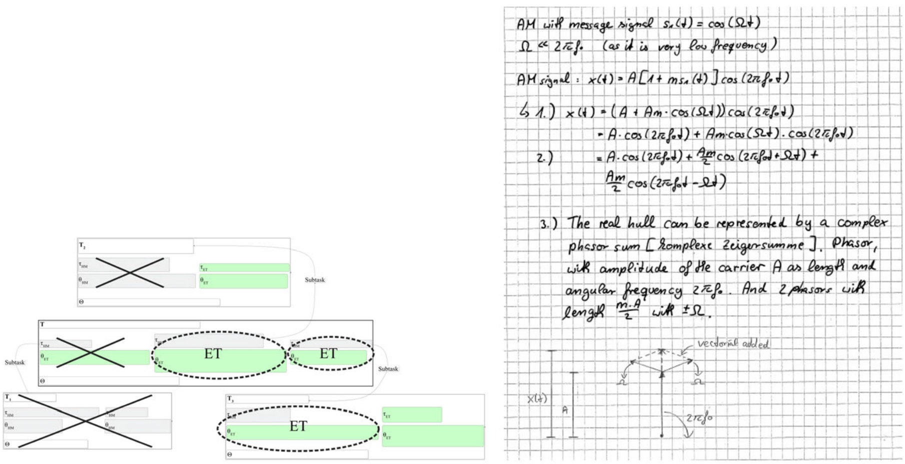
a **b**

Fig. 6 **a** Analysis result for C2. **b** Student solution C2

techniques themselves are certainly HM-typical. Furthermore, there is no aspect here that suggests a location of the techniques within the ET-discourse.

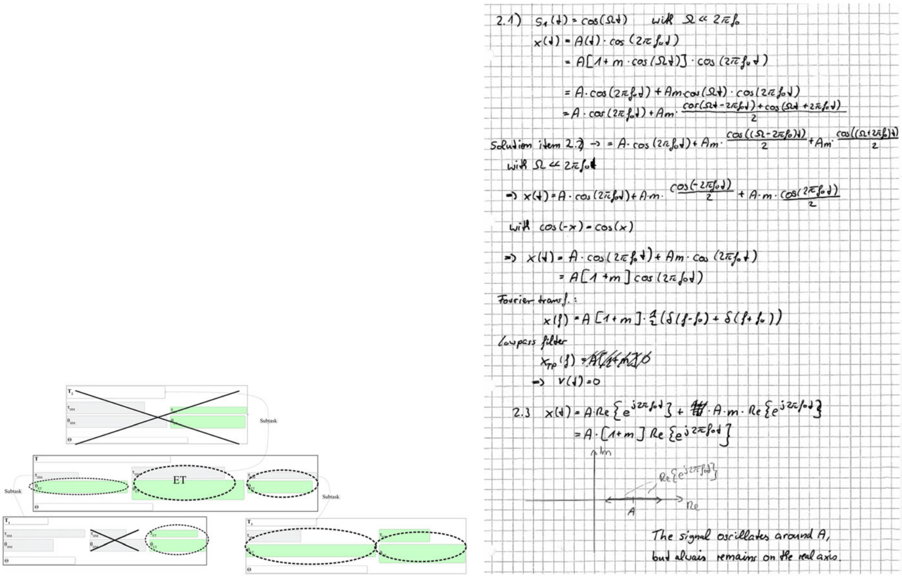
Second, we consider the corresponding technologies: We have located the corresponding technologies within the ET-discourse. Here, aspects of scope and purpose and elements of justification arise that can no longer be located in the HM-discourse. In the following we will go through steps [1] to [3] and discuss the corresponding technologies in more detail:

[1] The signal $x(t)$ must first be transformed in lines (1) to (3) (τ_{HM}). Then $x(t)$ can be interpreted as a real part of a rotating carrier phasor with a time-dependent amplitude $A(t)$ and aspects of amplitude modulation could be visualised in the diagram. This scope of the technique is based on the idea of graphically representing the principles of amplitude modulation, therefore this first technique is justified within the ET-discourse (θ_{ET}). This justification, that is linked to the overall aim of the task and also already links to steps [2] and [3] of the solution process, can be further focused on the first step: In particular the calculation step



a **b**

Fig. 7 **a** Analysis result for I1. **b** Student solution I1



a b

Fig. 8 **a** Analysis result for I2. **b** Student solution I2

from line (2), in which $x(t)$ could be interpreted as a real part of three rotating phasors drawn in the origin, to line (3), in which $x(t)$ can be interpreted as a rotating carrier phasor with time-dependent amplitude $A(t)$, is central here. Only if $x(t)$ is represented as in line (3), $x(t)$ can be interpreted as amplitude modulated and a phasor diagram can be drawn in which the amplitude modulation of the signal $x(t)$ can be displayed graphically. There is no justification within our reconstructed HM-discourse, that justifies the step from line (2) to line (3).

[2] In the next step the mathematical expression must be interpreted in order to draw the diagram (τ_{HM}). The central point here is that the components of $x(t)$ must be

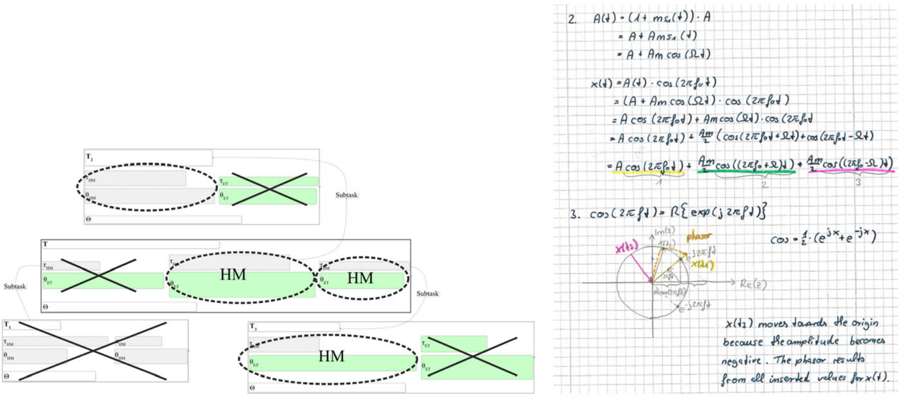


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Fig. 9 **a** Analysis result for I3. **b** Student solution I3

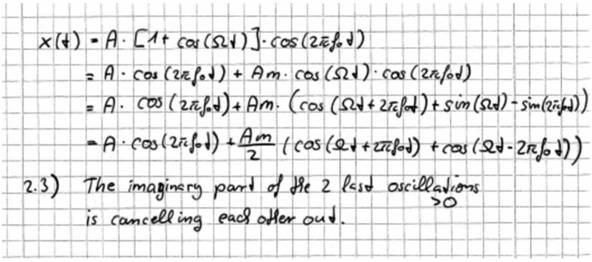

Fig. 10 Student solution N1

interpreted as belonging to the respective frequencies and thus to the respective signals (θ_{ET}).

[3] Finally, the third technique is the drawing of the phasor diagram (τ_{HM}). Here, however, this HM technique is embedded in the ET-discourse, since it is a geometric illustration of a modulation process (θ_{ET}). The phasor diagram is an alternative representation of the amplitude modulation in Fig. 2 (right). It has the advantage over the representation in Fig. 2 (right) that some effects relevant to amplitude modulation can be displayed.

In total, three techniques with corresponding technologies arise for steps [1] to [3], each of which is characterised by an embedding of HM-techniques in the ET-discourse. Relevant theoretical aspects come from theory of modulation and theory of complex numbers. In Fig. 4 this part of the analysis is illustrated in the bold framed rectangle. How the embeddings look like in each case will be clarified in a next step of analysis: Each of the three techniques will be considered as a separate subtask, T_1 to T_3, with its own techniques and technologies. In Fig. 4 these parts of the analysis are illustrated in the light framed rectangles.

In T_1, to transform $x(t)$, the identity $\cos(2\pi ft) = \Re\{\exp(j2\pi ft)\}$ given in the problem definition must be applied (τ_{HM}). For the justification (θ_{HM}) of this step, the relation between the representation of a complex number in polar form and in exponential form is relevant. In the following, calculation rules for complex numbers (θ_{HM}) are applied, namely factoring out of the real part and of $\exp(j2\pi f_0 t)$ (τ_{HM}). These are techniques that occur in higher mathematics courses and are also correspondingly justified inner-mathematically. There are no references to ET aspects.

In T_2, to interpret $x(t)$ as a phasor that could be drawn in the Argand diagram, two relevant techniques play a role: First, the expression in line (3) must be interpreted as a

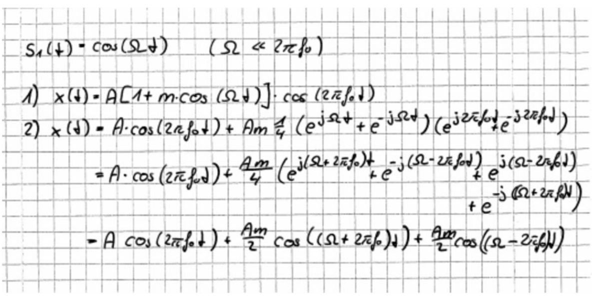

Fig. 11 Student solution N2

projection of a phasor onto the real axis (τ_{HM}). This technique is assigned to the HM-discourse, since also in higher mathematics complex numbers are understood as phasors, represented in Argand diagrams. The projection of the phasor on the real axis is interpreted as a real part of the complex number (cf. Fig. 3 and Strampp, 2012). This is justified in the introduction of complex numbers via the connection of complex numbers, phasors and exponential representation (θ_{HM}). And then the expression in the square brackets, $A(t)$, must be interpreted as modulation (τ_{ET}). Here knowledge about how the different frequencies are assigned to the different signal types, how carriers and message signals are related by the modulation process, and how this is reflected in the multiplication and the additions in the mathematical expression is relevant (θ_{ET}).

In T_3, the drawing of the phasor diagram, first the phasors must be drawn (τ_{HM}). This technique is located in the HM-discourse. But the concrete way in which the phasors have to be drawn, sideband phasors on top of carrier phasors, is explained by the fact that modulation is a superposition of signals (θ_{ET}). The corresponding technology is thus located in the ET-discourse. Here again a HM-technique embedded in the ET-discourse appears, which could be differentiated in a third step of analysis, which we will refrain from because it is not necessary for the analysis of the students' solutions. As a second technique, rotational aspects have to be marked by drawing curved arrows, which are labelled with the respective frequencies (τ_{ET}). This does not occur in higher mathematics courses and refers to electrical engineering aspects, hence our location in the ET-discourse. The markings also indicate that these are general periodic signals: This is based on the electrical engineering conception of the representation of signals by rotating phasors and that the length of the phasor $\exp(j2\pi f_0 t)$ changes time-dependently according to $A(t)$. So, the technology is also to be located in the ET-discourse.

Our analysis regarding the role of the two mathematical discourses in the three steps of the solution show that the embeddings take different forms in each case (see also Fig. 4): For the first step the embedding is formed only from HM-discourse aspects. In this case an interpretation in the sense of applying mathematics in the engineering context is plausible. For the second step the embedding is formed from both HM-discourse aspects as well as ET-discourse aspects. And for the third step the embedding contains another embedding of a HM-technique in the ET-discourse and its aspects. Except for step one, the view that mathematics is (simply) applied in electrical engineering is not adequate. Instead, diverse transitions between the two mathematical discourses appear. They constitute breaks in the sense that they each follow a different rational. These breaks often remain implicit, although they represent an important aspect. The breaks indicate places that are not accessible from a single discourse and its techniques, and thus mark something additional to be learned.

Steps 2 to 4: Analyses of Students' Solutions

Following the institutional analysis which showed that the shift of representation from the symbolic to the graphic form is the core of the solution and involves specific transitions between mathematical discourses, we will analyse the student solutions especially with regard to this shift of representation and focus our explanations on transitions between the mathematical discourses. We use the graphical representation of the institutional analysis as a reference for analysing the student solutions and as a tool to graphically represent the result of our analyses: Praxeological aspects that cannot be identified in the student solutions, because the data does not provide this, are crossed out in the corresponding diagram. Aspects that can be identified in the student solution and correspond to the aspects in the lecturer sample solution are displayed without specific marking. Aspects that are present but different are circled with dashes.

In total 15 students handed in their solutions for this exercise. According to the assistant's marking we categorised them as follows:

- **Correct diagram**: phasor diagram with no or minor corrections, e.g. added arrows indicating the direction of rotation or angle labels (4 solutions). See also C1 and C2 in Figs. 5b and 6b.
- **Incorrect diagram**: phasor diagram with major corrections, e.g. added arrows indicating phasors or adding a whole correct phasor diagram (5 solutions). See also I1 to I3 in Figs. 7b, 8b, and 9b.
- **No diagram**: student solutions, that contain calculations but no diagram (6 solutions). See also N1 and N2 in Figs. 10 and 11.

To protect the students' privacy, we have rewritten the solutions without reproducing the assistant's marking. All student solutions considered contain correct solutions for items 1 and 2 of the exercise, possibly with the exception of minor sign errors. In the remainder of this section we follow, thesis by thesis, steps 2 to 4.

Thesis 1: The Sample Solution Is Realised in Student Solutions

The student solution C1 in Fig. 5b largely follows the sample solution. Of the three steps in the solution process, steps [1] and [3] can be identified in C1. Step [2] is not identifiable, therefore it is crossed out in Fig. 5a. However, the exercise assignment did not ask for an explicit interpretation, so the absence is not a deficit.

In step [1] the transformation of $x(t)$ can be recognised as a HM-technique embedded in the ET-discourse. In particular, the step from line (2) of the sample solution to line (3), which is especially associated with the ET-discourse, is present. The change from line (2) to line (3) in itself represents, in the sense of our approach, a subject-specific observation correlate for the interplay of HM- and ET-discourse. In step 3 both aspects, the drawing of the phasors in their specific relations as well as the indication of rotational aspects are present. The validation of the observation correlates here essentially follow the institutional analysis; there is nothing in the student solution that suggests otherwise. With regard to the transitions between the mathematical discourses, we can state that, except for step two, the transitions from the institutional analysis occur. This supports our explanation of the student work C1 with regard to thesis 1 on the basis of the discourse-related observation correlate: The student solution C1 realises a correct solution in the sense of the sample solution. This does not imply that arguments and justifications can be found in the individual considerations of the students that realise further ideal typical aspects in the context of the change of presentation. However, the text-related analysis does at least point to this possibility.

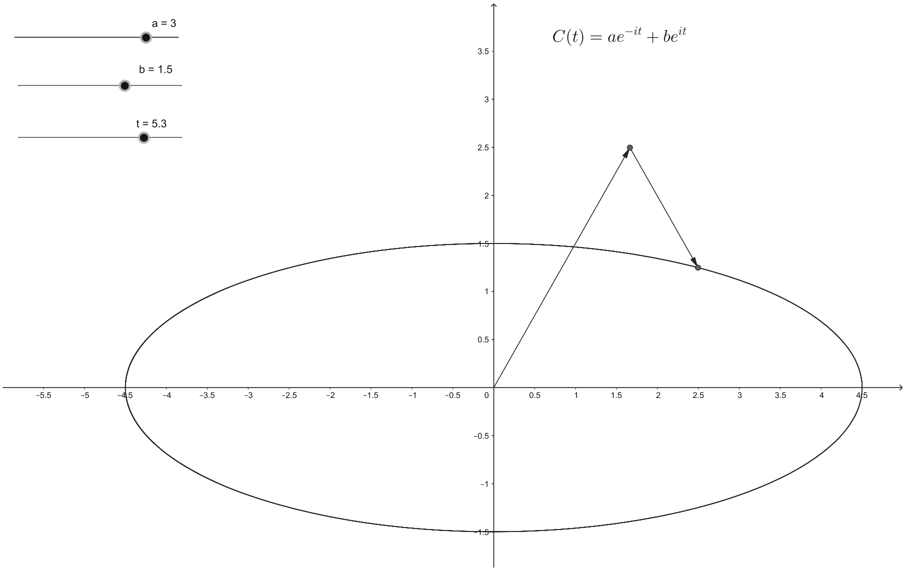

Fig. 12 Visualisation of curve and corresponding phasors with GeoGebra

Thesis 2. There Are (Almost) Correct Diagrams in Student Works, without the Step from Line (2) to Line (3) from the Sample Solution

The student solution C2 in Fig. 6b contains a correct diagram. Steps [1] and [2] are not present. The corresponding parts in Fig. 6a are crossed out. Especially the transformation step of the sample solution to line (3) is not realised. The diagram can be related to the solution for item 2: $A\cos 2\pi f_0 t + \frac{Am}{2}(\cos(2\pi(f_0 - F)\,t) + \cos(2\pi(f_0 + F)\,t))$. There, the

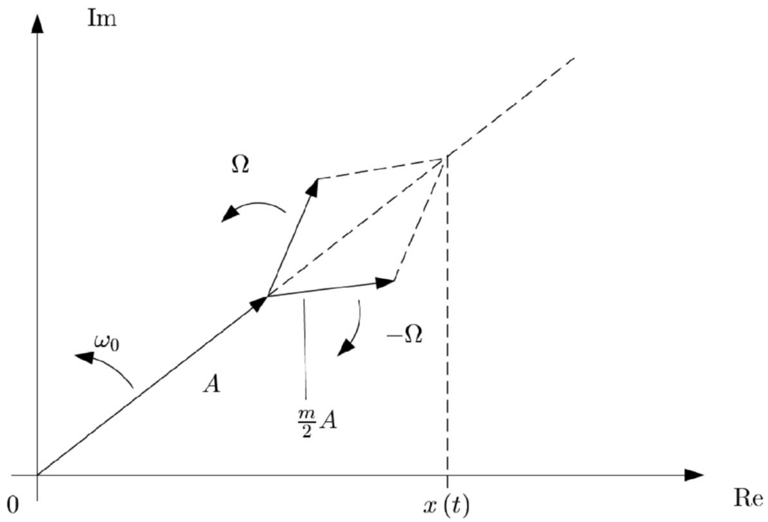

Fig. 13 Representation of $x(t) = A[1 + m\cos(\Omega t)]\cos(2\pi f_0 t)$ as the real part of a rotating phasor $A(t)\exp(j2\pi f_0 t)$ with $\omega_0 = 2\pi f_0$

cosine representation does not allow to separate the different frequencies of carrier and message signal, which is, however, at the core of the representation in line (3) of the sample solution. This student work also does not contain any verbal justifications, the observation of which would allow further interpretations regarding the individual rationale. The correct drawing of the diagram, however, allows an institutionally supported interpretation possible within the ET-discourse: concepts from basic lectures in electrical engineering, which associate elementary signals and their cosine representations with a corresponding phasor representation (see Fig. 1), are used to create the diagram. This provides a justification within the ET-discourse that differs from the institutional analysis, thus the corresponding parts in the diagram are circled in dashed lines. It allows the correct diagram to be drawn without having to grasp what appears to be essential in the institutional context of amplitude modulation. This follows directly from the institutional analysis and is compatible with an explanation of the student solution, which identifies it as the realisation of the indicated ideal typical mathematical discourses: The drawing of the phasor diagram can be interpreted as HM-technique that is embedded in the ET-discourse, but with ET-aspects that differ from the institutional analysis and bridge the void that is formed by the missing amplitude modulation related ET-aspects. So, the step from line (2) to line (3) in the sample solution is not necessary to create a correct diagram since other ET-aspects, in particular the relationship between cosine and phasor (cf. Figure 1), could constitute a bridging.

Thesis 3: In Student Solutions that Do Not Contain a Diagram or a Wrong Diagram, both Obvious and no ET-Discourse Aspects Occur. In both Groups of Works, References to the HM-Discourse Occur. In Other Words, the Relationship between the ET-Discourse and the HM-Discourse Can Be Very Different in Student Solutions

We first deal with the solutions with incorrect diagrams (I1, I2 and I3 in Figs. 7b, 8b and 9b) and then move on to the student works without a diagram (N1 und N2 in Figs. 10 und 11). We do not give a diagram for the analyses results for N1 and N2.

In the student solution I1, see Fig. 7b, steps [2] and [3] can be identified. The result from item 2 is not transformed further, so the aspects that correspond with step [1] are crossed out in Fig. 7a. The student solution contains an interpretation with references to carrier phasor, two other phasors (representing the sidebands) and the respective frequencies but not to the corresponding mathematical expressions. Especially aspects, that focus the relationship of complex numbers, exponential representation and phasors are missing. Therefore, the HM-aspects in subtask T_2 are crossed out and the praxeological aspects in step [2] are solely located within the ET-discourse. In step [3] both aspects, drawing of phasors in their specific relation as well as the indication of rotation are present. But the phasors are not drawn in the Argand diagram. The student solution essentially contains correct and relevant aspects for graphically representing amplitude modulation. However, the assistant corrected this solution by providing the diagram from the sample solution. This indicates that the student's solution is not an adequate representation in terms of the institutional teaching-learning context. Student solution and sample solution differ in the following sense: the diagram in the sample solution contains phasors drawn into the Argand diagram and thus contains not only the link between amplitude modulated signal and phasors, but also the link to the mathematical description by complex numbers. This link to mathematisation is missing in the student solution I1. So, the HM-aspects in subtask T_3 are replaced by ET-aspects. Step [3], like step [2], is solely located in the ET-discourse.

This solution can thus be explained as a mathematically informal realisation of the ET-discourse: phasors are graphical representations of signals, which can be handled without a mathematical description (see also the introduction of phasors and complex numbers in electrical engineering in section 2). Transitions between discourses are not present.

The student solution I2 in Fig. 8b contains all three steps from the sample solution, but with significant deviations. Therefore, all three steps in T are circled in dashed lines, see Fig. 8a. In step [1] a variety of techniques are tried out: The given relationship $\Omega \ll 2\pi f_0$ is used to eliminate the Ω-part in the cosine. The attempts with Fourier transform and low-pass filter are represented in subtask T_1 by an additional ET-technique and –technology block that is not present in the schema of the institutional analysis. In the end the relation given in the problem definition is used to attempt finally a graphical representation of the term $A[1+m]\ Re\left\{e^{j2\pi f_0 t}\right\}$. All ET-discourse references appearing in this work, were covered in previous lectures of the course. The interpretation of the mathematical expression in step [2] focusses ET-discourse aspects like signals and oscillation. References to the Argand diagram, HM-discourse on complex numbers, or ET-aspects such as the connection of cosine or complex exponential function with the phasor representation are not present. Therefore step [2] is located solely in the ET-discourse. In I2 the Argand diagram is present, but no phasors. $x(t)$ is drawn as double arrow on the real axis. Therefore, the HM-technique embedded in an ET-discourse is present, but different from the institutional solution. Rotation resp. oscillation aspects are indicated, but these again differ from the sample solution. In summary both, HM-aspects and the ET-discourse, are clearly present, but not in a coherent and, in terms of the institutional solution, goal-oriented manner. In particular, the link between mathematical terms and their graphical representation in the spirit of modulation principles is missing.

The student solution I3 in Fig. 9b does not contain step [1]. Steps [2] and [3] are present, but differ from the institutional analysis. Therefore, the corresponding aspects in Fig. 9a are crossed out or circled in dashed lines.

Several elements in this solution indicate aspects of step [2]. First, each of the three cosine-terms is underlined with a different colour. Each term is thus individually interpreted as something to be drawn. These colours can also be found in the diagram, the respective phasors are marked accordingly. The ET-aspects from subtask T_2 are not present. The text explains the drawing, so it also gives hints on how to interpret the mathematical expression (in order to draw it). It contains functional aspects: “all values for x(t)”, insertion aspects, and “x(t1)”, “x(t2), x(t3)” for the three phasors. The Phasor diagram from the HM repetition in the first lecture of the signal theory course is reproduced (cf. Fig. 3). Step [2] is therefore located in the HM-discourse. The last aspect is also relevant for Step [3], that is also located in the HM-discourse. The three cosine terms are drawn as three different phasors each. In addition to the three phasors, the diagram also contains elementary properties of complex numbers: the connection between the cosine and the complex exponential function as phasor and the complex conjugate. Rotational aspects are missing. Subtask T_3 is solely located in the HM-discourse. The student solution I3 can be explained as a realisation of a pure HM-discourse. It mirrors the student solution I1, but with HM- instead of ET-discourse. Aspects indicating a connection to amplitude modulation are missing and transitions to the ET-discourse do not occur.

The solution to item 3 in N1 contains only an interpretation, that could be associated to step [2], cf. Fig. 10. The part of the calculation in item 2 to which this statement refers is "…$(\cos(\Omega t + 2\pi f_0 t) + \cos(\Omega t - 2\pi f_0 t))$". The arguments of the cosine terms are each interpreted as a complex number in Cartesian representation, whose imaginary parts cancel each other out due to different signs. This ignores the fact that there are no complex numbers and that the terms under consideration are arguments of cosine functions that cannot simply be added. We assign this part of the solution to the ET-discourse: the relation to oscillations and the relation between time-dependent sinusoidal functions, i.e. oscillations, and complex numbers is relevant in the ET-discourse. In the solution N1, the cosine is obviously also not interpreted as a time-dependent function, but as a somewhat unclear mixture of oscillation and complex number.

While the solution N1 has ET-discourse aspects, albeit incorrectly, the solution N2 does not contain ET-discourse aspects. In this work, the addition theorems are not used to transform the cosine terms. Instead, cosine terms are rewritten using the complex exponential function, the multiplication is performed according to the calculation rules, and then converted back into cosine terms. This corresponds to the rationale of the HM-discourse to use the complex exponential function to simplify calculations.

With regard to thesis 3, we can therefore conclude that here too, the student work could be explained on the basis of praxeological aspects and, in particular, the ideal typical mathematical discourses. In contrast to theses 1 and 2, the explanations are much more diverse, depending on the complexity of thesis 3. With regard to a transfer of these subject-specific explanations to individual and subjective justifications, analogous considerations apply, as they were formulated in the concluding discussion of thesis 1 and will be taken up again in the following final chapter.

Discussion

The praxeological approach enabled us to explain student solutions of an exercise in the context of amplitude modulation. The detailed analyses of the exercise and the student work were based on an identification of different ideal typical mathematical discourses within the signal theory course. Moreover, we have described a systematic sequence of methodical steps enabling a well-founded and productive connection between on the one hand institutional and ideal type analyses and on the other hand individual student work. The extensive use of a graphical tool for representing praxeological structures allow us to understand deeper possible relationships between the interrelated mathematical discourses and their effects, which transcends the vision of the role of mathematics in engineering as something essentially to be applied. This refers both to the sample solution and thus to institutionalised taught knowledge, as well as to individual student solutions with their very own discourse configurations. The latter results goes beyond analyses in which praxeological models are used as a reference to prove that student solutions differ from this reference.

In view of the (possibly non-existent) correspondence between subject-specific explanations and the subjective considerations of the students, we like to share the following remark: The ideas and considerations on which our analyses are based are compatible with actual-empirical research approaches in the field of subject science, which aim to reconstruct subject-related patterns of reasoning (Holzkamp, 1985, chapter 9). Studies concerning the extent to which the explanations presented in section 4 fit with students' individual considerations would of course require further empirical data, for example interviews. But their analyses presuppose, in terms of research logic, such praxeological analyses and subject-specific explanations that we have presented.

Our outcomes potentially enable HM- and ET-lecturers to make didactic decisions about whether or not to explicate various mathematical discourses that are effective in electrical engineering courses. Here, of course, appropriate didactic tools must be developed that do not use the research-related terms introduced in this paper. Simplified variants of the graphical representation of the praxeological analysis, see Fig. 3, could eventually be quite useful for lecturers to identify changes and breaks in mathematical discourses. Analogously, one could argue with regard to our analyses of the student work: Here, on the one hand, discourse-relevant observations or diagnoses would be possible and, on the other hand, these could of course form a basis for appropriate feedback to the students. Both suggestions could be useful and effective for teaching and learning, even if our explanations do not correspond one-to-one with the ideas of the teachers when creating the sample solution or of the students when working on the exercises, since the aspects we identified refer to some extent to the institutional knowledge to be taught.

Finally, we give an example of how our findings can be used to develop concrete suggestions for modifying exercises: In the course on higher mathematics, which our sample students have attended, nearly all exercises concerning complex numbers cover standard topics including change of representation, calculations with complex numbers and determining the roots of polynomials. The following exercise is an exception:

Which curves are described in the complex plane by

$$ae^{-ti} + be^{ti}, a, b \in \mathbb{R} \text{ constant}, t \in \mathbb{R}?$$

This exercise was hardly worked on by students of the course, and was labelled with "too difficult" in the students' lecture notes. The exercise immediately changes its character when software such as GeoGebra can and may be used. Then one can see, among other things, that circles and ellipses appear as curves (see Fig. 12).

The illustration of the terms as phasors gives rise to a connection between algebraic and geometric aspects of complex numbers (see our characterisation of the HM-discourse in section 2): By representing the two complex numbers as aligned phasors, the peak of their sum always moves on the curve. Moreover, this exercise can further be adapted so that a similar type of curves appears as the one relevant to the complex phasor diagram of the exercise we examine in this contribution.[14] In other words: By using software, an otherwise essentially unprocessed task can be extended in such a way that it becomes connectable to the mathematical ET-discourse of our considered exercise.

As far as we know, such an approach of modifying the teaching in higher mathematics has been little studied up to now. Generally, application problems from the main subjects are supplemented to show that mathematics can be applied in a meaningful way, which mainly addresses mathematics *for* engineering. In our suggestion of an exercise, the HM-discourse would be expanded with regard to the mathematical ET-discourse, or rather its practices, in order to establish connections to signal theory, which addresses elements of mathematics *in* engineering within HM.

[14] Here we also refer to the work of De Oliveira and Nunes (2014) who investigate rotating phasor pathways derived from different standard amplitude modulation systems.

Declarations

Conflict of Interest On behalf of both authors, the corresponding author states that there is no conflict of interest.

Appendix: Exercise and sample solution

The exercise under consideration is structured in three items:

1. A message signal $s(t) = \cos(\Omega t)$ has to be amplitude modulated. The result is $x(t) = A[1 + m\cos(\Omega t)]\cos(2\pi f_0 t)$
2. The result of item 1. Has to be written as the sum of three harmonics. The result is

$$x(t) = A\cos(2\pi f_0 t) + \frac{Am}{2}\cos(2\pi f_0 t + \Omega t) + \frac{Am}{2}\cos(2\pi f_0 t - \Omega t)$$

3. The result of item 2. Has then to be displayed graphically in the complex plane as a rotating phasor with varying amplitude.

Our analysis focusses item 3. of the exercise. The exact problem definition of item 3 is:

3. Graphically display $x(t)$ in the complex plane as a rotating phasor with varying amplitude using the relationship $\cos(2\pi ft) = \Re\{\exp(j2\pi ft)\}$ and the result under item 2.

Sample solution:

One first writes

$$\begin{aligned} x(t) &= A\cos(2\pi f_0 t) + \frac{Am}{2}\cos(2\pi f_0 t + \Omega t) + \frac{Am}{2}\cos(2\pi f_0 t - \Omega t) \\ &= A\Re\{\exp(j2\pi f_0 t)\} + \frac{Am}{2}\Re\{\exp(j(2\pi f_0 t + \Omega t))\} \\ &\quad + \frac{Am}{2}\Re\{\exp(j(2\pi f_0 t - \Omega t))\} \\ &= \Re\left\{ \exp(j2\pi f_0 t) \underbrace{\left[A + \frac{Am}{2}\exp(j\Omega t) + \frac{Am}{2}\exp(-j\Omega t)\right]}_{A(t)} \right\} \end{aligned} \tag{1}$$

and interprets the expression in the square bracket as a real-valued time-dependent amplitude $A(t)$, which modulates the carrier phasor $\exp(j2\pi f_0 t)$ rotating at frequency f_0 in Fig. 13.

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