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Tacit Models that Govern Undergraduate Reasoning about *Sub*spaces

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Abstract This study is concerned with the reasoning that undergraduates apply when deciding whether a prompt is an example or non-example of the subspace concept. A qualitative analysis of written responses of 438 students revealed five unconventional tacit models that govern their reasoning. The models account for whether a prompt is a subset of a vector space, whether the zero vector is included, the structure of vectors, their number in the formula for the general solution to the system of linear equations, and the corresponding coefficient matrix. Furthermore, a conception was identified in students' responses, according to which the algebraic structure of a vector space passes from a 'parent' space to its subset, turning automatically it into a subspace. For many students this conception of an inheriting structure was instrumental for identifying and reasoning around *sub*spaces. Polysemy of the prefix 'sub' and students' prior experiences in identifying concept examples are used for offering explanations for the emergence of the conception.

Keywords Identification reasoning · Polysemy · Subspace · Tacit models · University mathematics

Introduction

Let us consider some of the tasks that Sarfaty and Patkin (2013), Vinner and Dreyfus (1989), and Wawro et al. (2011) offered to the participants of their studies:

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- Sarfaty and Patkin (2013) presented “standing” and “lying” solids to second-graders and asked them to identify cylinders, cones and pyramids. The students were also asked to explain their answers.
- Vinner and Dreyfus (1989) provided first-year college students with graphs of three curves: one contained a “loop”, one was discontinuous and another turned from a straight line into a curve. The question was whether there exist functions with such graphs.
- Wawro et al. (2011) worked with first-year university students who were asked, “Consider the vectors <1,2,3,5>, <2,5,-3,1>, and < 2,4,5,-7>. Do these vectors form a subspace?[1]”.

On the one hand, the presented tasks engage students with different mathematical concepts having little in common. On the other hand, Vinner and Dreyfus (1989) refer to their questions as “identification tasks”, which puts the spotlight on the act of identification rather than on the identified concept. Building on Vinner and Dreyfus’s approach, I propose that identification tasks call for *identification reasoning*, which is providing argumentative judgments for whether a prompt is an example or non-example of a particular concept.

Clearly, at different points in their mathematics education landscape, students apply qualitatively different models of identification reasoning. For example, Sarfaty and Patkin (2013) report that many young students reasoned their judgments with attributes of real-life objects that were conceived as prototypes for solids. Undergraduates’ reasoning in Wawro et al. (2011) was shown to be a mixture of personal concept images and attributes of a formal definition of a subspace. Despite these differences, some characteristics can be considered canonical to identification reasoning. For instance, confirmation of examples requires identification of all critical attributes of a concept in the prompt, whereas the lack of a single attribute is sufficient for refuting non-examples (Bruner et al. 1956). Thereby, identification reasoning can be viewed as a cross-mathematical activity, some characteristics of which evolve together with students’ progression through the curriculum, when other characteristics remain fixed (Kontorovich 2018a).

The studies mentioned above illustrate the value of identification reasoning, for instance when developing understanding of particular concepts and making sense of the role of definitions in mathematics. Furthermore, identification reasoning may constitute a necessary and crucial step in other mathematical practices, such as problem solving. Indeed, in their qualitative analysis of 820 scripts of matriculation exams, Movshovitz-Hadar et al. (1986) classified approximately a third of students’ mistakes as “theorems and distortions”. The category accounted for situations where students misidentified concept examples in the assigned problems and pursued their solutions with theorems in which the concepts were involved. In this way, significant parts of what the students considered as “solutions” turned out to be irrelevant to the assigned problems.

The study reported in this paper is concerned with undergraduates’ identification reasoning about subspaces; a fundamental concept in linear algebra, the understanding of which has been rarely explored. The described cross-curricular perspective is used in

[1] The imprecise formulation of the question was intentional.

the study for offering plausible explanations for the reasoning models that students use when their previous experiences with identification reasoning serve as a baseline.

Background

In this section, I briefly review some aspects of complicated relations between concepts and examples (Relations between Concepts and Examples section) and present findings on students' understanding of subspaces (Students' Understanding of Subspaces section).

Relations between Concepts and Examples

The role of concept examples and non-examples has been widely acknowledged in the mathematics education community. For instance, Sinclair et al. (2011), Watson and Mason (2005) and others suggest that the space of concept examples that one can generate provides insights about her or his understanding of the concept. Indeed, richness of a personal example space can be associated with a rich concept image (Tall and Vinner 1981). Alongside this view, a classical body of research is concerned with the prototype effect – students' tendency to associate concepts with a limited number of examples (e.g., Hershkowitz 1989; Schwarz and Hershkowitz 1999). For instance, quadratic and non-constant linear functions were often found to be prototypes of the function concept (e.g., Tall and Bakar 1992; Schwarz and Hershkowitz 1999). Furthermore, students tend to extract common features from their examples and treat them as critical attributes of the concept. Indeed, in the study of Tall and Bakar (1992), many students suggested that functions should be continuous, non-constant and representable with a single formula. These findings resonate with Sinclair et al. (2011), who propose accounting not just for scarcity and density of one's example spaces but also for their structures in terms of generativity, connectedness and generality.

From the cross-curricular perspective, the tension between particularity of examples and generality of concepts is ingrained in the teaching and learning of mathematics. For instance, the concept of number is introduced through natural numbers and gradually expanded to rational, integer, real and complex numbers. Research shows that each one of these transitions entails significant difficulties to the students as it requires a radical conceptual change (e.g., Kontorovich 2018b, 2018c; Vamvakoussi and Vosniadou 2004). Compatible difficulties were identified in the case of other concepts, such as tangent lines (e.g., Biza and Zachariades 2010), exponents (Pitta-Pantazi et al. 2007) and equivalence (e.g., McNeil 2007).

In the context of university mathematics, the relations between concepts and examples were often explored with a special attention to students' usage of formal definitions. A considerable body of knowledge pertains to the concepts of limits and derivatives (e.g., Cornu 1991; Giraldo et al. 2008; Jones and Watson 2017; Monaghan 1991; Roh and Lee 2017; Tall and Vinner 1981). An overall finding emerging from these studies suggests that students rarely operate with formal definitions and tend to rely on alternative mental models, which are shaped by accessible examples and non-examples (e.g., Dorier et al. 2000). In regard to the limit concept, Cornu (1991) notices that the majority of students do not master this idea even at the advanced stages of their studies, which does not necessarily prevent them from succeeding in problem solving and course examinations. The situation echoes with a famous phrase of George Box proposing that all models are wrong but some are useful (Box and Draper 1987). Accordingly, the study at hand was targeted at identifying models that students find useful for operating with the concept of subspace.

Students' Understanding of Subspaces

While the students' understanding of vector spaces has drawn the attention of the research community (e.g., Maracci 2008; Parraguez and Oktaç 2010), the findings on subspaces come from a handful of studies. Harel and Kaput (1991) assigned students the question: "Let V be a subspace of a vector space U, and let β be a vector in U but not in V. Is the set $V+\beta=\{v+\beta|\ v$ is a vector in $V\}$ a vector space?". The researchers reported that students' responses were clearly divided between those who engaged in checking the whole list of vector-space axioms and those who considered $V+\beta$ as a shift of a vector space V. The researchers clearly prioritized the latter approach due to its effectiveness in the assigned question requiring refutation of $V+\beta$. Indeed, the shift corrupts the fulfillment of some axioms, such as the inclusion of the zero vector and closure. Furthermore, $V+\beta$ fulfills many of the vector-space axioms "by default" simply because of it being a subset of a vector space U. Accordingly, the adherents of the former approach could have concentrated on validating selected axioms of a vector space rather than checking all of them.

Students' imageries of subspaces were the focus of the study of Wawro et al. (2011). Based on individual interviews with eight students, researchers distinguished between three imageries of the concept: a geometric object – when a subspace was associated with familiar geometrical figures, such as a plane or a line; an algebraic object – when a subspace was described through algebraic terminology of vectors, dimensions and matrices; and a part of a whole – when a subspace was related to some larger "parent space". The students were persistent with their imageries when engaging with a formal definition and related concepts, such as closure under addition and scalar multiplication.

Another interesting finding of Wawro et al. (2011) is the *nested subspace conception* – a view, in which the spaces $\mathbb{R}^n$ are conceived as nested, for instance $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$, which is a subspace of $\mathbb{R}^4$, and so forth. Notably, the researchers chose to view the conception as an early image of isomorphism rather than a misconception. The researchers reported that they were unable to identify any obvious source for all the participants to exhibit some variation of the conception. The conception of nested subspaces is leveraged in the findings of this study in "Findings section" and some ideas on its genesis are offered in "On the Meanings of 'sub' and Identified Tacit Models section".

Conceptual Framework

The conceptual framework of the study contains constructs of a tacit model and polysemy. The constructs are traditionally associated with distinct schools of thought: cognitivism and linguistics. However, Prediger et al. (2008) propose that combining the constructs that are rooted in fundamentally different theories is possible if the resulting framework is used for analysing a concrete empirical situation rather than for constructing a coherent complete theory. In their words, "[c]ombining theoretical approaches does not necessitate the complementary or even the complete coherence of the theoretical approaches in view" (p. 173). To clarify, the concrete empirical situation that was in the focus of this study was undergraduates' identification reasoning in the case of the subspace concept.

Tacit Models

Fischbein (1989) argues that mathematical concepts are abstract and formal constructs formed by axiomatic constraints. However, "[t]he main psychological problem is that we are not *naturally* equipped to manipulate concepts and operations, the consistency of which is not supported by some empirical evidence" (p. 9, italic in the origin). He suggests that we often resolve the problem by producing *tacit models* that turn original abstract concepts into more concrete and accessible structures.

According to Fischbein's definitions, a system B can be considered a model of a system A if there is a certain isomorphism between them. The adjective 'tacit' indicates that an individual can be not aware of the influence of the model B or the extent of it in situations in which A is involved. For example, Fischbein et al. (1985) asked middle-school students to choose an operation for each one of the following problems without performing the computation:

1. From 1 quintal of wheat, you get 0.75 quintal of flour. How much flour do you get from 15 quintal of wheat?
2. 1 k of a detergent is used in making 15 k of soap. How much soap can be made from 0.75 k of detergent?

Both problems require multiplication 15×0.75 (System A). However, nearly 80 % of the students responded correctly to the first problem; and only 25 % of the students were correct in the second problem. Fischbein et al. (1985) explain the difference with the tacit model in which multiplication is conceived as repeated addition (System B): multiplication is a commutative operation but the model distinguishes between the operator and the operand. Then, the model is effective as long as the operator is a whole number, like in the first problem (i.e. *fifteen times* 0.75). The operator is a decimal in the second problem (i.e. 0.75 *times* fifteen), which is indigestible for the model.

Research proposes that tacit models can develop at initial stages of the learning process and can influence learner's interpretations and reasoning "behind the scenes" even at advanced stages (e.g., Fischbein 2001; Stavy and Tirosh 2000). Fischbein (1989) maintains that "a main task of the psychologist interested in mathematics education is to identify such models and to suggest the means by which the student may become able to control their influence" (p. 9). Accordingly, *the goal of this study was to detect tacit models that govern students' identification reasoning in the context of the subspace concept.*

From a theoretical perspective, tacit models are developed by learners but identified by researchers. Accordingly, it is worth considering the conditions under which a model can be labeled as identified (see Simon 2017 for a compatible discussion).

Fischbein (1989) proposes a list of six common characteristics of the models: (i) a model is a *structural entity* that provides a meaningful interpretation that explains clusters of misconceptions exhibited by a substantial number of learners; (ii) it has *concrete, practical, behavioral nature* that enables learners to operate with an abstract concept; (iii) it has a *simple* and even trivial character; (iv) it *imposes a number of constraints* that prescribe under which circumstances the model can be applied; (v) if the circumstances are met, the model becomes *autonomous* and its behavior does not depend on external constraints; and (vi) it is *robust* to tensions with formal knowledge acquired by an individual. I apply these characteristics in "Summary and Discussion section" for evaluating the models in "Findings section" that were identified in this study.

Polysemy

Durkin and Shire (1991) used the constructs of homonymy and polysemy for explaining some of the challenges in mathematics teaching and learning. They use *homonymy* to capture a property of words having distinct and unrelated meanings; for example, "the bank was located two stations away from the bank of the river". *Polysemy* is used to refer to words the meanings of which are different but related; for example, "she enjoyed reading the newspaper that the Pravda newspaper published". The words 'bank' and 'newspaper' were used twice but the context induced an appropriate meaning for each appearance.

In mathematics education, the role of polysemy cannot be overstressed as it can be found in many mathematical terms (e.g., Presmeg 1992; Zazkis 1998; Kontorovich 2018a) and symbols (e.g., Kontorovich 2016, 2018b, 2018c; Mamolo 2010). Two types of mathematical polysemy can be distinguished. The first type is relevant to terms and symbols with a specialized use in a mathematics register, a use which does not align perfectly with everyday language (cf. Pimm 1991). Allow me to illustrate this type of polysemy with a problem:

> Three girls each had a cup of coffee. Each girl put an odd number of lumps of sugar in her coffee – twelve in total. How many lumps of sugar could each girl take?

Some might maintain that the problem is unsolvable because the even number 12 cannot be accumulated from three *odd* numbers. However, 1, 1 and 10 is a solution because 10 is an *odd* number of lumps of sugar to put in a coffee.

This is the place to explicate that this study grows from the assumption that the categorization of a mathematical term as polysemous or homonemous is a prerogative of the categorizer rather than a matter of "truth" (Kontorovich 2018a). In the above problem, the two meanings of 'odd' were probably not related for the believers in its insolvability, when for me, this mathematical joke is the one to bond between the meanings. From the research perspective, learners' recognition of connections between meanings in different registers can be leveraged by a researcher for arguing for the learners' polysemous view of the word; the lack of evidence of recognition might be not sufficient for building the case for learners' homonemous view. This approach has been extensively used for making sense of students' approaches to mathematical limits that are often viewed as "constraints that are not reached", just as speed limits in the everyday register (e.g., Cornu 1991; Monaghan 1991).

The second type of polysemy occurs within the mathematics register itself (Zazkis 1998). For instance, Zazkis (1998) focused on polysemy of 'quotient' and 'divisor' in the context of whole and rational numbers. At the request to determine the *quotient* in the division of 12 by 5, some of her pre-service teachers argued for 2, when others suggested 2.4. The reasoning of the former group can be explained with applying the term's meaning in the context of whole numbers, where a quotient is a number x such that $12 = 5x + r$ for $0 \leq r < 5$. The reasoning of the latter group is valid in the context of rational numbers, where a quotient is associated by a result of dividing 12 by 5.

Confusions with polysemy of the first type were found in the context of young students. For instance, Durkin and Shire (1991) asked 4- and 5-year-olds to write *high* and *low* numbers. The researchers reported that typical responses included lengthy and shrinked numbers, respectively. Evidence of students' struggles with polysemy of the second type have been indicated in the context of university mathematics. Kontorovich (2016) found that undergraduates can get confused with the symbol 'f^{-1}', which pertains to reciprocal and inverse functions. Specifically, his analysis of students' written responses revealed misidentifications of the intended concept and even occurrences in which both concepts were used interchangeably.

According to some theoretical frameworks, terms and symbols are tools for meaning-making (c.f. Sfard 2008). Then, the described approaches of children and undergraduates to polysemous terms can be encapsulated with an act of borrowing meanings from one context and applying them to another context. Polysemy of the prefix 'sub' is used in "On the Meanings of 'sub' and Identified Tacit Models section" for reviewing the tacit models of *sub*spaces identified in this study in "Findings section".

Method

In light of the scarcity of research on students' understanding of the subspace concept, I decided to collect data from a large number of students. It became possible through integrating the research prompt (see Fig. 1) in the final exam of a large mathematics course. The course was given at a New Zealand university and it was intended for undergraduates majoring in computer science, economics, statistics and finance.

For students participating in the course, it was the second encounter with university mathematics with a focus on two variable calculus and topics in linear algebra. In linear algebra, the students were introduced to the concept of vector spaces, which was connected to matrices and linear equations from their first course. Specifically, the course lecturer introduced a formal definition of a vector space and presented several examples and non-examples. In the following lecture, a subspace was first defined as a subset of a vector space that is a vector space on its own. Then, a discussion was made about the vector-space axioms that are necessary and sufficient for concluding that a subset is a subspace. This lecture ended with the standard conclusion that non-emptiness and closure under addition and scalar multiplication are necessary and sufficient for identifying subspaces. In the following lectures, the lecturer proved that

Let $A = \begin{bmatrix} 1 & -3 & 4 & -3 & 2 \\ 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \end{bmatrix}$ and $\vec{b} = \begin{pmatrix} 5 \\ -5 \\ 9 \end{pmatrix}$. You may use the fact that the matrix $\left[A|\vec{b}\right]$

row reduces to $\left[\begin{array}{ccccc|c} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$.

(i) Find the general solution to $A\vec{x} = \vec{0}$. Is this a subspace of $\mathbb{R}^5$?

(ii) Find the general solution to $A\vec{x} = \vec{b}$. Is this a subspace of $\mathbb{R}^5$?

Fig. 1 Prompting problem

a general solution to a system of linear equations $A_{m \times n}\overrightarrow{x_n} = \overrightarrow{b_m}$ is a subspace of $\mathbb{R}^n$ if and only if $\overrightarrow{b_m}$ is the zero vector (*the Subspace Theorem* in the rest of the paper). Accordingly, the lecturer expected the students to reference the Subspace Theorem in their solutions to the problem in Fig. 1. During the course, students had multiple occasions to engage with subspaces and the Subspace Theorem through at least ten problems that appeared in tutorials, homework assignments, and online quizzes. Furthermore, tutorials, assignments and quizzes were used for communicating the importance of argumentation and justification when working on mathematical problems.

The prompting problem in Fig. 1 was developed in collaboration with the course lecturer and its design was shaped by three considerations. *First*, the problem was intended for solution by hundreds of students as part of their final exam, the results of which counted for 60% of their course grade. Therefore, the course lecturer was interested in relatively standard problems that evaluate students' understanding of the course material. *Second*, the problem should have engaged the students with a formal definition of the subspace concept. Accordingly, we chose to contextualize the problem in $\mathbb{R}^5$ for preventing students from applying geometrical reasoning. Indeed, in $\mathbb{R}^2$ and $\mathbb{R}^3$ subspaces can be effectively associated with lines and planes passing through the origin. *Three*, Questions (i) and (ii) were designed for accessing students' confirming reasoning and refutation. The explicit request for reasoning appeared in the written guidelines for the exam that stated, "You must give **full** working and **reasons** for your answers to obtain full marks" (bold in the origin).

The data for the study came from four hundred thirty-eight students who worked on the prompting problem. The findings emerged from an iterative inductive process (Denzin and Lincoln 2011): At the first stage, all written responses were reviewed and their mathematical correctness was evaluated. Then, the responses were classified according to the identification reasoning that students demonstrated. This stage was focused on the distillation of attributes that students conceptualized as critical for identification of subspaces. The attributes were structured in four preliminary models: *subset of a vector space, subset of a vector space containing the zero vector*, *number of* *vectors in the formula for general solution*, and *coefficient matrix*. The models were reapplied to the whole data corpus.

For enhancing the clarity of the developed categories, a graduate student at the final stages of her studies in mathematics education was involved. The preliminary models were discussed with her and reapplied on the random sample of about 5% of responses. Then, the student categorized another random sample of about 25% of the data by herself. We independently agreed on 92% of responses. The remaining responses were discussed, which led to refinements in the developed categories. Specifically, the category "*n-dimensional vectors*" was distinguished from the "*subset of a vector space*". The resulting five categories are presented in the next section with illustrations and commentary.

Findings

I start with a general overview of students' responses and continue with the five reasoning models that emerged from data analysis. Table 1 summarizes the evaluations of students' responses to Questions (i) and (ii). The table shows that the vast majority of students were successful with determining general solutions to the assigned systems of linear equations. The evaluations of identification reasoning were less favorable: while more than 90% of the students correctly identified their general solutions in Question (i) as a subspace, the identification of approximately 10% was correct in Question (ii). A significant portion of students did not provide any reasoning for their identification in both questions. The reasoners ($n = 338$ in Question (i) and $n = 141$ in Question (ii)), in their turn, were more successful with refuting non-examples (22.7%) rather than with confirming examples (4.4%).

Overall, it is worth mentioning that no one of almost four and a half hundred students applied the subspace Theorem that was recurrently highlighted in course instruction. Moreover, the identification reasoning of only two students involved all the three critical attributes of subspaces: non-emptiness, closure under addition and multiplication by a scalar. Fifteen students referenced the two latter attributes in their responses but misidentified them in their general solutions in Question (ii) (see Fig. 2a and b for examples). This may suggest that the reasoning of these students was rote-driven.

Table 1 Evaluations of students' responses

Evaluation question		Correct	Incorrect	No response
Question (i)	Determination of a general solution	416	21	1
	Identification of a subspace	405	4	28
	Reasoning for identification	15	323	99
Question (ii)	Determination of a general solution	362	21	54
	Identification of a subspace	43	236	104
	Reasoning for identification	32	109	168

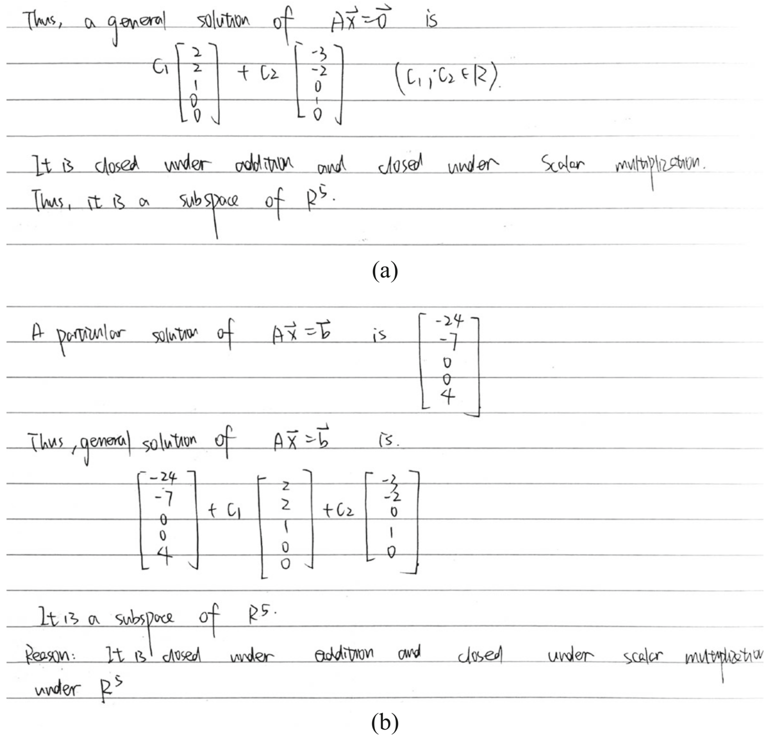


Fig. 2 **a** – Example of a response to Question (i). **b** – Example of a response to Question (ii)

The evaluations presented in Table 1 support Fischbein and Dorier's perspectives on the challenges that formally defined concepts can entail. Table 2 contains the distribution of models that were identified in students' responses, models that served students as alternatives to formal definitions and theorems. While students' responses to the two parts of the problem in Fig. 1 were analysed independently, it turned out for each student that her or his reasoning in Question (i) and Question (ii) was characterized with the same model. In the cases when a student explained her or his answer in a single question (Question (i) mostly), her or his identification in the second question aligned with the model that was constructed based on the reasoning from the first question.

Subset of a Vector Space

In their responses, the vast majority of the students correctly computed general solutions to the assigned systems of linear equations and obtained sets of five-coordinate vectors. Approximately one third of the students identified the sets as subspaces and reasoned that the solutions "are in", "belong to" or "are a part of" $\mathbb{R}^5$. Figure 3a and b illustrate that for many students the critical attribute for identification was the number of rows that the vectors in the computed solutions had. This reasoning can be associated with a model in which a subspace is conceptualized as a subset of a vector space. Indeed, any vector in the general solution with five entries belongs to the

Table 2 Distribution of identified models in students' responses

Identified models	Distribution in the responses to question (i)	Distribution in the responses to question (ii)
Subset of a vector space A set of vectors from a vector space is considered as a subspace.	121	50
Subset of a vector space containing the zero vector A set of vectors from a vector space containing the zero vector is considered as a subspace.	30	11
n-dimensional vectors Vectors with n coordinates are considered as a subspace of $\mathbb{R}^n$. The terms of dimension and basis appear in the reasoning.	41	17
Number of vectors in the formula for general solution The general solution to the system of linear equations is considered as a subset of $\mathbb{R}^k$, when k is the number of vectors appearing in the solution formula.	92	38
Coefficient matrix A general solution to the system of linear equations is considered as a subset of $\mathbb{R}^3$, when 3 is the number of rows in the coefficient matrix.	54	25

vector space $\mathbb{R}^5$ and together the vectors form a subset. Conventionally, the attribute of being a subset of a vector space is critical for the subspace concept. For the adherents of this model, the attribute is necessary and sufficient.

Subset of a Vector Space Containing the Zero Vector

The general solutions of a smaller group of students also consisted of vectors with five coordinates, however, the inclusion of the zero vector was a critical attribute in their reasoning. Figure 4a and b provide an illustration of such reasoning: as a response to Question (i), a student highlighted that the solution is a subspace because the zero vector can be obtained by setting c_1 and c_2 to zero. As a response to Question (ii), s/he wrote that the solution is not a subspace of $\mathbb{R}^5$ because "it does not pass through the origin/does not contain a 0 vector". The student did not elaborate on how s/he arrived to this conclusion, which illustrates a typical level of elaboration that the students exhibited in their responses. Possibly, the student assumed that it is obvious that any variation of c_1 and c_2 will not nullify the bottom coordinate in the general solution in Question (ii).

Overall, the reasoning that the students in this category demonstrated can be framed with a model in which a subset of a vector space containing the zero vector is considered as a subspace. Accordingly, the solutions' inclusion or exclusion of the zero vector determined students' identification of an example and non-example of the subspace concept. In terms of evaluation marks, adhering to this model was rewarding for the students because their identification of concept examples was correct in Questions (i) and (ii), and their refutations deserved full marks in Question (ii). The

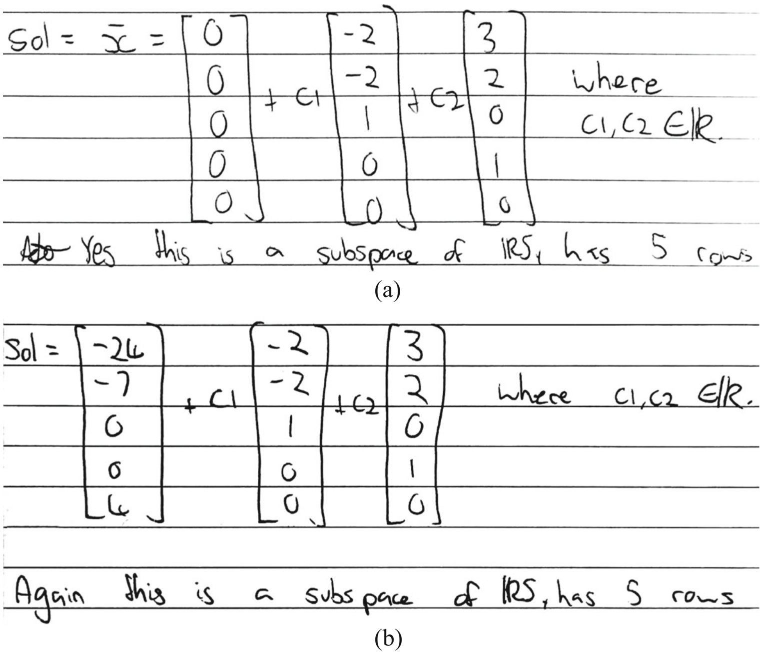


Fig. 3 **a** – Response to Question (i) from the *Subset of a Vector Space* category. **b** – Response to Question (ii) from the *Subset of a Vector Space* category

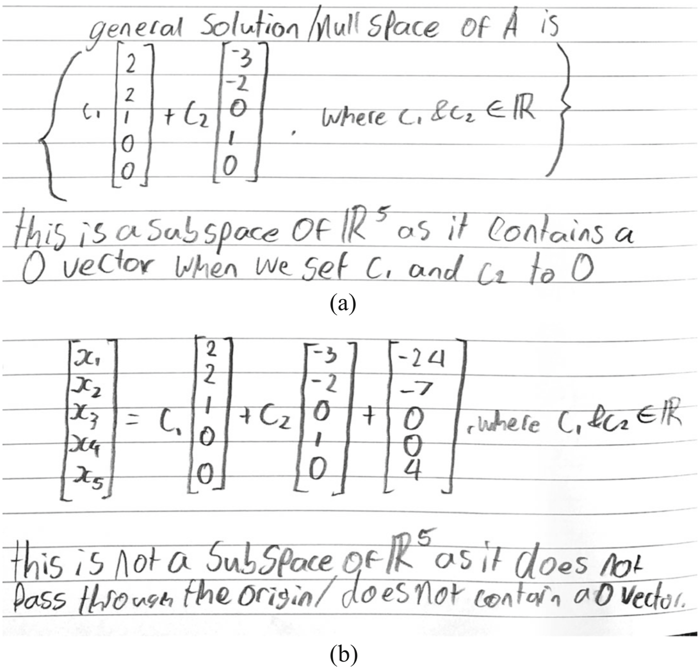


Fig. 4 **a** – Response to Question (i) from the *Subset of a Vector Space Containing the Zero Vector* category. **b** – Response to Question (ii) from the *Subset of* $\mathbb{R}^5$ *Containing the Zero Vector* category

confirming reasoning in Question (i) was partial as a closure under addition and scalar multiplication were also expected to be addressed.

n-Dimensional Vectors

The model in the focus of this section can be considered as a variation of the model in "Subset of a Vector Space section" with the difference that "Dim(5)" and the notion of basis appeared in students' reasoning. For instance, in her/his responses presented in Fig. 5a, a student wrote that the determined general solution is a subspace of $\mathbb{R}^5$ "because the basis is $\left\{\begin{bmatrix}2\\2\\1\\0\\0\end{bmatrix},\begin{bmatrix}-3\\-2\\0\\1\\0\end{bmatrix}\right\}$ giving us Dim(5)". The student also used "Dim(5)" in Question (ii) for confirming that the general solution is a subspace (see Fig. 5b). In this way, it seems reasonable to propose that "Dim(5)" is associated with the number of

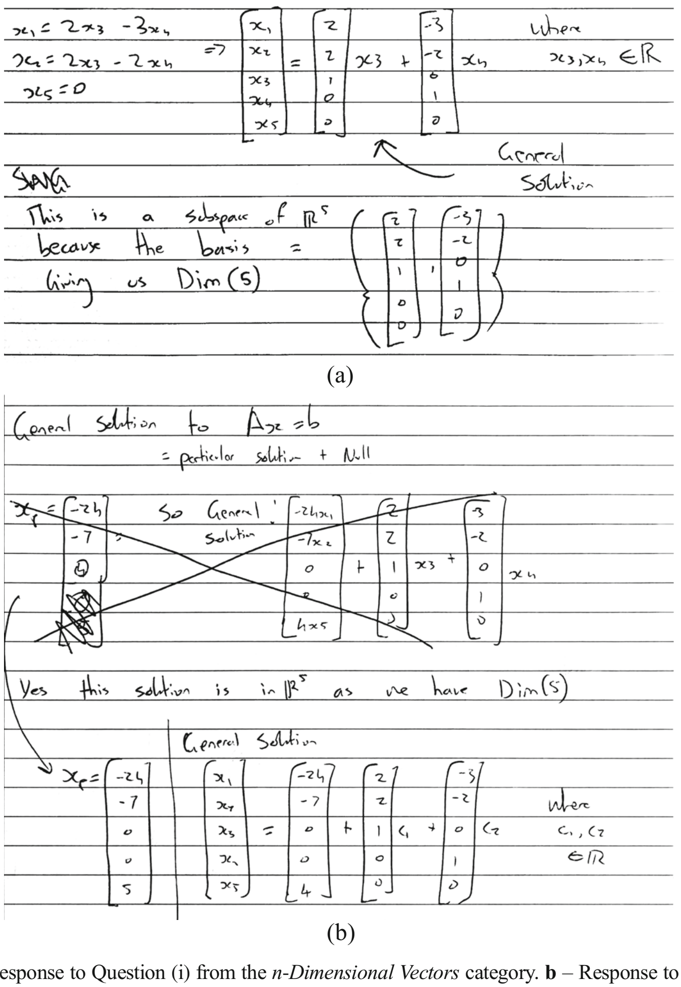


Fig. 5 **a** – Response to Question (i) from the *n-Dimensional Vectors* category. **b** – Response to Question (ii) from the *n-Dimensional Vectors* category

rows had by the vectors in the general solution. Then, similarly to the *subset of a vector space* model, the students identified their solutions as subspaces of $\mathbb{R}^5$.

I distinguish the current model from the one presented in "Subset of a Vector Space section" due to polysemy of the terms 'basis' and 'dimension' that the students in this category used. Colloquially speaking, 'basis' is often associated with a source that generates something else. Linear Algebra accounts for this meaning by considering the vectors in the basis as a source that generates a vector space through spanning. However, additional attributes appear in the formal definition of the term, for instance linear independence of the vectors. The student in Fig. 5a is reticent about it, which may signal borrowing the everyday meaning of a 'basis' for Linear Algebra (see "Polysemy section" for polysemy of the first type). On the other hand, it is also possible that the student assumed that the independence of the two vectors is obvious in this case.

Polysemy within the mathematics register (see "Polysemy section" for details) can offer an explanation for students' adherence to "Dim(5)". Overall, the usage of 'dimension' was three-fold in the course: (1) the number of linearly independent vectors in the basis of a non-trivial vector space; (2) a property of geometrical objects, for instance "one-dimensional line", "two-dimensional plane" and "*n*-dimensional $\mathbb{R}^n$", and (3) the number of vector coordinates in $\mathbb{R}^n$, for example "two-dimensional vectors". While in the two former usages 'dimension' indicates that the spanned object is a vector space, in the later usage it describes a structure of vectors in $\mathbb{R}^n$. Clearly, when $\mathbb{R}^n$ is under discussion, it can be described as "*n*-dimensional" from the three perspectives. The responses of all the students in this category, however, contained 'dimension' or 'Dim' in conjunction with the number 5, which allows proposing a general model: the students acknowledge an *n*-dimensional structure of vectors in $\mathbb{R}^n$ and interpret it as sufficient for the vectors to form a subspace. Thereby, the meaning-borrowing mechanism can be discerned in this model.

Number of Vectors in the Formula for General Solution

In their responses, approximately a quarter of students associated the general solutions that they determined to the systems of linear equations in Question (i) and Question (ii) with $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. Figures 6 and 7 provide examples of such responses from two students: in Fig. 6a, the association with $\mathbb{R}^2$ can be explained either by the number of vectors or with the number of free variables appearing in the formula for the general solution. In Fig. 6b, the general solution in Question (ii) was represented with three vectors and two free variables but the solution was considered by both students as being 'in' $\mathbb{R}^3$. Based on the similarity of all students' responses in this category, it seems plausible to propose that their association was driven by the number of vectors appearing in the formula for the general solution.

In terms of an eventual identification of subspaces, the reasoning of the students in this category can be clearly divided between two sub-models: in the first model, $\mathbb{R}^2$ and $\mathbb{R}^3$ are considered as parts of $\mathbb{R}^5$, which yields the decision that the determined solutions are examples of subspaces (38%, e.g., Fig. 6). In the second model, $\mathbb{R}^2$ and $\mathbb{R}^3$ are conceptualized as not included in $\mathbb{R}^5$, and then both solutions are identified as its non-example (62%, e.g., Fig. 7). In this way, the first model accepts the conception of nested subspaces (Wawro et al. 2011), when the second model rejects it. Furthermore,

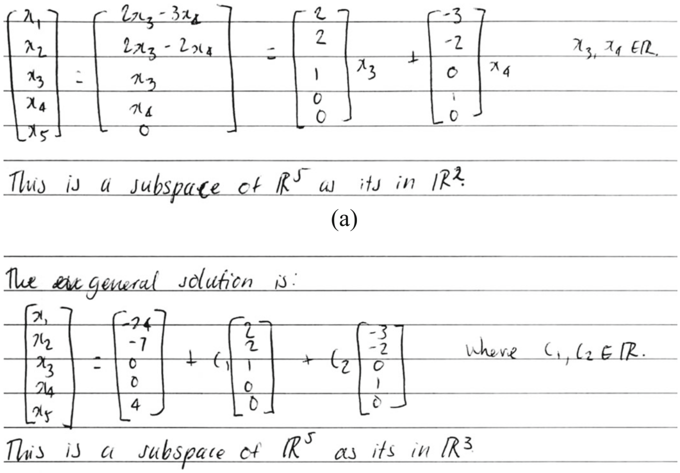


Fig. 6 **a** – Response to Question (i) from the *Number of Vectors in the Formula for General Solution* category. **b** – Response to Question (ii) from the *Number of Vectors in the Formula for General Solution* category

the confirming reasoning of the adherents of the first model proposes that a general solution could be any part of $\mathbb{R}^2$ and $\mathbb{R}^3$ without additional constraints. The data was not sufficient for inferring whether non-emptiness and closure were important for the adherents of the second model as their refutations were based on the mismatch between $\mathbb{R}^5$ and $\mathbb{R}^2$ or $\mathbb{R}^3$.

Coefficient Matrix

In the previous subsection, the students reasoned their identification of examples and non-examples of subspaces with attributes of general solutions $\overrightarrow{x}$ that they determined to the assigned system of linear equations $A\overrightarrow{x} = \overrightarrow{b}$. The identification reasoning of the students in the current category was based on the number of rows of the coefficient matrix A, which was three in the case of the prompting problem. Figures 8 and 9 illustrate some typical responses in which students claimed that the matrix is 'in' $\mathbb{R}^3$ and "$A \in \mathbb{R}^3$". The eventual identification of examples and non-examples of subspaces may be explained with students' rejection or acceptance of the conception of nested subspaces (Wawro et al. 2011). Indeed, $\mathbb{R}^3$ in Fig. 8b seems to be viewed by a student as not a part of $\mathbb{R}^5$, which led to the conclusion that the obtained solution is not a subspace of $\mathbb{R}^5$. Notably, the student referred to her or his general solution in Fig. 8a as

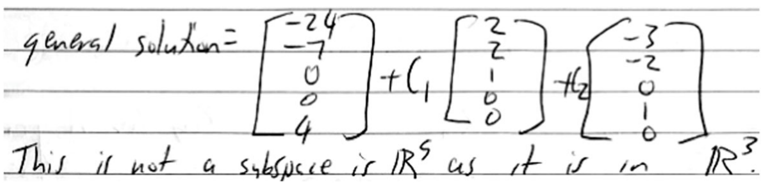
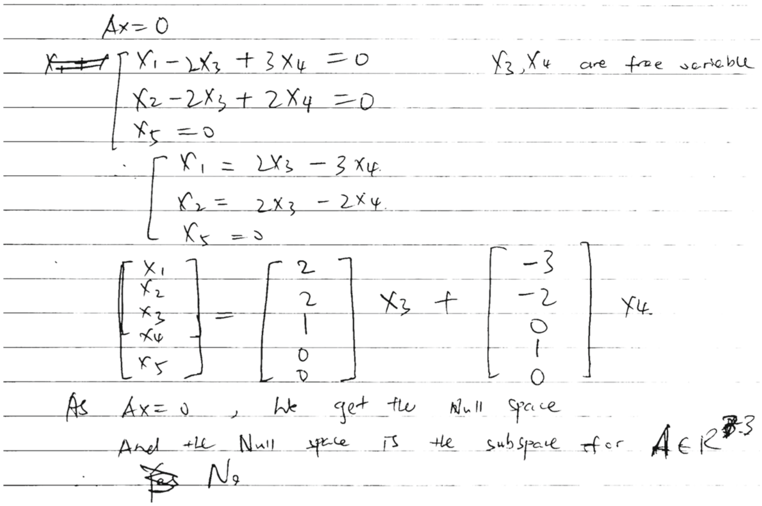


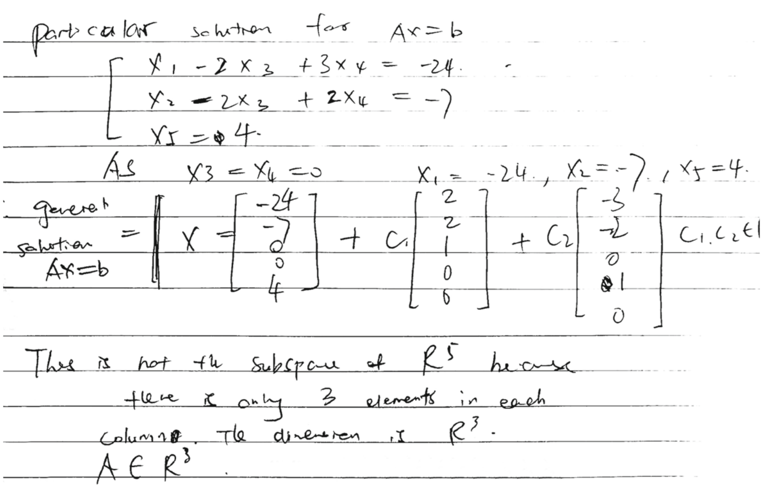
Fig. 7 – Response to Question (ii) from the *Number of Vectors in the Formula for General Solution* category

"*Null space*" and explicated that it is a subspace of $\mathbb{R}^3$. Another student in Fig. 9, identified the general solution as a subspace of $\mathbb{R}^5$ since "the matrix is in $\mathbb{R}^3$".

It is worth noticing that the described model is not totally invalid. In alignment with the Subspace Theorem, an identification of a general solution to $A_{m\times n}\overrightarrow{x_n} = \overrightarrow{b_m}$ as a subspace of $\mathbb{R}^n$ is predetermined by the vector $\overrightarrow{b_m}$ and could be reasoned without detecting the solution itself. Unfortunately, the responses of the students in this category suggested that the coefficient matrix A is the one that predetermines whether the solution is a subspace or not.



(a)



(b)

Fig. 8 **a** – Response to Question (i) from the *Coefficient Matrix* category. **b** – Response to Question (ii) from the *Coefficient Matrix* category

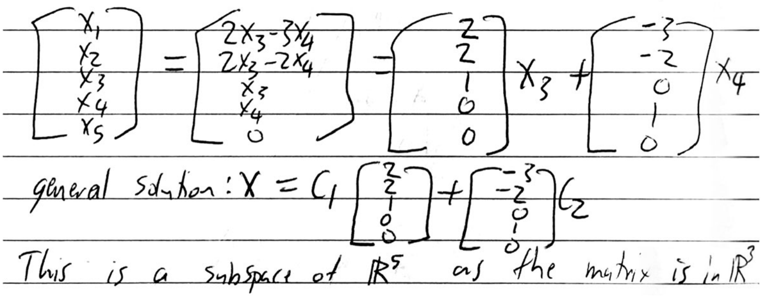


Fig. 9 – Response to Question (i) from the *Coefficient Matrix* category

Summary and Discussion

This large-scale study was aimed at contributing to the empirical body of knowledge on students' grasp of the subspace concept – a fundamental algebraic structure, students' understanding of which has been rarely explored. Specifically, the study focused on an identification reasoning that undergraduates apply for judging whether a prompt is an example or non-example of a subspace. The data showed that only 15 students out of 438 referred to non-emptiness and closure of subspaces in their reasoning. This finding aligns with a well-documented phenomenon of students avoiding formalism and resorting to more accessible approaches (e.g., Dorier et al. 2000; Fischbein 1989; Vinner 1991). Accordingly, the data analysis concentrated on alternative models that students developed for coping with the abstract concept.

Identification reasoning that the undergraduates demonstrated in this study was captured with five models: *subset of a vector space*, *subset of a vector space containing the zero vector*, *n-dimensional vectors*, *number of vectors in the formula for general solution*, and *coefficient matrix*. The names of the models highlight the attributes that the students considered critical for identification and reasoning about subspaces; the models themselves are summarized in Table 2. Four (out of six) general characteristics of implicit models (Fischbein 1989) are particularly notable in the identified models: *First*, they emerged from the analysis of a large pool of responses to questions that invited confirming reasoning and refutation. Accordingly, the models can be viewed as structural entities that relate and explain clusters of misconceptions. *Second*, the models equipped students with concrete and practical approaches to identification and reasoning about an abstract concept. *Third*, for many participants the same model was found to be suitable for describing her or his reasoning in both questions appearing in the prompting problem. In this way, confirming reasoning and refutation of many undergraduates were captured with the same model. Others seem to use the same model for confirming their identification of subspaces in two problem situations. Accordingly, it may be proposed that the identified models are autonomous as they emerged from situations with different external constraints. *Fourth*, the models were identified through analysing students' responses in a final course exam. Then, the models can be considered as robust and sustainable to conventional approaches that were promoted through course instruction and students' extensive engagement with subspaces.

The fifth and sixth characteristics of implicit models (simplistic nature and circumstances for application) are more arguable in the current study. The first model, *subset* *of a vector space,* is rather simplistic as it proposes that vectors' inclusion in a 'parent' vector space is sufficient for vectors to constitute a subspace. The remaining four models required students to carry out multiple-step processes for confirming examples and refuting non-examples. The second model, for instance, required noticing that the obtained general solution encompasses vectors that belong to a vector space and determining whether the zero vector is included. Thereby, these models cannot be considered simplistic in the sense of enabling an immediate identification of concept examples and non-examples. The simplicity, however, may be viewed in the fact that even when the students went through a multiple-step process, the reasoning that they provided accounted for a single attribute only. For instance, in their responses, the adherents of the second model concentrated on the ex- or inclusion of the zero vector and they were reticent about the general solution being a subset of a vector space. Generally speaking, a reasoning based on a single attribute reduced the level of complexity that a conventional identification reasoning entails by eliminating the conceptual difference between confirming concept examples and refuting non-examples (cf. Bruner et al. 1956).

In regard to the sixth characteristic, the constraints of the identified models could not be fully addressed due to the methodology of the study. Indeed, the findings emerged from the analysis of students' written responses to single questions that were assigned under exam conditions. Thereby, additional and methodologically diverse research is needed for constructing an empirically-driven body of knowledge on students' reasoning about subspaces. Hopefully, the models identified in this study will provide a useful foundation for the forthcoming construction. In the meanwhile, traces of the identified models can be recognized in the findings of Wawro et al. (2011). Indeed, their findings on students' imageries of a subspace as part-of-a-whole resonate with the *subset of a vector space* model. In the imagery of a subspace as an algebraic object, the participants of Wawro et al. operated with vectors and matrices that were expected to satisfy some conditions in relation to the dimension of their 'parent' vector space $\mathbb{R}^6$. These approaches are compatible with the reasoning of many participants in *n-dimensional vectors* model, who focused on the *coefficient matrix* and expected general solutions to relate to 5. Lastly, the nested subspace conception was constitutional in the *number of vectors in the formula for general solution* and the *coefficient matrix* models. Similarities between findings that emerged from the studies conducted in different countries and with different methodologies may suggest that the proposed models are more widespread than initially assumed.

Fischbein's perspective grows from the gap between formally defined concepts and implicit models that learners develop. A complementary view may highlight pedagogically desired similarities between the two. In this view, the *subset of a vector space* and *subset of a vector space including the zero vector* models can be considered as students' internalization of fundamental attributes of a formal concept definition; *n-dimensional vectors* that generate solutions can be viewed as an emerging understanding of a spanning set; rejection of the nested subspaces conception in *the number of vectors in the formula for general solution* can be assigned to students' recognition of a vector's structure in $\mathbb{R}^n$; the *coefficient matrix* model may be associated with students' understanding that some properties of a general solution are predetermined by the linear system of equations. Hopefully, in their further studies the students will put these models' characteristics in use for developing fluency with conventional approaches to the subspace concept. The concluding section of the paper contains some ideas on why this hopeful development has not occurred yet and how it can be further promoted.

On the Meanings of 'Sub' and Identified Tacit Models

The evaluations of students' identification reasoning presented in Table 1 might be interpreted as a call for reformation of course instruction in the topic of subspaces. However, what reformations are beneficial, at least potentially? In my perspective, before pedagogical strategies are considered, the particular concepts that give students difficulty need to be analysed epistemologically. Accordingly, this section contains a polysemous analysis of the subspace concept and offers an explanation for the genesis of a conception that can be discerned in all identified models.

Merriam-Webster's (online) dictionary proposes several meanings for the prefix 'sub', including: under, beneath, below; secondary, next lower than; subordinate portion of. The meanings are exemplified with such words as '*sub*species' and '*sub*-committee': focusing on some part of the map of species results in subspecies; when some members of the committee leave the room, a subcommittee stays. The examples illustrate a fundamental property that holds for many everyday 'subs': a 'sub' constitutes a part of some 'parent' category and inherits its structure. Indeed, subspecies are species on their own right and a subcommittee often functions as a committee that is focused on particular issue. These examples support the *inherited structure conception* – one's assumption that properties of a 'parent' category pass on to its 'subs'.

The inherited structure conception can be recognized in the models identified in this study. In the models presented in "Subset of a vector space section" and "*n*-Dimensional Vectors section" the students noticed that their general solutions constitute a 'sub' of the 'parent' vector space $\mathbb{R}^5$, which was sufficient for concluding that the 'sub' is a *sub*space, in the sense of a part of a (vector) space. The students in "Subset of a Vector Space Containing the Zero Vector section" assumed that 'subs' with the zero vector inherit the structure of a vector space. By claiming in "Number of Vectors in the Formula for General Solution section" and "Coefficient Matrix section" that their general solutions are subspaces of $\mathbb{R}^2$ or $\mathbb{R}^3$, the students accepted the inherited structure conception as well. In this way, the participants' reasoning echoes with participants' of Durkin and Shire (1991) who borrowed meanings for polysemous terms from the everyday register and used it in mathematics.

Durkin and Shire (1991) explored polysemy among preschoolers. Then, it may seem naïve for undergraduates to apply everyday meanings in the mathematics register. Indeed, how can it be that university students with more than a decade of experience in mathematics did not developed a shield to this type of polysemy? At this point, I rely on students' previous experiences with identification reasoning to argue that the inherited structure conception is valid for many concepts in school curriculum. For instance, if f is a function on the domain I_1 and I_2 is a *sub*set of I_1, then f is a function on the domain I_2; squares, rhombi and rectangles are *sub*categories of parallelograms and they possess all the properties of the parallelograms. The validity of the conception stems from the structure of these concepts, a structure which is determined solely by the properties of elements in the ‘parent’ category rather than by the relations between the elements.

A vector space is probably the first instance of an algebraic structure that students encounter, and then, it is hardly surprising that they embark on it with conceptions and models that worked for previously studied structures. Further research can explore whether spotlighting the differences between algebraic and non-algebraic structures has an impact on students’ understanding. Specifically, the differences can be spotlighted with concepts in school mathematics for which the inheriting structure conception does not hold. For instance, odd and even functions can be used to illustrate that not all reduction of functions’ domains preserves oddness and evenness; also, not all subsets of an interval are intervals.

Some instructional reformations are easier to implement than the others. For example, it might be useful to make students aware of tacit models and conceptions that can develop (cf. Fischbein’s 1989, 2001). Such a preventive pedagogy at first glance can have proactive consequences: Poole (2011), a popular Linear Algebra textbook that was used in the scrutinized course, introduces subspaces in the following way:

> “We have seen that, in $\mathbb{R}^n$, it is possible for one vector space to sit inside another one, giving rise to the notion of a subspace. For example, a plane through the origin is a subspace of $\mathbb{R}^3$. We now extend this concept to general vector spaces.
>
> **Definition:** A subset W of a vector space V is called a ***subspace*** of V if W is itself a vector space with the same scalars, addition, and scalar multiplication as V” (pp. 451-452, bold in the origin).

The introduction preluding the definition highlights the inherited structure conception and the inclusion of the zero vector, which resonances with the tacit model presented in “Subset of a Vector Space Containing the Zero Vector section”. In light of the findings of the study, it can be asked whether Poole’s attempt to make the concept more accessible can have a side effect of empowering some tacit models. With this concern in mind, lecturers may decide to review their course materials and teaching approaches in a search for possible sources of unconventional tacit models.

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