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Conceptions of Angles, Trigonometric Functions, and Inverse Trigonometric Functions in College Textbooks

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Abstract Using expository text and examples available in 10 college textbooks we identify two conceptions of angles, trigonometric functions, and inverse trigonometric functions that rely on either a static or a dynamic definition of angle. Although the textbooks favor a conception of trigonometric functions that is based on a dynamic conception of angle, they split in their definition of inverse trigonometric functions. We argue that transparency in making explicit how these conceptions can be bridged might be useful in understanding difficulties that emerge when solving problems with inverse trigonometric functions.

Keywords Ratio trigonometry · Circle trigonometry · Trigonometric functions · Inverse trigonometric functions · Conceptions

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Trigonometry in the United States has traditionally been a high-school course, taught either as an independent course or as part of a pre-calculus course. Many American universities and colleges offer trigonometry as part of a sequence of preparatory courses that lead to a calculus sequence (Lutzer et al. 2007). The enrolment in trigonometry courses at 2-years colleges in the United States[1] has fluctuated between 46,000 and 56,000 students between 1990 and 2010 (Blair et al. 2013, p. 137), yet we know very little about how this topic is taught. The few scholars who work on trigonometry have investigated students' understanding of radian measure (Moore 2010), the advantages and disadvantages of using a either ratio or a functional approach in teaching about trigonometric relationships (Kendal and Stacey 1997; Weber 2005), and future teachers' knowledge of trigonometry (Fi 2003). As part of a larger project on community college mathematics instruction, we collected various explanations of mathematical ideas. Some of the explanations of inverse trigonometric functions raised questions about the way in which textbooks portray the topic. In this paper we describe what we found as we sought to understand how textbooks present ideas related to inverse trigonometric functions that may stem from distinct conceptions of angles and of trigonometric functions. We focus on the presentation found in 10 college textbooks as a first step to understand an instructor's language in an explanation. We organize the paper into five sections. First we describe the theoretical framework that guided the investigation; second we discuss the literature that informed the study including the specific research questions that guided the study; third we describe the study itself, its context and the instructor's explanation that prompted this investigation, the data we collected, and the analysis we performed; fourth we present our findings; and fifth, we conclude with a discussion of the findings.

Theoretical Framework

In our research we assume that teaching and learning are phenomena that occur with people enacting different roles—those of teacher or students—aided by particular resources, and constrained by specific institutional requirements. We are not necessarily concerned with the knowledge, beliefs, or attitudes of the individual teacher who delivers an explanation on inverse trigonometric functions that is difficult to follow; nor are we concerned that when we interview the students of said teacher they indicate having little recollection of being in the room when the explanation was delivered. Rather we seek to see whether the teacher's difficult-to-follow explanation can be justified by how knowledge is built in the available resource (the course textbook) to argue that the lack of transparency in how the knowledge is constructed can indeed make it more difficult to present an organized and coherent explanation of how to solve problems on inverse trigonometric functions, and that it is therefore possible for students to not recall any of the said explanation. To build this argument, we use Balacheff's model of mathematical conceptions (Balacheff and Gaudin 2010) to analyze textbook content. Balacheff defines a *conception* as the interaction between the cognizant subject and the milieu (those features of the environment that relate to the knowledge at stake). His basic proposition is that conceptions of mathematical

[1] Two-years colleges are post-secondary institutions in the United States that offer the first two years of a college degree. In addition they offer remediation courses in English and mathematics, vocational certificates, and job training. They also offer courses for community enrichment.

notions are tied to particular problems in which those conceptions emerge. Thus Newton's conception of function was substantially different than Dirichlet's because each was working with a different phenomenon (Balacheff and Margolinas 2005). As mathematicians solve new problems, our conceptions about particular notions evolve. The combination of all these different conceptions is what constitutes a persons' knowledge (*knowing*) about a particular mathematics notion. This way of understanding conceptions allows for potentially conflicting ideas about a mathematical notion to coexist without creating a dissonance in the knower. Indeed, discrepancies are only such for observers of the situation, as the knower might be using specific problems to conceptualize the mathematical ideas. Conceptions can be distinguished from each other because they have different manifestations of four key components defining a conception:

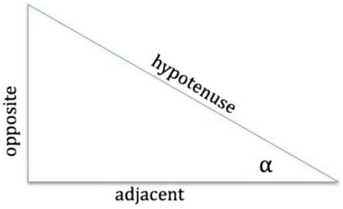
- P, the set of problems in which the notion at stake is present;
- O, the specific operations that are carried out to solve the problem
- R, the systems of representations used or associated with the problems, and
- Σ, the control structures, the organized set of criteria that helps the knower decide what to do in a given situation, determine that an answer has been found, and establish that the answer is correct (see Balacheff and Gaudin 2010, p. 190).

We use this model to describe how conceptions of mathematical notions emerge from written text. Written text in this context refers to one of three types present in mathematics textbooks: exposition, examples, or exercises (Love and Pimm 1996). Exposition text "directs the reader… expound[ing] its subject matter in a discursive fashion, [it] may [use] devices such as questions, visual materials, or tasks as assisting concept formation." (p. 387). Examples are "intended to be 'paradigmatic' or 'generic', offering students a model to be emulated in the exercises which follow" (p. 387). In the exercises, the students are encouraged to actively engage with the text, by working through tasks that mimic previous examples or with a more varied collection of problems. Two prior studies have used Balacheff's model with textbook content, one pursued an analysis of conceptions of functions as present in the exercises of middle school textbooks from several countries (Mesa 2004) and the other pursued an analysis of the conceptions associated with initial value problems from examples in university calculus textbooks (Mesa 2010). These analyses revealed the structure of the potential conceptions fostered by the textbook presentation by attending to these four elements of the model. The study presented herein analyzes both the exposition sections and the examples of post-secondary textbooks on angles, trigonometric functions, and inverse trigonometric functions.

Literature Review

Two main approaches to teaching trigonometry have been documented in the literature, ratio (or triangle) trigonometry and circle trigonometry.[2] In the first approach the sine and cosine of an angle α, are presented as ratios between the sides of a right triangle that has one angle labeled α (see Fig. 1).

[2] Authors use triangle or ratio interchangeably as they describe this approach. We use prefer ratio, but use the author's preferred terminology when describing the work.



$$sin\alpha = \frac{opposite}{hypotenuse}$$

$$cos\alpha = \frac{adjacent}{hypotenuse}$$

$$tan\alpha = \frac{opposite}{adjacent}$$

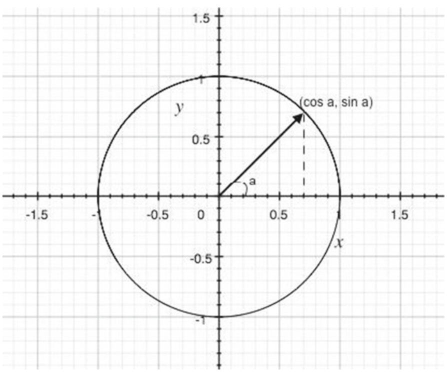
α is restricted by the geometry of the triangle.

Fig. 1 A ratio representation of the simple trigonometric functions. The initials for the names of the function and the names of the sides (Sine is Opposite over Hypotenuse, etc.) are used in the United States as a mnemonic, SOHCAHTOA, and it is very popular in teaching

In the second approach, sine and cosine of α refer to the coordinates of a point on the circumference of a unit circle, whose ray determines the angle α (See Fig. 2). It is thought that triangle trigonometry provides an easier path for students' understanding of trigonometric concepts as a transition to circle trigonometry with which students can fully study periodic functions. Interestingly, historically, the development of trigonometric ideas followed the opposite route (Bressoud 2010).

According to Matos (1990) the Babylonians used angles to classify oscillating celestial bodies suggesting a reliance in a circle trigonometry approach. In the pure geometry developed by the Greeks, however, a relational representation of an angle was used, implying a triangle-based trigonometry. It seems as if parallel tracks were kept, with some scholars relying on triangle trigonometry (e.g. Hipparchus in 2nd century BCE and Al-Khwarizmi, 9th century, see Bressoud 2010, p. 108) and most using circle trigonometry. Triangle trigonometry achieved prominence in the 16th century thanks to the publication of Johann Müller's *De Triangulis Omnimodis* (On Triangles of Every Kind). The instructional switch from circle to triangle trigonometry occurred in "the mid to late 19th century" (p. 107). We owe the use of the unit circle to Euler who in the 18th century "decided that the radius should be fixed at 1" (p. 110).

Scholars do not agree on what is the best approach to use in teaching trigonometry. For example, Kendal and Stacey (1997) stated that the ratio method of teaching trigonometry was deemed more effective than the circle method. Their conclusion was based on their



The coordinates of a point on the circle correspond to the sine and cosine of the angle that is created by a ray starting in the origin and sweeping an angle α.

Fig. 2 A unit circle representation for trigonometric functions. In this representation the trigonometric functions of any angle determined by the rotation of the ray over the origin can be formulated study of two groups of Year 10 students in Australia who were randomly assigned to two conditions: one group (88 students) received instruction using the unit circle approach whereas the other (90 students) received instruction via the ratio (triangle) approach. The teachers received materials for 20 lessons (each lesson was 45 min long) for the trigonometry unit that emphasized conceptual development and had identical exercises. The groups were comparable at the onset of the study. At the end of the course the gain in trigonometry scores and the retention in course were greater for the students receiving instruction with the ratio method than for the students in the unit circle approach; the ratio-instruction students made significant gains in other dimensions analyzed (e.g., attitude, solving algebraic equations). Moreover, low ability students gained the most. The authors attributed the results to the easiness of the ratio method for identifying appropriate formulas (specifically the simplicity of the SOHCAHTOA mnemonic); in their view, the unit circle method was "fraught with multiple opportunities for mistakes" (p. 327). It appeared that another advantage of the ratio method was that it allowed for algebraic procedures to be more readily used. They acknowledge, however, that their tests did not assess conceptual understanding. The authors recommend using the unit circle to introduce basic trigonometry, connect these to ratios, and then use the ratio method for solving triangles.

In contrast, Weber (2005) criticized textbooks for their stress on using a ratio approach to teach basic trigonometric relationships before a functional approach. By focusing on ratios, he claims, students are hindered in learning to interpret trigonometric relationships as functions. In his teaching experiment, Weber instructed students to use a protractor and a unit circle to geometrically understand the sine and cosine of an angle. After going through this process, students demonstrated a better understanding of trigonometry compared to students who were taught by conventional textbook methods that used ratios.

While it is not sufficiently clear what accounts for the differing conclusions across these two studies (there is not sufficient detail about the specifics of the interventions) we can say that these approaches are useful for learning something about trigonometric functions. Bressoud argues for a presentation that follows the historical development, circle trigonometry followed by triangle trigonometry, which, he proposes, might better meet the needs of our students who are less likely to need to solve navigational problems and more likely to encounter problems in biology, physics, and social sciences that require the modeling of periodic phenomena, for which circle trigonometry is better suited. In any case, this work suggest the need to better understand how the knowledge about these notions is laid out in textbooks, as such investigation can shed light about the way in which potential conceptions can support or hinder students' understanding of trigonometric functions. As a first step in addressing this need, we proposed the following research question: what are the conceptions of angles, trigonometric functions, and inverse trigonometric functions present in the expository sections and examples of university textbooks?

The Study

Context

The question driving the analysis presented here emerged as an exploration related to an incident captured in our observations of instruction in trigonometry and calculus courses taught at a large community college in the Midwest of the United States (Mesa et al. 2014). Some of the faculty we observed as part of a research grant (Mesa 2008) were teaching trigonometry and, through video-recall interviews, we documented difficulties students had following the explanations the teachers gave in class. Studying these teachers' explanations we noticed that the language used contained references to the circle and the triangle approaches to trigonometry that were never explicitly discussed. We present part of an explanation that illustrates the conflicts we saw.

Elizabeth's Explanation

In this explanation Elizabeth is working on the problem: "Find $\sin^{-1}(\frac{\sqrt{2}}{2})$". She starts by restating the problem as "The sine of what angle is root two over two? [On the board she had written: $\sin^{-1}(\frac{\sqrt{2}}{2}) = (\theta)$, then: $\sin(\sin^{-1}(\frac{\sqrt{2}}{2})) = \sin(\theta)$, and then: $\frac{\sqrt{2}}{2} = \sin(\theta)$]." This is a natural reformulation of the problem; up to this point, trigonometric functions had been being applied to angles although the transition from 'angles' to 'real numbers' had not been made explicit. She then asks: "So where must this angle [points to theta] live for the inverse sine? In other words, the sine of what angle is root two over two?" With these questions Elizabeth seems to be making a connection to the unit circle, which she had used earlier in the lesson to draw the inverse sine and cosine functions. She had used the language "live" to signal that for inverse trigonometric functions to be well defined, their input values had to be restricted. After numerically checking that $\frac{\sqrt{2}}{2}$ was a number between $-\pi/2$ and $\pi/2$,[3] she says:

T: But you have to be careful that this is only quadrant one. Because what's the sine of three-quarters pi? Square root of two over two, right? [writes $\theta = \pi/4$]. Let's get a unit circle up here to look at. [Draws a circle, with Quadrants I and IV shaded]. This [points to a point on the circle at 45°] is one quarter, square root of two over two, square root of two over two [the coordinates of the point on the unit circle]. This [pointing to the point on the unit circle at 135°, in Quadrant II] is three-quarters. This is minus root two over two, root two over two [the coordinates of the second marked point on the unit circle]. And what is the sine of ($3\pi/4$)? Square root of 2 over 2! So now, because of our restricted domain on the sine… we're only on the right hand side of the circle, we don't get this [points to the half un-shaded circle in Quadrant II].

S: So, it the reference… It is, it is…

T: Well, it's quadrants one and four.

S: x isn't positive.

T: I don't know, because I don't see any xs. So I don't know. Keep in mind, this [points to the whole statement of the problem] is an angle. It's positive, so

[3] The verification should be that the argument, $\frac{\sqrt{2}}{2}$ is a number between -1 and 1. The angle that will be obtained will be between $-\pi/2$ and $\pi/2$.

17 I go onto quadrant one, for all the trig functions are positive in quadrant one.
18 They all use quadrant one.

Elizabeth starts by warning the students that "this," meaning the angle, is "only Quadrant I" which is an implicit reference to the restriction of the inverse sine function (Line 1). She then draws a unit circle to identify a possible angle (π/4) that would give "square root of 2 over two" (Lines 2-4). She imposes a coordinate system on the circle, identifies the coordinates for the sine and cosine of 3π/4, and points out that the sine of this new angle (she does not point to the angle, only to the coordinates of the point on the unit circle) would also be "square root of two over two" (Lines 8-9). She then concludes, "we don't get this" (Line 10). Yet in the explanation, Elizabeth had found an angle that satisfies the equation but that needs to be discarded. This is justified with the statement: "It's [the ratio] positive, so I go onto quadrant one, for all the trig functions are positive in quadrant one," (lines 16-17) a clear reference to the ratio calculation of the function instead of a reference to the restriction of the function. Over the course of the lesson, the question of which angles could be kept and why multiple answers weren't allowed emerged several times but we did not see an argument for accepting or rejecting them that did not cross the boundaries between circle and triangle trigonometry. We came across to similar episodes in other lessons by other teachers.

Our theoretical approach to research on instruction acknowledges the crucial role that resources play in what unfolds in classrooms (Cohen et al. 2003). Rather than focusing on the instructors and their knowledge of the material—a possible study one can pursue—we chose to focus on how the textbook used in the course (McKeague and Turner 2008) addressed the content. In this way we could identify and trace the possible origins of the language used and use this information, in a later study, to probe instructors' understanding and management of these conflicts in the classroom (Mesa 2014). We complemented the analysis by including additional college textbooks to identify how they were arguing for domain restriction for inverse trigonometric functions and how those arguments were connected to triangle and circle trigonometry.

Data

We analyzed 10 post-secondary textbooks: the trigonometry and the pre-calculus textbooks used at the study's college (Larson and Hostetler 2007; McKeague and Turner 2008), four calculus textbooks used either in this college or at one of the several transfer institutions in the state (Hughes-Hallett et al. 2008; Ostebee and Zorn 2002; Stewart 2012; Thomas et al. 2001), three classic or honors textbooks for mathematics majors (Apostol 1967; Spivak 1976; Zenor et al. 1999)[4], and one pre-calculus textbook that had a graphical approach to calculus and was developed during the calculus reform of the 90s (Hungerford 1997). As a group the textbooks represent a continuum of courses the students would take (trigonometry, pre-calculus, and calculus) and a wide range of options and perspectives to study how textbooks, in general, approach trigonometry.

[4] Students at 2-years institutions can enroll in courses that may transfer to a 4-years degree program in a different institution.

Analysis

To conduct the analysis we identified all chapters or sections that corresponded to inverse trigonometric functions, and as the analysis of these sections was pursued, we realized the need to include all chapters and sections related to two key notions involved in the exposition, angles and trigonometric functions. We analyzed the expository text and the examples. In all we analyzed 140 pages across the 10 textbooks (14 pages on angles, 55 on trigonometric functions, and 71 on inverse trigonometric functions) and 78 examples (12 on angles, 34 on trigonometric functions, and 32 on basic inverse trigonometry[5]). Of these examples, 13 were from the trigonometry textbook, 26 were from pre-calculus textbooks and the remaining 39 were from calculus textbooks.

We parsed the expository text and examples seeking to identify each aspect of Balacheff's model of conceptions: the problems (P) in which angles, trigonometric functions, and inverse functions are needed, the operations (O) that are called for (e.g., restrict the domain of sine), the representations used (R) (e.g., unit circle, triangle, Cartesian plane), and the control structure (Σ) (e.g., verifying that the solution is in the correct interval). We then constructed flowcharts for each set of data in order to identify trends among the different conceptions. Figure 3 shows an example of this coding for the expository text for trigonometric functions in one of the calculus textbooks. Each line of text is numbered and the code we assigned is on the right hand side. Lines 5 and 6 are coded as the Problem at stake: "to calculate the values of the cosine and sine directly from the coordinates of *P*." Lines 7, 12, and 13 are coded as the operations that need to be done to solve this problem: (1) "Dropping a perpendicular from *P* to the *x*-axis." (2) "Read the magnitudes of *x* and *y* from the triangle's sides." (3) "Put the sign according to the quadrant in which the triangle lies." The representation used is the unit circle (Line 8); Line 13 also includes a control: negative for the *x* coordinate and positive for the *y* coordinate—this can be seen also in the Cartesian plane. Although there is a triangle used in this representation, the triangle is not the one that one would use with ratio trigonometry: the angle that this problem is working on is not inside the triangle, as ratio trigonometry would require.

The second author coded all the text and examples; the first author checked a random selection of 10 % of the pages. There was full agreement in the coding.

Findings

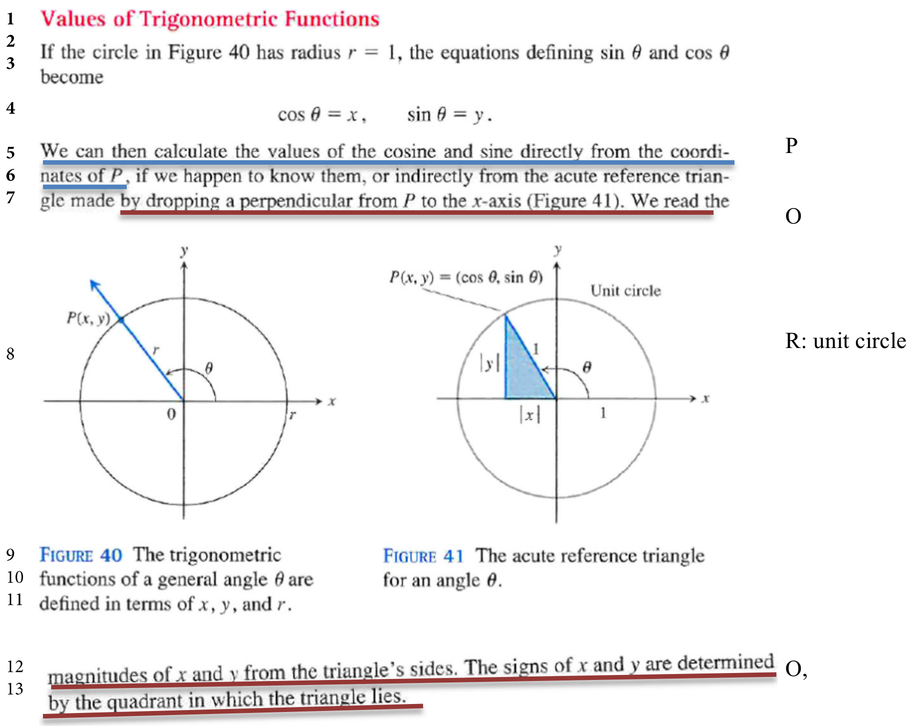
For the purpose of this paper we present conceptions around three central problems (P),[6] *What is an angle*?, *What is sine of an angle*?, and *How is the inverse sine defined*? that arose from the analysis. We use these to describe the conceptions identified across the textbooks for each of the three notions, angles, trigonometric functions, and inverse trigonometric functions, by presenting each of the features of the model.

[5] We excluded examples and sections regarding integration and differentiation of these functions because they were not pertinent to this analysis.

[6] Throughout this section we use the letters P, O, R, and Σ to refer to the elements of the quadruplet that define a conception in Balacheff's model. We keep Σ which is the original notation proposed by Balacheff.

Angles

Three textbooks did not include a definition of angles (Larson and Hostetler 2007; Ostebee and Zorn 2002; Zenor et al. 1999) and only four included examples related to the notion (Hungerford 1997; McKeague and Turner 2008; Spivak 1976; Stewart 2012). The textbooks that define what an angle is, each give a definition that emphasizes *how to construct* or *measure* the angle (an action that must be executed). Angles are defined in two ways. First, by referring to the arc length of a unit circle that is intersected by two rays rotated about a point at the center of that circle (O). For example, Hungerford (1997) describes an angle as "being formed dynamically by *rotating* a half-line around its endpoint (the **vertex**)" (p. 365, emphases in original). This definition of rotation of a ray on an end point lends itself to a representation over the unit circle (R), in which the angle increases as its ray rotates counter-clockwise or decreases as its ray rotates clock wise. Second, an angle is defined by its relation to the lines encompassing it. For example, McKeague and Turner (2008) state, "An angle is formed by two rays with the same end point" (p. 3). This definition of angle as 'the space' between the rays sharing a same endpoint lends itself to a representation in which the angle is fixed (R); by adding a perpendicular line to one ray (O) the trigonometric ratios can be established. None of the textbooks address the issue of checking what is or what is not an angle (Σ).



Values of Trigonometric Functions

If the circle in Figure 40 has radius $r = 1$, the equations defining $\sin\theta$ and $\cos\theta$ become

$$\cos\theta = x, \qquad \sin\theta = y.$$

We can then calculate the values of the cosine and sine directly from the coordinates of P, if we happen to know them, or indirectly from the acute reference triangle made by dropping a perpendicular from P to the x-axis (Figure 41). We read the

P

O

R: unit circle

FIGURE 40 The trigonometric functions of a general angle θ are defined in terms of x, y, and r.

FIGURE 41 The acute reference triangle for an angle θ.

magnitudes of x and y from the triangle's sides. The signs of x and y are determined by the quadrant in which the triangle lies. O,

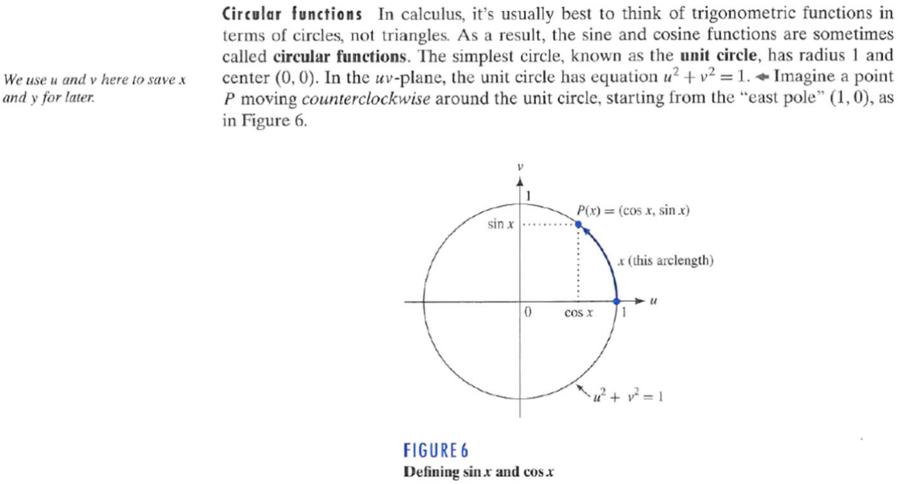
Fig. 3 Coding sample. (Thomas et al. 2001, pp. 44–45)

Trigonometric Functions

We found two distinct conceptions of trigonometric functions. First, the trigonometric functions of sine and cosine are obtained as a direct read (O) from the x- and y-coordinates of the point that is making a particular angle on the unit circle (R, see Fig. 4a). Second, trigonometric functions result from identifying ratios between the sides of a right triangle (O, R, see Fig. 4b).

The unit circle approach seems to be preferred ("it's usually best to think of trigonometric functions in terms of circles, not triangles") as it fits better the real world situations that involve periodicity. When finding values for the sine of angles (P), the unit circle (R) helps make visible two things:

- The angles $\pi/4$ to $9\pi/4$ are different: the second angle can be obtained from the first angle by a full rotation of a point on the unit circle (O), and
- The value of the function is the same (Σ): Because the end point *is* the same, the value of the sine for these two different angles must be the same.

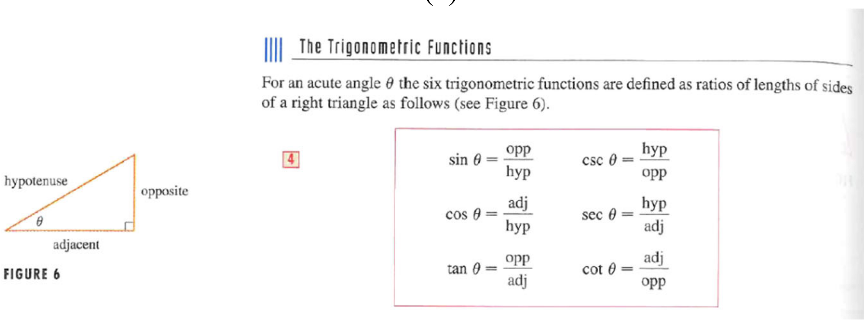


Circular functions In calculus, it's usually best to think of trigonometric functions in terms of circles, not triangles. As a result, the sine and cosine functions are sometimes called **circular functions**. The simplest circle, known as the **unit circle**, has radius 1 and center (0, 0). In the uv-plane, the unit circle has equation $u^2 + v^2 = 1$. ← Imagine a point P moving *counterclockwise* around the unit circle, starting from the "east pole" (1, 0), as in Figure 6.

We use u and v here to save x and y for later.

FIGURE 6
Defining sin x and cos x

(a)



The Trigonometric Functions

For an acute angle θ the six trigonometric functions are defined as ratios of lengths of sides of a right triangle as follows (see Figure 6).

FIGURE 6

4

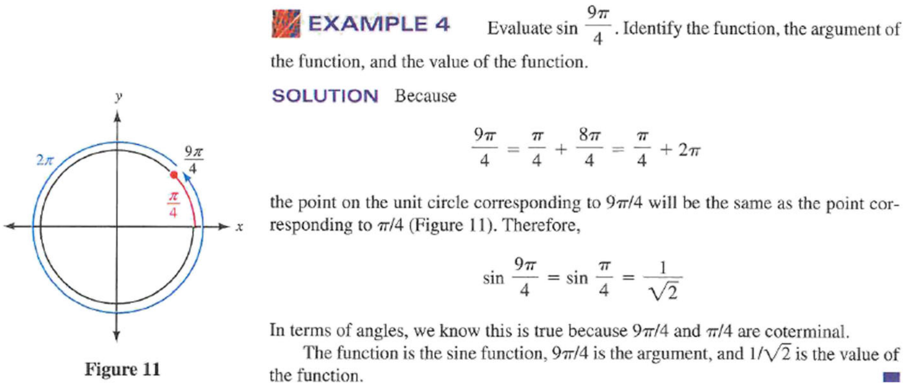
$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

(b)

Fig. 4 **a** Definition of trigonometric functions as coordinates of points on the unit circle (Ostebee and Zorn 2002, p. 30), **b** Definition of trigonometric functions as ratios (Stewart 2012, p. A26)



EXAMPLE 4 Evaluate $\sin \frac{9\pi}{4}$. Identify the function, the argument of the function, and the value of the function.

SOLUTION Because

$$\frac{9\pi}{4} = \frac{\pi}{4} + \frac{8\pi}{4} = \frac{\pi}{4} + 2\pi$$

the point on the unit circle corresponding to $9\pi/4$ will be the same as the point corresponding to $\pi/4$ (Figure 11). Therefore,

$$\sin \frac{9\pi}{4} = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

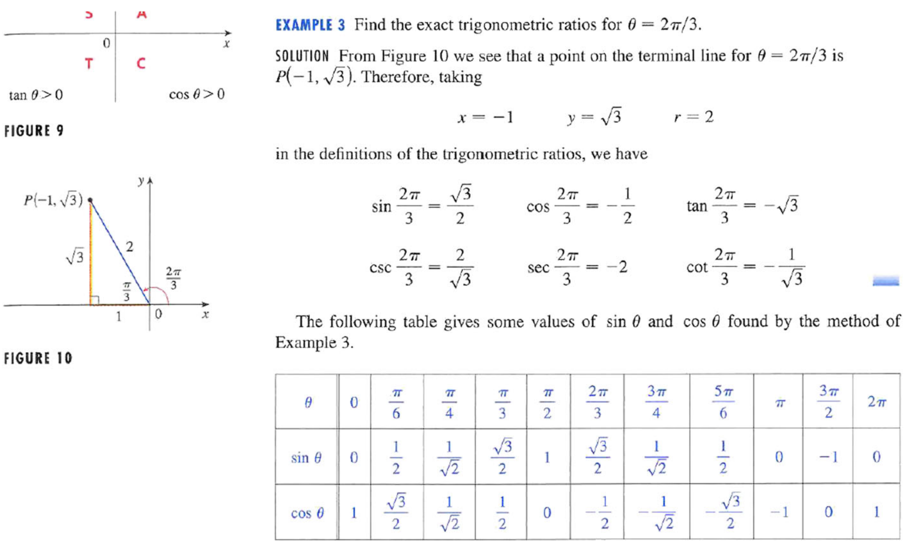
In terms of angles, we know this is true because $9\pi/4$ and $\pi/4$ are coterminal.

The function is the sine function, $9\pi/4$ is the argument, and $1/\sqrt{2}$ is the value of the function.

Fig. 5 Evaluating sine of angles (McKeague and Turner 2008, p. 134)

In contrast, when using the ray definition of angles, the representation for the sine of the angles (i.e., sin(9π /4) and sin(π/4)) would be the same, but the triangle would need to be oriented differently. Textbooks manage this problem by situating the triangle on a Cartesian plane.

This is illustrated in Fig. 5 with angles 2π/3 and π/3. The problem of finding the values of specific trigonometric functions or ratios demands different approaches depending on whether one is using a triangle representation or a circle representation. Placing the triangle on the Cartesian plane, as in Fig. 6, seems arbitrary, one can identify the lengths of the segments using the Pythagorean theorem. It is unclear why one would immediately recognize the point (-1, √3) as a terminal point for the 2π/3 angle. How these decisions are made is not explicitly addressed by any of the textbooks.



EXAMPLE 3 Find the exact trigonometric ratios for $\theta = 2\pi/3$.

SOLUTION From Figure 10 we see that a point on the terminal line for $\theta = 2\pi/3$ is $P(-1, \sqrt{3})$. Therefore, taking

$$x = -1 \qquad y = \sqrt{3} \qquad r = 2$$

in the definitions of the trigonometric ratios, we have

$$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2} \qquad \cos\frac{2\pi}{3} = -\frac{1}{2} \qquad \tan\frac{2\pi}{3} = -\sqrt{3}$$

$$\csc\frac{2\pi}{3} = \frac{2}{\sqrt{3}} \qquad \sec\frac{2\pi}{3} = -2 \qquad \cot\frac{2\pi}{3} = -\frac{1}{\sqrt{3}}$$

The following table gives some values of $\sin\theta$ and $\cos\theta$ found by the method of Example 3.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	0	1

Fig. 6 Evaluating sine of angles. The triangle for identifying ratios for 2π/3 is the triangle used to identify ratios for π/3. The triangle is however reflected over the y-axis (Stewart 2012, p. A27)

EXAMPLE 14: The sine function does not have an inverse, since the horizontal line $y = 1$ passes through the graph of the sine function in infinitely many places. To resolve this problem, we restrict the domain of the sine function so that the resulting function has an inverse. (See Figure 10.)

We let Sin be the function defined by $\mathrm{Sin}(x) = \sin(x), -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. The Sin function has an inverse Sin^{-1}, which is also called the Arcsine function. Thus,

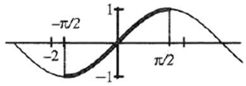


Figure 10. *A portion of the sine function with an inverse.*

$$\sin(\mathrm{Sin}^{-1}(x)) = \sin(\mathrm{Arcsin}(x)) = x.$$

Fig. 7 Defining an injective trigonometric function and its inverse (Zenor et al. 1999, p. 79)

Inverse Trigonometric Functions

Defining inverse trigonometric functions (P) is an interesting problem on its own, and it arises because regular trigonometric functions are not injective (1-to-1) on their domain. The definition of inverse sine requires restricting the domain of sine to ([–π/2, π/2]). We identified two approaches for defining the inverse sine. In one approach authors create a new invertible function called "Sine" which is sine restricted from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ (O, see Fig. 7) and that in one case, appealed to a new representation, the Cartesian plane for the representation of the sine.

This is the approach taken by (Spivak 1976) who also uses *arcsin* exclusively to refer to the inverse of this function, indicating that $\sin^{-1}$ is not used because "arcsine is not the inverse of sin but of the restricted …Sin" (p. 307). Apostol (1967) emphasizes that one is working with a "new" function (p. 253) and names arcsine as the inverse of the new injective (restricted) function. Notably, with this approach finding y such that $\sin^{-1}x = y$ is not a valid problem, as the function $\sin^{-1}$ does not exist.

In the second approach authors comment on the need to restrict the domain of the function so that "for each value of x between -1 and 1, there will be one and exactly one value of y" (McKeague and Turner 2008, p. 240) and describe the process as "surgery" (Ostebee and Zorn 2002, p. 186) and the selection of the interval "natural" (p. 186) or as "universally agreed on by mathematicians" (Hungerford 1997, p. 528). There is no attempt to use a specific notation to differentiate the functions even though they may call them "new" (Thomas et al. 2001, p. 50).

Hughes-Hallet et al.'s (2008) example, propose the problem in Fig. 8.

Having multiple solutions for the equation is what makes defining the inverse function problematic. The authors propose that the solutions that are

On occasion, you may need to find a number with a given sine. For example, you might want to find x such that

$$\sin x = 0$$

or such that

$$\sin x = 0.3.$$

The first of these equations has solutions $x = 0, \pm\pi, \pm 2\pi, \ldots$. The second equation also has infinitely many solutions. Using a calculator, we get

$$x \approx 0.305, 2.84, 0.305 \pm 2\pi, 2.84 \pm 2\pi, \ldots.$$

Fig. 8 Finding a number with a given sine. (Hughes-Hallett et al. 2008, p. 33)

Properties of Inverse Sine ▶

$$\sin^{-1}(\sin u) = u \quad \text{if} \quad -\frac{\pi}{2} \le u \le \frac{\pi}{2}$$

$$\sin(\sin^{-1} v) = v \quad \text{if} \quad -1 \le v \le 1$$

EXAMPLE 4 Since $\sin \pi/6 = 1/2$, we see that

$$\sin^{-1}(\sin \pi/6) = \sin^{-1}\left(\frac{1}{2}\right) = \pi/6$$

because $\pi/6$ is between $-\pi/2$ and $\pi/2$. On the other hand, $\sin 5\pi/6$ is also 1/2 so an expression such as $\sin\ ^{1}(\sin 5\pi/6)$ *is* well defined. But

$$\sin^{-1}\left(\sin \frac{5\pi}{6}\right) = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}, \qquad \textit{not} \quad \frac{5\pi}{6}.$$

The identity in the box is valid only when u is between $-\pi/2$ and $\pi/2$. ■

Fig. 9 Finding the inverse function of an angle outside of the range of the sin function. (Hungerford 1997, p. 530)

within the interval [−π/2 , π/2] are the "preferred" solutions (O, p. 34), then proceed to define arcsine as the function that provides the preferred solution. Thus, with this approach it is possible to raise the question of finding the many values for which sin $x = ½$ (e.g., π/6, 5 π/6, etc.) requiring an additional operation that adds all the other possible solutions because of the periodic nature of the function. Establishing the properties of inverse functions is now a different problem as indicated in Example 4 in Fig. 9.

Because the domain of sin(x) is all real numbers, it is possible to find the value of sin(5π/6). In this case a symbolic representation (R, the value, π/6, belongs to the corresponding interval) is used to map (O) the value into the accepted range. But clearly π /6 and 5π /6 are not equal although their sin is the same (Σ), thus to make the first statement in the colored box true one cannot consider 5π /6 as a valid input (O, Σ). In all these definitions the representation used is symbolic. There are no explicit references to a triangle or a circle representation.

In Fig. 10 the statement about angles that "come from" certain quadrants (Σ) alerts that any other angles can't be included, and it involves both a circle and a triangle representation. The language "come from" is reminiscent of Elizabeth's "We don't get this." Notably however, the circles used *are not* unit circles.

The restrictions leads into a situation in which circles are not unit circles anymore (one has radius 2 the other has radius √2) and triangles are oriented in very convenient ways. The process of deciding how to construct the triangles or how to over impose the circles is not addressed (is it perhaps left to the teacher?). A navigation problem in which inverse functions are needed appeals only to the triangle representations and avoid the issue of considering angles outside of the expected domain (see Fig. 11).

The solution of the flight problem is done using the symbolic representation, with aid of a diagram that includes a triangle with various labels. The actual triangles used are not depicted, however, and the verification that the argument for the inverse

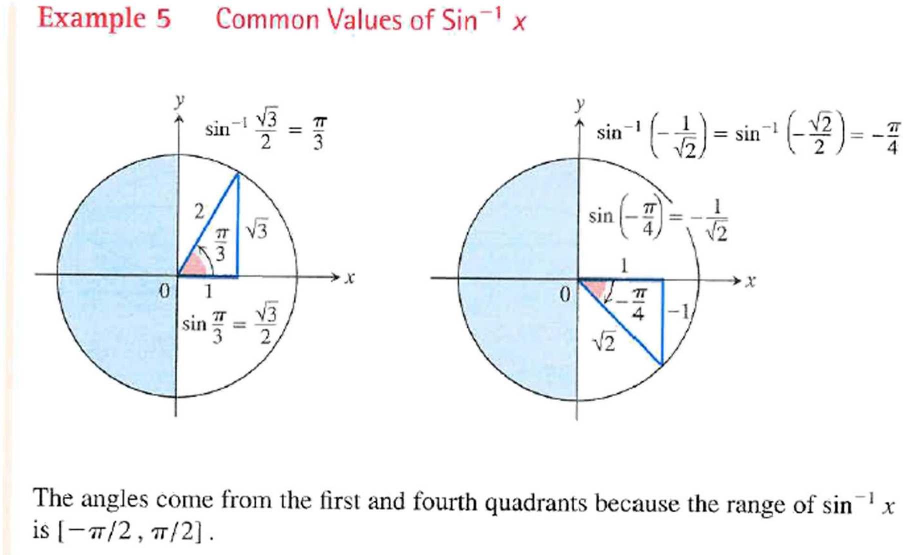


Fig. 10 Angles "come" from the first and fourth quadrant (Thomas et al. 2001)

function is absent (although by inspection the reader can verify that both arguments are numbers that are less than 1). The caveat that the drawing is not to scale is useful, as the angle c is clearly larger than 15°.

The analysis presented reveals different ways in which angles, trigonometric functions, and inverse trigonometric functions are conceptualized in college trigonometry, pre-calculus, and calculus textbooks. These two main conceptions observed are synthesized in Table 1; the number of textbooks that exhibited each conception is given in Table 2.

As is apparent from Table 2, the preferred conception of angles in these textbooks is one compatible with a "dynamic" conception, as a rotation represented on the unit circle, with only two textbooks addressing a "static" conception, which was almost immediately discounted. The definition of angles provided referred directly to their measurement in radians. Such definition is then used to define the trigonometric functions. Cosine and sine of the angle θ are defined as the coordinates on a Cartesian plane of the point that has rotated an arc subtending θ over a unit circle centered in the

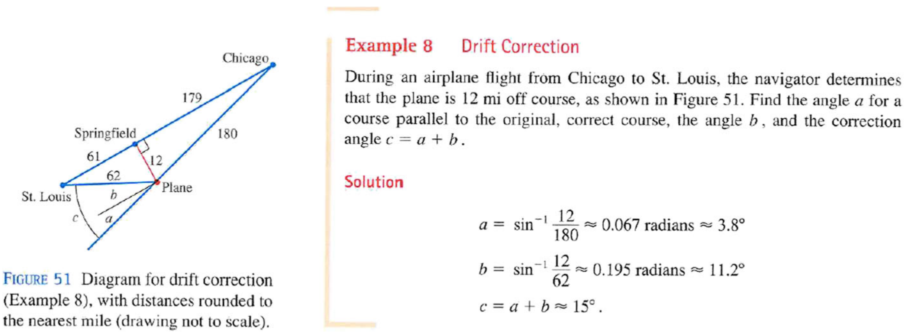


Fig. 11 A problem requiring the computation of inverse sin. (Thomas et al. 2001, p. 54)

Table 1 Conceptions of angle, trigonometric functions, and inverse trigonometric functions in the textbooks

Elements of the conception	Conception 1	Conception 2
Problem 1	*What is an angle?*	
Representations	• Line and pivot point	• Rays
Operations	• Rotation around the point located on the x-axis	• Intersection of two rays
Control	• Not included	• Not included
Problem 2	*What is sin of an angle?*	
Representations	• Unit circle	• Right triangle
Operations	• Identify coordinates of points on the unit circle. Sin of the angle is the y coordinate of the point.	• Identify the length of the sides of the triangle • Sin of the angle is the ratio of the opposite side to hypotenuse of the triangle
Control	• The representation for $\sin(\pi/4)$ and $\sin(9\pi/4)$ are different (another rotation) but they have the same value	• The representation for $\sin(2\pi/3)$ and $\sin(\pi/3)$ are the same (triangle is oriented differently)
Problem 3	*How is* $\sin^{-1}$ *defined?*	
Representations	• Unit circle	• Triangle
	• Cartesian plane	• Cartesian plane
Operations	• Define $y=\sin^{-1}x$ as satisfying two conditions: $x = \sin y$ and $\pi/2 \le y \le \pi/2$	• Define a new function Sin on the interval $[-\pi/2, \pi/2]$, Sin^{-1} is its inverse
Control	• Accept the existence, but not inclusion, of possible out-of-range solutions as consequence of rotation	• Out-of-range solutions do not exist.

origin. This definition is "preferred" and as indicated by the table, favored by seven of the eight textbooks that included it, with only the trigonometry textbook including the two conceptions. Finally, four textbooks define sine^{-1} as a the inverse of a new injective function that avoids a circle representation, whereas the rest only state the need to restricts the domain of the sine function in order to find an inverse, and suggest that this decision is arbitrary.

Table 2 Number of textbooks exhibiting each conception

	Conception 1	Conception 2	Topic not Discussed
Angle	7	2[a, b]	3
Trigonometric functions	7	2[c, d]	2
Inverse trigonometric functions	6	4	

[a] Two textbooks (McKeague and Turner 2008; Spivak 1976) included the definition of angles as rays, ("An angle is simply the union of two half-lines with a common initial point," Spivak, p. 300) in addition to the definition as rotation, but quickly advised not to "[scrutinize it] too carefully, as [it] shall soon be replaced by the formal [definition] which we really intend to use" (Spivak, p. 300)

[b] Three textbooks (Larson and Hostetler 2007; Ostebee and Zorn 2002; Zenor et al. 1999) did not include this topic

[c] Only the trigonometry textbook (McKeague and Turner 2008) included both conceptions

[d] Two textbooks (Larson and Hostetler 2007; Zenor et al. 1999) did not include this topic

Discussion

The analysis of expository text and examples in the 10 textbooks identifies features of the conceptions of angles, trigonometric functions, and inverse trigonometric functions that might create a space for cumbersome justifications in answering problems that call for these conceptions. The two conceptualizations of angles as either static (via triangles) or dynamic (via unit circles) may help us notice potential pitfalls when working on problems that ask for the inverse trigonometric function of a value that is in the range of a trigonometric function. Defining a new function might be a good solution but it strips the original function of its periodicity, a main reason to prefer a unit circle trigonometry. The approach that acknowledges periodicity results in cumbersome language of restriction of angles one can get and angles that one can't. Elizabeth's original problem, "Find $\sin^{-1}(\frac{\sqrt{2}}{2})$," is unnaturally forced into having a single solution and requires a series of steps that invoke circle and ratio conceptualizations of the trigonometric function that are difficult to manage. Had her example involved a different argument, such as $\sqrt{3}/2$, the representation would probably not be a unit circle anymore. Textbooks do not make these transitions explicit (see Fig. 10). Stacey and Vincent (2009) claimed that the unit circle method was "fraught with multiple opportunities for mistakes" (p. 327), we agree, but add that this approach actually respects the periodic nature of the functions, which is only apparent within circle trigonometry.

Our position is not necessarily that each of the approaches seen is unwarranted or problematic. Rather that more effort needs to be made to make explicit how the various elements of the conceptions result in different classes of problems. The problems that each of these approaches solve are of different nature by virtue of the different operations, representations, and means of controls available to deal with them. Navigation problems (that might be approached with ratios) are quite different from oscillation problems (that might be approached with circle trigonometry).

In this study we found that textbook exposition and examples as related to angles can potentially originate different conceptions of angles, and that features of these conceptions can be traced into the definition of trigonometric functions, and that the definition of inverse trigonometric functions create difficulties that result either in arbitrary steps or in definitions that avoid periodicity. A more thorough treatment of angles, and a careful explanation of how to bridge these conceptions might be useful to include, either as part of the exposition, or within examples in the textbooks that address trigonometric functions.

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