Int. J. Res. Undergrad. Math. Ed. (2017) 3:225–242
DOI 10.1007/s40753-016-0041-2

Structural Reasoning

Guershon Harel[1] · **Osvaldo Soto**[2]

Published online: 23 August 2016

Abstract In this paper we give a general definition of structural reasoning, followed by a typology that instantiates the definition in mathematical practices. Five major categories of structural reasoning, some with subcategories, are offered: (a) pattern generalization, (b) reduction of an unfamiliar structure into a familiar one, (c) recognizing and operating with structure in thought, (d) epistemological justification and (e) reasoning in term of general structures. The first four categories of this typology are illustrated by field-based events of secondary mathematics teachers' mathematical behaviors. The potential contribution of the typology to research in mathematics education is that its elements can be used as indicators and conceptual labels of empirical observations. The potential contribution of the illustrative field-based events is that they can serve as initial models for structural reasoning phenomena to be examined by researchers.

Keywords Structure · Structural reasoning · Intellectual need · Epistemological justification · Structural method · DNR-based instruction

Background

Although the term "structural reasoning" is suggestive, it is not easy to define due to its numerous manifestations in mathematical practice. For some, it is synonymous with formal deductive reasoning. For example, Küchemann and Hoyles (2009) state: "A

✉ Guershon Harel
harel@math.ucsd.edu

Osvaldo Soto
osoto@ucsd.edu

[1] University of California, San Diego, 9500 Gilman Drive, Applied Physics and Mathematics BLDG, Room 7420, La Jolla, CA 92093-0112, USA

[2] University of California, San Diego, 9500 Gilman Drive, Social Science Research BLDG, Mail Code 0036, La Jolla, CA 92093-0036, USA

major challenge in mathematics education is to develop students' abilities to reason mathematically, that is to make inferences and deductions from a basis of mathematical structures, henceforth referred to as structural reasoning …" For others, it is a way of thinking related to but different from deductive reasoning. For example, the Common Core State Standards in Mathematics (CCSSM 2010) state eight foundational mathematical practices. One of these practices concerns deductive reasoning ("Construct viable arguments") and the other structural reasoning ("Look for and make use of structure"). Still for others, structural reasoning pertains to a meta-cognitive activity of re-organizing acquired knowledge into structured schemata (Harel 2013b). There are also debates on structural reasoning in mathematics in the philosophy of mathematics, in relation to "mathematical understanding" (Reck 2009) and "mathematical explanation" (Steiner 1978).

The launch of the CCSSM and the national debate on their potential efficacy led the first author of this paper to examine cognitive and instructional aspects of the CCSSM, focusing in particular on their eight constituent mathematical practices (Harel 2013a, 2014). As part of this work, Harel (2013a) offered a definition of structural reasoning along with a limited, preliminary typology of its manifestation in mathematical practice. In this paper, we expand on this definition and offer a more extensive typology along with field-based illustrative events. The earlier preliminary typology was revised and subsumed under the new typology.

The definition of structural reasoning offered in Harel (2013a) was based on the common meaning of its source noun, "structure". Structure, according to the American Heritage dictionary, can be thought of as "something made up of a number of parts that are held or put together in a particular way". In mathematics the way these "parts" are held together is not restricted to physical or mental spatial configurations. Rather, they can be relation(s) one conceives among different objects. For example, an algebraic expression, say an equation, is of a particular structure when it is viewed as a string of symbols put together in a particular way to convey a particular meaning. Likewise, different word problems containing similar words and phrases may appear to an individual as of a similar form, whereby forming a common textual structure, even though their story lines differ (what is known as "problem isomorph," Hayes and Simon 1977; Gick and Holyoak 1983; Reed 1989). One may also recognize that operations across different domains (e.g., multiplication of matrices and multiplication of integers) share common properties. With this image of "structure", Harel (2013a) offered the following definition, slightly reworded here:

> *Structural reasoning* is *a combined ability to: (a) look for structures, (b) recognize structures, (c) probe into structures, (d) act upon structures, and (e) reason in terms of general structures.*

In this paper, this definition was expanded to include one additional aspect of structural reasoning: *(f): the ability to see (be aware of) how a piece of knowledge acquired resolves a perturbation experienced*, what is dubbed in Harel (2013b), epistemological justification. Of importance is the recognition that this addition is related to Leron's (1983, 1985) notion of structural proof. Notice that there is no specific reference to mathematics in this (expanded) definition. Analyses of actual mathematical behaviors illustrating the various aspects of this definition led to a typology of structural reasoning as it may manifest itself, at least partially, in *mathematical* practice.

We have here three nested levels of specifications in the process of articulating structural reasoning. In the first level, structural reasoning is merely an act of relating (i.e., putting parts together in a particular way). In the second level, this broad characterization is narrowed into six general abilities (i.e., abilities a–f above). In the third, and final, level, these abilities are instantiated through specific mathematical practices illustrated by a series of behavioral events. By focusing on these six abilities, we lay no claim to completeness. We hope that our analysis will generate interest for further exploration of structural reasoning, which may, in turn, lead to revision or expansion of our typology.

While most of the behavioral events we discuss in this paper were taken from a teaching experiment with secondary teachers, this study is not an empirical study. Rather, it is an analytic-theoretical study, accompanied with illustrative events taken from a large body of empirical data. The potential importance of this typology to research in mathematics education is two-fold. First, the various categories of the classification can potentially be used as basic units of data analysis in the sense of Corbin and Strauss' (1990) canons and procedures of grounded theory; namely, the elements can serve as indicators and conceptual labels of empirical observations. Second, and of equal importance, the field-based events can serve as initial models, concept images, for structural reasoning phenomena, to be substantiated through accepted methodologies.

Structural Reasoning: A Theoretical Perspective

In this section, we discuss categories of structural reasoning, which instantiate the various abilities constituting the above definition of this way of thinking. The categories are not in one-to-one correspondence to these abilities; rather, they are characterized by different combinations of the abilities. Nor are the categories mutually exclusive, as is unavoidably the case in cognitive studies.

Pattern Generalization and Reduction of Unfamiliar Structure into a Familiar One

Since structural reasoning (as any other way of thinking) is developmental in nature, it is expected that these abilities evolve gradually with individuals and communities through various social and cultural interventions. Recognizing that the formation of students' structural reasoning must begin long before they enter college, the CCSSM (2010) included it as a major expectation. Specifically, abilities (a), (b), and (d) in our definition are akin to those called for by one of the eight CCSSM Practice Standards: "*Look* for and *make use of structure*". The CCSSM give a few examples to convey the intention of this standard:

> Mathematically proficient students look closely to *discern a pattern* or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according

> to how many sides the shapes have. Later, students will see 7 × 8 equals the well-remembered 7 × 5 + 7 × 3, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2 × 7 and the 9 as 2 + 7. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

While these are illuminating examples, they do not capture the full scope of structural reasoning in mathematics. Several scholars have investigated various elements involved in structural reasoning. Before we discuss their studies, let us extract the characteristics of structural reasoning intended by this CCSSM (2010) statement. There are three:

(a) *Pattern generalization*. The examples the CCSSM gives for this type of structural reasoning are limited to empirical generalization.
(b) *Construction of desirable structures*. The drawing of an auxiliary line in a geometric figure is an example.
(c) *Formation of conceptual entities*. Here the examples are limited to entities formed in working with algebraic expressions for the purpose of investigating or claiming a certain property of the expression.

The last two aspects of structural reasoning may be subsumed under the more general characteristic of *reducing an unfamiliar structure into a familiar one.* For the ultimate purpose in these forms of reasoning is to identify or create a familiar structure in which, or with which, a mathematical investigation can be pursued.

This form of structural reasoning was the focus of Hoch and Dreyfus' (2004) study. They describe *structure sense,* as it applies to high-school algebra, as

> … the [abilities] to see an algebraic expression or sentence as an entity, recognize an algebraic expression or sentence as a previously met structure, divide an entity into sub-structures, recognize mutual connections between structures, recognize which manipulations it is possible to perform, and recognize which manipulations it is useful to perform.

It is reasonable to assume that structure sense, as described by Hoch and Dreyfus (2004), is what students should acquire in school to develop more advanced forms of structural reasoning. Alas, their findings are that structure sense is weak among eleventh-grade students enrolled in intermediate to advanced mathematics tracks. They do not explicitly address the cognitive source of students' weaknesses, however. Kirshner and Awtry (2004) report an interesting finding related to structural reasoning among students as they are introduced to algebra, which might account for these weaknesses. They report that students' errors in algebra symbol manipulation stem from "visual salience of a rule, rather than just from its declarative complexity" (p. 231). The nonrepresentational view of cognition Kirshner and Awtry hold free them from the need to define the notion of "visual salience" in precise terms and definitive algorithms; in fact, such a definition is arguably not feasible. However, their examples of "visually salient rules" (e.g., $(xy)^2 = x^2y^2$) versus "non-visually-salient rules" (e.g., $x^2 - y^2 = (x - y)(x + y)$) are sufficient to provide an image for their intention. Thus, the abilities depicted by Hoch and Dreyfus may be accounted for by the fact that their tasks lack visual salience in the eyes of their subjects.

Back to the first characteristic of structural reasoning, pattern generalization, it is crucial to distinguish between two ways of thinking associated with this characteristic, *result pattern generalization* (RPG) and *process pattern generalization* (PPG) (Harel 2001). For example, observing that 2 is an upper bound for the sequence $\sqrt{2}$, $\sqrt{2+\sqrt{2}}$, $\sqrt{2+\sqrt{2+\sqrt{2}}}$, … because the value checks for the first several terms, is RPG. On the other hand, it is PPG to deduce that there is an invariant relationship, $a_{n+1} = \sqrt{a_n + 2}$, between any two neighboring terms of the sequence; and therefore since $\sqrt{2} < 2$, all the terms of the sequence are bounded by 2. Thus, PPG is a way of thinking in which one attends to regularity in the process, though it might be initiated by regularity in the result, whereas RPG is a way of thinking in which one attends solely to regularity in the result—obtained by substitution of numbers, for instance.

In sum, this discussion suggests that the CCSSM's (2010) practice "attention to structure" can be interpreted in terms of two categories of structural reasoning, each with two subcategories:

1. *Pattern generalization*:
   - (a) *Result Pattern Generalization (RPG)*
   - (b) *Process Pattern Generalization (RPG)*
2. *Reduction of an Unfamiliar Structure into a Familiar One*
   - (a) *Construction of Desirable Structures* and
   - (b) *Formation of Conceptual Entities*

Recognizing and Operating with Structure in Thought

Piaget's (1971) notion of anticipation implies a mathematical ability whose presence or absence may define, in part, mathematical maturity. This is the ability to carry out in thought algebraic operations without actually performing them, what Dubinsky (1991) called "process conception". The following example illustrates this ability or the absence thereof. In the process of carrying out a particular task concerning the cubic formula, students had to determine the sign of the expressions,

$$u = \sqrt[3]{\frac{Q}{2} + \sqrt{\left(\frac{Q}{2}\right)^2 - P^3}} \text{ and } v = \sqrt[3]{\frac{Q}{2} - \sqrt{\left(\frac{Q}{2}\right)^2 - P^3}},$$

knowing that P and Q are positive and satisfy the condition,

$$\left(\frac{Q}{2}\right)^2 - p^3 \geq 0.$$

Some of these students were able to both form a nested structure and make a logical conclusion from it in thought. They expressed their reasoning without accompanying it with written symbolic representation. Through their utterances, it was evident that they mapped u and v into the structures $\sqrt[3]{a+\sqrt{c}}$ and $\sqrt[3]{a-\sqrt{c}}$, respectively, where $c = a^2 - b \geq 0$. In doing so, it was stipulated, these students conceived several expressions (e.g., $Q/2$ and $(Q/2)^2 - P^3$) as conceptual entities, objects in the sense of APOS theory (Dubinsky 1991). Furthermore, reasoning quantitatively they concluded that a must be greater than $\sqrt{c}$, and therefore u and v must be non-negative. Other students were unable to follow this sequence of arguments and needed the presence of actual symbolic representations and manipulations to arrive at the same conclusion.

The ability to carry out operations in thought without performing is indispensable throughout the various domains in mathematics, often without an alternative. The following example may suffice to illustrate its indispensability. Consider the concept of "span", the set of all linear combinations of a given set of vectors in a finite-dimension vector space. One must imagine forming all the possible linear combinations of, say, the vectors $v_1, v_2, \ldots, v_n$, creating, in turn, a set comprising them, i.e., span $(v_1, v_2, \ldots, v_n)$. There is no way to actually list these combinations. One must carry out their construction and maintain it in thought, while solving a related problem for example. Anyone who taught this concept would likely recognize the difficulties students encounter in constructing it.

A combination of structural reasoning abilities is present in this category: look for structures, recognize structures, probe into structures, and act upon structures. Furthermore, the event described here includes an instance of conceptual entity formation, a characteristic of one of the aforementioned categories.

Epistemological Justification

By the constructivist theory of learning, for any piece of knowledge K possessed by an individual or community, there exists a problematic situation S out of which K arose. S, prior to the construction of K, is referred to as an individual's *intellectual need:* S is the need to reach equilibrium by acquiring a new piece of knowledge (Harel 2013b). If an individual's perturbational state S has led her or him to construct a piece of knowledge K *and* the individual has seen how K resolves S, then we say that the individual possesses an *epistemological justification for the creation of* K (ibid). Thus, epistemological justifications concern the genesis of knowledge, the perceived reasons for its birth *in the eyes of the learner*. As such, it may be considered as meta-cognitive knowledge. Consider, for example, the following scenario: A student with knowledge of abstract algebra might consider the *ideal of polynomials annihilating a given operator* $\boldsymbol{T}$ *over an n-dimensional vector space,* and recognize that such an ideal is not empty, since it contains an annihilator polynomial of degree $\boldsymbol{n^2}$. In doing so, a problematic situation S consisting of the following questions may occur: What is a generator for this ideal? While the degree of such a polynomial is definitely not greater than n^2, can it be n? Is there a polynomial of degree n that annihilates T? If the student is aware of a particular piece of knowledge that resolves this problematic situation for her or him, then we say that that knowledge is an epistemological justification for that student. Cayley-Hamilton Theorem ("Any linear operator on a finite-dimensional vector space is annihilated by its characteristic polynomial") might be that knowledge.

This example deals with the emergence of *K* in the form of a conjecture, i.e., Cayley-Hamilton Theorem before its proof. In other cases, *K* might be a definition emerging from a particular need *S*. For example, the concept of linear independence may emerge as a resolution for the question, when does Gaussian Elimination lead to a loss of equations (i.e., zero equations in a system obtained through the application of elementary operations). Still in other cases, *K* might be an axiom. For example, the incidence axioms in Euclidean geometry may emerge as a need to communicate humans' spatial imageries (See Harel 2014).

Epistemological justification, unlike the other characteristics of structural reasoning, necessarily involves a meta-cognitive process, in that one is aware of how an acquired piece of knowledge *K* resolves an experienced intellectual perturbation *S*. It involves awareness of a causal relation between two crucial experiences in knowledge construction. This is particularly the case in reading and comprehending proofs. Leron's (1983, 1985) idea of *structural proof* is akin to this aspect of epistemological justification. The pedagogical goal of structural proof, as stated by Leron, is to make the learner *aware* of the ideas hidden in the traditional, linear presentation of proofs. The source "structure" here intends to convey the essential act of the method: the process of *restructuring* a linear presentation of a proof, as is commonly presented in textbooks or research papers, to a multi-dimensional presentation that conveys possible thought processes involved in the construction of a proof; how one comes up with a piece of knowledge *K* to resolve a problematic situation *S*.

Reasoning in Terms of General Structures

We conclude our typology with a characteristic of structural reasoning that we have neither expected nor observed in our teaching experiment, but it has been discussed in the literature (e.g., Dorier and Sierpinska 2001). It refers to one's ability to *reason in terms of general structures, not only in terms of their instances*. With this ability one reasons in terms of abstract mathematical structures, such as "group," "ring," and "field." Two combined abilities are surmised to be cognitive prerequisites to this way of thinking: *reasoning in terms of conceptual entities* and *reasoning in terms of operations on conceptual entities* (Harel 2013a). The basis for this hypothesis is the work of Tall and Vinner (1981) on concept image and Dubinsky and McDonald (2001) APOS theory. One's conceptualization of a general mathematical structure rests on the nature of the concept image (Tall and Vinner 1981) one has of that structure. Initially, a general structure is abstracted by a person from concept images concrete to that person, whereby they become *instance structures* for her or him. A necessary condition for this to happen, according to Dubinsky and McDonald (2001) is that these objects are conceived by the person as conceptual entities (see also Greeno 1983). For example, if a teacher illustrates the meaning of the theorem in the vector space of functions but the students have not yet encapsulated function as a conceptual entity, it is unlikely that the students will understand the teacher's illustration.

Dorier and Sierpinska (2001) better explicate the underlying epistemological processes that are brought to bear on one's ability to reason in terms of general structures: These are (a) the ability to recognize similarities between objects, tools and methods, which is what brings one to unify and generalize concepts; and (b) the ability to make the unifying and generalizing concepts explicit as objects, which is what induces a reorganization of old competencies and elements of knowledge. As an example of the first ability, consider the following three equations, offered by Dorier (2000):

1. $xf(x) - f''(x) = x^2 + 1$
2. $u_{n+2} - 3u_{n+1} = u_n + 1$
3. $\begin{cases} 3x + 4y \qquad\quad - w = 0 \\ \qquad\quad 2y + 5z + w = 2 \end{cases}$

Dorier points out that while "each of these equations has to be solved with specific tools, with respect to very distinct fields of mathematics, …, in all three cases, it is possible to state that the set of solutions is *an affine two-dimensional linear manifold*" (pp. 3–4). As an example of the second ability, based on Dorier (2000), Dorier and Sierpinska (ibid) discuss the genesis of the theory of vector space, which began to emerge in the late nineteenth century, but was institutionalized after 1930. It refers to formalization of the underlying structure of linear algebra through axiomatization, by which linear problems were reconstructed using the concepts and tools of a new axiomatic theory.

Summary

Five major categories of structural reasoning, some with subcategories, are offered in this section (Fig. 1): (a) *pattern generalization*, (b) *reduction of an unfamiliar structure into a familiar one,* (c) *recognizing and operating with structure in thought,* (d) *epistemological justification* and (e) *reasoning in term of general structures*.

The typology instantiates in mathematical contexts a broad definition of structural reasoning—as a cluster of six abilities: *(a) the ability to look for structures, (b) the ability to recognize structures, (c) the ability to probe into structures, (d) the ability to act upon structures, (e) the ability to reason in terms of general structures, and (f) the ability to form epistemological justifications.*

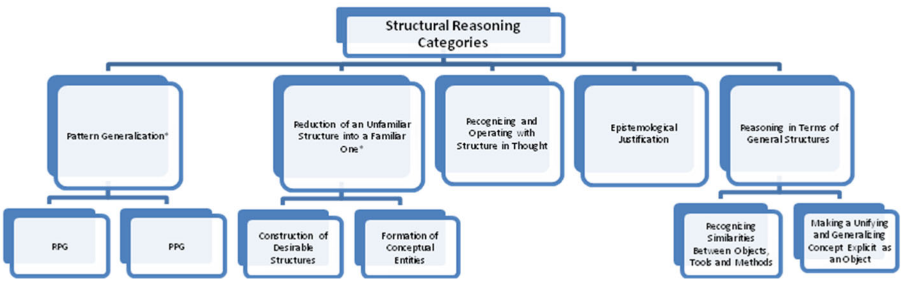

Fig. 1 A Typology of structural reasoning

With the exception of the *reasoning in terms of general structure categories,* each of the other subcategories was illustrated with field-based events, as we will see in the next section.

Events of Structural Reasoning

The goal of this section is to illustrate the aspects of structural reasoning discussed earlier through events from a teaching experiment with secondary mathematics teachers. We highlight that these events do not constitute claims that particular instances of structural reasoning actually took place; rather, they are initial models for observations that might be considered occurrences of structural reasoning. A researcher using these models would need additional data in the form of clinical interviews, video recording, and the like, to ascertain their actual presence in learners' mathematical behaviors.

An Investigation into the Quadratic Equation and Graphs of Functions

During an instructional segment focusing on the quadratic equation, the following in-class problem was assigned:

> Problem: *For what values of k does the equation* $(k-1)^2x^2+k=(k^2-1)x$ *have (a) real roots? (b) positive roots? (c) negative roots?*

As expected, upon seeing a problem involving a quadratic equation, teachers would invoke the quadratic formula, and in this particular case would examine the sign of the discriminant; namely, when $(1-k^2)^2-4(k-1)^2k\geq 0$. The teachers' first spontaneous action was to transform the given equation into a standard form, $(k^2-2k+1)x^2+(1-k^2)x+k=0$, which is a manifestation of the reduction of an unfamiliar structure to a familiar one.

Kaitlyn, one of the teacher participants, followed this action by representing the discriminant $(1-k^2)^2-4(k^2-2k+1)k$ and expanding it to the form $k^4-4k^3+6k^2-4k+1$. She then noticed that the coefficients of this polynomial match Pascal's triangle, to conclude that $k^4-4k^3+6k^2-4k+1=(k-1)^4$. What is significant here is that Kaitlyn encountered an impasse in dealing with the polynomial inequality, $k^4-4k^3+6k^2-4k+1\geq 0$, and as a consequence she probed into the structure of the polynomial as it is expressed in its coefficients. This indicates, according to our definition of structural reasoning, the presence of elements of structural reasoning; specifically, those of probing and observing structure.

While Kaitlyn's success in utilizing Pascal's Triangle to restructure the determinant polynomial is notable, it impressed her group mates as highly esoteric and even "accidental". Furthermore, it is significant that Kaitlyn's initial, spontaneous action, as was observed by the teacher-researcher during the small-working group, was to actually expand the determinant polynomial, rather than carry out, in thought, goal-driven actions. To illustrate a contrast to this action by Kaitlyn, consider the following alternative approach to dealing with the same polynomial brought up by another class member.

In essence, the approach consists of observing that an entity, $(k-1)^2$, is common to the two expressions comprising the polynomial, despite the fact that this entity is not explicitly present in the polynomial, as is shown in the following manipulation:

$$\left(1-k^2\right)^2-4(k-1)^2k = (1-k)^2(1+k)^2-4k(k-1)^2 = (k-1)^2\left((1+k)^2-4k\right) = (k-1)^2(k-1)^2 = (k-1)^4$$

Several structural reasoning characteristics are key to this process. First, the observation, in thought, that $(k-1)^2 = (1-k)^2$. Second, the presence of $(k-1)^2$ in $(1-k^2)^2$ was elicited by carrying out the factorization in thought, $(1-k^2)^2 = (1-k)^2(1+k)^2$. Third, it is not that the problem solver first performed this factorization with no attention to the entity, $(k-1)^2$, and then observed its appearance in the two expressions comprising the polynomial. Rather, the solver seemed to look for the entity, $(k-1)^2$, in the expression $(1-k^2)^2$ and derived its presence there through factorization in thought. Fourth, and last, the solver conceived of $(k-1)^2$ as a conceptual entity.

The aspect of structural reasoning involving the formation of conceptual entities is further illustrated by the following event.

Problem: Graph the function

$$y = \frac{\sqrt{\frac{9+x^2}{3x}+2}+\sqrt{\frac{9+x^2}{3x}-2}}{\sqrt{\frac{9+x^2}{3x}+2}-\sqrt{\frac{9+x^2}{3x}-2}}$$

The class as a whole (almost) spontaneously observed that the function consists of the fraction

$$y = \frac{A+B}{A-B}.$$

Most then proceeded along a line similar to the following development.

$$A = \sqrt{\frac{9+x^2}{3x}+2} = \frac{|x+3|}{\sqrt{3x}}, \qquad B = \sqrt{\frac{9+x^2}{3x}-2} = \frac{|x-3|}{\sqrt{3x}},$$
$$A+B = \frac{|x+3|+|x-3|}{\sqrt{3x}}, \quad \text{and} \quad A-B = \frac{|x+3|-|x-3|}{\sqrt{3x}}.$$

And so,

$$y = \frac{A+B}{A-B} = \frac{|x+3|+|x-3|}{|x+3|-|x-3|}.$$

Specifically, by considering the domain of the function, teacher participants transformed the latter function structure into a split-function structure: $y = \frac{3}{x}$ if $0 < x \leq 3$ and $y = \frac{x}{3}$ if $x > 3$, which in turn facilitated the drawing of the function.

Here the need to graph the function seems to have been a driving force for the participants to reduce the given expression into a familiar structure. The following is an example where the need to avoid computation seemed to have been an intellectual motivation to form a structure.

Problem: Solve the following system

$$\begin{cases} u\,v \;=\; -\dfrac{p}{3} \\ u^3+v^3=-q \end{cases}.$$

To solve this system, one arrives through substitution of variable at the system,

$$\begin{cases} v=-\dfrac{p}{3u} \\ 3^3\left(u^3\right)^2+3^3qu^3-p^3=0 \end{cases}.$$

The quadratic equation in the latter system has two solutions:

$$u=\sqrt[3]{-\frac{q}{2}+\sqrt{\left(\frac{p}{3}\right)^3+\left(\frac{q}{2}\right)^2}}\ \text{and}\ u=\sqrt[3]{-\frac{q}{2}-\sqrt{\left(\frac{p}{3}\right)^3+\left(\frac{q}{2}\right)^2}}.$$

An alternative to substituting these values of u in the original system is to utilize the symmetry between u and v in the system to obtain the values of v:

$$v=\sqrt[3]{-\frac{q}{2}+\sqrt{\left(\frac{p}{3}\right)^3+\left(\frac{q}{2}\right)^2}}\ \text{and}\ v=\sqrt[3]{-\frac{q}{2}-\sqrt{\left(\frac{p}{3}\right)^3+\left(\frac{q}{2}\right)^2}}.$$

We see in these three episodes several characteristics of structural reasoning. We began by illustrating how transforming an equation into a standard form exemplifies reducing an unfamiliar structure to a familiar one. Of more interest is the difference between expanding an expression first versus pausing, reflecting on the structure of an expression, and carrying out, in thought, goal-driven actions. While expansion can lead to solutions in many cases, as our illustration pointed out, often this approach is carried out haphazardly. In these examples we also hypothesize potential intellectual needs for constructing structures (i.e., a need to handle an unfamiliar structure, the desire to visualize the graph of a function or the desire to avoid computation). These hypotheses can be useful starting points for research on pedagogical practices aimed at the refinement of structural reasoning.

The following contrasting examples illustrate the distinction between RPG and PPG as forms of structural reasoning.

An Investigation into the Arithmetic Sequences

The goal of promoting structural reasoning among the participants in the teaching experiment took many forms. Here we discuss one of the many in-class problems that aimed specifically at eliciting PPG (process pattern generalization) and contrasting it with RPG (result pattern generalization).

Problem: *Consider the following two arithmetic sequences of sets of positive integers.*

(a) *(1), (2, 3), (4, 5, 6), (7, 8, 9, 10), …*
(b) *(1), (3, 5, 7), (9, 11, 13, 15, 17), …..*
How many terms are there in the n^{th} *set in each sequence?*
What is the sum of the terms in n^{th} *set in each sequence?*

In this particular problem, the teachers began by computing the sum of the terms in the first several sets of the sequence, hoping to observe from these particular cases a pattern that relates the first (or last term) of a respective set to the position of the set in the sequence, and then generalize empirically the required relationship—a typical RPG approach. The following solution by Alden, a teacher participant, is an example.

Alden arranged the data in a table coordinating the position of the set in the sequence with the value of the first term in the set. He did so for the first five sets (first two columns in the Table 1).

Next (in the third column), he decomposed the terms additively in the second column as a sum of *1* and the sum of an arithmetic sequence (e.g., for the set in the 4th position, the first term is *7,* which is the sum of *1* and the sum of the arithmetic sequence *1 + 2 + 3*). From these five cases, Alden generalized that the value of the 1st term of the n^{th} set could be expressed as $1 + [1 + 2 + \ldots + (n - 1)]$ and noted that $1 + 2 + \ldots + (n - 1)$ was an arithmetic series whose sum is $[n(n - 1)]/2$, concluding that the first term in the n^{th} set is $(n^2 - n + 2)/2$. Alden's approach definitely includes elements of attention to structure, as can be seen in the purposefulness of the organization of the five particular cases. However, the way of thinking that governed his structuring activity is empirical, not deductive, in that he generalized from the result of the pattern (RPG), rather than from the process underlying the pattern (PPG).

The persistence of RPG as a proof scheme among learners, both teachers and students, is well-documented (see for example, Harel and Sowder 2007 and Stylianides and Stylianides 2009). However, one of the pedagogical achievements of our teaching experiment is that despite the well-documented persistence of RPG as a proof scheme, its counterpart, the PPG way of thinking, became part of many participants' repertoire of ways of thinking. The following solution approach to Item (b) by another teacher participant, Gilbert, provides an example of the PPG way of thinking.

Table 1 Alden's arrangement of data

Set position	Value of first term	Additive decomposition of first term
1	1	1
2	2	1 + 1
3	4	1 + 1 + 2
4	7	1 + 1 + 2 + 3
5	11	1 + 1 + 2 + 3 + 4
…		
n		1 + 1 + 2 + … + (n − 1)

For the first four sets Gilbert, one of the teaching experiment's participants, created a table coordinating six elements (as shown in Table 2):

- the position of the set in the sequence (1st column),
- the number of terms in the set (2nd column),
- the terms comprising the set (3rd column),
- the sum of the terms (4th column),
- the value of the first term in the set (5th column), and
- the value of the last term in the set (6th column).

He then observed a relationship among the 1st column, 2nd column, and 6th column (as is depicted in Table 3) and observed the following pattern in the data by attending to regularity in the results.

Gilbert observed that the last term of each set can be computed by adding twice the number of terms in the set to the last term of the preceding set. Unlike Alden, Gilbert was able to explain why the last term of the n^{th} set must be $(2n-1)(2)$ larger than the last term of the *n-1st* set—because, according to him, there are $2n-1$ terms in the n^{th} set and each successive term within a set increases in value by 2.

Thus, the primary distinction between Gilbert and Alden's solutions is a matter of RPG versus PPG: While Gilbert provided an explanation, a cause, for the underlying structure of the pattern as it continues ad infinitum, Alden's merely accepted this structure at face-value, without attending to its underlying cause.

An Investigation into Triangle Inequality

The construction of auxiliary lines in geometry problems is a form of structural reasoning if one constructs such lines for the purpose of creating a desirable structure. The following episode is an example.

Problem: *CM is a median to the side AB in a triangle ABC. Prove that CM < ½(AC + BC).*

Ray, one of the teacher participants, began by simplifying the given inequality into the inequality $2\,CM < AC + BC$ as a new goal. To express the presence of $2CM$ geometrically, he extended CM to $CC' = 2CM$ as pictured below (Fig. 2).

He then realized that in order to investigate the inequality, $2\,CM < AC + BC$, the segments comprising this inequality should somehow be comprising a recognizable structure; in this case, a triangle. It was only natural for him to construct the segment AC' (Fig. 3).

Table 2 Gilbert's arrangement of data

Set position	Number of terms	Terms in set	Sum of terms	Value of first term	Value of last term
1	1	1	1	1	1
2	3	3, 5, 7	15	3	1 + 3(2)
3	5	9, 11, 13, 15, 17	65	9	1 + 3(2) + 5(2)
4	7	19, 21, 23, 25, 27, 29, 31	175	19	1 + 3(2) + 5(2) + 7(2)

Table 3 Gilbert's pattern generalization

Set position	Number of terms	Value of last term
1	1	1
2	3	1 + 3(2)
3	5	1 + 3(2) + 5(2)
4	7	1 + 3(2) + 5(2) + 7(2)
…		
n	2n − 1	1 + 3(2) + 5(2) + … + (2n − 1)(2)

His cognitive schemes of parallelograms and triangle inequality seems to have triggered the construction of BC', forming a parallelogram, from which he (a) concluded that $AC' = BC$ ("A quadrilateral in which the diagonals bisect each other is a parallelogram", and "In a parallelogram, the opposite sides are equal"), and (b) applied the triangle inequality to conclude that $2\,CM = CC' < AC + AC' = AC + BC$.

Thus, the extension of the median and the construction of the parallelogram were driven by a need, the intellectual need to express objects and conditions geometrically for the hope of forming a manageable, familiar structure. If there were evidence that the solver was aware of the causal relationship between the initial intellectual perturbation and its resolution via this construction of auxiliary lines, we could conclude that this is a case of an epistemological justification.

To illustrate what such evidence might look like, consider the following example.

Problem: *The points P and M bisect the sides CD and CB of the parallelogram ABCD. AM and AP intersect with the diagonal DB at F and E respectively. Prove that* $DE = EF = FB$ (Fig. 4).

The solution commenced by constructing diagonal AC. The solver then denoted the intersection of the two diagonals by O, the intersection of AM and BO by F, and the intersection of AP and DO by E (Fig. 5).

He then observed that BO and AM are medians in $\triangle BCA$, and DO and AP are medians in $\triangle DCA$, and therefore, $BF = 2\,FO$ and $ED = 2\,EO$. But since O is the intersection of the two diagonals of the parallelogram, $BO = DO$, from which he easily concluded that $DE = EF = FB$.

What are the solver's epistemological justification considerations in this solution? The solver expressed his reasoning as follows: He sought ways of transforming the figure into one that includes familiar structures. One such structure is a parallelogram with *both* its diagonals, and so the solver constructed diagonal AC and the point O as its intersection with the already-present diagonal. This, in turn, triggered him to consider the observed facts, $BF + FO = OE + ED$ and $FO + OE = FE$. To achieve the desired goal, i.e., showing that $DE = EF = FB$, he realized that he needed: (1) $FO = OE$, (2)

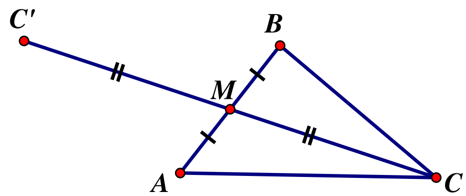

Fig. 2 Ray extends the median (created with Geometer's Sketchpad)

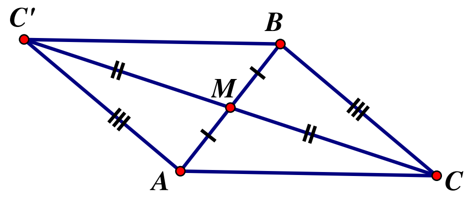

Fig. 3 Ray adds auxiliary lines (created with Geometer's Sketchpad)

$BF = 2\,FO$, and (3) $ED = 2OE$. His cognitive schemes for triangles with the ratio of intersecting medians had then been activated, from which he concluded the required relation.

Summary

This section offered field-based illustrative events of the first four categories of structural reasoning in our typology. The fifth category, *reasoning in terms of general structure*, was beyond the scope of the data we have from the teaching experiment; it was illustrated in the previous section by examples pertaining to the historical development of linear algebra.

These illustrations include the following:

1. Transformations of objects, such as changing forms of a given equation or altering geometric figures by adding auxiliary lines, carried out for the purpose of reducing an unfamiliar structure to a familiar structure, for example in order to be able to graph a function.
2. Transformation involving the formation of structures conceived as conceptual entities, what Dubinsky (1991) calls an object conception, for example when a common element to different expressions is acted upon as an input for a particular process.
3. Formation of structures through transformations carried out in thought. This ability is akin to Dubinsky's (1991) notion of process conception. It is in contrast to the limited ability of forming structures with the aid of manipulations expressed visually on paper.
4. The formation of different kinds of regularities (i.e., structure); that is, the process pattern generalization (PPG) and the result pattern generalization (RPG), for example in the process of justifying a particular argument. While the former belongs to the deductive proof scheme, the latter belongs to the empirical proof scheme.

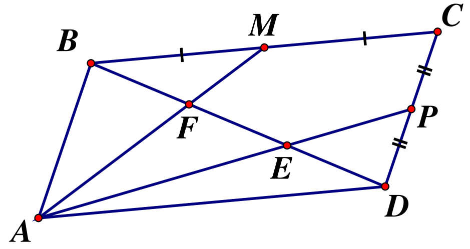

Fig. 4 A parallelogram with given segments (created with Geometer's Sketchpad)

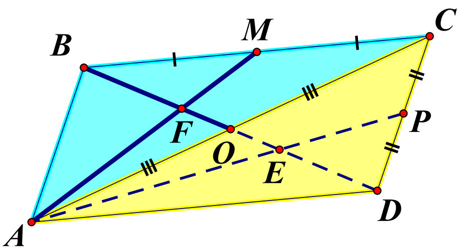

Fig. 5 One problem solver adds the second diagonal focusing on two triangles (created with Geometer's Sketchpad)

5. Epistemological justifications—justifications that aim at explaining cause, the cause of the birth of an idea, such as when one is aware that the creation of auxiliary lines in a geometric figure was a response to a need to investigate congruency of two triangles.

Research Questions

Given the importance of structural reasoning in mathematical practice, a natural question to ask is how to promote it in the classroom, and especially how to plant the seeds for its growth among students in early grades? After all, it is clear that structural reasoning cannot begin with students when they enter college, but it must start its formation in elementary and secondary school, as the CCSSM have recognized. Evidence exists to indicate that young students at the elementary level are capable under certain instructional conditions to construct certain aspects of structural reasoning. For example, research has shown that in student-centered environments, young children observe regularities and are able to share their observations with others; and that they can even recognize that RPG-based arguments are insufficient as justifications (Schifter 2009). Research has also shown that some pedagogical approaches developed by Davydov et al. (1994) advance PPG ways of thinking among elementary school students (Morris 2009); and that "a careful mixture of free play, collaboration, questioning, revisiting the problem, and a call for justification" lead young students to not only observe structures but also to create structures (Maher 2009). On the other hand, Kirshner and Awtry's (2004) work mentioned earlier suggests that there might be cognitive factors involving "visual salience" that would unavoidably cause difficulties for students to construct and observe structures among algebraic expressions. More research is needed to better understand the cognitive process and epistemological and didactical obstacles (in the sense of Brousseau 2002) involved in cultivating structural reasoning among students.

Our own research (Harel et al. 2014) has shown how older learners can be helped to transition from RPG to PPG. Against common practice and intuitive instructional tendencies, we have shown that the encouragement of skepticism (as to whether an assertion is true beyond the cases evaluated) is instructionally ineffective. Instead, a more effective approach is to compare RPG-based solutions to those that are PPG based. The conceptual account for this approach was that while RPG solutions, such as the one by Alden, provide certainty, PPG solutions, such as Gilbert's, provide enlightenment; an understanding of the *cause* for the pattern being the way it is. The pedagogical idea of shifting learners' attention from *certainty* to *causality,* whereby advancing their epistemological justifications, requires more research, especially since in mathematics not all justifications might be conceived as causal (e.g., "proof by contradictions" and "if and only if" statements are examples, Mancosu 1999).

Aspects of APOS theory are relevant to structural reasoning, specifically, the notion of encapsulation, where a process is reconceptualized into an object, or a conceptual entity. But here too more research is needed to better understand how to translate this research into the pedagogical practice of helping students engage in encapsulation for structural purposes.

We neither expected nor observed instances of the last category (*reasoning in term of general structures*) in our teaching experiment. However, the relation between those we have illustrated and those that define this category is rather apparent. For example, the ability to form mathematical objects, i.e., conceptual entities, is well recognized as a cognitive pre-requisite for dealing conceptually with structures, such as vector spaces of functions (Dubinsky 1991). Likewise, historically, the ability to create and explicitly state unifying and generalizing concepts—such as the concept of vector space, linear transformation, and linear functional—are related to epistemological justification, in that they came about as a response to particular intellectual needs to resolve certain problems in functional analysis (Dorier and Sierpinska 2001). From an analytic point of view, the epistemological justification category raises the following question. Does the construction of epistemological justifications differ across axioms, definitions, theorems, and proofs? For example, are epistemological justifications created for the incidence axioms different, cognitively and epistemologically, from those that arise for, say, the definition of linear independence or the proof of Lagrange Theorem? We do not have field-based events that might illustrate these differences or the absence thereof. Nor, is it likely that empirical data would be sufficient to answer this question in relation to its philosophical aspect.

References

Brousseau, G. (2002). The didactical contract: The teacher, the student and the milieu. *Theory of Didactical Situations in Mathematics: Didactique des Mathématiques, 1970–1990* (pp. 226–249). Netherlands: Springer.

Corbin, J. M., & Strauss, A. (1990). Grounded theory research: procedures, canons, and evaluative criteria. *Qualitative Sociology, 13*(1), 3–21.

Davydov, V. V., Gorbov, S. F., Mikulina, G. G., & Savel'eva, O. V. (1994). *Mathematics*. Moscow: First grade.

Dorier, J. L. (Ed.). (2000). *On the teaching of linear algebra* (Vol. 23). Springer Science & Business Media.

Dorier, J., & Sierpinska, A. (2001). Research into the teaching and learning of linear algebra. In D. Holton et al. (Eds.), *The teaching and learning of mathematics of university level: an ICMI study* (pp. 255–273). Dordrecht: Kluwer Academic Publishers.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 95–126). Dordrecht: Kluwer Academic Publishers.

Dubinsky, E., & McDonald, M. (2001). APOS: a constructivist theory of learning in undergraduate mathematics education research. In D. Holton et al. (Eds.), *The teaching and learning of mathematics at university level: an ICMI study* (pp. 273–280). Dordrecht: Kluwer Academic Publishers.

Gick, M. L., & Holyoak, K. J. (1983). Schema induction and analogical transfer. *Cognitive Psychology, 15*(1), 1–38.

Greeno, J. G. (1983). Conceptual entities. In D. Gentner & A. L. Stevens (Eds.), *Mental models*. Hillsdale: Lawrence Erlbaum Associates.
Harel, G. (2001). The Development of Mathematical Induction as a Proof Scheme: A Model for DNR-Based Instruction. In S. Campbell & R. Zaskis (Eds.), *Learning and Teaching Number Theory.* Journal of Mathematical Behavior (pp. 185–212). New Jersey: Ablex Publishing Corporation.
Harel, G. (2013a). DNR-based curricula: the case of complex numbers. *Journal of Humanistic Mathematics, 3*(2), 2–61.
Harel, G. (2013b). Intellectual need. In *Vital directions for mathematics education research* (pp. 119–151). Springer New York.
Harel, G. (2014). Common Core State Standards for geometry: an alternative approach. *Notices of the AMS, 61*(1), 24–35.
Harel, G., & Sowder, L. (2007). Toward a comprehensive perspective on proof. In F. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning*, National Council of Teachers of Mathematics.
Harel, G., Fuller, E., & Soto, O. (2014). Determinants of a DNR expert's teaching. In Y. Li (Ed.), *Transforming Mathematics Instruction: Multiple approaches and practices* (pp. 413–437). Springer.
Hayes, J. R., & Simon, H. A. (1977). Psychological differences among problem isomorphs. In N. J. Castellan Jr., D. B. Pisoni, & G. R. Potts (Eds.), *Cognitive theory* (Vol. 2, pp. 21–41). Hillsdale: Erlbaum.
Hoch, M., & Dreyfus, T. (2004). Structure sense in high school algebra: The effect of brackets. In *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 49–56).
Kirshner, D., & Awtry, T. (2004). Visual salience of algebraic transformations. *Journal for Research in Mathematics Education, 35*(4), 224–257.
Küchemann, D., & Hoyles, C. (2009). From empirical to structural reasoning: tracking changes over time. In D. A. Stylianou, M. L. Blanton, & E. J. Knuth (Eds.), *Teaching and learning proof across the grades: A K-16 perspective*. New York: Routledge.
Leron, U. (1983). Structuring mathematical proofs. *The American Mathematical Monthly, 90*(3), 174–185.
Leron, U. (1985). Heuristic presentations: the role of structuring. *For the Learning of Mathematics, 5*, 7–13.
Maher, C. A. (2009). Children's reasoning: discovering the idea of mathematical proof. In D. A. Stylianou, M. L. Blanton, & E. J. Knuth (Eds.), *Teaching and learning proof across the grades: A K-16 perspective*. New York: Routledge.
Mancosu, P. (1999). *Philosophy of mathematics and mathematical practice in the seventeenth century.* Oxford: Oxford University Press on Demand.
Morris, A. (2009). Representations that enable children to engage in deductive argument. In D. A. Stylianou, M. L. Blanton, & E. J. Knuth (Eds.), *Teaching and learning proof across the grades: A K-16 perspective*. New York: Routledge.
National Governors Association Center for Best Practices & Council of Chief State School Officers (2010). *Common Core State Standards for Mathematics.* Washington, DC: Authors.
Piaget, J. (1971). *Biology and knowledge* (Trans: Walsh, B.). Chicago: University of Chicago Press. (Original work published 1967).
Reck, E. H. (2009). Dedekind, structural reasoning, and mathematical understanding. In B. van Kerkhove (Ed.), *New perspectives on mathematical practices* (pp. 150–173). Singapore: World Scientific.
Reed, S. K. (1989). Constraints on the abstraction of solutions. *Journal of Educational Psychology, 81*(4), 532–540.
Schifter, D. (2009). Representation-based proof in the elementary grades. In D. A. Stylianou, M. L. Blanton, & E. J. Knuth (Eds.), *Teaching and learning proof across the grades: A K-16 perspective*. New York: Routledge.
Steiner, M. (1978). Mathematical explanation. *Philosophical Studies, 34*(2), 135–151.
Stylianides, G. J., & Stylianides, A. J. (2009). Facilitating the transition from empirical arguments to proof. *Journal for Research in Mathematics Education, 40*, 314–352.
Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151–169.