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In the Pursuit of Relevance – Mathematicians Designing Tasks for Elementary School Teachers

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Abstract In this paper we investigate task design in an unusual professional development course for elementary school teachers, conceived and taught by research mathematicians. Prior analysis singled out *relevance for teaching* as a critical design issue for engaging teachers in effective learning. The aim of the current research is to uncover how relevance for teaching was achieved without compromising mathematical rigor and depth. Findings are based on an analysis of three representative cases of task designing in which the authors where involved – one as instructor and task designer, the other as participant observer. Our analysis reveals a designing model that first addresses purely mathematical concerns and then refines tasks, taking into consideration a series of constraints imposed by the requirement of relevance for teaching. Using Schoenfeld's Resources-Orientations-Goals framework for decision-making, we show how the mathematicians drew on their special knowledge of mathematical content to achieve such relevance in ingenious ways. We find that tasks were best aligned with Knowles's principles of Adult Learning in cases where the designers appropriated the teachers' point of view, no longer seeing the need for relevance as a constraining imposition, but rather as an opportunity to combine and merge knowledge specific for teaching and purely mathematical knowledge.

Keywords Task design · Elementary school mathematics · Mathematicians · Teacher professional development · Adult learning · Decision-making

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Introduction

Designing professional development (herein referred to as PD) for in-service teachers can be a challenging endeavor. Teachers expect PD to be relevant for their teaching practices. However, curriculum for PD is often guided by what experts believe teachers need to know, and its practical relevance may not always be obvious. If teachers do not see how the PD activities will contribute to their teaching, they are unlikely to engage in effective learning. This raises a general question that is central to this paper – what is required for PD tasks to be perceived as relevant by the participating teachers?

The current research addresses this question in a particularly challenging setting. The PD under investigation was the initiative of a mathematics professor at a leading research university in Israel, and was taught by mathematics Ph.D. students. The overarching goal of the PD was to broaden and deepen elementary school teachers' knowledge and understanding of the mathematics they teach. Such a goal might not appear relevant to the teachers. After all, the mathematical content of elementary school, especially in the lower grades, is perceived as quite straightforward. Teachers may well wonder what there is to learn about counting and about the four basic operations that they do not already know. It is more common for elementary school PD programs to focus on didactical issues such as how to best teach the content and what difficulties the students are likely to encounter, the relevance of which is more obvious.

And indeed, the goal of focusing on content, combined with the choice of teacher educators - young Ph.D. students of mathematics with no experience in elementary school teaching - caused some discontent on the part of the teachers. They were particularly dissatisfied when the mathematicians touched on content that went beyond the scope of elementary school mathematics, which appeared to be particularly irrelevant for their teaching practice. This discontent threatened to derail the whole PD.

Fortunately, there was a large degree of flexibility in the curriculum and there was no prepared material for the PD, a fact that positioned the young mathematicians not only as teacher educators but also as task designers. They were preoccupied with motivational issues - addressing the mathematical content in their tasks in ways the teachers would deem relevant for their teaching. The mathematicians were in a position to "learn on the job", and did eventually find ways to reconcile the teachers' demand for relevance with their own goals of addressing mathematical content in a deep and meaningful manner. The PD was eventually considered successful, by the instructors, the participating teachers, and by officials from the Ministry of Education. The careful crafting and designing of tasks was found to be central in bridging the differences in expectations.

The special nature of the learning that took place in this unusual PD is discussed by Cooper elsewhere (2014, 2015a, b). The focus of this paper is the task design in the PD. For us, task design refers to both the designing process and the resulting tasks. We believe that the full significance of a designed product emerges only when we know something about the intentions and goals of the designer, which for their part evolve during the designing process. We will present a model of the task designing process, and analyze some of the tasks that emerged from this process in terms of the ways in which they addressed the mathematical content and the ways in which they achieved relevance for teachers.

We present three cases of task design. The third case is presented only after the first two cases are analyzed and discussed. The reason for this is that we consider the third case to be more mature - achieving a deeper sense of relevance - but this can only be appreciated after the other cases have been explored.

Literature Review and Research Questions

The research described in this paper is problem-driven in the sense used by Arcavi (2000). The situation under investigation - a professional development course for primary school teachers run by mathematicians - occurred spontaneously; it was not driven by research. Relevance for teaching emerged as a central issue, as described in the introduction, and this paper describes the designers' pursuit of relevance through task designing and the nature of relevance-for-teaching they achieved in their tasks. Accordingly, the main goal of our literature review is to situate the notion of relevance with respect to existing research literature on mathematics teacher education and task design. We review the mathematical knowledge that is considered relevant for teaching, and its implications for task design[1] for teacher education. This review considers the perspectives of three communities: research mathematicians, mathematics education researchers, and teachers.

Research mathematicians are gradually becoming more involved in school mathematics education (Bass 2005). Bass himself is a research mathematician who has been extensively involved in teacher education, and Goldenberg (1999) stressed the importance of including university mathematicians on teams that develop curriculum, such as "Connected Geometry". The topic of ICMI study 22 (Margolinas 2013) was Task Design in Mathematics Education, and the study dedicated one of its five themes to task design within and across communities, acknowledging in the introduction to the theme that "various design communities, such as those consisting of researchers, teachers, professional developers and teacher trainers, or textbook writers, have different aims and agendas for task design" (p.420). We found it curious that the community of research mathematicians was not explicitly mentioned in this introduction, nor was it addressed in any of the papers in the study proceedings. This may be due, in part, to the deep and profound difference between mathematics as practiced by mathematicians and school mathematics (Watson 2008), which suggests that the mathematical knowledge of research mathematicians might not be relevant for school mathematics education. Bass rejects this assertion, arguing that "[T]he knowledge, practices, and habits of mind of research mathematicians are not only relevant to school mathematics education, but… this mathematical sensibility and perspective is essential for maintaining the mathematical balance and integrity of the educational process" (Bass 2005, p. 418). However, Bass also warns that research mathematicians often lack mathematical knowledge that is specific for teaching, which may be essential for engaging productively and meaningfully in school mathematics education.

The term "mathematical habits of mind", as used by Bass, was coined by Cuoco et al. (1996) in an attempt to describe modes of thought that transcend content

[1] We use the term "task" to refer to any material intended to promote complex mathematical activity in the PD, and in particular planned discussions to be led by the instructor.

knowledge, and are taken for granted by mathematicians. Some of these habits are described as guiding principles for the design of tasks and curriculum. One such guiding principle is a focus on "core ideas" or "big ideas" (Goldenberg 1999). The importance of "big ideas" in teacher education was also stressed by Hsu et al. (2007). They believe that focusing teachers on big ideas that explain a number of procedures may help change teachers' common classroom pattern of stressing procedures for getting correct answers. Schoenfeld, another mathematician who turned to mathematics education, writing with Kilpatrick on proficiency in teaching mathematics (Schoenfeld and Kilpatrick 2008), stressed the importance of breadth and depth of teachers' mathematical knowledge, which allows them to organize content so that students are introduced to big ideas. Chazan (1999) described how a particular approach to algebra, organized around the big idea of functions, served him as a teacher in two ways: helping students understand the goal of problems and the characteristics of desired solutions, and shifting the locus of authority in the classroom away from the teacher as a judge and towards the teacher and students as inquirers.

Wu, another research mathematician involved in school education, explicates his attitudes toward mathematics through five fundamental principles (Wu 2011): 1. Every concept is precisely defined, and definitions furnish the basis for logical deductions; 2. Mathematical statements are precise. At any moment, it is clear what is known and what is not known; 3. Every assertion can be backed by logical reasoning; 4. Mathematics is coherent; it is a tapestry in which all the concepts and skills are logically interwoven to form a single piece; 5. Mathematics is goal-oriented, and every concept or skill in the standard curriculum is there for a purpose. Wu believes that the mathematics of the school curriculum must be presented in a way that is consistent with these fundamental principles.

If we accept the importance of breadth and depth of teachers' mathematical knowledge, and of "big ideas" as an organizing principle, mathematicians' involvement in task and curriculum design seems relevant, and indeed is called for by mathematicians (e.g. Bass 2005; Ralston 2004; Wu 2011) and mathematics educators (e.g. Goldenberg 1999) alike. Furthermore, Zaslavsky (2007), in summarizing the Journal of Mathematics Teacher Education's special issue on task design, singles out some desirable characteristics of teacher educators revealed in the issue's papers: "a deep interest and ongoing involvement in mathematics" (Zaslavsky 2007, p. 433) and "disposition to do mathematics, awareness and appreciation of mathematical structures and connections within and across domains" (p. 435), characteristics which are likely to be abundant in mathematicians.

We now turn to the perspective of mathematics education researchers, and to a particular aspect of teacher education - task design. Zaslavsky notes that "mostly [in the JMTE special issue] there is no specific curriculum for mathematics teacher education that needs to be "covered"" (2007, p. 436), a fact that positions teacher educators as task designers. Various researchers have discussed and analyzed the nature of tasks that are particularly suitable for developing the kind of mathematical knowledge that is particularly relevant for teaching. Many of them draw on Shulman's work on pedagogical content knowledge – PCK (1986) – and more recently, the works of Ball and her associates who developed a more comprehensive framework of mathematical knowledge for teaching – MKT (Ball et al. 2008). Suzuka et al. (2009) list some design principles for what they call "MKT tasks". Tasks well suited for developing

MKT should: create opportunities to develop a flexible understanding of central mathematical ideas, provoke a "stumble", open opportunities to build connections among mathematical ideas, support multiple representations and solution methods, and provide opportunities to engage in mathematical practices that are central to teaching. The last item on this list is likely to be beyond the expertise of research mathematicians, who have little knowledge of practices that are central to school teaching. Stylianides and Stylianides (2010) differentiate between two types of tasks for teacher education: typical mathematics tasks and pedagogy related (P-R) mathematics tasks. Typical mathematics tasks have a primary mathematics object (e.g. generation/validation of conjecture, comparison of solution methods, etc.), and focus on important elements of mathematical knowledge for teaching, including ideas that are fundamental and hard to learn. P-R tasks have all the features of typical mathematics tasks with the additional feature of having a secondary yet substantial pedagogical object (e.g. how would you respond if a student made the following claim), situated in a particular pedagogical context (assumptions regarding grade level, institution, etc.).

The concepts of MKT tasks and P-R mathematical tasks do not completely overlap, yet they both highlight the affordances and the limitations of the resources that research mathematicians may bring to task designing for teacher education. Mathematicians may well recognize and focus on ideas that are central in the curriculum, encourage and develop connections among mathematical ideas, and encourage a multiplicity of representations and solution methods, since these are fundamental principles in university mathematics as well. However, Stylianides & Stylianides comment that "The pedagogical demands implicated by the design, implementation, and solution of P-R mathematics tasks [require] not only good knowledge of mathematics but also some pedagogical knowledge. … it may be hard to require or expect that [research mathematician] instructors …have knowledge of pedagogy" (Stylianides and Stylianides 2010, p. 170). Goldenberg's solution is design teams that include members with varied expertise, including mathematicians and professional teacher educators. Bass sees mathematics for teaching as a field of applied mathematics, and argues that "the first task of the mathematician who wishes to contribute in this area [of school mathematics education] is to understand sensitively the domain of application, the nature of its mathematical problems, and the forms of mathematical knowledge that are useful and usable in this domain" (Bass 2005, p. 418). However, these strategies may not always be practicable, and may end up driving away mathematicians who wish to contribute to school mathematics education.

We now turn to the perspective of mathematics teachers. Prospective teachers, and to a greater extent in-service teachers, may have strong opinions regarding what they need to learn for their professional development. Furthermore, these strongly held ideas may not coincide with researchers' ideas about what teachers need to know. Particularly in the lower primary grades, where the content is often considered straightforward (e.g. counting, addition and subtraction of whole numbers), teachers may not see the point in dwelling on the depth and breadth and big ideas of the mathematical content they teach. Stylianides & Stylianides touch on the issue of teacher motivation when they suggest that P-R tasks may have the potential to motivate prospective teachers' engagement "… by helping them see and appreciate why the mathematical ideas… might be important for their future work as teachers of mathematics" (p. 165). This hypothesis gains weight in light of the seminal work of Knowles (1990) on Adult Learners, who found that adults typically need to know the reason or reasons for learning something. Knowles also found that adults tend to have a problem-solving orientation to learning - they are more likely to engage in learning if it promises to solve real-life problems. This is consistent with Watson and Mason (2007), who note that teachers in professional development are forever on the lookout for something they can use in their classrooms, perhaps immediately. Classroom-ready tasks can be seen as addressing teachers' real-life problem of what to teach tomorrow morning. They add that teachers may reject tasks as "not relevant to my teaching" if they are not classroom-ready. And indeed, centering at least a part of a professional development course on such tasks is arguably the most obvious way of achieving relevance for teachers and motivating them, however it is not the only one.

We now come to the process of designing tasks. Zaslavsky (2007) lists two components of this process, as they emerged in the JMTE special issue: 1. An initial choice or design of tasks by a teacher educator, inspired by personal knowledge, professional literature or textbooks, a knowledge of relevant theory, or familiarity with teacher practice. 2. An iterative process of design and modification, including a predictive analysis, trial, reflective analysis based on teachers' work, and adjustments. Literature on task design tends to focus more on the conscious and explicit iterative process, which is often explored in the context of design experiments or design-based research, and less on the more tacit aspects of the process: the inspiration for the initial development of an idea, and the process by which it evolves into a task.

The literature we have reviewed has shown that mathematicians' orientations toward mathematics (mathematical habits of mind, a focus on big ideas) may be relevant for teacher education. Their mathematical knowledge is broad and deep, yet it is not necessarily Mathematics for Teaching. If they engage in professional development for in-service teachers and design their own tasks, they will need to reconcile their orientations with the teachers' conception of relevance. This leads us to the following research questions:

1. What nature of relevance-for-teaching did the mathematicians achieve in their tasks for the PD?
2. What was the designing process by which relevance-for-teaching was achieved?

Theoretical Frameworks

In this paper we will examine both task *designing* – the process of creating tasks, and the tasks that emerged. For this purpose we adopt two theoretical perspectives. We consider task designing to be a case of "goal-oriented activity", and adopt Schoenfeld's ROG (resources, orientations, goals) model of decision making (Schoenfeld 2010) as a theoretical framework for describing, analyzing, and explaining the designers' decisions at crucial stages of the design process. We also draw on Andragogy - the theory of Adult Learning (Knowles 1990) - as a theoretical framework for exploring relevance-for-teaching by analyzing how tasks in the PD met, or failed to meet, the needs and expectations of the teachers.

ROG is a model of goal-oriented activity that was developed by Schoenfeld (2010) as a tool for explaining how and why teachers make the instructional decisions they make as they teach. Out of the multitudinous factors influencing instructional decisions, Schoenfeld identified three categories that he found to be most significant, in the sense that understanding them is both necessary and sufficient for explaining decisions. These categories are the knowledge and other resources available for the teacher, his/her orientations (e.g. beliefs, views and attitudes) and goals - the conscious or unconscious aims the teacher is trying to achieve. The relationships between resources, orientations and goals for the decision-making process are that orientations "…shape the prioritization of the goals that are established for dealing with those situations and the prioritization of the knowledge that is used in the service of those goals" (Schoenfeld 2010, p. 29).

While ROG is most commonly used to analyze teachers' in-the-moment teaching decisions in the classroom, (e.g. by Paterson et al. 2011; Thomas and Yoon 2013; Törner et al. 2010), it has also been used for analyzing the design decisions made by teachers prior to their lessons (e.g. Pinto 2013, 2015; Thomas and Yoon 2013). In the most common application of the framework, pre-designed tasks, along with the teacher's *lesson image* (Schoenfeld 2000, p. 250), are taken as the point of departure from which in-the-moment teaching decisions are made. Yet, Schoenfeld does consider ROG general enough to account for a much broader range of activities, as he exemplifies in his book (Schoenfeld 2010).

Phrased in terms of Schoenfeld's framework, the PD instructors brought a set of ROG that is not commonly seen in the teaching of elementary school PD. Regarding resources, their education and experience provided them with extensive subject matter knowledge, but with little pedagogical content knowledge (PCK). The orientations and goals that they brought were also not typical of elementary school PD. Their orientations toward mathematics and learning reflected those of the community of research mathematicians, and their overarching goals were to focus mainly on mathematical content, much less on issues of how to teach it in school.

Much of the task designing in the PD was conducted as a group effort. While each of the instructors brought his own individual set of ROG to the designing process, the decisions we examine in this paper were made in group discussions, and were shaped and determined by orientations and goals shared by the mathematics students involved in the PD. Accordingly, in the analyses of the designing processes, we often refer to the ROG of the group as a whole. In our attempt to phrase the mathematical orientations that shaped the designing process, we take inspiration from the works of Cuoco et al. (1996); Bass (2005); Schoenfeld (1992) and Wu (2011), and refer to certain orientations as if they are common to the whole community of mathematics graduate students, or even to research mathematicians in general.

We use Knowles's theory of adult learning (1990) for analyzing the tasks designed in the PD. This theory is based on some principles or assumptions regarding adults' motivation to learn. We list all six principles, though we only found principles 1–4 to be relevant for our analysis:

1. Adults need to know the reason or reasons for learning something, a principle herein referred to as *need to know*. Unlike school children, they cannot be expected to learn something just because their teacher told them they must.

2. One of the most obvious reasons to learn something is that it promises to solve life-problems related to the learner's work. Adults tend to have a problem-solving orientation to learning, a principle herein referred to as *life-problem orientation*.
3. Adults need to be ready to learn something new, a principle herein referred to as *readiness*. This principle emphasizes that motivation to learn may be related not only to *what* is being learned, but also *when* it is learned. In our context this will usually have to do with having pre-requisite knowledge to understand and appreciate new knowledge.
4. Adults' identity is shaped by experience, and as a result adults may be motivated in situations where their experience (professional and personal) is used as a resource in learning, a principle herein referred to as *experience*. Furthermore, adults' identity may be so strongly linked to their experience that ignoring their professional experience in learning situations may be perceived by the learners as rejecting them as individuals.
5. Adults have a psychological need to be perceived as being capable of self-direction.
6. Adults' motivation to learn is primarily internal. Attempts to apply extrinsic motivation are likely to backfire. Most adults are naturally motivated to grow, but this natural motivation may be blocked due to a negative self-conception as a student, lack of resources necessary for learning, or if the learning environment violates any of the Andragogy principles listed above.

We chose the theory of adult learning as a framework for analyzing tasks in the PD not only because its principles are widely accepted in the field of adult education, but also because the teachers' feedback, both formal (expectation and feedback questionnaires) and informal (discussions, complaints, compliments), clearly showed that these principles apply to the present context and thus are central to understanding their motivation to learn. Two illustrative examples are *tasks that I can use in my classroom* (from an expectations questionnaire), and *it made me think about what multiplication is*, *how to teach it for deep understanding*, *and possible student errors* (from a feedback questionnaire following a PD activity). See (Cooper and Touitou 2014) for a detailed analysis of teacher feedback.

We stress that by using the theory of adult learning we do not imply that the designers had any knowledge of this theory. However they did have two implicit models of adult learning from which to take inspiration – their own experience as adult graduate students, and their experience as teaching assistants in courses for undergraduate mathematics students, who are also (young) adult learners. The instructors' natural tendency was to model their PD teaching and task designing after their own university teaching. This model is quite well aligned with Knowles's principles of adult learning for the target population of undergraduate mathematics students, but we will show that it violates some of these principles when applied to in-service teachers, and can be expected to create some serious motivational issues. In order to address the teachers' expectations (both explicit and implicit), the instructors eventually adjusted their teaching and designing in ways that are more aligned with Knowles's principles, as they pertain to elementary school teachers.

Setting and Methods

The PD under investigation was first initiated and taught in 2009 by a mathematics professor at a leading research university in Israel. The program was quite popular, and in following years he recruited a team of approximately 10 mathematics Ph.D. students to manage the growing demand. Since then about 90 in-service elementary school teachers participate every year in ten 3-hours sessions taught in parallel in 4–6 tracks for various grades. The program was endorsed by the Ministry of Education, and is recognized for professional development credit. In this paper we discuss three cases of task design, expanding on a short account previously given by Cooper and Arcavi (2013). The tasks described in Cases 1 and 2 were designed and taught during the 2012 academic year by students newly recruited to the PD program - Case 1 in the second session and Case 2 in the sixth. The tasks in Case 3 were designed and taught at the beginning of the 2011 academic year by students in their second year of involvement in the PD. This represents a very small portion of the tasks designed in the PD over the years. Thus, a few words on our selection of these tasks are in order. Our selection of cases was by and large subjective. We selected tasks that intrigued us in their use of mathematical, meta-mathematical, or didactical ideas that were new or surprising for us. We were guided by an aesthetical sense, and we hope readers will share our feeling that these examples have some beauty. However, in terms of representativeness, this selection is also valid; the three cases relate to the work of a variety of instructors, working with three different grade level teachers, and exemplify a designing scheme that was common to many of the tasks designed for the PD. The quest for relevance, while remaining faithful to the mathematics, was a major design issue throughout the PD, as reflected in numerous discussions and email exchanges between the designers during the PD. Indeed, this quest for relevance is a central theme in works that analyze other aspects of the PD, (e.g. Cooper 2015a, b, 2016). The cases that we analyze here typify different ways in which relevance was achieved (or not).

The authors of this paper were both intimately involved in the PD. Pinto was an instructor - at the time a mathematics Ph.D. student, now a mathematics-education researcher. The description of Case 3 is his first-hand account of the group designing process in which he was involved, and at the same time a reflective observation of his own actions and those of his colleagues. Cooper, who holds an M.Sc. in mathematics, was investigating the PD as part of his doctoral research in Mathematics Education. He was a participant observer of the task design process, sitting in on many of the designers' meetings, was included in their email discussions, and was occasionally asked for advice regarding specific ideas.

Sources of data for this study were:

- The authors' notes taken while observing or participating in design meetings
- Detailed lesson plans which were the outcome of task designing
- Email exchanges between instructors, voicing their ideas, concerns and opinions
- Observations, audio recordings and field notes from the PD sessions
- Teacher expectation and feedback questionnaires

The description of Case 3 is more detailed than Cases 1 and 2. This is due to the nature of the design process, which in Case 3 was a more collective process than in

Cases 1 and 2, and thus left more traces (emails, design documents, notes from meetings).

We describe task designing as a two-phase process, where the product of the first phase is what we call a *naive task*, since it is based only on the mathematicians' ROG and disregards many constraints of the teaching situation in which it was to be used. The second phase takes into consideration constraints of the teaching situation. This is inspired by the fact that many of the tasks we observed, or were involved in the designing thereof, were in fact redesigns of tasks that were found to be problematic in previous years. We suggest this two-phase process as a model of the task designing, and use this model even when there was no actual naive version of the task. The designers often had an imaginary naive task in mind, against which they measured potential tasks. Contrasting an actual task with such an imaginary task is useful for showing how the environment (elementary school PD) constrained the process, and how the designers addressed these constraints, thus revealing the role of their resources, orientations and goals in the designing process. These goals and orientations were in some cases made explicit by the designers in design meetings or in correspondences. When this was not the case, we inferred goals and orientations through an analysis of the tasks. These interpretations were later checked and confirmed by the designers.

Cases

In this section we present two cases that exemplify the mathematics Ph.D. students' task designing. A third case is presented after the discussion of the first two cases. All three cases are described in terms of our two interpretive frameworks – ROG and Adult Learning – and their unfolding follows a recurring structure:

- How overarching goals, together with the designers' orientations and resources, shaped specific learning goals for the tasks
- A naive task satisfying these goals, but disregarding the constraints imposed by the learners' expectations (mainly of relevance for teaching)
- An analysis of the naive task from the point of view of adult learning
- Redesigning the task, taking into account the constraints of the learning situation
- An analysis of the new task or tasks from the point of view of adult learning

Case 1: a Spotlight on One-to-One Correspondence (Grade 1–2 Teachers)

Many children master the skill of rote counting (reciting *1*, *2*, *3*…), yet have difficulty counting how many objects there are in a set (Mathematics Learning Study Committee 2001). How can this be? What are the similarities and the differences between counting numbers and counting objects? This cognitive question has mathematical underpinnings - how is the skill of counting the objects of a set connected to rote counting? Unraveling this presents a teaching opportunity – a topic of the utmost relevance for early grade teachers, where the mathematicians may bring some fresh insight.

In unpacking the mathematical content of counting the objects of a set, the principle of 1–1 correspondence is central (ibid.). Children need to set up such a correspondence between the objects of the set and the sequence of natural numbers by synchronizing rote counting with some gesture related to the objects (e.g. pointing). Any flaw in setting up the correspondence will lead to an error. However, 1–1 correspondence, and its foundational role in counting, is not mentioned in any of the Ministry of Education's curricular documents for elementary school (Pedagogical secretariat of the ministry of education in Israel 2009).

The designer of the following task decided to address the topic of counting. He realized how foundational the concept of 1–1 correspondence is for this basic skill, and decided that illuminating it would be a worthy teaching goal for the PD, and one for which he had the necessary resources - a deep understanding of 1–1 correspondence, and the ability to unpack how it features in developing the skill of counting objects. Furthermore, this goal also addressed an important meta-mathematical goal. For this designer it was important to present the correspondence as more fundamental than counting. According to his orientations, mathematics has a hierarchical structure, and revealing a part of this structure was a teaching goal shaped by his orientations.

How might one go about showing that 1–1 correspondence is more fundamental than counting? The problem of comparing cardinalities of sets was recognized as a suitable setting. Although a natural strategy is to count the number of objects in each set, this is not necessary. Instead one can try to set up a 1–1 correspondence between the members of the sets, and see if the objects of one set are exhausted before the objects of the other. Comparing cardinalities without counting would clearly reveal the 1–1 correspondence as fundamental.

For mathematicians, a natural context for this problem is comparing cardinalities of infinite sets, where the existence of a 1–1 correspondence between the members of two sets is taken to be the definition of equal cardinalities. A "naive" task that follows this approach was actually tried out in the PD in previous years, rather unsuccessfully. The theory of adult learning can help us understand why - such an approach violates some of the theory's principles. To begin with, the cardinality of infinite sets is a difficult topic, for which elementary school teachers are not *ready* - they may not have the necessary background for tackling such an abstract topic. Furthermore, the topic is far removed from the teachers' world of content, and many teachers complained that it was not relevant for their teaching, since learning it would not address any actual teaching situations they were likely to encounter in their work (violating the principle of *life-problem orientation*). Lacking any practical reason to learn this particular piece of mathematics, this task does not address the teachers' *need to know* - what reason could there possibly be for them to need this knowledge? The designer found himself with a teaching opportunity, guided by worthwhile mathematical and meta-mathematical goals, yet constrained by the teaching context. The design challenge was to find a task that invites comparing the cardinality of sets by means of a 1–1 correspondence, yet will still be relevant for the PD teachers. What is it about comparing infinite sets that makes such a compelling case for 1–1 correspondence? Obviously counting is not possible in this context! Thus the designer realized that what he needed was a task requiring the comparison of *finite* sets that somehow rules out the counting strategy. He came up with idea to use objects in motion. He designed an applet[2] where players need

[2] An English translation of the original applet can be found at http://stwww.weizmann.ac.il/g-math/cooper_project/english/index.html. Use Google Chrome or Firefox (MSIE is not supported).

to determine whether there are more blue discs or red discs on the computer screen. All the discs constantly move around randomly, so counting is not a realistic strategy. On the other hand, players may stretch a line between a red disc and a blue disc (setting up a correspondence between them). When this is done, the two discs drop off the screen. If repeated enough times, at least one of the colors will be exhausted.

We now use the theory of adult learning as a framework for highlighting to what extent this task (along with some ideas about how to implement it in the PD) addresses adult learning principles. *Need to know*: If teachers do not understand the underpinnings of counting, they will not be effective in their efforts to help children who are "not getting it". This point may have provided the necessary motivation for learning this mathematical content, but it was not made explicit in teaching, suggesting that it was not the designer's reason for teaching this piece of content. According to his orientations towards mathematics, the real reason for learning this content is "in the mathematics". Teachers are more than *ready* for this insight - they have all the prerequisite knowledge. A *life-problem orientation* is satisfied on two counts - the applet may be used in class, and even if it is not, the alternate strategy for comparing cardinalities without counting may provide a useful teaching resource.

Case 2: Making Sense of Multiplication Properties (Grade 3–4 Teachers)

According to the Ministry of Education's curriculum (Pedagogical secretariat of the ministry of education in Israel 2009), students should begin to make computational use of the commutative and the associative properties of addition and multiplication in grade 2. The ministry's documents do not suggest how these properties should be explained to students, or what sense they should make of them. From a computational perspective, the two rules taken together tell us that in calculations involving a sequence of multiplications, we may change the sequence in any way we wish. This encourages students to start by seeking out pairs of numbers that they can multiply mentally. For example, in calculating $25 \times 17 \times 4$ it makes sense to calculate $(25 \times 4) \times 17$. In this context it is not at all clear why these two rules should not be replaced by a simpler *multiply in any sequence* rule. This section recreates the selection of this piece of content for the PD, and the designing process leading to a task that aims to explain why the two rules should be taught as distinct.

We note that this content can be seen as meta-mathematical, as it relates to the structure of the mathematical body of knowledge. How did the goal to teach this particular content come about? Once the designer had decided on the general mathematical content (multiplication properties), he asked himself what added value he could bring to such a seemingly straightforward topic, and recognized this subtle point as opportunity to take advantage of his resources - his deep understanding of the underpinnings of multiplication properties. This goal was also shaped by the designer's orientations toward mathematics; focusing on these properties and on their origins may serve meta-mathematical goals, such as highlighting that they are not arbitrarily imposed rules, but rather direct consequences of the definition of multiplication.

Having decided on this piece of content and its meta-mathematical entailments as the teaching goal, the instructor considered various approaches. From the perspective of university mathematics, a natural way to justify two distinct properties would be to show an algebraic structure similar to whole number multiplication, in which one of the two properties (associativity, commutativity) holds and the other does not, for example, a non-commutative group. Such groups can be found even within school mathematics, for example the group of congruency-preserving planar transformations (translations, reflections, rotations and their compositions). It is simple enough to design a task that will show that associativity holds and commutativity does not (e.g. clockwise rotation followed by a horizontal reflection does not yield the same result as a horizontal reflection followed by a clockwise rotation).

Had he adopted this approach, the designer would have been true to his orientations regarding mathematics. However, there are some severe constraints that make the relevance of such a naive task questionable in the context of elementary school PD. Ignoring these constraints would have violated some of the principles of adult learning. It would be very difficult to convince the teachers that the topic of non-commutative groups is something they *need to know* about. It is also questionable whether the teachers have the *readiness*, i.e. if their mathematical background is such that they can appreciate the similarities between the two multiplicative groups - integers and transformations. Perhaps most significantly, such an approach is not aligned with adults' *life-problem orientation* to learning. Investigating these properties in a domain where the objects are transformations (not numbers) and the operation on these objects is composition of transformations is quite a stretch. Not only does the example require some mathematical work, it is also not clear how a conclusion reached in this context applies to multiplication of numbers. How could such an understanding of multiplication properties be relevant for elementary school teaching?

The constraint of relevance led the instructor to search for an approach that would faithfully address the meta-mathematical goal, but in a context that would be comprehensible and relevant for teachers. The designer realized that the bare context of pure calculation could not show the distinction between the two properties as intended, but there is another context in which teachers deal with these properties - they need to demonstrate and explain them. An important meta-mathematical principle is that every mathematical fact can be justified. Perhaps the distinction between the two properties can be explained on the basis of this need for justification. The designer decided to focus on the associative property, drawing on the abundant "design resource" of word problems. Word problems are used in instruction to model and apply calculations to real world situations, which may be meaningful for learners. The designer sought a problem that would demonstrate the associative rule but would not demonstrate the commutative rule or the compound rule - multiply in any sequence. He came up with the following: *Five buses left on a school tree-planting activity. Each bus carried 40 students, and each student took two tree-saplings. How many tree-saplings were there in total? Explain how you calculated your answer.*

The designer intended to discuss three different calculation procedures, one beginning with 5×40, one beginning with 40×2, and one beginning with 5×2. The instructor mapped these three options into the following expressions:

A. $(5 \times 40) \times 2$;
B. $5 \times (40 \times 2)$;
C. $(5 \times 2) \times 40$

At this point the teachers would be requested to explain their calculation in terms of the word problem, referring to buses, children and saplings.

Calculations A and B can easily be grounded in the situation – start by finding the total number of students (5×40), or start by finding the number of saplings per bus (40×2), but expression C is different. It is difficult to say what 5×2 represents in the context of the problem. Viewed as mathematical expressions, the equivalence of A and B is implied by the associative property, but their equivalence to C is not. To show this last equivalence we need the commutative property as well. Although it is natural and perfectly legitimate to begin the calculation with the easy multiplication fact 5×2 (based on the *multiply in any sequence* rule), this calculation removes us from the context of the problem. Thus, this task presents a situation that explains only part of the *multiply in any sequence* rule - the associative property. The designer saw the fact that commutativity and associativity have distinct explanations as a possible indication that these two properties should indeed be viewed as distinct.

Comparing the naive task and the final task, we see that the teaching goals did not change. The designer remained true to his goal of justifying why the two multiplication properties should be taught as distinct. However, the nature of the justification did change. The designer found quite an ingenious way (relating operations to a meaningful context) to make his point in a manner that is relevant for the teachers, even though it may not be mathematically rigorous.

Here again we make use of the theory of adult learning to understand the advantages of the final task over the naive task. Most notably, the final task addresses the teachers' *readiness* to know. Unlike the naive approach (non-commutative group) the task eventually chosen is grounded in elementary school mathematics. The teachers have declarative knowledge of commutativity ($a \times b = b \times a$) and of the associative property, which provides all the background necessary for a deeper understanding. The fact that the task was based on a realistic word problem was bound to contribute to the teachers' internal motivation to participate in this activity and to learn from it. The word problem was one that teachers could take to their own classroom, helping them to apply their new understanding to their life-problems of teaching (*life-problem orientation*). Furthermore, this problem addresses teachers' life-problem of needing to feel comfortable and confident with the mathematics in order to teach it well. However, the *need to know* principle was not addressed explicitly in this task. Teachers may not be aware of a need to understand this delicate mathematical point, and from their perspective the task did not provide any real incentive to go beyond the basic procedural aspects of the multiplication properties.

Before we proceed, we would like to discuss how, if at all, the naive task and the final task managed to show the sense in having two distinct multiplication properties. Since the justifications the designer considered were meta-mathematical and pedagogical in nature, they could not be achieved by means of a mathematical proof. Each of the tasks shows the distinction between the properties as productive, but in very different senses. The naive task addresses a mathematician's point of view – multiplication of numbers is not an isolated mathematical idea, it is an instance of a more general notion, namely multiplication in groups, and as such it inherits a set of terms and properties. Making up new language just for multiplication of numbers is unproductive. For teachers, productiveness needs to be judged according to their life-problems, in this case, explaining and demonstrating mathematics. The setting in the task demonstrates and explains the associative property, but provides no insight into the commutative property. Other situations may demonstrate and explain the commutative property (e.g. grid model of multiplication). However, neither of these situations demonstrates or explains the unified property (multiply in any sequence). While, strictly speaking, this kind of argument does not formally justify the distinction between the properties, it does provide some sense of why this distinction may be productive in the context of teaching.

Discussion - The Designing Process "Unpacked"

We have described two cases of mathematicians designing tasks for professional development. Our description of the cases followed a common scheme, which we will elaborate and reflect on in this section: (1) Local teaching goals for the task are shaped and prioritized by overarching PD goals and the designers' ROG; (2) the designers' knowledge, particularly mathematical content knowledge, is activated and inspires an initial naive task consistent with these goals; (3) an inner conflict emerges in the designers between their beliefs and goals and their perception of relevance for the teachers; (4) the designers unpack and refine their orientations and goals in an attempt to resolve this conflict; and finally (5) they redesign the task, which we analyze using the framework of adult learning. We now take a closer look at this process.

ROG Shaping Local Goals

The local goals for each of the tasks were shaped by a number of factors: They were all consistent with the overarching goals of the PD - to deepen and broaden the teachers' understanding of the mathematical content they teach. They were also closely aligned with the mathematicians' orientations towards mathematics and its big ideas. This is evident in goals such as disentangling the commutative and the associative properties of multiplication, which derives from the orientation "Every assertion can be backed by logical reasoning… every concept or skill in the standard curriculum is there for a purpose" (Wu 2011, p. 379) and focusing on the big idea of 1–1 correspondence and its fundamental nature, which is related to "*mathematics is coherent; it is a tapestry in which all the concepts and skills are logically interwoven*" (ibid.). We also see how the mathematicians' knowledge resources constrained their choice of goals - none of the described tasks aimed to address or develop Pedagogical Content Knowledge (Shulman 1986), a type of knowledge outside the designers' expertise. However in the two cases we described, the designers' resources did not only constrain their choice of goals. The designers were not required to cover any pre-defined content. A question they often voiced was *what can I teach these teachers that they don't already know?* To answer this question the designers actively sought out teaching goals where their university conception of mathematics would provide significant added-value, for example their axiomatic approach to multiplication properties or their hierarchical understanding of counting competencies. In this we see evidence of a designing goal - to capitalize on the designers' resource of deep mathematical understanding. This presents an unusual relationship between resources, orientations and goals. In the ROG literature, goals are typically shaped by orientations, and possibly constrained by limited resources. As far as we know, this role of resources in inspiring goals has not been reported.

Faced with the constraints imposed by the teaching situations, the designers added a new overarching goal - to make sure the tasks would be perceived by the teachers as relevant. This overarching goal often forced them to modify their initial teaching goals. In this process they needed to be explicit about their orientations in order to prioritize their goals for the task, and decide which of the goals were critical, and which could be "sacrificed" in the interest of greater relevance for the teachers. For example – addressing multiplication properties in the concrete context of a word problem was found to be more important than the orientation to be true to an axiomatic approach to mathematics. Resolving the conflict between critical teaching goals and the goal of relevance called for some ingenuity, drawing on the designers' mathematical and meta-mathematical resources - their ability to chart a broad space of approaches to making a mathematical point, and choose one that takes the constraints into consideration. This is particularly evident in the applet that prevents the counting of a finite set of objects, forcing the player to find an alternate strategy.

Relevance, Teaching Models, and the Theory of Adult Learning

Based on expectation and feedback questionnaires, there are a number of different modes of relevance that are pertinent for the teachers. They are mostly concerned with mathematics pedagogy - tasks that can be taken as-is to the teachers' classrooms, activities from which teachers gain new insight into student difficulties or innovative ways to teach particular topics. Along with this pedagogical mode of relevance, there is a less prominent yet significant strand in the feedback questionnaires that indicates, at least for some teachers, the relevance of tasks and activities from which they can gain new insight into the mathematical content.

The instructors had different notions of relevance, grounded in their university teaching. What can we expect the theory of adult learning to contribute to our understanding of relevance in the PD? We will use it as a means to unpack the notion of relevance, first to show how tasks that appear relevant for mathematicians may not appear relevant for teachers, and then to gauge how well the redesigned tasks actually address the issue of relevance for teachers.

What are university mathematics students' *needs to know* and *orientations to learning*? A strong mathematical orientation is that "every assertion can be backed by logical reasoning" (Wu 2011, p. 379). University instructors do not discuss explicitly with their students why they *need to know* the proofs underlying mathematical facts and procedures. The *orientation to learning* mathematics at university is shaped by mathematicians' life-problems – proving theorems as part of mathematical research. On the other hand, elementary school teachers' *needs to know* are related to their life-problems of teaching. Teaching mathematical procedures entails a set of life-problems that is completely different from those of mathematicians. Of course, an important part of teaching procedures should be understanding and explaining why these procedures work the way they do, which is actually quite closely related to proving. However, teachers' *life-problem orientation* to learning proofs derives from questions related to teaching, such as: *is this an explanation I can use in my classroom? Do my students need to understand why this procedure works? Are my students ready to understand why this procedure works?* It is interesting to note that these pedagogical questions of relevance for the classroom mirror the principles of adult learning - students' *need to know* and their *readiness* to learn. If the answer to any of these questions is "no", the teachers' *need to know* may not be satisfied. As for understanding mathematics, Ma (1999) has shown the importance of a profound understanding of fundamental mathematics (PUFM) for teaching in elementary school, and teachers obviously strive to understand the content they teach. However, as Ma demonstrated, teachers may not always be aware of the mathematical depth of elementary topics, or how such depth relates to their teaching. Thus, tasks that focus on a deep understanding of the content may not always appear to the teachers as addressing their life-problems.

An important factor in *readiness* to learn is the difficulty of a task – one may not be ready if the task or the target knowledge is too difficult for the learner. We have seen cases where the naive task was more difficult for the teachers than the mathematicians may have anticipated. For example, the notion that there are different types of infinities and that some infinities are considered larger than others, is counter-intuitive and difficult to comprehend.

Need to know is one source of motivation to learn. What else may motivate learning? An appreciation of the aesthetics of mathematics is a central orientation of mathematicians, whereby showing the beauty of some piece of mathematics provides intrinsic motivation to learn. It is not clear to what extent the teachers and the mathematicians share the same sense for aesthetics of mathematics.

The discussion above shows why the naive stage of tasks was not likely to address the constraint of relevance. We now discuss how the redesigned tasks addressed this issue in spite of the differences in the parties' orientations. In both presented cases, a feature of the redesigned task is that it confines itself to elementary school mathematics. The tasks do not assume familiarity with topics that the teachers may not feel comfortable with, thus addressing the teachers' *readiness* to learn. Furthermore, the first task (1–1 correspondence applet) can be seen as an attempt to address the teachers' *life-problem orientation*, by offering an activity that can be used in the classroom. However, we are left with the feeling that the call for relevance was addressed in these tasks in a somewhat superficial manner. Even when the mathematical content was arguably relevant for teaching, this relevance was not made explicit, and the *need-to-know* principle was not addressed. The ROG analysis of the designing process in these cases suggests that the designers retained their original goals and orientations, and regarded the need for relevance as an external constraint imposed on their work. This will be even more evident when these tasks are contrasted with the following tasks (Case 3), described in the next section, where we will argue that the designers appropriated the teachers' point of view, and integrated the teachers' needs into their own orientations and goals.

Case 3: New Modes of Relevance (Grade 3–6 Teachers)

We now describe the designing process of a sequence of tasks that focus on a conceptual understanding of the arithmetic mean and alternative strategies for computing it. The initial designing stages are similar to the cases presented earlier, where the need for relevance is perceived as a constraint. However, as the designing proceeded, the designers' pursuit for relevance took on a different nature. The need for relevance was no longer a constraint that hindered the designers' teaching goals, but rather something that generated new authentic teaching goals.

The arithmetic mean (herein referred to as mean) is first encountered in Israeli mathematics classes in grade 5. The Ministry of Education's curriculum presents it both conceptually - as an intermediate value of some data (no greater than the largest and no smaller than the least) - and procedurally - "the mean of a list of numbers is their sum divided by the length of the list" (Pedagogical secretariat of the ministry of education in Israel 2009, p. 109). Research shows that whereas many students know how to compute the mean, they are often unable to handle its conceptual aspects and fail to define or even describe the mean without relying on the standard algorithm (e.g. Russell and Mokros 1996; Watson and Moritz 2000). In particular, many students fail to develop high-level conceptions of the mean such as: mean as point of balance, mean as typical value, mean as fair-share and mean as midpoint. The mean appealed to the designers since there was a substantial gap between how they perceived it - as mathematically deep, conceptually rich and beautiful - and how they perceived that it is taught in school. They believed that defining the mean algorithmically draws attention to how to calculate the mean, and detracts attention from the meaning of the mean and how the mean may represent a numerical data set. Furthermore, seeing definitions as a foundation on which conceptual understanding may be built, the designers strongly believed it is vital that teachers and students be familiar with a variety of definitions and understand the connections among them.

For these reasons, the designers saw in this topic an opportunity to capitalize on their mathematical knowledge in the pursuit of a worthy teaching goal, while keeping the mathematical content in the forefront. They sought to design tasks based on various conceptualizations of the mean, which would engage the teachers in producing and investigating a variety of alternative definitions. This plan opened a host of mathematical and meta-mathematical teaching goals in line with the designers' orientations. On the mathematical level it opened an opportunity to deepen the teachers' understanding of the standard algorithmic definition and to enrich their conceptual understanding of the mean when new conceptions are compared to old ones. Alternative definitions support a variety of alternative methods to compute the mean, providing opportunities for the teachers to seek alternative solutions for problems, an activity highly valued by the designers. On the meta-mathematical level, weighing the advantages and disadvantages of alternative definitions provides opportunities to reflect on the nature of definitions in general and to draw attention to questions like: *What is the role of definitions in mathematics? What do we expect of definitions? What constitutes a good definition?*

Having decided on these teaching goals, the designers gathered their resources and used their skills and familiarity with different uses of the mean in real life situations as inspiration for the production of a variety of precise definitions of the mean. The tasks described in this case focused on the following definition: *The mean is the point about which the sum of (signed) deviations is zero*. This perspective on the mean is not explicitly stated in the curriculum but it suited the designers' goals and orientations: it is concise, elegant, stated in precise yet elementary mathematical terms, it highlights an important property of the mean and it generalizes and makes precise the notion of the mean as the mid-value of a numerical data set, providing a bridge between the standard algorithmic definition and the conception of the mean as a measure of a typical value of a data set.

However, designing a task that would guide the teachers towards rediscovering this definition turned out to be a serious challenge. Establishing that the proposed property indeed qualifies as a definition requires proof. Namely, given a list of numbers - must there exist a number such that the sum of deviations from it is zero? If so, why is this number unique? How do we know this unique number is indeed the same as the one obtained by applying the standard algorithm? The mathematician-designers felt compelled to raise and resolve these issues, which are at the core of mathematical expertise. It is feasible to design a “naive” task that will guide the solver through the stages of some formal proof showing that the new definition is equivalent to the standard one. This approach was appealing to the designers, but at the same time it raised concerns that the teachers would reject it as being irrelevant for their teaching practices. Indeed, considering this task in the context of the principles of adult learning supports these concerns. It is not clear why the teachers *need to know* a precise proof of equivalence, particularly when the proven claim is not part of the curriculum. The *readiness* of the teachers to learn or to produce a formal proof is also questionable. The algebraic and logical aspects of this proof are beyond the scope of elementary school mathematics and may not make sense to the teachers. In order to state and apply the property of deviations in their classrooms, the teachers need only know *that it is so* and not *why it is so*. Thus it is not clear how learning a proof of the equivalence solves life-problems, i.e. problems related to teaching. An alternative solution that seemed likely to be more relevant for the teachers was to “prove by example” that the sum of deviations from the mean is zero. However such a path would violate the designers’ strong commitment to mathematical validity.

Thus the challenge the designers faced was to design a task that would lead the teachers towards the alternative definition while satisfying the seemingly incompatible requirements of a) being mathematically valid and b) maintaining relevance for elementary school teachers. Looking for a mathematically valid and simple explanation why the sum of deviations from the mean is zero, the designers took tacit inspiration from a well-known problem solving heuristic: “If you cannot solve the proposed problem … try to solve first some related problem … Could you imagine a more accessible related problem? You should now invent a related problem…” (Pólya 1973, p. 114). The simpler related problem was obtained by restricting the discussion to the mean of two numbers. In this case the alternative definition takes on an intuitive meaning - the sum of deviations is zero at the midpoint on the number line, and only there. Thus, uniqueness and existence of the alternative definition are immediate, and proving the equivalence of the two definitions is reduced to acknowledging that the mean of two numbers, taking either definition, is represented by their midpoint on the number line. The designers considered the goal of showing this equivalence both worthy and feasible.

Letting go of the general case and restricting the discussion to the mean of two numbers did not come easily to the designers. Their orientation to state and prove mathematical facts in the most general setting possible was not satisfied. However many of the orientations that shaped the original goals for the task were still satisfied in the context of the mean of two numbers. The alternative definition is still elegant and mathematically precise, and it leads to a simple alternative algorithm for calculating the mean: *find the difference between the two numbers*, *and add half the difference to the smaller of the two* (*or subtract it from the larger*). Furthermore, restricting to the case of two numbers allowed the designers to rely on visual geometric intuition, making the midpoint definition more natural, and a mathematically valid proof of the equivalence of the definitions more accessible. Faced with the dilemma between generality on one hand and precision and clarity on the other, the mathematicians-designers chose precision and clarity. Thus the first stage of the designing process led to the following task:

Task 3.A: For each of the following pairs of numbers mark the two numbers and their mean on the number line. What do you notice about the mean of two numbers?
{6, 8}, {15, 19}, {48, 54}, {96, 106}, {163,177}

Note that the task directs the teachers' attention to the number line while leaving room for the teachers to realize the geometric meaning of the mean on their own, making the revelation more powerful. The pairs of numbers in the task were chosen so that finding their midpoint would be a more appealing strategy than adding them and dividing by two. The designers' intention was to follow the task with a discussion of the shared solutions and encourage the proposal of alternative ways to define and find the mean of two numbers, with the implicit message that alternatives can be rewarding both conceptually and practically. They did not intend to prove the equivalence of these alternatives; rather, they believed that discovering the alternative definition on their own would afford the teachers an intrinsic conviction of the equivalence, which might stand in lieu of a formal algebraic proof.

Had the designers stopped here, this example would be similar to the two examples considered earlier in this paper - task designing under the constraints of a need for relevance. However, at this point a second shift occurred in the designing process, as the designers' quest for *relevance* took the form of a genuine wish that the teachers should make some use of the task in their classrooms. The designing of the following sub-tasks can be seen as an attempt to fathom just what it would take to make the tasks useful for the teachers.

Prior experience indicated that the teachers would be reluctant to make use of such a task in their classrooms, even if they agreed in principle with its didactical value. The teachers professed on several occasions strong beliefs that alternative definitions and solution methods are confusing for their students. Such an argument cannot be overcome by mere mathematical justifications, or by explicitly conveying the designers' own beliefs. The designers realized that only the teachers could convince themselves, and decided to make this issue an explicit goal for a follow-up task.

Task 3.B: Suppose we decide that before introducing the mean in class we want to start by first teaching the mean of two numbers as a separate topic. What definition would you use in your own classroom? Would you teach one of the definitions or more? What would your considerations be?

The task was meant to inspire a discussion that in its essence is not mathematical but pedagogical. Its point of departure was that the mean of two numbers should be taught, avoiding the possible objection that this topic is not part of the curriculum. The focus is on the alternative definition: Should it be taught? Why? The designers hoped that the answers would entail mathematical ideas, and aimed to lead this discussion to what they saw as the desired conclusion, but they did not know what kinds of arguments were likely to be raised. They naturally asked themselves what reasons the teachers might come up with to teach the alternative definition. One answer they came up with was so compelling that they decided to address it with yet another follow-up task, this time returning to the mean of many numbers.

Task 3.C: Ilana, one of your top students, challenged herself to find the mean of the numbers 138, 140, 149, 151 and 152. Eventually, after a long struggle (the use of calculator is forbidden in her class) she found the mean to be 144. However Rachel, her close friend, did her own calculations and found the mean to be 146.

a) Ilana double-checked her calculations and found no mistake. She concluded that the mean is indeed 144. How would you respond?
b) How would you advise Ilana to check her answer? How would you check Ilana's answer?

Rachel is right, the mean is indeed 146. How can Ilana find her error? Since the mean is the outcome of her calculation, all she can do is repeat it, in all likelihood repeating her error. How can the teacher advise Ilana in this situation? An independent correctness check is called for. If the context had been the mean of two numbers, then finding the midpoint on the number line could be such a check. For the mean of five numbers finding the sum of deviations provides such a check. The sum of deviations from 144 is 10, not zero, which indicates that Ilana was wrong.

Discussion of Case 3

We have described a sequence of four sub-tasks. We outlined a naive task that was meant to lead the teachers to rediscover an alternative definition and suggest a formal and general proof of equivalence. In Task 3.A the discussion was restricted to the mean of two numbers, to simplify the precise handling of the alternative definition. Task 3.B encouraged the teachers to discuss possible motivation for teaching the notion of the mean of two numbers as their midpoint on the number line. Task 3.C provided motivation for the teachers to learn, and perhaps also teach, the alternative definition of the mean in its general form. Taken together, the tasks suggest both a practical need for an alternative definition of the mean, and an intuition as to where to start looking for such a definition.

Compared to the naive task, Task 3.A is better aligned with the principals of adult learning. Most notably, this is a task that the teachers can take to their classrooms (addressing *life-problem orientation*), and if not the task itself, then certainly the concept of the mean as a midpoint, and the alternative algorithm for computing the mean of two numbers that it entails. In the examples given in the task the alternative algorithm proves to be more efficient than the standard algorithm, making it a worthwhile piece of knowledge for the teachers themselves, even if they choose not to teach it (addressing *need to know*). An explanation that the mean of two numbers is represented by their midpoint on the number line can be based on intuitive visual or geometric considerations, making it natural and accessible to the teachers (addressing *readiness*). Tasks 3.B and 3.C take one step further. Set in a hypothetical fictitious classroom situation, they appeal to the teachers' experience. Both tasks serve to increase intrinsic motivation by making explicit why the teachers *need to know*, each one addressing a different type of life-problem. Task 3.B touches on the issue of planning a teaching sequence, and Task 3.C is set in an imaginary life-problem.

Compared to Case 1 (one-one correspondence) and Case 2 (multiplication properties), Case 3 is seen to be better aligned with the principles of adult learning, suggesting that it achieves a higher degree of relevance for the teachers. This relevance was not achieved simply by adhering to these principles (of which the designers were not explicitly aware), but rather, we see a shift in the designers' attitude to the teachers' call for relevance. The ROG analysis of the designing processes helps explain how this shift came about. At the most basic level (Tasks 1, 2, and 3.A), relevance for teaching functioned in the designing process as a constraint that needed to be satisfied. The designers considered relevance as a goal, but did not unpack pedagogical implications of the mathematical ideas they wished to convey. Consequently, the goal of relevance was operationalized as a conflict that needed to be resolved, and was achieved by avoiding content that teachers may perceive as not relevant. At the next level (Task 3.B), the designers set themselves a new goal - to convince the teachers to take the content taught in the PD to their classrooms. This moved them away from their comfort zone - the mathematics. Still perceiving themselves as lacking adequate resources for making pedagogical claims, they invited the teachers to discuss practical implications of taking the task to their classrooms. At the highest level (Task 3.C), we see the designers attempting to understand the concept of relevance for teachers, and coming up with insights of their own. We note that the resources the designers turned to in search of relevance for teaching were mathematical – in the case of Task 3.C the designers' insight stemmed from their knowledge of different roles of mathematical definitions. This "mathematical" nature of relevance for teaching is likely to be different from the teachers', yet when we consider the kind of questions the designers were asking themselves about the mathematics they were trying to convey, we see in Case 3 an indication that they had started to appropriate the teachers' orientations, and were integrating them into their own ROG. Their own orientation towards relevance changed as new teaching goals emerged. The resources that they utilized were no longer strictly mathematical – they developed some pedagogical insight (how a teacher might use the alternate definition of mean) and made use of it in their designing process.

Recall that the designers' initial goal in Case 3 was to engage the teachers as math-doers - exploring alternate definitions of the arithmetic mean, comparing definitions, and in the process reflecting on the nature of mathematical definitions - leaving it up to the teachers to work out the pedagogical implications of the tasks and to transfer the content to their classrooms. The teachers' call for relevance led to the emergence of new goals that took into account the nature of teachers' mathematical needs. The three tasks that emerged in Case 3, taken as a whole, provide not only the motivation for an alternate definition (addressing *need to know*), but also an intuition as to where to look for such a definition (addressing *readiness*). This suggests that the pursuit of relevance can generate new mathematical and meta-mathematical design goals, if such relevance is genuinely integrated in the designers' ROG.

Coda: were the Tasks MKT Tasks? Were they Pedagogy-Related?

Our first research question asked about the nature of relevance-for-teaching that was achieved in the designed tasks. In this paper we have taken the teachers' perspective on relevance, which directed and constrained the task designing. We now reconsider the tasks from two other perspectives discussed in the literature review: "MKT tasks" (Suzuka et al. 2009) and "Pedagogy-Related" (P-R) tasks (Stylianides and Stylianides 2010).

The tasks described in Cases 1 and 2, as well as Task 3.A, are typical mathematical tasks in the sense used by Stylianides and Stylianides (2010). The primary mathematical objects in each task - strategies for comparing quantities, illustrating why the associative property of whole number multiplication is separate from the commutative property, and representing the mean of two numbers as their midpoint on the number line - all constitute important mathematical knowledge for teaching. Furthermore, the three tasks satisfy many of the features of MKT tasks in the sense used by Suzuka et al. (2009): the task in Case 1 creates an opportunity to develop a more flexible understanding of the central idea of comparison and connects this idea to 1–1 correspondence; it provokes and builds upon a stumble (the initial tendency to count objects fails); it directly addresses the idea of multiple solution methods. The task in Case 2 creates an opportunity to develop a more flexible understanding of multiplication properties, which underlie the central topic of algebraic manipulations; it provokes a stumble - a tempting calculation strategy that is difficult to justify; it opens an opportunity to build sophisticated connections between word problems and formal algebra; and, it supports multiple representations (algebraic and verbal). Task 3.A creates an opportunity to develop a flexible understanding of a central mathematical idea by introducing a new conception of the mean; it provokes a stumble - the tempting familiar approach of calculating the mean is more tedious than finding the midpoint; it supports multiple representations of the mean and suggests multiple solution methods; it provides opportunities to engage in mathematical practices that are central to teaching, by inviting the teachers to discover the alternative definition on their own. Both Tasks 2 and 3.A address this last feature of MKT tasks in providing teachers the opportunity to engage in the pedagogical practice of explaining. However, neither of these tasks is pedagogy-related (P-R), lacking well defined pedagogical object and pedagogical space.

Tasks 3.B and 3.C are quite a different story. At first sight, Task 3.B is not mathematical, as it seemingly calls for a discussion related to teaching approaches and methods. In this sense, Task 3.B is different from the typical examples of MKT and P-R tasks. However, we note that one of main purposes of Task 3.B was to follow-up and explicate mathematical and meta-mathematical ideas touched on in Task 3.A, such as the affordances of multiple representations and solutions methods - an important element of mathematical knowledge for teaching. Thus, Task 3.B complements Task 3.A, and the two taken together can be seen as a P-R task. Task 3.C was designed to unpack and explicate these mathematical ideas even further. It is aligned with the principles of P-R tasks. It is situated in a pedagogical context but requires a mathematical resolution – coming up with an alternative method for finding or expressing the mean. It builds on various conceptions of the mean, and by guiding teachers to an alternative definition it also underlines the importance of having alternative representations and explanations - valuable mathematical knowledge for teaching. Task 3.C is also well aligned with the features of MKT tasks. It creates a stumble - the teachers are 'stuck' if they do not come up with an alternative method for finding the mean; it creates opportunities for teachers to engage in mathematical practices (e.g. conjecture, generalize and reason); it calls for multiple solution methods and highlights the virtue of such variety; and, it continues the line of the first two tasks in opening opportunities for teachers to develop a more flexible understanding of mean.

Discussing P-R tasks, Stylianides & Stylianides note the importance of a pedagogical *space* in addition to a pedagogical *object*, to help engage (prospective) teachers in mathematical activity from the perspective of an adult who is preparing to become a (better) teacher of mathematics. Task 3.C appears to have such a pedagogical space in the contextual details it provides: Ilana is a top student, and calculators are forbidden. However, we are not certain that this is what Stylianides and Stylianides had in mind. It seems that calculators are forbidden for mathematical reasons - in order to direct the solution towards a mental calculation. We speculate that Ilana was presented as a strong student in order to preempt the teachers' common objection that such a calculation "isn't in the curriculum". We wonder if the idea of pedagogical space might delineate a limitation of research mathematicians' knowledge for task design. Perhaps the design of authentic pedagogical spaces requires a familiarity with classrooms that can only be achieved through experience.

Summary

In this paper we have analyzed three tasks that were designed for a PD for in-service elementary school teachers, focusing our analysis on the designers' pursuit of relevance throughout the designing process, and its impact on the tasks that emerged. Issues of relevance underlie the surface of many PD programs, where experienced teachers are often required to take courses for credit, and may feel that they are unlikely to learn anything that will have an impact on their practice. The issue of relevance was particularly acute in this PD, where the teacher educators - mathematics Ph.D. students - had an overarching goal to focus on broadening and deepening the teachers' mathematical understanding. Relevance could not be achieved simply by adopting the teachers' conception of relevant tasks (for example tasks they could use "as is" in their own classrooms), since the instructors were strongly committed both to their conception of what it means to understand mathematics, and to addressing elementary mathematics in a deep and meaningful manner in the PD.

This unusual setting proved to be a valuable research opportunity for what Streefland has called *vertical didactising* - "the activity of developing new tools and principles for instructional design" (Yackel et al. 2003). Bridging the gap between the content experts and the teachers, which was crucial for designing tasks that would be perceived as relevant by both parties, can be seen as a general design challenge for any PD that aims to address mathematical content without renouncing pedagogical relevance. Bridging this gap was not straightforward, and required some ingenuity on the part of the designers. Tackling this challenge as a two-phase problem appears to have been a useful strategy.

Using the ROG framework, we have shown that at the very least, the designers needed to adopt the goal of satisfying the demand for relevance as a constraint that kept them within the realm of elementary mathematics (e.g. Cases 1 and 2); however in some tasks (e.g. Case 3) we found that the pursuit of relevance did not constrain the designers, but rather inspired the growth of new knowledge and the formation of new teaching goals that addressed both the mathematics and teaching practices. We consider this evolution of the designers' ROG to be evidence of a higher level of appropriation of the teachers' conception of relevance.

We used the theory of Adult Learning as an interpretive framework for analyzing modes of relevance in various tasks, both prospective tasks and tasks that were actually designed and used. We feel that this framework proved to be well suited for analyzing relevance, and suggest that it may be useful in future research for articulating some design principles for PD tasks. This framework also helps highlight the motivational virtues of P-R and MKT tasks. *Need to know* may be achieved by provoking a stumble (you need to know this because your students may also stumble here); the pedagogical object of P-R tasks and the opportunities to engage in mathematical practices central to teaching both address adults' *life-problem orientation*, but additionally may appeal to teachers' *experience*, if the pedagogical object is of the nature "how would you respond to the following teaching situation".

Our main focus in this paper has been on relevance-for-teaching as perceived by the teachers, due to its strong influence on the designing process. We find it interesting that this mode of relevance seems to have converged with relevance as perceived by mathematics education researchers, in the sense that the tasks that were better aligned with the principles of adult learning were also better aligned with the concepts of MKT tasks and P-R tasks. This suggests that mathematics educators, who tend to focus on mathematical and pedagogical aspects of task design, might benefit from a greater awareness of research that attends to motivational aspects, such as the theory of Adult Learning.

Throughout this paper we have not directly addressed the mathematical knowledge for teaching that was taught and learned in the PD. Indeed, the designers were not overly concerned with the nature of the mathematics that the teachers were expected to learn from engaging in the tasks. Yet we feel that both the teachers and the instructors enhanced their knowledge of mathematics for teaching. The mathematicians, through their encounter with elementary school mathematics and teachers, gained insight into questions that were initially outside their field of expertise. The act of designing tasks (rather than making use of tasks designed by others) appears to have contributed to their learning. The teachers, through engaging with these tasks and discussing them with each other and with the mathematicians, had opportunities to develop both mathematical and pedagogical aspects of their knowledge for teaching. Exactly how new knowledge for teaching emerged as a result of this encounter, through the pursuit of relevance, has been the focus of separate analyses of this PD conducted by Cooper (2014, 2015a, b, 2016).

As a final note, we would like to comment on how gratifying it was to observe the mathematics students and the teachers cooperating, with genuine mutual respect for each other's expertise. Such cooperation is not common, and we hope it will provide inspiration for future initiatives that bring together the communities of mathematicians and mathematics teachers. For example, we think it would be interesting to have mathematicians and teachers work together on task design. We do not know if the productive pursuit of relevance that we described in this paper would transfer to such a setting; we speculate that it would.

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