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# Instructional Technology Tetrahedron and Network Visualization: Conceptualizing Online Teaching Through a Lens of Reflective Noticing

**Jinqing Liu**[1,2] · **Orly Buchbinder**[1]

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## Abstract

Online learning and teaching, accelerated by the global pandemic and rapid advancement of digital technology, require novel conceptual and analytical tools to understand better the evolving nature of online teaching. Drawing on the classical model of the instructional triangle and previous attempts to extend it, we propose the *Instructional Technology Tetrahedron* (ITT)—a conceptual framework that integrates technology into the instructional triangle to represent the role of technology, as a learning tool and a mediator between teachers, students, and content. Combining the ITT framework with network visualization strategies allowed for representing the intensity of interactions within the tetrahedron. We illustrate the affordances of the ITT framework by analyzing reflective noticing patterns of three prospective secondary teachers (PSTs) who reflected on the video recordings of their own online teaching, with each PST teaching four online lessons to groups of high-school students. We demonstrate the utility of the ITT framework to characterize individual noticing patterns, in a particular lesson and across time, and to support a variety of cross-case comparisons. The discussion sheds light on the broader implications of the ITT framework.

**Keywords** Online teaching · Instructional Technology Tetrahedron · Reflective noticing · Instructional triangle

✉ Jinqing Liu
jqliumathedu@gmail.com

1 Department of Mathematics and Statistics, University of New Hampshire, Durham, NH, USA

2 Education Department, University of Scranton, Scranton, PA, USA

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## Introduction

The on-going advancement of digital technologies and their use in mathematics classrooms dramatically increased with the development of online learning environments. Even before the global pandemic, which accelerated the shift to online learning, Engelbrecht et al. (2020) described the integration of digital technologies as revolutionizing classroom interactions. The term "educational technology" has been constantly evolving from referring to handheld devices like calculators, through digital learning software like dynamic geometry, to online learning platforms and video conferencing platforms like *Zoom*, *Google Meeting*, and *Microsoft Team Meeting*, popularized during the pandemic (Clark-Wilson et al., 2020). We use the term *technology* broadly and inclusively to indicate that online teaching may involve multiple digital technology tools and platforms.

These new technological environments have given rise to unique discourses and interactions between students and teachers, reshaping the nature of mathematical knowledge (Borba, 2012; Borba et al., 2024). Consequently, researchers (e.g., Clark-Wilson et al., 2020; Sinclair & Robutti, 2020) identify a need for novel theoretical and conceptual tools to characterize the new online format of mathematics classrooms.

One classical model for conceptualizing interactions in mathematics classrooms is an instructional (or didactic) triangle (Cohen et al., 2003; Goodchild & Sriraman, 2012). The basic model consists of a triad: students, teacher, and mathematical content as vertices in a triangle; the mutual interactions between them are represented as the sides of a triangle. Cohen et al. (2003) place the instructional triangle within concentric circles representing multiple environments (see Fig. 1), in which classrooms are situated. In online classrooms, digital technologies serve both as the environment in which classroom interactions take place and as a mediating learning tool. This is apparent when educational technologies, like computer algebra systems or dynamic geometry environments, are integrated within online teaching and video conferencing platforms.

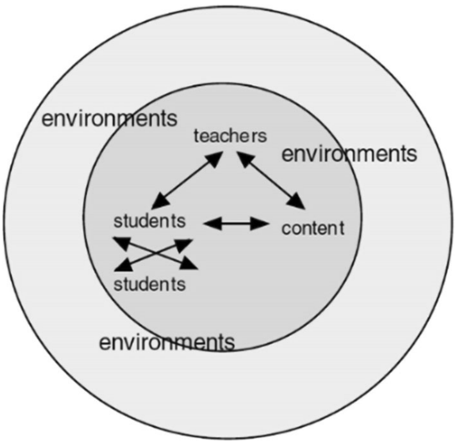



**Fig. 1** The instructional triangle (Cohen et al., 2003, p. 124)

In this article, we describe an *Instructional Technology Tetrahedron* (ITT) framework, modified from the instructional triangle, to capture the interactions in an online classroom environment. Technology is added to the original triangle as a central vertex while keeping all the vertices in a single plane, essentially generating a tetrahedral net[1] (see Fig. 6). This positioning conveys the role of technology as a central link between teacher, students, and content, and as an environment enabling these interactions.

The framework was developed as a part of the larger research examining how prospective secondary teachers (PSTs) learn to integrate reasoning and proving in teaching mathematics (Buchbinder & McCrone, 2020, 2023). The PSTs participated in a specially designed capstone[2] course, *Mathematical Reasoning and Proving for Secondary Teachers*, in which they developed and taught in local schools four proof-oriented lessons, recorded them, and then reflected upon their teaching. Originally, the PSTs used 360-degree video cameras to capture classroom interactions, but during the pandemic, in the Fall of 2020, the course and the practicum switched online, with the PSTs teaching and recording their lessons via *Zoom* video conferencing tool. This transition motivated the need for an analytic framework for capturing PSTs' learning from teaching in an online environment (Liu et al., 2022). The resulting ITT framework is the product of our theoretical conceptualizing and empirical analysis.

In this conceptual article, we present the ITT framework and its theoretical underpinnings and illustrate its use for studying PSTs' reflective noticing (which will be defined below) (Buchbinder et al., 2021; Moore-Russo & Wilsey, 2014) in an online setting. We discuss the affordances of the ITT framework as an analytic and representational tool beyond the illustrated example.

## The Instructional Triangle and Its Extensions

Instructional (or didactic) triangle (Fig. 1) is a classic model for conceptualizing classroom instruction as a series of interactions between teacher, students, and content (Cohen et al., 2003). The roots of this triadic conceptualization can be traced to Brousseau's (1997) theory of didactical situations, which describes the mutual responsibilities and expectations, i.e., the "didactical contract" between the teacher and students with respect to a particular mathematical content. The instructional triangle, as a fundamental model representing the relationships in mathematics classrooms, has been extensively used by mathematics education researchers ever since (e.g., Goodchild & Sriraman, 2012; Herbst & Chazan, 2012; Hiebert et al., 2005; Nipper & Sztajn, 2008).

The instructional triangle highlights the dynamic nature of instruction as a system and a product of interactions between the teacher, students, and content within environments (Goodchild & Sriraman, 2012). Teachers and students are active agents

[1] Hereafter, we use the word *tetrahedron* for simplicity to refer to a *tetrahedral net*.

[2] A course taken by students toward the end of their degree, which serves as a culminating experience of their academic training, and links students' academic competence to their future professional occupation.

in the system, and their knowledge, beliefs, identities, and emotions impact their mutual interactions and their interactions with mathematical content. The latter refers to mathematical concepts, procedures, tasks, problems, and questions, which constitute "mediating artifacts" the teachers choose and enact in class to enable and facilitate student learning. Cohen et al. (2003) depict the teacher, students, and content as vertices of the instructional triangle, with their interactions as arrows connecting the vertices. The triangle is situated within circles representing environments and external influences, such as school leadership, parents, educational policies, and historical and sociocultural factors. Cohen et al., (2003, p. 127) maintain that, "teachers and students shape environments by what they notice and how they respond, but environments shape attention and response."

Over the years, there have been multiple proposals to modify the instructional triangle, to reflect the evolving thinking of mathematics educational researchers, as well as the cultural, pedagogical, and technological developments affecting mathematics classrooms. Some of these modifications to the instructional triangle were described in a special issue of *ZDM: The International Journal on Mathematics Education* edited by Goodchild and Sriraman (2012). Two general types of modifications can be identified: changing the substance of the existing nodes or adding nodes to the instructional triangle. An example of the first approach is Nipper and Sztajn's (2008) use of the instructional triangle to represent various professional development situations, where the vertices of the triangle correspond to organizers, participants, and content (Fig. 2a). The latter can correspond to teacher developers, teachers, mathematics, and the original instructional triangle, resulting in a sequence of nested triangle models (Fig. 2b).

An example of the second type of modification is Jaworski's (2012) model, which adds a vertex of didacticians (Fig. 3a). The didacticians' role is similar to that of teacher developers in Nipper and Sztajn's model, but they are directly connected only to teachers or a community of teachers.

Another example of this kind is Rezat and Sträßer's (2012) model, which expands the instructional triangle to a tetrahedron by adding a vertex of mediating artifacts, such as textbooks, manipulatives, or digital technologies (Fig. 3b). This representation emphasizes the connectivity of all components and the key role of artifacts in mathematics instruction (Rezat & Sträßer, 2012).

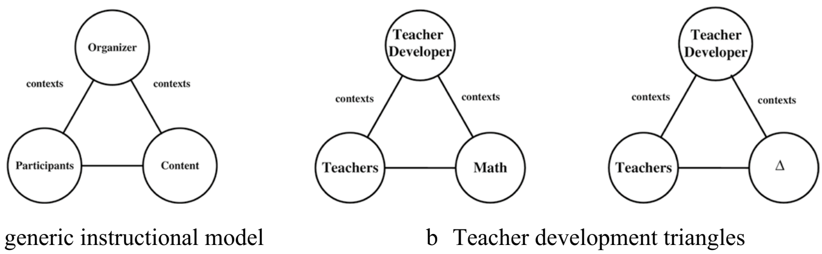
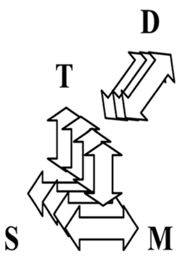
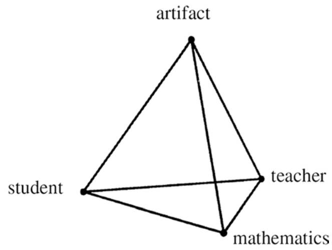



**Fig. 2** Instructional triangle modified models by Nipper & Sztajn, 2008 (pp. 336–337)




a The didactic triangle for multiple teachers (T), students (S), mathematics (M) and didacticians (D) (Jaworski, 2012, p. 624)

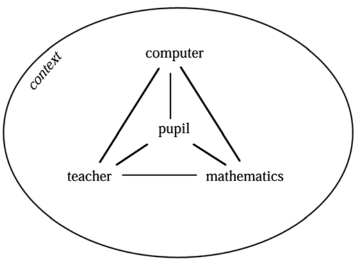



b Tetrahedron model of the didactical situation (Rezat & Sträßer, 2012, p. 645)

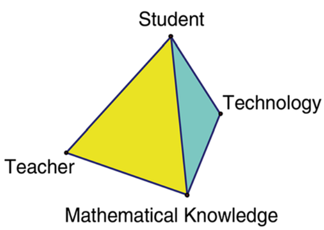
**Fig. 3** Instructional triangle modified models extended with an additional vertex

Reflecting on how digital technologies affect the relationships in the instructional triangle, Goodchild and Sriraman (2012) ponder, "Does the technology introduce another "vertex" such that it is necessary to refer to a didactic quadrilateral?" (p. 582). Several modifications went in this direction. For example, as early as 1986, Tall included "computer" as a vertex in the instructional triangle, creating a tetrahedral net model (Fig. 4a). Similarly, Olive et al. (2010) integrated technology as the fourth vertex in the didactical tetrahedron (Fig. 4b).

Both models in Fig. 4 place the student at the center (or at the top) and place technology as a vertex connected to all other ones in the learning environment. This structure appropriately represents the role of technological tools in face-to-face settings. However, in online settings, technology is more than an element in the learning environment. In such settings, different types of technology come into play simultaneously, some serving as artifacts mediating learning and some as the environment itself (Freiman, 2020). Some modifications of the instructional triangle attempted to represent this duality. For example, Cao et al. (2021) examine the instructional dynamics in an online classroom from the teacher's perspective. The authors use a four-component

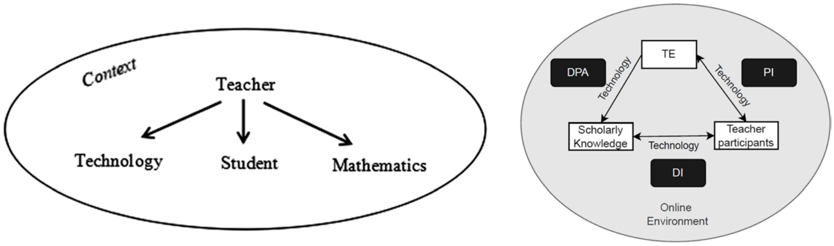



a The Didactic Tetrahedron (Tall, 1986, p. 6)




b The didactic tetrahedron (Olive et al., 2010, p. 136)

**Fig. 4** Please check if figures and captions are captured/presented correctly.The instructional models that integrated technology




a Four-component model of online mathematics teaching (Cao et al., 2021, p. 159)

b The use of technology in the didactic system. Pedagogy integration (PI), Didactic integration (DI), Didactic Pedagogy Anticipation (DPA) (Marfuah et al., 2022, p. 70)

**Fig. 5** The instructional models integrating online technology

model (Fig. 5a), representing how the teacher manages online interactions with students, mathematics, and technology. However, the model does not represent how the teacher manages students' interactions with technology or how students interact with mathematics that is being mediated through technology.

This mediating role of technology is foregrounded in Marfuah et al.'s (2022) didactic system for an online learning environment developed during the study of teachers' online learning of matrix multiplication (Fig. 5b). This triadic model consists of a teacher educator, scholarly knowledge, and teacher participants (akin to Nipper & Sztajn, 2008). Technology is diffused throughout the triangle and positioned along the interaction edges. This representation captures various interactions *through* technology, but not necessarily interactions *with* technology directly, e.g., the teacher educator manipulating *GeoGebra* applets.

This overview illustrates several points. One is that scholars seem to consider the simplicity of the instructional triangle as a strength as they attempt to preserve its basic structure. At the same time, researchers seek to represent the profound changes that digital technology and online tools bring to mathematics classrooms. Second, different models focus upon and elevate different aspects of the triangular (or tetrahedral) instructional model based on the particular research foci (Goodchild & Sriraman, 2012). We followed both tendencies in our Instructional Technology Tetrahedron (ITT), focusing on highlighting technology as a mediating mechanism in an online environment.

## Instructional Technology Tetrahedron (ITT)

Our modification of the instructional triangle was motivated by a desire to conceptualize multiple interactions in an online environment. We extended the instructional triangle by placing technology as a central vertex of the triangle and connecting it to all other vertices: teacher, students, and mathematical content (Fig. 6).

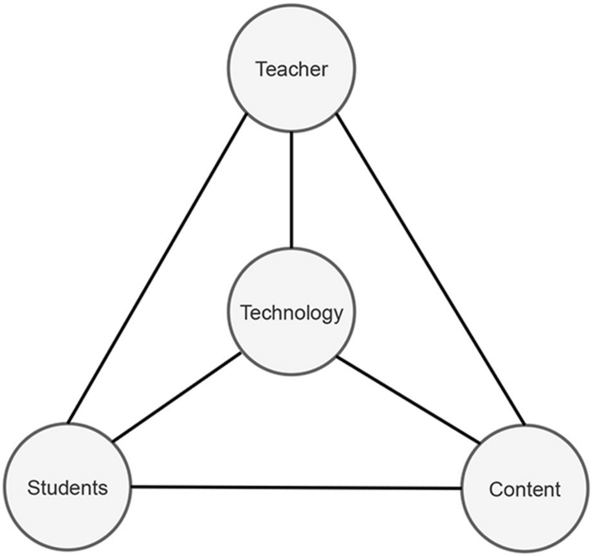



**Fig. 6** The Instructional Technology Tetrahedron (ITT) model

The connecting lines represent pairwise relationships: students–content, students–technology, teacher–technology, teacher–students, teacher–content, content–technology. The resulting model—*Instructional Technology Tetrahedron* (ITT)—resembles Tall's (1986) Didactic Tetrahedron (Fig. 4a), but with technology at the center. This positioning aims to capture the nature of online instruction, in which mathematical content is represented *through* technology and where teachers and students interact with technology, with mathematical content, and with each other *exclusively* through technology (Seaton et al., 2022). Additionally, by keeping the ITT model in 2D, we aim to represent the role of technology as an environment in which all interactions occur and as a tool that supports teaching and learning.

According to Borba and Villarreal (2006), digital technology is saturated with humanity through its design and conception to the extent that it can be seen as having its own agency. Technological tools are shaped by the human mind while simultaneously amplifying and reorganizing human mental functioning (Borba et al., 2024). Ruthven (2012), who used Tall's Didactic Tetrahedron model, justified adding technology as a vertex to represent "the significant role of technology in mediating the relationships between content, student, and teacher [and that] technology vertex can be interpreted at several levels, from that of material resources present in the classroom to that of fundamental machinery of schooling itself" (p. 627). Technology affects how students interact with and learn mathematics, and how teachers design learning opportunities and facilitate student learning. These relationships are represented in our model by pairwise relationship lines connecting technology to content, students, and the teacher. We will further elaborate on and illustrate these connections below.

The ITT framework allows capturing interactions between teachers, students, and content in technologically rich settings, which can be either face-to-face or online environments. For the latter, our modification of Tall's Didactical Tetrahedron by repositioning technology at the center of interactions carries a conceptual meaning of representing the ever-present and unavoidable influence of technology on the nature of interactions between the teacher, students, and mathematical content. Moreover, as we will show later in the article, by combining the ITT framework with the novel tools of *network visualization*, researchers can represent the intensity of various types of classroom interactions in online settings.

It is important to note that classroom interactions can be captured from the perspective of an observer or of a participant within the interactions. Our study adopted the second approach, as we used the ITT framework to analyze the *reflective noticing* of prospective secondary teachers teaching in an online environment. In the following sections, we elaborate on the concept of reflective noticing; describe the study in which the ITT framework was developed; illustrate how we used the ITT framework to analyze PSTs' reflective noticing.

## Reflective Noticing

The concept of *noticing* or *professional vision* has been extensively explored (e.g., Dindyal et al., 2021; König et al., 2022; Sherin et al., 2011). Noticing refers to "specialized ways in which teachers observe and make sense of classroom events and instructional details" (Choy & Dindyal, 2020). There have been several conceptualizations and definitions for the concept of noticing, which may include two, three, or more components, such as attending to elements of practice, perceiving classroom events, interpreting them, connecting to broader pedagogical principles, making inferences about classroom events and making instructional decisions (Dindyal et al., 2021; Sherin et al., 2011). Nevertheless, all definitions of noticing seem to share two common aspects: *attending* (perceiving) aspects of classroom practice and *interpreting* (making sense of) them. Therefore, along with other researchers (e.g., Stockero, 2021), we adopt a definition of noticing comprised of these two elements only.[3]

Moreover, according to Scheiner (2021), attending to and interpreting occur simultaneously and instantaneously, even if they are verbalized sequentially by an observer. Thus, in the sense of attending to and interpreting, noticing may occur tacitly and unconsciously. However, reflecting on what one notices requires a goal-oriented process of *reflection*. Schon (1983) distinguished between *reflection-in-action*, which occurs in the moment of teaching, and *reflection-on-action*, which is a process of deliberate retrospective thinking after the teaching act. We focus upon reflection-on-action, which has been considered as one of the key processes of teacher professional growth (e.g., Hollingsworth & Clarke, 2017). For the reflection process to be productive and conducive to teacher professional learning, it must be intentional, deliberate, critical, and forward-looking. It must also attend to multiple aspects of teaching and interactions between them, interpret and analyze these

[3] For example, we refrained from including decision-making as a component of noticing (cf., Jacobs et al., 2010), since teacher decision-making itself is a complex construct that relies upon multiple individual and institutional factors (e.g., Blömeke & Kaiser, 2017; Schoenfeld, 2010).

multiple aspects, and connect them to prior experiences and educational principles (Davis, 2006; Jay & Johnson, 2002; Moore-Russo & Wilsey, 2014). This means that reflection is always a goal-oriented process, which is intrinsically connected to noticing through the aspects of attending to and interpreting.

Combining the processes of noticing and reflecting together, in our prior work (Buchbinder et al., 2021), we introduced the concept of *reflective noticing*, as a process of teachers deliberately reflecting upon the aspects of classroom interactions they noticed. We used reflective noticing to analyze PSTs' learning from reflecting on video recordings of their own student teaching practice.[4] Reflecting on the video of one's own teaching (whenever it is available) has a high potential to stimulate productive reflection with PSTs (e.g., Blomberg et al., 2013; Liu & Buchbinder, 2022; Moore-Russo & Wilsey, 2014; Nagro & deBettencourt, 2018). Below, we illustrate how we used the ITT framework to characterize PSTs' reflective noticing patterns.

## The Setting, Participants, and Data Sources

As mentioned above, the ITT framework was developed as a part of the larger project, which explored how prospective secondary teachers' expertise to teach mathematics via reasoning and proving evolves over time. The project followed several cohorts of PSTs starting in the senior year of their university-based mathematics education program and three years after graduation. The data for this study came from three undergraduate PSTs in the final year of their program, who participated in a specially designed capstone course, *Mathematical Reasoning and Proving for Secondary Teachers* (Buchbinder & McCrone, 2020, 2023).

The course aimed to enhance PSTs' content and pedagogical knowledge of proof and to connect it to secondary mathematics teaching. The course comprised four modules, each dealing with one proof theme—a topic identified in the literature as challenging to teach and learn. The four proof themes were the following: (1) direct proof and argument evaluation; (2) conditional statements; (3) quantification and the role of examples in proving; (4) indirect reasoning. In each module, the PSTs refreshed and strengthened their knowledge of the proof theme and connected it to pedagogical aspects of secondary teaching by analyzing students' proof-related conceptions and strategies for supporting student learning. A culminating task in each module was designing and teaching in local schools a lesson that integrated a particular proof theme with a topic from a secondary mathematics curriculum (the topics were requested by the co-operating schoolteachers). Usually, the PSTs recorded their lessons with 360-degree video cameras and reflected upon them (Buchbinder et al., 2021). In Fall 2020, due to the global pandemic, the practical component of the course switched online, with PSTs teaching and recording their lessons via *Zoom* video conferencing platform.

[4] Some but not all definitions of noticing (e.g., Sherin & van Es, 2005) include reflective aspects, like connecting to broader pedagogical principles and reasoning about classroom events. Our concept of reflective noticing emphasizes reflection in a broader sense, not just making connections to pedagogical principles.

In this article, we focus upon three PSTs who participated in the capstone course during that semester: Nancy, Olive, and Diane (pseudonyms). They were in the last year of their program, pursuing high-school mathematics teaching certification. The three participants were chosen for the analysis since they were the only PSTs who, after graduation, continued to participate in the follow-up study, which aimed to examine their long-term professional development. During the capstone course, Nancy and Olive taught different groups of students in the same high-school geometry class: however, each PST developed her own lesson plan. Diane taught a group of students in an algebra class, with a different cooperating teacher from Nancy and Olive.

Each lesson lasted about 50 min, with 6–12 students participating. After the lesson, the PSTs watched the *Zoom* recording of their lesson and about every five minutes of the video wrote a comment reflecting upon something they noticed in the video; resulting in about eight or nine comments per video. The exact prompt for this assignment was: "Briefly describe what happened in the lesson during this five-minute episode and reflect on what you found interesting in this episode. That is, I want you to analyze what happened in this episode." It should be noted that this was a mathematical course with pedagogical applications (Wasserman et al., 2023); the PSTs did not receive any training in noticing (cf., Stockero, 2021), nor were they introduced to any frameworks related to noticing or reflective practices. It was expected that PST would make their own decisions on what and how to reflect on their video. The commenting was done in the *Canvas* Learning Management System (Fig. 7). Next, the PSTs wrote a reflection

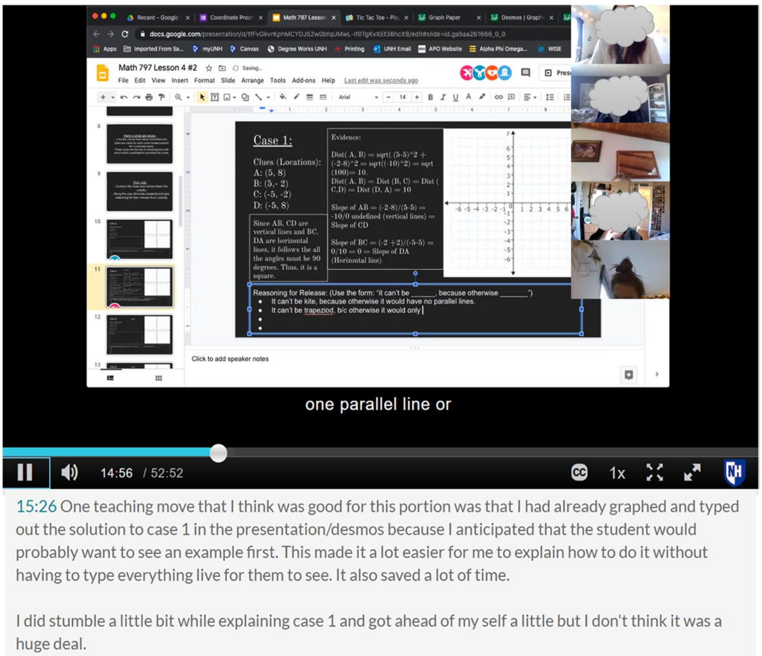



**Fig. 7** Illustration of PSTs' commenting on their own video

essay on their lesson based on five prompts, e.g., "How do you know whether the students understood the content of your lesson?".

The primary data source for this analysis was the PSTs' comments on their video. Supplementary data sources were the PSTs' lesson plans and reflection essays.

## Applying the ITT Framework to Analyze PSTs' Reflective Noticing

In what follows, we present our conceptualization of the ten components of the Instructional Technology Tetrahedron (ITT) framework (Fig. 6) in the context of reflective noticing using illustrative examples from the data. We use "teacher" in a broad sense when describing the categories and "PST" when presenting data quotations. The quotations are brought verbatim, with the following changes: PSTs' and students' names were changed to pseudonyms, the names of the software tools were capitalized and italicized, and whenever the content or the context of the quote was unclear, we added clarifications in square brackets. Text in round parentheses within the quotations is the PST's original text.

### Descriptions of Categories

The *Teacher* category considers the teacher's reflection on their personality, behavioral characteristics, voice pitch, thoughts, emotions, and self-impressions. The teacher pays attention to who they are, and how they look, sound, and behave, rather than focusing on the specific things they do or say in class. Consider this quotation by Olive: "I was mad at myself watching this clip. Stop clicking on the screen so much, Olive! I think the prolonged silences were just getting to me and I fidgeted to make them more comfortable for myself." In this quotation, Olive reflects on her emotional discomfort with the "prolonged silence," and being upset about her unconscious response to silence—fidgeting and clicking on the screen. We identified a range of instances of PSTs noticing and reflecting on themselves in both negative and positive ways, e.g., "I personally think I sound quite calm and collected," "I am saying 'um' a lot," "I was consciously trying to slow down … I'm naturally a fast talker."

The *Students category* refers to the teacher reflecting on students' behavior, feelings and personalities, interactions with peers and classroom participation. For example, Nancy's comment, "I think the students enjoyed having the math memes in the presentation," expresses her attention to how students feel about certain mathematical elements of the lesson.[5] Another example is Olive's reflection on student participation as a group, e.g., "students are talking far less," or as individuals, "she [the student] is always fantastic about participation and contributes some great

[5] For a warm-up activity, Nancy had students share their favorite mathematical mems, i.e., amusing or interesting captions or pictures about mathematics from online or social media sources (Merriam-Webster.com Dictionary, Merriam-Webster, https://www.merriam-webster.com/dictionary/meme).

stuff." The *Students* category captures the instances of the teacher noticing the general behavior of an individual or a group of students, rather than their mathematical thinking, which is captured in the *Students–Content* category, described later.

The *Content* category refers to teachers' reflection on the mathematics of the lesson, including reasons or actions for including or highlighting certain mathematical ideas, or learning opportunities provided by certain content. For example, in the quotation below, Diane reflected on the mathematical objects—the types of graphs she chose to include in her lesson, and her rationale for including graphs of non-linear functions to address a potential misconception:

> This example [a parabola] was a good graph to bring up. Although they [students] are learning about linear functions, I didn't want them to think that linear functions are the only functions out there.

Reflections on *Content* may also include analyzing mathematical content that was added to the lesson "in-the-moment" or should be removed in the future (e.g., "I recognise that I probably could have done without a few of these highlighted examples—they get repetitive quickly"), reflections on the relationship between the school curriculum content and the proof themes of the course, their own minor mathematical errors "hiccups" (e.g., "There was a minor hiccup on two of the slides' co-ordinates"), or new mathematical understanding. Thus, the *Content* code captures the teacher's reflective noticing of the mathematical content of the lesson, often with related rationale for keeping, adding, withdrawing or correcting the planned content of the lesson.

The *Technology* category involves reflecting on technology as a tool, and its functioning (well or not), without relating it to a specific teaching move or students' mathematical thinking, e.g., "I had some technical issues with a link for the exit ticket," "It was nice using this platform [*Google Slides*] to complete the similarity proof, because it didn't require me to change applications mid-lesson." The teacher may notice the challenges of teaching online in general or the affordances or challenges of using a particular software, e.g., *Zoom*, *GeoGebra*. For example, Nancy reflected on the challenge of not being able to see the students' faces and names while they were sharing their screen with students in *Zoom*. She wrote:

> The only thing that wasn't so good in this part of the lesson was that when I was sharing my screen, I couldn't see the students' names and so, when I didn't know a student's name, I felt awful about it. I think I handled it alright though.

The main focus of this quote is on the technology itself and its function for enacting the lesson.

The category *Teacher–Content* refers to the teacher's reflection upon how they taught a particular mathematical content, or their instructional decisions related to teaching certain mathematical content. As opposed to the *Content* category above, the *Teacher–Content* category describes PSTs' reflective noticing of the interaction between teacher and content, including planned teaching moves around the content, e.g., "I gave the example of rectangles having opposite sides congruent to get the juices flowing."

PSTs also noticed and reflected on their instructional decisions around particular mathematical content, e.g., "I think it was a successful teaching move to include some exposition about indirect reasoning at the end of the lesson instead of the beginning. It brought everything together really nicely for the students." The PSTs also reflected on teaching moves they were unsatisfied with and possible alternatives, e.g., "I should have included an image to explain what I am talking about – an angle bisector is not necessarily a median. Although I explain it in words, a diagram would have been more helpful."

The *Teacher–Students* category indicates attending to interactions between the teacher and the students, including teacher strategies for building productive interaction norms, facilitating students' online participation and developing student-centered instruction. Unlike the *Students* code, which captures reflection on students' general characteristics or classroom participation, the *Teacher–Students* code captures teachers' reflecting on their *own* moves to facilitate students' participation and build productive classroom norms. For example, Olive noticed how using student names was potentially conducive to student participation. She wrote:

> I pull John's name into the problem [as the mathematician who made this claim]. I keep making a point to do this, and I do think students appreciate it. Sometimes, when I mention their names, I catch them smiling. […] Hope it encourages participation from everyone else.

The PSTs noticed that students' participation online was diminished compared to what they would have expected in a face-to-face setting, and reflected on their strategies to increase students' participation. For example, allowing and encouraging non-verbal communication like a head nod or shake, encouraging students to respond via chat or vote with thumbs up or down, and randomly selecting students to answer teacher questions.

The category *Students-Content* describes the teachers reflecting on students' mathematical thinking as they interact with a mathematical task or respond to a question. The teacher may notice students' understanding of a particular mathematical idea or acknowledge their contribution. For example, in the context of proving triangle similarity, Olive reflected on students' attempts to generalize a particular type of configuration of triangles, which she called "bowtie" (⋈). Olive wrote:

> Sarah adds, "Won't there always be vertical angles?" and John adds the likelihood of having "a pair of alternate interior angles." Students were able to extend similarity concepts beyond the concrete and consider abstractly how many of the 'bowtie' shaped pairs of triangles could have these two properties and easily be similar. This was an important moment in my opinion.

In other instances, PSTs reflected upon challenges they observed in students' mathematical understanding and background, which they had not anticipated when planning the lesson.

The *Teacher–Technology* category captures teachers' reflections on how they managed technology for teaching effectively. This could cover a wide range of situations, such as reflecting on one's decision to use (or not) a certain technology, e.g., "I liked the use of the *Google Slides* here because I was able to see the students' progress through the similarity proof and get an understanding of what parts they found confusing." The PSTs could reflect on handling technology in the context of making in-the-moment pedagogical decisions and drawing conclusions on how to use technological tools more effectively in the future. For example:

> One teaching move that I found challenging occurred when I wanted to keep the problem up, but also wanted to show the list of ideas that the students were coming up with. I had to make a hard choice, but decided that the list would be more beneficial. I think if I were to redo this, I would write in *GeoGebra* (which I actually did the second time around). It was a little clunky, but at least they could see both things at once.

In this quotation, Nancy describes her struggle with technology, but contrary to the *Technology* category, the focus here is on Nancy's dilemma and eventual resolution of how to use technology in support of her teaching goals. Hence, we coded it as *Teacher–Technology*.

The *Students–Technology* category captures teacher's reflective noticing of how students interact with technology. For example, Olive noticed students' comfort with using the chat feature: "Students are super comfortable in the chat and write some really funny conditional statements!" Diane noticed students' online collaboration, "a few students were working together to move the lines around on the graph." All PSTs noticed the challenges of students collaborating in an online setting. For example, Nancy wrote:

> When I told the students that they could work together, no one did. I think *Zoom* makes this hard because it's not like you can turn to your neighbor and discuss. Instead, if you want to talk, you end up talking in front of everyone which can make people nervous. Also, as a student, it can seem daunting to start up a discussion with your peers. This is something I need to keep in mind for the future.

In this quotation, Nancy reflected on how interacting with peers through technology may feel "daunting" or uncomfortable for the students. She consequently made a note to consider this in her future teaching.

The *Content–Technology* category refers to a teacher's reflection on how technology is useful or not in representing a particular mathematical content. For example, Nancy reflected on her use of an online game of *Tick–Tack–Toe* to illustrate how indirect reasoning is involved in choosing various game moves. She wrote about this online game: "It was applicable to what the goals of the lesson were about (indirect reasoning), and I think the students enjoyed it." Here, Nancy reflected on the alignment between the technology tool and the lessons' objectives. The focus of reflection is neither on the technology itself, nor on Nancy's teaching moves, nor on how students interacted with it, but rather on the relationship between mathematical content and the appropriateness of the technological tool for attending to content-related mathematical goals. Thus, we categorized it as a *Content–Technology* category.

### Additional Considerations Related to Coding Categories

The ten coding categories described above correspond to the ten elements of the Instructional Technology Tetrahedron (ITT) framework, namely, the four vertices:

teacher, students, content, and technology, and the six pair-wise connections between the vertices (see the summary in Appendix: Table 1). These codes captured the richness of PSTs' reflective noticing. In some instances, PSTs' comments touched upon several aspects, encompassing more than one or two vertices. For example, Nancy's quotation above—"I liked the use of the *Google Slides* here because I was able to see the students' progress through the similarity proof and get an understanding of what parts they found confusing"—mentions student confusion with mathematical content, that of similarity proofs.

We interpret this quotation as a reflection on how the technological tool of *Google Slides* supported her instructional practice allowing her to monitor student work. Hence, we coded this as *Teacher–Technology*. While we recognized the potential to deepen the analysis by considering triadic or manifold relationships among the vertices, our primary goal in the data analysis for this study was to extract the core ideas of the PSTs' reflective noticing comments, which led us to focus on the four vertices and the six connections between them (ITT model in 2D). Future studies can expand upon our current exploration by examining the faces of the tetrahedron (ITT model in 3D).

Another theme that emerged in the PSTs' reflections was time management. Before the capstone course, the PSTs had never taught full-length lessons to school students, surely not online. Also, the PSTs had no interaction with the students outside the four lessons spread across the semester; their planning relied on the information provided by the co-operating teacher. With the limited lesson planning experience and limited information on students' knowledge, the PSTs often had to deviate from their lesson plan and make in-the-moment modifications due to time constraints. As we analyzed PSTs' reflective comments on time management, we were able to categorize them within other ITT categories, such as *Teacher–Content* or *Teacher–Student*. For example, in the comment below, Olive criticized herself for taking over the discussion and reflected upon the benefits of allowing students to explain their thinking despite a tight lesson schedule:

> I see myself picking up more teacher dialogue than I like [...]. When students are unsure, it's good to discuss the question at length – no matter how much of a schedule I am trying to keep. I would rather take the time to explain this than just provide the answer and continue.

This time-related quotation was coded as *Teacher–Student*, since its focus is on the importance of teacher–student mathematical dialog. Time-related comments could also be coded as *Teacher–Content*. For example, Diane wrote: "I made a decision at this point to skip even discussing what a domain and range of a function were because I wanted time to do the activity on conditional statements." Here, the focus of the reflection is on the teacher's decision to prioritize a particular content—conditional statements—over another due to time constraints. Thus, we did not include "time" as a separate category.

## Data Analysis and Visualization

We used the ITT framework and the coding scheme described above to analyze the reflective noticing of Olive, Nancy, and Diane in their comments on the video recordings of their lessons. Given the nature of the ITT framework, our analysis primarily focused on capturing what aspects the teachers attended to within the context of “reflection on action.” We used the teachers’ interpretations predominantly to determine the primary focal points of their reflective comments. The lengths and the richness of these comments varied. Some could be assigned a single code, while others addressed multiple coding themes. In these cases, longer comments were broken into shorter thematic units and assigned separate codes. Overall, 304 codes were assigned across the 12 reflections (three PSTs, four reflections each), with Olive contributing 143 codes, Diane 108 codes, and Nancy 53 codes.

The coding process began with the two authors individually coding portions of the data (one reflection from each PST), using initial general descriptions of the codes, which we based on the initial theoretical assumptions on what each code (e.g., *Teacher–Technology, Students–Technology, Content*) could mean in the context of our study. The researchers met weekly to discuss and reconcile the codes and refine the coding scheme. In this process, we enriched the initial coding scheme using examples from the data and clarified the distinctions between various coding categories (e.g., between *Technology* and *Teacher–Technology*). Next, a third researcher, who was working on the large project, was trained on using the coding scheme and then independently coded three complete lesson reflections – one from each PST.

The average agreement across the three coders on these three reflections was 86%, indicating a relatively high coding consensus. All discrepancies in the remaining 14% of codes were resolved through discussion. This process led to refining further the coding scheme and fine-tuning the categories, resulting in the final coding scheme of ten codes and the additional coding considerations described above (see Appendix: Table 1 for the summary of the coding scheme). With this refined scheme, we re-coded the rest of the data, after which the first author reviewed and verified the accuracy of the coding again for the entire data corpus. Any remaining discrepancies were resolved through discussion. Once the data was coded, for each PST, we calculated the frequency of codes in each lesson reflection, both as numeric counts and in percentages. Using percentages allowed for identifying data trends across the three PSTs and across the four lessons.

To visualize the outcomes, we drew inspiration from *network visualization*—a process of visually representing data points and connections between them. In this process, the data points are nodes of a graph, and connections between the nodes are the graphs’ edges. The size of the nodes and/or the thickness of the edges represent various data characteristics, like edge strength and frequency. Network mapping and visualization have been increasingly used in mathematics education research to visualize relationships in complex systems. For example, Weinberg et al. (2016) have used network graph methodology to visualize the structure of lectures in abstract algebra as a network map of mathematical narratives, events, and connections between them. Valero (2010) used network visualization to represent the whole

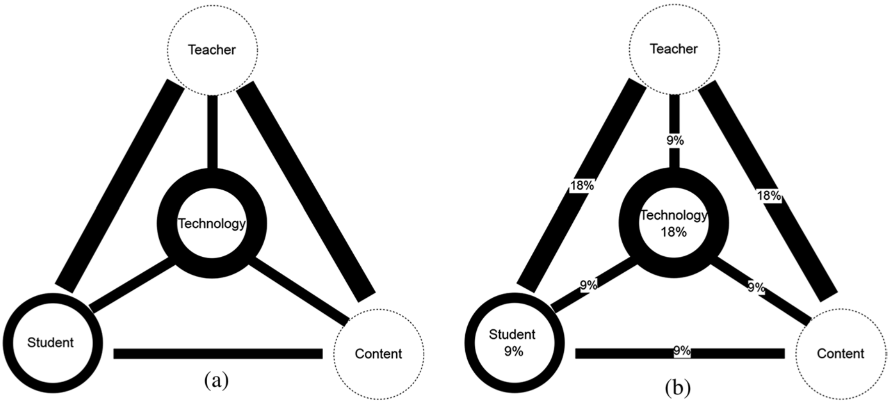



**Fig. 8** **a**–**b** ITT network map of Nancy's reflective noticing in lesson 2

field of mathematics education as a "network of social practices." Working from the theory of embodied mathematics, Mowat and Davis (2010) used network analysis to represent mathematical knowledge as networks of concept as nodes and conceptual metaphors as links among them.

We visualized the PSTs' reflective noticing in each lesson using the ITT model (Fig. 6) as a network. The percent of each code is represented by the width of the line edge, i.e., 1% code corresponds to 1 pt. line width on the diagram. Codes that did not appear, i.e., 0% are represented by dotted lines. For example, Fig. 8a shows a network map of Nancy's reflective noticing in lesson 2; Fig. 8b shows the same map with the marked percentage of each code.

In lesson 2, Nancy integrated the proof theme of conditional statements within the geometry content of Isosceles and Equilateral Triangles. Nancy used various technological tools in her lesson. She created slides in *Prezi* to introduce definitions and examples of conditional statements and the converse. Next, she had students work individually on a *GeoGebra* task, containing three conditional statements: (1) [In triangle ABC] if $\overline{BD}$ is a median, then it dissects angle B; (2) if a triangle is equilateral, then it is isosceles; (3) in ΔHIJ, if K and L are mid-points of $\overline{HI}$ and $\overline{HJ}$ respectively, then $\overline{KL}$ is a mid-segment. For each statement, the students were asked to: (a) identify the hypothesis and conclusion; (b) decide if the statement is true or false and if false, construct a counterexample; (c) formulate the converse statement; (d) decide if the converse is true or false, and if false construct a counterexample. Figure 9 shows the *GeoGebra* screen with the first statement and Nancy's solutions (in red). To summarize the lesson, Nancy created an "exit ticket" in *Google Forms,* eliciting student feedback about their understanding of the topic of the lesson.

The two versions of Fig. 8 show that in lesson 2, compared with other vertices of the tetrahedron, Nancy reflected quite extensively on *Technology* (18%) and how it worked for her in the lesson (e.g., "the transition from Prezi presentation to *GeoGebra* was pretty smooth and I think I did a good job explaining the key aspects of *GeoGebra*. One really important thing that I mentioned here was how to scroll! [in *GeoGebra*, to get to all the statements on the screen]." Nancy also reflected on all types of

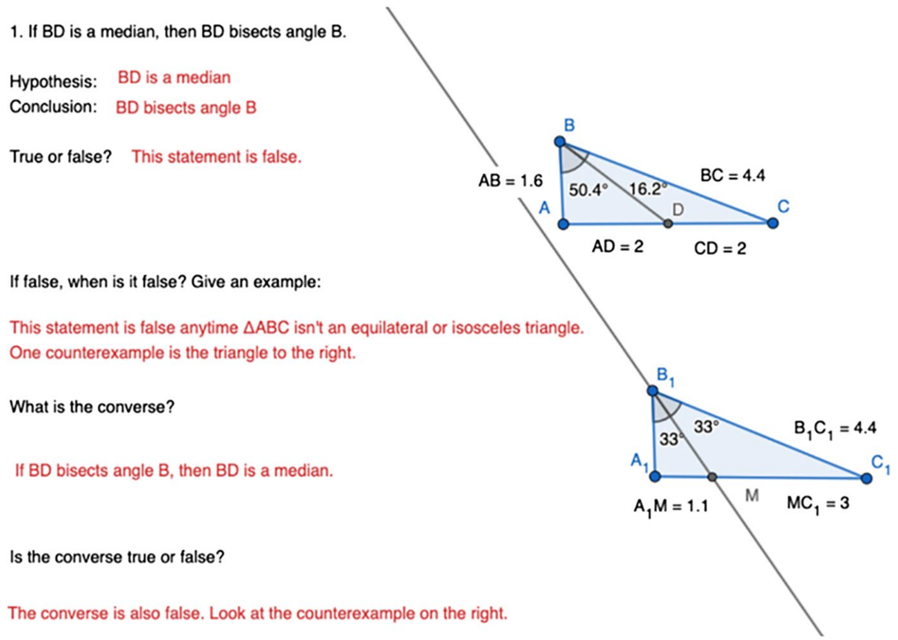



**Fig. 9** Nancy's solutions to students' GeoGebra activity, Statement 1

interactions involving technology (9% each, see Fig. 8). For example, concerning *Teacher–Technology*, Nancy was pleased with herself for creating solution screens in advance to share with the students during the lesson summary. She wrote: "I used my solutions to show a counterexample. It saved time because I didn't have to construct a counterexample from scratch on the fly." This shows her reflecting on the effective use of technology to support educational activity and maintain the flow of the lesson.

Additionally, 36% of Nancy's reflections (at 18% each) concerned *Teacher–Students* (e.g., "I liked giving students a chance to work on the problems on their own … because then you'll have a better idea of what you need to focus on later on"), and *Teacher–Content* interactions, (e.g., "I asked them [students] what 'bisecting' means. This is important because I wanted to make sure they understood what a conditional statement is saying."). Contrary to the first lesson, Nancy did not reflect upon herself, perhaps due to getting more used to hearing herself on video; nor on the mathematical content of the lesson, about which she felt quite confident. In her post-lesson reflection, Nancy rated her performance as five out of five, and wrote that she felt "really good about the lesson and how it went" and doing "a good job of explaining everything."

This visualization of reflective noticing on the ITT image captures the foci of teacher noticing and their intensity, i.e., frequency in a particular lesson. Using this process, we created 12 ITT visual network maps, with and without marked percentages (Appendix: Fig. 14). In what follows, we illustrate the affordances of the ITT framework and its visualization as an analytical tool.

## Illustrating the Affordances of the ITT Framework

### Portraying an Individual PST's Trajectory of Reflective Noticing: The Case of Nancy

The ITT framework allows examining and visualizing individual PSTs' trajectory of reflective noticing across multiple points in time. Figure 10(a–d) shows Nancy's reflective noticing map in each lesson and across the four online lessons, shown without the percentage markings for a more holistic visual perception of the data patterns.

Nancy's online lessons were on the topics in high-school geometry: supplementary and vertically opposed angles; isosceles and equilateral triangles; triangle similarity theorems; analytic geometry proofs about quadrilaterals. Each lesson focused on one topic respectively.

As Fig. 10a shows, in lesson 1, Nancy reflected extensively on herself (*Teacher* 21%), noticing her speaking habits (e.g., "I am saying 'um' a lot."), her fast-speaking pace and reflecting on the need to slow down: "I need to slow down and not rush through explanations. I'm naturally a fast talker and when I get nervous, I tend to talk even faster … I need to work on slowing it down in the future." Nancy seldom returned to these points in the following lessons (Fig. 9 b–d), suggesting that the heightened focus on herself was due to the novelty of the online teaching experience, and of watching herself on video. Nancy's focus in lessons 2–4 shifted away from herself toward various types of interactions between the teacher, students, technology, and content, with a slightly different reflection focus in each lesson.

There is a strong prevalence of *Teacher–Content* codes in lessons 1 and 4 (43% and 35% respectively) (Fig. 10 a, d). In lesson 1, Nancy's main concern was making sure the lesson achieved her teaching goals. For example, this is how Nancy critically reflected on her pre-labeling the vertically opposite angles in a diagram, in advance of the students' proving the congruence of vertical angles. She wrote:

> I realized after the fact that I had already prelabeled the angles 1, 2, 3, and 4. I probably should not have done that because one of my goals for the lesson was to have students realize we needed to generalize the angles to prove all cases, but by already labeling the angles, I kind of defeated the purpose.

Nancy's tightened focus on *Teacher–Content* resurfaced in lesson 4, the last one of the semester. As opposed to all previous lessons, where Nancy designed her own tasks, in lesson 4, she relied heavily on the activity designed by her cooperating teacher. It

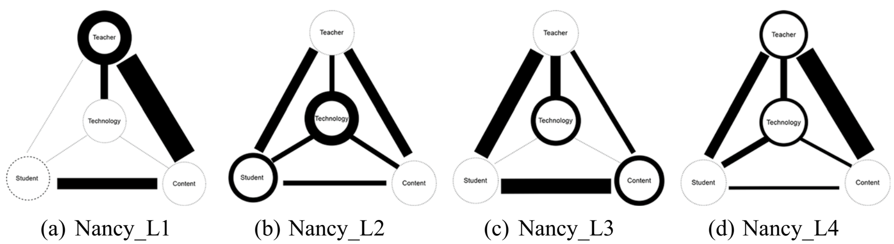



**Fig. 10** **a**–**d** Nancy's reflective noticing patterns across the four online lessons

was a game called "The Quadrilateral Detective," where students used analytic geometry proofs to determine the type of quadrilateral when given the coordinates of its vertices. To integrate indirect reasoning in this activity, Nancy had students formulate statements, such as "The quadrilateral cannot be a kite because otherwise, it would not have parallel sides," and justify these statements with calculations. In the summative course essay, Nancy shared that this lesson was most challenging for her content-wise. She wrote: "The only proof theme that I found to be difficult to incorporate was indirect reasoning. I wasn't sure how to create an organic environment that fostered this kind of thinking on top of relating it to the material that the students were currently studying." Nevertheless, Nancy did her best to move forward with this online lesson. She led students through solving one analytic geometry proof example as a group, and then asked students to work in pairs on other proofs. This appeared to be too difficult for students to do on their own and for Nancy to manage online. She became nervous about not being able to achieve her content goals related to indirect reasoning:

> The students took longer on the co-ordinate proofs than expected. [...] I just wanted them to have something written down for the indirect reasoning piece so I could use it later. [...] I decided [...] they could finish the co-ordinate proofs later.

This helps to explain Nancy's increased focus on *Teacher–Content* interactions in lesson 4.

Across lessons 2–4, the *Students–Teacher* codes were prominent (18%, 27%, and 18% of codes, respectively), as seen in Fig. 10 b–d. Nancy reflected on her interactions with students and on supporting their mathematical engagement. For example, Nancy thought about how she handled one student's mistake by asking her to recall definitions of isosceles and equilateral triangles, which helped the student to correct herself:

> Instead of telling her that this was wrong, I asked her what the definitions of an equilateral and isosceles triangle are. This was key because by asking her questions, she was able to amend her answer later on.

Nancy's reflective noticing of *Technology*, with various interactions, is present in lessons 2–4, while in lesson 1, the only technology-related component was *Teacher–Technology* (14%). In lesson 1, Nancy was the one manipulating a *GeoGebra* applet, while students watched and contributed verbally or by writing responses in the chat. However, as the semester progressed, Nancy delegated more responsibility to students in handling digital tools by themselves. For example, in lesson 2, Nancy had students themselves interact with a *GeoGebra* applet; in lesson 3, students completed two-column proofs about triangle similarity by writing steps in shared *Google Slides*; in lesson 4, students used *Desmos* to plot the vertices of quadrilaterals and to verify the resulting quadrilateral's type. Nancy's reflective noticing patterns in Fig. 10 reflect her attention to delegating responsibility to students for interacting with various technological tools.

## Cross-Case Comparisons

In addition to depicting individual PSTs' reflective noticing trajectories, the ITT network maps provide important visual tools for comparing across cases. The comparison can be made at the level of trajectories or the level of individual lessons. For example, Fig. 11 (a–d) shows Diane's reflective noticing trajectory across the four online lessons. Visually, Diane's pattern of reflective noticing is very different from that of Nancy's. It seems to be quite consistent across the four lessons, skewed toward the content vertex of the tetrahedron and the teacher or student interactions with the content.

The categories of *Content* and *Students–Content* are dominant in all lessons, except lesson 2, where the main focus was on *Teacher–Content* interaction. In that lesson, Diane discovered that students' knowledge of functions did not match her expectations, so she had to adjust her planned activities accordingly, on the spot. This was a stressful experience on which she reflected extensively, e.g., "considering that their teacher asked me to do a review of linear functions and rate of change, I was a little surprised. I made a decision at this point to skip even discussing what a domain and range were because I wanted to do the activity on conditional statements."

Diane's reflective noticing pattern is characterized by a strong *Students–Content* focus, suggesting that she paid careful attention to students' mathematical thinking. For example, in lesson 4, which dealt with indirect reasoning, Diane asked students to explain whether the quotient rule of exponents can be used to simplify $\frac{5^6}{7^3}$ or not and explain their reasoning. She then reflected upon how students responded when she pressed for an explanation:

> When I asked the students "why," one student told me that the "x's could not be different." When I asked him to clarify what he meant, he said that, "you cannot cancel 7's and 5's." I appreciated that he was thinking about the meaning of the rule here, and what it actually does mathematically when giving his explanation.

Reflective noticing of *Technology* is visible in the first two lessons (Fig. 11 a, b) when Diane was getting acquainted with teaching online. She reflected on the challenges of not being able to see all student names while sharing the screen, e.g., "I wish that I had been able to see their names," and internet connection issues. These

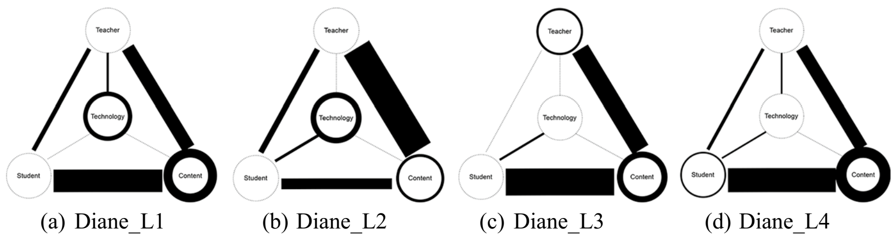



**Fig. 11** **a**–**d** Diane's reflective noticing patterns across the four online lessons

comments disappeared in the later lessons (Fig. 11 c, d), as Diane became comfortable with using and coordinating technology— *Zoom* and *Google Slides*.

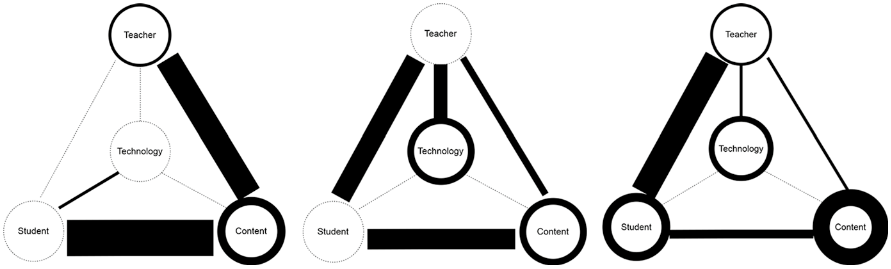
Diane's lessons were on Algebra 1 topics: solving equations with variables on both sides, the concept of function and rate of change, systems of linear equations and the division rule of exponents. For each lesson, Diane created interactive *Google Slides*, which contained the tasks for students to solve and some explanations. In lesson 1, Diane was the only person to manipulate the slides while students followed along, but in lessons 2–4, students were given opportunities to manipulate some aspects of the slides, like moving a vertical line to see where it crosses a graph of a curve. Consequently, Diane reflected on *Students–Technology* interactions, e.g., "a few students were working together to move the lines around on the graph," and on how she utilized technology (*Teacher–Technology*), e.g., "As usual, I had a hard time getting an explanation out of the students, so I encouraged that they give reasons in the chat."

Still, these reflections were minimal compared with Diane's main focus on *Teacher–Content* and *Students–Content* interactions. Overall, comparing Diane and Nancy's reflective noticing patterns across the four lessons, we see very different trajectories over time. What is being compared here is not the mathematical lessons themselves, nor how the two PSTs used technology. Rather, the ITT networks allow us to visualize and examine what aspects of the technology-rich online mathematical lessons that the PSTs noticed and reflected upon in each lesson and how the foci of their reflective noticing changed across the four time points.

Another form of cross-case comparison with ITT can be made on the level of a single lesson across several PSTs. For example, Fig. 12 shows reflective noticing patterns for Diane, Nancy, and Olive in lesson 3. By this time in the semester, each PST had taught three online lessons and had opportunities to experiment with different types of technology, like *Google Slides*, *Google Forms*, *Prezi*, and *GeoGebra*. In the course sessions, the PSTs were provided with a space to share and discuss their online experiences and give each other feedback on how to use different technological tools and how to address low student participation via *Zoom*. In addition, in lesson 3, all three PSTs integrated some opportunities for students to manipulate technology, like writing solutions and moving objects (graphs, triangles) on the shared slides. Thus, in terms of the PSTs' relative comfort with online teaching and level of student involvement with technology, lesson 3 represents an optimal time-point for cross-case comparisons.

Another common thread for these lessons is the proof topic—quantification and the role of examples in proving, which all PSTs had to integrate into their online lessons; this was dictated by the course structure (Buchbinder & McCrone, 2023). Additionally, Nancy and Olive had a common cooperating teacher, and although they developed their own lesson plan and had a different group of students, the geometry topic was the same, and students came from the same class. In other words, there are many commonalities justifying cross-case comparisons.

With this background in mind, Fig. 12 provides an interesting window into the three PSTs' reflective noticing styles. Diane paid almost no attention to technology, although she reflected upon how students interacted with it (Fig. 12a). She reflected mainly on her own and on students' interactions with mathematical content, which is consistent with her longitudinal pattern (Fig. 11). Both Olive and Nancy reflected upon *Technology* and *Teacher–Technology* interactions. For




(a) Diane_L3 (b) Nancy_L3 (c) Olive_L3

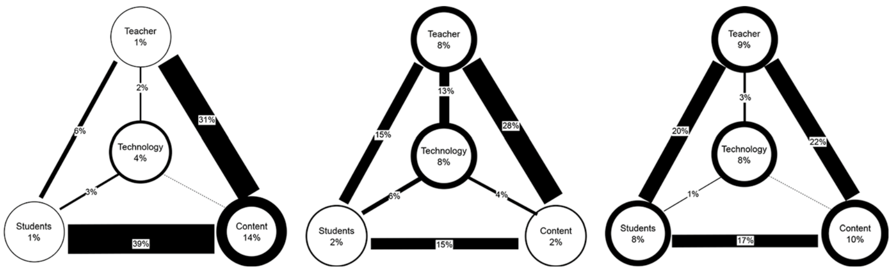
**Fig. 12** **a**–**c** Reflective noticing patterns, lesson 3—Diane, Nancy, and Olive

example, Olive attended to the fact that, while she created an opportunity for students to mark a diagram in the shared slides by moving markers on the screen, she did not account for the size of the diagram, making it difficult for students to manipulate. The main focus of Olive's reflective noticing was on the mathematical *Content* and *Teacher–Students* interactions, meaning on her pedagogical moves to support student mathematical engagement (Fig. 12c). Nancy, in this lesson, attended mainly to *Teacher–Students* interactions and *Students–Content* interactions, while also reflecting upon her own handling of technology (Fig. 12b).

## Aggregated Cross-Case Comparison

Another type of affordance of the ITT network map involves aggregating reflective noticing scores across all four lessons for each participant. Figure 13 shows an example of such representation, with the percentages of each category overlayed on the ITT map.

Figure 13 shows that across four online lessons, all three PSTs reflected extensively on *Teacher–Content* interactions (22–31% of all comments). For Olive and Nancy, this was the modal category of noticing, while Diane's modal category was *Students–Content* interactions. Among the three PSTs, Olive's reflective noticing was most evenly distributed among the four vertices of the tetrahedron,




(a) Diane Total (b) Nancy Total (c) Olive Total

**Fig. 13** **a**–**c** Aggregated patterns of reflective noticing for the three PSTs with *Technology* receiving a similar focus as other categories of the "original" instructional triangle of students, teachers, and content (e.g., Cohen et al., 2003). Interestingly, while Olive reflected extensively and rather evenly on the pairwise interactions among the teacher, students, and content, she paid the least attention, among the three PSTs, to interactions with technology. Comparatively, Olive and Nancy had almost the same total percent of reflective noticing codes for the pairwise interactions between the teacher and students (Nancy 58%, Olive 59%), and the same percent of *Technology* codes (8%). However, Nancy also had the highest percentage of codes related to *interactions* with technology, in particular *Teacher–Technology* (13%). Indeed, among the three PSTs, Nancy was the one who experimented the most with different types of technological tools in her lessons and with different modes of engaging students with these tools, like having students explore *GeoGebra* applets by themselves and post the results of their investigations in shared *Google Slides*.

Overall, comparing the data for the non-technological and technological aspects in Fig. 13, it is evident that PSTs tended to reflect predominantly upon the non-technological aspects. Additionally, of the technological aspects, the PSTs merely reflected on students' interactions with technology. One possible explanation for this could be that, due to the novelty of the online teaching experience for the PSTs, their attention to the technological aspects of online teaching was still in an emerging stage. Future studies can examine this aspect more in-depth.

## Discussion

In this article, we introduced the Instructional Technology Tetrahedron (ITT) Framework as a conceptual and analytic tool for characterizing interactions in an online mathematics classroom. This contribution is in response to the call articulated in the research literature (e.g., Cao et al., 2021; Clark-Wilson et al., 2020; Marfuah et al., 2022) for the need to extend the basic instructional triangle to represent classroom interactions in technology-rich and online settings. This need is motivated by the broad acknowledgment in mathematics education that technology profoundly changes the ways teachers and students interact with each other and with mathematics: Even more so, when the whole learning process is moved online (Engelbrecht et al., 2020; Seaton et al., 2022; Sinclair & Robutti, 2020). Hence, there is a need for conceptual and analytic tools for capturing these unique classroom interactions.

Like other researchers before us, we used the basic instructional triangle framework as a starting point for our conceptualization, while referring to various extensions of the Triangle proposed in the literature (Goodchild & Sriraman, 2012). The resulting ITT framework closely resembles Tall's (1986) Didactic Tetrahedron (Fig. 4a), but with technology at the center and in the same plane as the other vertices. Keeping all the vertices of the ITT framework in one plane (a tetrahedral net, rather than a 3D tetrahedron) aims to emphasize that we see all the vertices interacting with each other in the same plane, so to speak, rather than a certain vertex being above others. Placing technology at the center is a minor change from Tall's Didactic Tetrahedron model, but it carries a conceptual meaning, specifically, for capturing classroom interactions in an online setting, where all interactions between the interlocutors occur *exclusively* through technology.[6]

The word "technology" may mean various entities: digital tools for learning mathematics like *GeoGebra* and *Desmos*, and non-mathematics-specific, video-conferencing platforms like *Zoom* and *Microsoft Teams* (Freiman, 2020). Distinguishing between technology as a *tool* and a *medium* may be useful for exploring certain research questions (Borba, 2012), and is worthy of future exploration. We can envision, that such distinctions can be represented on the ITT map, for example, using colour, different line styles, vertex shapes, or other methods of network visualization (e.g., Mowat & Davis, 2010). However, making such distinctions was outside our objectives in this study.

For our goal of conceptualizing different types of interactions in an online learning setting, the relative simplicity of the ITT framework is an advantage. Combining the ITT framework with network mapping and data visualization techniques constitutes unique and novel affordances of ITT network maps for representing the intensity (or frequency) of various classroom interactions in online (and/or technology-rich) settings.

The second aim of our article was to illustrate how the ITT framework with network mapping can be used as a conceptual, analytic, and visualization tool in an empirical study. In our study, we used these tools to analyze the reflective noticing patterns of three PSTs who commented on video recordings of their online lessons. To examine how the PSTs learn to teach online, we captured the foci and the intensity of their reflective noticing on the ITT network maps.[7]

We illustrated several affordances of ITT maps, for example, for tracing an *individual PST's reflective noticing trajectory over time* (i.e., at different points in time). Using the cases of Nancy (Fig. 10) and Diane (Fig. 11), we illustrated how the reflective noticing patterns of these PSTs changed from one lesson to another, as they gained more experience with online teaching. The stark contrast between these two cases also shows the utility of the ITT network maps for *cross-case comparisons*. These comparisons can be made *across several time points*, as we showed for Nancy and Diane, or for a *single time point*, like in Fig. 12, which juxtaposes all three PSTs' reflective noticing patterns in their third lesson. Additionally, the PSTs' reflective noticing can be examined in *aggregate across several lessons* (Fig. 13). This can provide information about general tendencies in the noticing patterns of a particular PST and compare across several PSTs based on their aggregated data.

Examining the reasons behind these tendencies is an empirical question bearing further exploration. Our goal here was to illustrate how the configuration of the ITT framework (with technology as a central vertex) and the network visualization tools, allowed eliciting these various outcomes. Thus, our study contributes both to the body of knowledge on teaching with technology in online settings, an area which has grown significantly, especially in the post-pandemic era, and to the literature on teacher noticing (e.g., König et al., 2022; Santagata et al., 2024; Sherin et al., 2011).

[6] Indeed, we posit that the ITT framework, with network visualization in the form of ITT maps can be used to capture interactions in a face-to-face classroom where digital educational technology is used (just as Didactic Tetrahedron).

[7] It is beyond the scope of this article to present the findings of that study, and we do it elsewhere (Buchbinder & Liu, in preparation).

Some limitations of the ITT framework and potential ways to mitigate them need to be acknowledged. For example, the ITT framework does not capture other dimensions of noticing (e.g., Stockero, 2008), like a positive or negative stance toward the object of reflection; nor whether a teacher reflected on a specific event or on a general aspect of teaching. Further, the ten categories of the ITT framework may be insufficient for capturing more fine-grained themes, like certain aspects of student mathematical thinking (Sherin & van Es, 2005). To mitigate these limitations, the ITT framework may be used in conjunction with other tools from the noticing literature or by introducing sub-categories to respond to a particular research question. For instance, in the *Content* category, we additionally noted instances where the PSTs reflected upon specific proof-related topics, like conditional statements, counterexamples, or indirect reasoning, which was important in the context of examining PSTs' proof-related teaching practices (Buchbinder & Liu, in preparation).

Some uses of the ITT maps, like cross-case comparisons, require ensuring some basis that warrants such comparisons. Although educational research does not lend itself easily to fixing variables, we attempted to substantiate commonalities among cases. Specifically, our study was done in the context of a single course, where the PSTs had comparable mathematical and pedagogical backgrounds and similar educational experiences. Each PST taught the same group of students, in the same mathematical area (Algebra or Geometry) with the same proof themes, but the content of the lessons differed. Beyond these contextual factors, the basis for cross-case comparison stems from the fact that our research focused on the PSTs' *reflective noticing*, rather than the specifics of their lessons. Still, the ITT maps can serve as an even more robust analytic tool in studies of teachers' noticing which use *common* video artifacts, like video clubs or targeted interventions (e.g., Santagata et al., 2021; Stokero, 2021).

Despite these limitations, we see many other potential applications for the ITT framework with network visualization, beyond the illustrated study. While this framework was developed in the context of a particular study of the university capstone course focused on reasoning and proving, we assert that the ITT framework can be broadly applied, to almost any online teaching setting involving PSTs or practicing teachers in various institutional contexts. For instance, as already mentioned, future studies could explore the triadic connections among the vertices of the ITT framework, i.e., the metaphorical faces of the tetrahedron, to illustrate the richness of teachers' reflective noticing. Additionally, our focus on PSTs' reflective noticing was motivated by our desire to understand the emergent practices of online teaching from the PSTs' own perspective. As suggested by Spangler (2019): "We as teacher educators need to demonstrate the curiosity and intellectual humility that allows us to understand how and why something a teacher did or said came from a place that made sense to them" (p. 2).

The ITT framework can also be used to capture classroom interactions from the *researcher*'s perspective; furthermore, the two noticing perspectives can be compared. The ITT framework can be used to analyze and compare the online teaching practices of teachers with varied levels of online teaching experience or varied levels of technology use. The framework can be adapted for representing other agents, like teacher educators and teachers (cf., Nipper & Sztajn, 2008). In general, we consider that the ITT framework can be broadly utilized to conceptualize, analyze, and visualize the teacher, students, content, and technology interactions in online settings.

## Appendix

**Table 1** Summary of the ten ITT categories with examples

| Category description | Example |
|---|---|
| **Teacher:** the teacher's reflection on their personality, behavioral characteristics, voice pitch, thoughts, emotions, and self-impressions | "I type so aggressively!"; "This is me panicking about my missing slides!"; "I definitely ramble here a lot"; "I personally think I sound quite calm and collected." |
| **Students:** the teacher's noticing of students' behavior, feelings and personalities, interactions with peers and classroom participation | "She [the student] is always fantastic about participation and contributes some great stuff." |
| **Content:** the teacher's reflection on the mathematics, including reasons or actions for including or highlighting certain mathematical ideas, or the provided learning opportunities | "This example was a good graph to bring up. Although they [students] are learning about linear functions, I didn't want them to think that linear functions are the only functions out there." |
| **Technology:** the teacher's reflection on technology as a tool, and its functioning (well or not), without relating it to a specific teaching move or to students' mathematical thinking | "The only thing that wasn't so good in this part of the lesson was that when I was sharing my screen, I couldn't see the students' names and so when I didn't know a student's name, I felt awful about it." |
| **Teacher–Content:** the teacher's reflection on how they taught a particular mathematical content or their instructional decisions related to teaching certain mathematical content | "I made a decision at this point to skip even discussing what a domain and range were because I wanted time to do the activity on conditional statements." |
| **Teachers–Students:** the teacher's noticing of the interactions between the teacher and the students, including teacher strategies for building productive interaction norms, facilitating students' online participation and developing student-centered instruction | "I pull John's name into the problem [as the mathematician who made this claim]. I keep making a point to do this, and I do think students appreciate it. Sometimes when I mention their names, I catch them smiling. […] Hope it encourages participation from everyone else." |
| **Students–Content:** the teacher's reflection on students' mathematical thinking as they interact with a mathematical task or respond to a question. The teacher may notice students' understanding of a particular mathematical idea or acknowledge their contribution | "Students were able to extend similarity concepts beyond the concrete and consider abstractly how many of the bowtie-shaped pair of triangles could have these two properties and easily be similar. This was an important moment in my opinion." |
| **Teacher–Technology:** the teachers' noticing of how they managed technology for teaching effectively. Can involve reflecting on one's decision to use (or not) a certain technology, or handling technology at the moment in the context of making pedagogical decisions | "One teaching move that I found challenging occurred when I wanted to keep the problem up, but also wanted to show the list of ideas that the students were coming up with. I had to make a hard choice but decided that the list would be more beneficial." |
| **Students–Technology:** the teacher's reflective noticing of how students interact with technology | "Students are super comfortable in the chat and write some really funny conditional statements!" |
| **Content–Technology:** the teacher's reflection on how technology is useful or not in representing a particular mathematical content | "It [an online game of *Tick–Tack–Toe*]was applicable to what the goals of the lesson were about (indirect reasoning), and I think the students enjoyed it." |

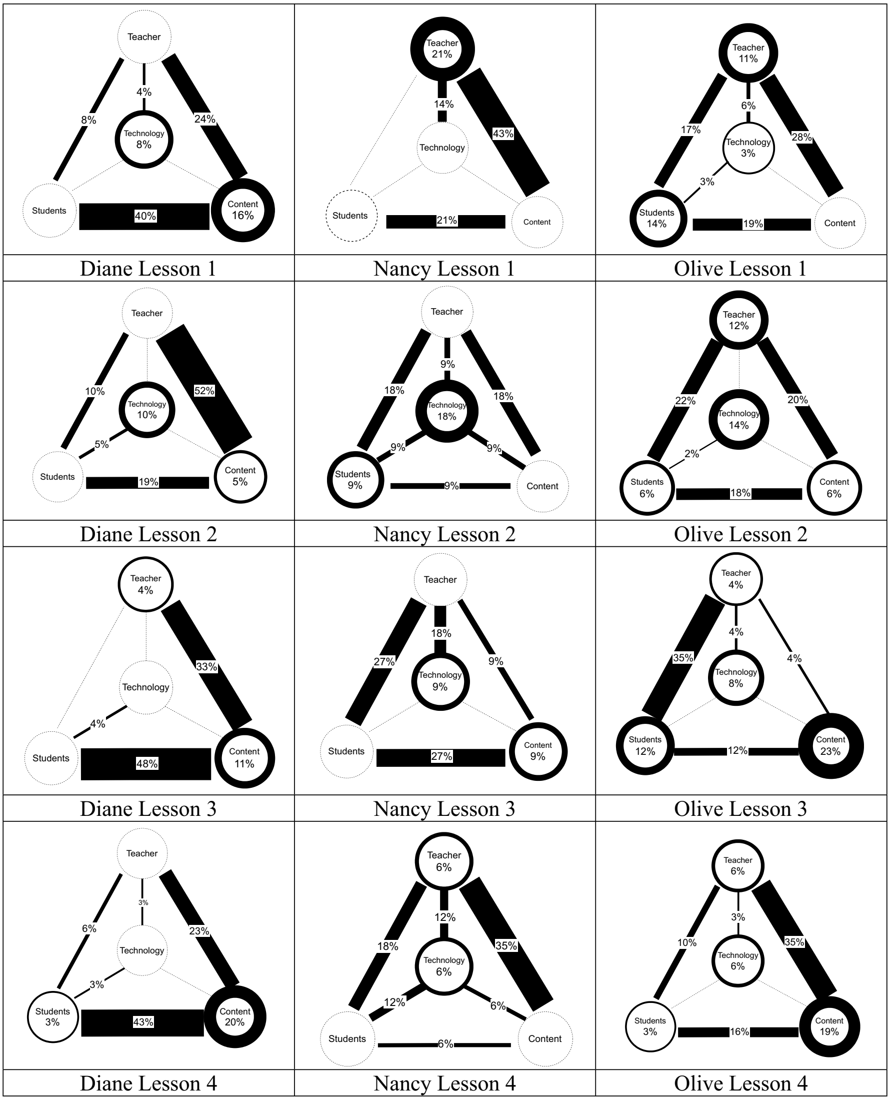



**Fig. 14** The ITT maps for the 12 lessons by the three PSTs—Diane, Nancy, and Olive

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**Data Availability** The authors will provide additional data excerpts upon demand, in accordance with data safety regulations.

## Declarations

**Conflict of Interest** The authors declare no competing interests.



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