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Duos of Digital and Tangible Artefacts in Didactical Situations

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Abstract

The duo of artefacts is a simplified model of the complex systems of various manipulatives (either tangible or virtual) that mathematics teachers and their students use in classrooms. It offers a means to study the complexity of the interweaving of the tangible and of the digital worlds in the teaching and learning processes. A duo of artefacts is defined as a specific combination of complementarities, redundancies and antagonisms between a tangible artefact and a digital artefact in a didactical situation. It is designed to provoke a joint instrumental genesis regarding both artefacts, and to control some of the schemes and mathematical conceptualizations developed by pupils during its use. This article exemplifies the model of a duo of artefacts, in the case of the pascaline and the e·pascaline for the learning of place-value base 10 notation of numbers. It details the design process of the e·pascaline (given the pascaline and its complementarities, redundancies and antagonisms), resulting from feedback of the digital environment and haptic properties of the tangible one. It provides examples of the evolution of pupils' conceptions of numbers when using the duo. It also shows how teachers transform the duo into a system of instruments, allowing them to manage the problem-solving strategies of their students, providing them with one or the other artefact, playing with their complementarities and antagonisms.

Keywords Pascaline · e·pascaline · Duo of artefacts · Place value · Virtual manipulatives

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Introduction

There is a growing interest in the interaction between physical and digital space in learning situations, due both to the new possibilities raised by recent evolutions in digital technologies (such as, gestures and multi-touch interfaces, robots, virtual worlds) and to the still slow evolution of teachers' practices in integrating these technologies. A solution to support teachers' digital practices might come from a better understanding of relationships and complementarities between digital resources and tangible ones in students' learning.

A particularity of primary school, at least in France, is the significant role of manipulatives in the teaching–learning process. Since the nineties, mathematics education research has studied the use of manipulation and the roles of the body and gestures in students' conceptualization of mathematics, following an older path of reflection initiated in Europe by Maria Montessori (1958), and Gattegno and Cuisenaire (1962). The seminal work on embodied cognition conducted by Lakoff and Núñez (2000) highlights the ways in which mathematical conceptualization is rooted in the body and the physical experience of the learner (Abrahamson 2019). The separation between body and mind or between physical action and mental activity is no longer a relevant grid of analysis for the learning and teaching of mathematics.

With regard to digital technologies, Papert (1980) had already emphasized the embodied roots of knowledge and mathematical reasoning with his notion of body syntonicity. He used this to explain why the use of computers is analogous with other mathematical activity, as it allows (seemingly) direct manipulation of mathematical representations. With the development of multi-touch tablets, robots and virtual reality, the distinction between manipulating tangible objects and manipulating digital representations has faded away and brings to the forefront the necessity to study their respective contributions. Furthermore, the close association between tangible objects and digital learning environments, which involve the body and the mind together in mathematical activity, is opening new possibilities for the design of didactical situations. Therefore, it raises new research questions.

Returning to the question of teachers' work and practices, the critical role of resources has been established and studied within the theoretical framework of the documentary approach to didactics (Gueudet and Trouche 2012). The structuring of teachers' resources into a system, which is associated with a scheme of use (a stabilized way of using the resources for a set of professional situations), can be used to characterize teachers' practices and expertise, as well as their possible evolution (Gitirana et al. 2018). Therefore, in order to understand how teachers' practices can evolve to integrate digital technology better, while benefiting from their expertise regarding manipulatives, we need to study the roles both of tangible and of digital resources in the complex systems of teachers' resources.

This article is a contribution to these issues. I present a model, called a duo of artefacts, resulting from previous work conducted with my colleague Michela Maschietto (Maschietto and Soury-Lavergne 2013, 2017; Soury-Lavergne and Maschietto 2015). It is both a model for describing learning and teaching processes involving tangible and digital tools and a methodology for designing and studying such learning situations. I argue that, in order to understand and study these complex situations of students or teachers interacting with a set of objects, tools and environments (either tangible or digital), there is a need for simplification. The duo of tangible and digital artefacts is a reduction of the resources system to two clearly distinct artefacts. I want to show how this simplification helps to produce knowledge, understanding and, along the way, concrete proposals for educational systems.

Interaction of Tangible and Digital Realities in the Research Literature

More than 30 years ago, Turkle and Papert (1992) studied the relationship between the digital and the tangible through different approaches to programming observed in apprentice programmers at the primary and university levels. They showed how *bricolage* (in reference to Levi-Strauss) and proximity to objects are the characteristics of an approach to programming that is as valid as more conventional ones, based on formalism and abstraction. What is interesting in their vision is the role they attributed to the computer: "The computer stands betwixt and between the world of formal systems and physical things; it has the ability to make the abstract concrete" (p. 4). They claimed that it is crucial for mathematics learning to re-evaluate the relationship to the concrete and gave to computers a key role in this process. This is like the role Papert gave to the gears he liked to play with during his childhood:

> The gear can be used to illustrate many powerful ‘advanced’ mathematical ideas, such as groups or relative motion. But it does more than this. As well as connecting with the formal knowledge of mathematics, it also connects with the ‘body knowledge’, the sensorimotor schemata of a child. You can be the gear, you can understand how it turns by projecting yourself into its place and turning with it. It is this double relationship – both abstract and sensory – that gives the gear the power to carry powerful mathematics into the mind. (1980, p. 20).

Papert wanted computers to play for children the role of a sensory-motor model, connecting the abstract to the senses and enabling mathematical knowledge to be apprehended and processed. Computers lie between the formal and the physical worlds, dealing with logical and abstract systems while themselves being tangible and concrete (Turkle and Papert 1992). But such a conception may lead to consider the computer as a sufficient whole, embedding and generalizing the properties of material artefacts, which could be a substitute for material artefacts.

With regard to mathematics in primary school, the question of the comparison between tangible objects and computer environments has been addressed directly. Resnick and his colleagues (Resnick et al. 1998) defined virtual manipulatives using objects and toys that are valued by children and that would be endowed with the computational and communicative properties of digital environments. Their starting point was not the digital environment, but rather the objects of the concrete world, whose properties would be enhanced by digitalization.

Later, Moyer-Packenham et al. (2002) proposed a definition of virtual manipulatives as digital entities whose manipulation on the screen makes it possible to represent a mathematical concept, a relationship or a procedure. Moyer-Packenham (2016) also subsequently specified that the most important thing is not the graphical appearance of the objects, but their dynamic behaviour in interaction, resulting from designers' choices. The coherence of the dynamic behaviour of the representation with mathematical knowledge is crucial. Her definition and classification of virtual manipulatives emphasises the behaviour in the interaction with the user, but does not question the link with the manipulation of tangible objects.

In the area of numbers and calculation, Manches et al. (2010) compared problem solving with physical and digital materials by students aged 4 to 8 years old. Their objective was not only to choose between physical and digital teaching materials, but also to design new learning environments, taking into account their respective contributions. They studied the relationship between the constraints on pupil action in each context and the developed strategies. This comparative study, from the point of view of the properties of the digital or tangible representations that are manipulated, did not conclude that one environment was preferable to the other. But it did show that possible actions, particularly at the interface, guided the ideas developed by the pupils. They concluded that what counts in the design of environments, whether material or digital, are constraints on manipulations and coherence with the targeted strategies.

This work highlights the complementarity between environments from the point of view of action. It justifies considering them both in a complementary way in learning situations. The more recent contribution from Sarama and Clements (2016) also concludes that the issue is not about concrete versus abstract, but rather is in the combination of physical and virtual manipulatives to help to construct knowledge. Knowledge can be very concrete for students having built rich connections between mathematics and personal experiences. Moreover, what matters is the sensory nature of the concrete manipulatives and their connection to the physical experience of the world.

In conclusion, this literature highlights the need to consider the articulation between the tangible and the digital world for the learning of mathematics.

Duo of Artefacts: A Theoretical Model for Designing and Studying Systems of Instruments

The question I want to respond to is how mathematical knowledge develops when a learner uses a system of tangible and digital resources. More specifically, what are the characteristics of this system of artefacts that allow mathematical knowledge to be constructed? Rabardel's instrumental approach (Verillon and Rabardel 1995) enables us to study the potentialities of a given artefact, either tangible or digital, for learning. I propose to study its application to a system reduced to two artefacts.

Conditions of a Joint Instrumental Genesis in a Duo of Artefacts

The instrumental approach makes the distinction between an artefact (i.e. an abstract or material object) considered independently of its use and an instrument (i.e. a two-fold entity combining an artefact component and a cognitive component). The artefact component of the instrument may be a part of, or an augmented version of, the initial artefact, resulting from the action of the subject. The cognitive component of the instrument is comprised of schemes. According to Vergnaud (2009), schemes have epistemic aspects and contain conceptual elements. They integrate operational and predicative forms of knowledge. The term *instrumental genesis* refers to the process by which a learner turns an artefact into an instrument that allows them to really achieve the task.

The schemes developed by the learner, along with the instrumental genesis, embed the knowledge, which results from actions with the artefact. Therefore, instrumental genesis can be seen as a process underlying the development of knowledge when using an artefact. Rabardel and Bourmaud (2003) pointed out that new instruments do not result from an isolated instrumental genesis but rather form a system that integrates previously developed instruments. Considering instruments in a system is a necessity. But this is very complex for researchers, especially when we want to design artefacts and analyze the learning processes involving these artefacts.

My proposal is to reduce the complex system of instruments that may result from instrumental genesis by selecting a duo of artefacts, a tangible entity and a digital one. The study of instrumental genesis related to duos is a means of understanding the more complex instrumental genesis of systems of instruments that occur in the classroom. “Science is reduction. But not all reductions are fruitful.” (Vergnaud 2009, p. 83).

My claim is that duos of digital and tangible artefacts are relevant to study mathematical learning situations. They are a reduced version of a complex system of objects and environments that constitute a didactical milieu for a given mathematical concept (Soury-Lavergne 2017). Under certain conditions, that I will describe below, the instrumental genesis of a duo of artefacts gives rise to a system of instruments and associated mathematical conceptualizations. Duos of artefacts make it possible to produce research results – for instance, to identify new instruments and new possible conceptualizations, as in Voltolini’s work (2018) about compasses and triangles. Additionally, they provide models to design new learning situations and new contributions to the education community, especially for teachers.

In order to support and control the emergence of a system of instruments, resulting from a given duo and their joint geneses, I studied the characteristics of the two initial artefacts and their relation. From the perspective of their use for teaching and learning, there should be continuities and ruptures between the two (Maschietto and Soury-Lavergne 2013). The highly realistic simulation of a tangible artefact in a digital environment is not a goal, especially since it is never entirely possible. The more realistic goal is to create two artefacts that can be the object of a combined genesis and turned into of a system of instruments. But, if the two artefacts are totally different, they may be involved in two isolated geneses, resulting in two instruments and not a system (see Fig. 1). The objective is the intertwining of the genesis related to each artefact.

Bourmaud (2007) mentioned that, “A system is a set of complementarities and antagonisms between the elements, a precise balance will be reached to allow their organization” (p. 65; *my translation*). Thus, a ***complementarity*** between two artefacts may be required to make necessary the use of each of them. However, complementarity alone does not prevent isolated instrumental geneses during the successive or separate use of the artefacts (see Fig. 2). The production of a system of instruments also requires some continuity between them. Thanks to continuity between the two artefacts, the user, either learner or teacher, can put them in relation to each other and see them as a potential duo.

Continuity may result from the ***redundancy*** of some characteristics of an artefact in the other. Redundancy in the system of instruments produces robustness and adaptability of the system. ***Antagonism*** between artefacts is also a characteristic that can lead to a system of instruments: “A system may be, conversely, less than the sum of the elements that compose it: it is the concept of constraints, i.e. qualities or properties inhibited by the organization of the set” (p. 65; *my translation*).

In the duo, antagonism stands for properties or functionalities of one artefact, which are inefficient or even divergent from the other. This opposition forces users to modify their development of schemes when passing from one artefact

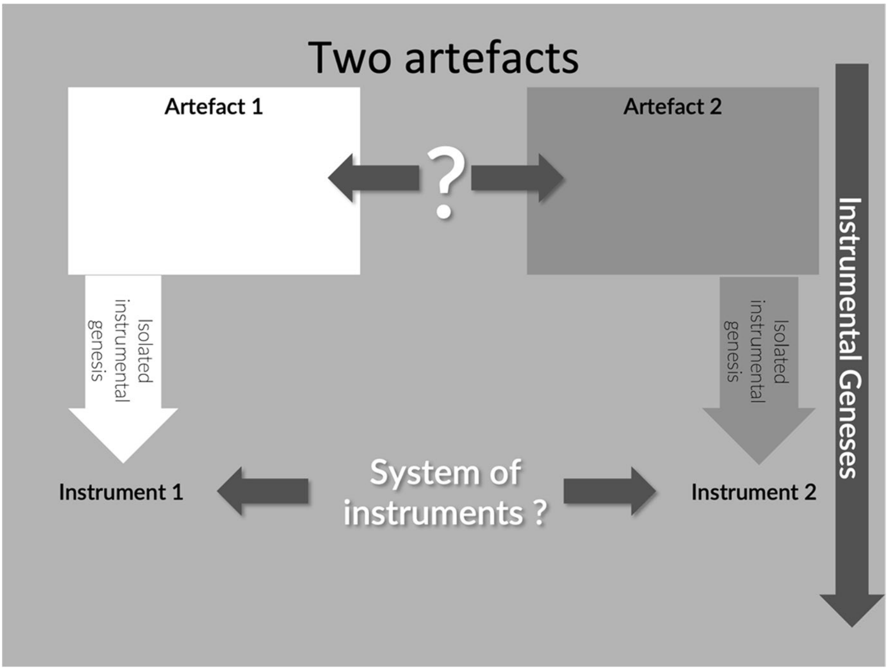

Fig. 1 Two isolated instrumental geneses produce two separated instruments

to the other. The creation of constraints in situations to which the schemes have to adapt is a powerful means for learning. Therefore, antagonisms between the two artefacts are a key characteristic to design duos for learning situations. As Rabardel (1995) declared, easing the action is not always the aim when it is about learning: "In an educational perspective, the criteria may be of quite of a different order. For instance, it can be desirable not to make action easier, but conversely, to create constraints on this action, so they lead the subject to operate the cognitive constructions that we want him/her to elaborate." (p. 154; *my translation*).

In conclusion, designing a duo is to conceive complementarities that justify the use of both artefacts and not just one, as well as redundancies that allow users to associate them in their activity and antagonisms to provoke evolutions.

Schemes, Gestures and Didactical Situations with a Duo of Artefacts

Gestures are both a means of learning and an outcome of learning. Schemes and gestures are cause and consequence of each of them. Schemes determine gestures by generating the activity of the subject in its two main dimensions: thought and behaviour. Gestures and verbalizations are elements of the behavioral dimension of the subject's activity: conceptualization belongs to the cognitive dimension. But gestures also determine schemes. Gestures are a means of acting on the milieu, thus to get feedback and, over time, to construct a

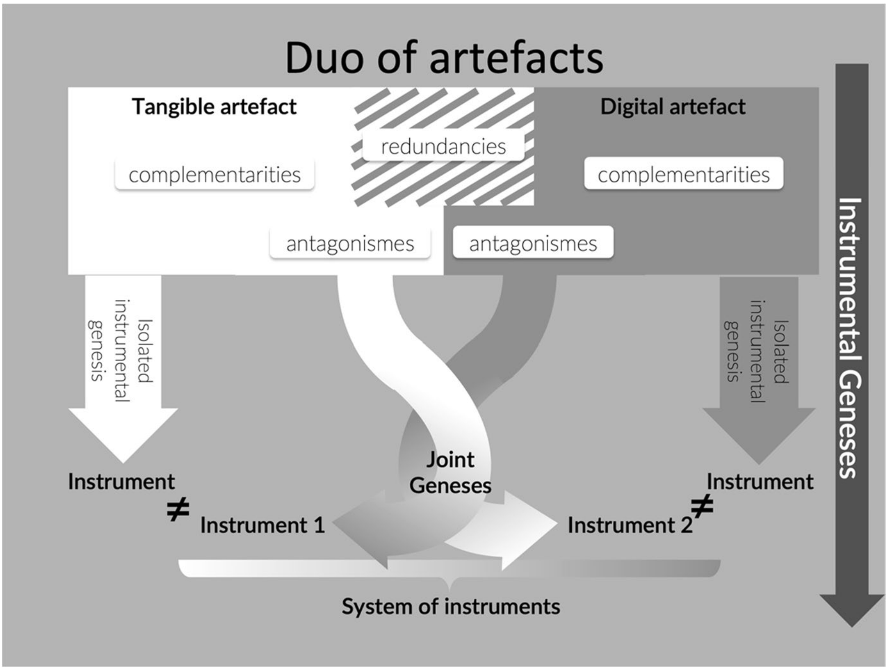

Fig. 2 Joint geneses with a duo of tangible and digital artefacts to obtain a system of instruments

stabilized and organized way of acting. The researcher accesses the schemes through observable behaviour of the subjects, among which include verbalizations and gestures.

Moreover, the meaning of the observable traces of activity lies in the situation, and didactic situations create the necessity to exploit both artefacts in the joint instrumental genesis. The theory of didactic situations makes it possible to conceive the duo as a constituent of a milieu (Brousseau 1997). Within the theory of didactic situations, mathematical knowledge is a property of equilibrium of a student–milieu system. Knowledge results from a system returning to equilibrium, through adaptation following a series of student actions on the milieu and feedback from the milieu.

A duo of artefacts provides a means to design a milieu for an a-didactic situation related to a specific piece of knowledge. It allows for the conceiving of the obstacles from one phase to the other, according to the artefact involved, leading to the necessity of adaptation for the student. It also offers continuity from one milieu to another, allowing the emergence and reinvestment of winning strategies during the situation. In order for the learner's interaction with the environment to be enriched by the use of both artefacts, they must have common characteristics and present elements of apparent continuity that call for the mobilization of the same strategies with both artefacts (development of schemes through assimilation). However, insofar as learning is targeted, antagonisms between the two artefacts must also exist, in order to provoke adaptations, reorganizations and thus the evolution of knowledge (development of schemes via accommodation).

The Pascaline and E·pascaline: An Exemplar Duo?

In a collaboration between the Italian laboratory of mathematical machines from Modena and the French Institute of Education in Lyon, my colleague Michela Maschietto and I have designed and studied a first duo, the pascaline and the e·pascaline, in order to evaluate the possible research outcomes from focusing on the articulation between the tangible and the digital for learning and teaching mathematics. The duo formed by the pascaline and the e·pascaline could be a possible 'exemplar' (in the sense of Kuhn 1990) of the idea of a tangible and digital duo of artefacts: "exemplars are solutions to concrete problems, accepted by the group as paradigmatic" (p. 397). They are known ways of dealing with a problem, which function as benchmarks and realize the mapping of scientific principles accepted by the community with the concrete world.

Without claiming it to be a proposal of the same scientific level, the idea of duo of artefacts allows a concrete address to some of the problems identified by the mathematics education community. One may consider the pascaline and e·pascaline duo as a possible exemplar to question and try to solve the problems of integrating technologies into teachers' practices and understanding mathematics learning with a complex set of artefacts.

Designing the Duo

The pascaline, which is the tangible part of the duo, is a small mechanical machine. It evokes the first geared calculating machines that spread in Europe from the seventeenth century onwards, including the famous Blaise Pascal machine. The design of the digital artefact, called the e·pascaline, was made in reference to the existing given tangible artefact, and aimed to provide continuities and discontinuities from one to the other (Maschietto and Soury-Lavergne 2013).

In designing the e·pascaline, we were looking for epistemic validity of situated interactions rather than visual fidelity (Wenger 1987; Balacheff 1993). The principles for the design of the e·pascaline in reference to the pascaline were:

- adding new functionalities to the digital artefact when designing it, to create complementarity between the two artefacts in order to make each of them useful from the point of view of the user, be they a teacher or a student;
- preserving some fidelity with common visual elements to ensure redundancy allowing the development of schemes by assimilation during the passage from one to the other;
- not implementing those characteristics of the mechanical machine which do not have an epistemic validity in relation to the targeted learning; leading to antagonism in order to bring about the development of schemes by accommodation.

The epistemic validity of the duo of artefacts results from the balance between the three principles associated with didactic situations that give a purpose to the use of each artefact.

The Pascaline to Teach and Learn Base-10 Place Value Notation

The pascaline consists of a gear of five toothed wheels (see Fig. 3). The pascaline makes it possible to display the numbers with their decimal position and to perform calculations. The ten digits from 0 to 9 are written on the teeth of the lower, yellow wheels, which represent units (right), tens (middle) and hundreds (left). The upper orange wheels transmit the rotational movement from one lower wheel to another, when necessary, thanks to the purple arrows on top of them. Each wheel of the pascaline can rotate clockwise or counter-clockwise. When the wheel of the units (or the wheel of the tens, respectively) has made a full turn, i.e. 10 units (or 10 tens, respectively), the orange wheel to its left rotates the tens wheel (or the hundreds wheel, respectively) by one notch. The rotation works in both directions. The spring-loaded purple ratchet to the left of each of the numbered wheels makes the rotation of the wheels discrete. The digits to be taken into account for the number displayed by the machine are to be read oppositely to the three red triangles at the bottom of the support.

Three procedures can be used to display a number N, written 'htu' on the pascaline (h stands for the hundreds digit, t for the tens digit and u for the units digit):

Procedure by **adjustment**: if users already know the digits to write the number they want to display, they just have to turn the three wheels so as to position the digits h, t and u in front of the triangles. Adjustments may be necessary when a digit already correctly positioned shifts when manipulating the other wheels. Users' control is over the final display, which must spatially match the one they anticipate.

Procedure by **iteration**: from the initialization of the pascaline to 000, the unit wheel is advanced by as many clicks/teeth as the number to be displayed, i.e. N. Users control the sequence of numbers up to N and the iteration of the units wheel.

Procedure by **decomposition**: starting from the initialization of the pascaline to 000, users rotate each of the three yellow wheels separately, to increment the units wheel by u notches, the tens wheel by t notches and the hundreds wheel by h notches, thereby controlling the base ten decomposition of N.

A main difference between the iteration and the decomposition procedures is the number of clicks it takes to display a number N, starting from the pascaline initialized to 000. With the iteration procedure, it takes exactly N clicks. With the decomposition procedure, it takes $h+t+u$ clicks, therefore necessarily less than or equal to N. This difference between the number of clicks of the two procedures creates the possibility of didactic situations in which the principles of base-10 place value notation are a key to a winning strategy.

Maschietto and her colleagues have studied the characteristics of the pascaline and its use in classrooms (Canalini Corpacci and Maschietto 2012). She and I have selected characteristics to be implemented in the digital artefact among those to which the pupils assigned mathematical meaning. We used pupils' gestures and drawings observed during classroom sessions (Maschietto and Soury-Lavergne 2013) to identify the elements carrying the meanings attributed to the machine by the pupils. Indeed, some signs produced during the pupils' activity can be interpreted as the observable part of usage schemes (Trouche 2004). From among the different pupil gestures appearing to turn the wheels, we have selected the one that supports the construction of the sequence of natural numbers by iteration of the unit. Our analysis of pupil drawings led us to select elements like the purple arrows on the auxiliary wheels to ensure some redundancy (even though it has no mathematical meaning) or to use arrow icons for buttons that trigger the rotation of the wheels in the e·pascaline.

The e·pascaline and the e-books

The e·pascaline (see Fig. 3) was designed as a complex object that looks like the tangible pascaline to ensure redundancy. Nevertheless, the implementation required additional choices fitting the digital editing environment constraints and those of our didactic analysis. This led to complementarities and antagonisms between the pascaline and the e·pascaline in order to support the joint instrumental genesis.

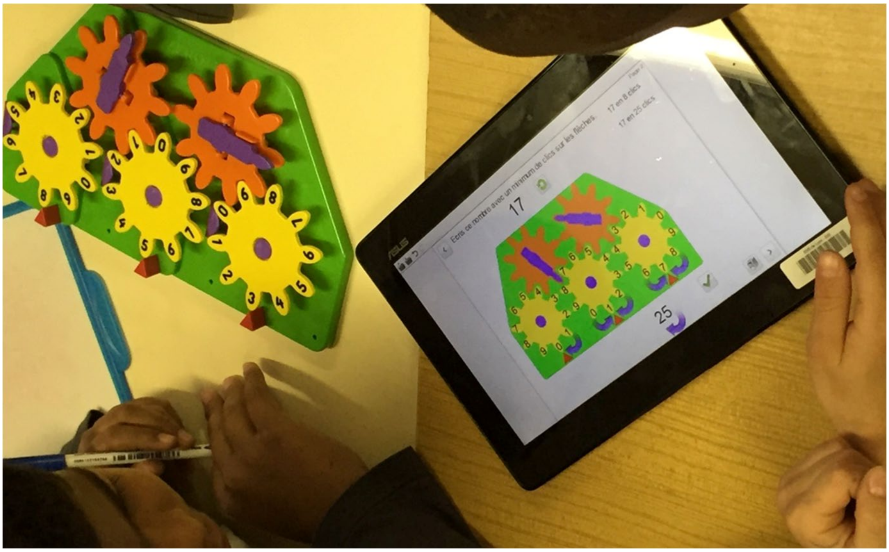

Fig. 3 The duo pascaline (left) and e·pascaline (right)

A first difference concerns the presence of buttons in the shape of a curved arrow, placed to the right and left of the red triangles, in order to operate the wheels. This feature is both a complementarity and an antagonism. Direct manipulation on the wheel has been avoided to prevent continuous movement of the wheels, like the one observed with the 'hand on the wheel' scheme on the pascaline. Other gestures, such as using the orange upper wheels or purple arrows, are also blocked. Thus, an antagonism is created by the impossibility of activating the wheels on the e·pascaline in the same way, as it is both possible and frequently done on the pascaline. We have made the assumption that a notch-by-notch activation of the e·pascaline would better support the conceptualization of natural numbers (though that still needs to be confirmed). Moreover, it strengthens the difference between the iteration and the decomposition procedures.

We chose an arrow as a metaphor for the action on the wheel. Arrows are present in students' drawings as a sign of the movement of the wheel, they become a means of action to provoke the same movement on the wheel. When launched, the wheel rotates in the direction of the arrow and stops on the next tooth. The 'one tooth at a time' scheme of use, although based on a different gesture (a button instead of a wheel tooth), is one that can assimilate the new situation. Moreover, the curved arrow shape of the button not only indicates the direction of rotation of the wheel, but also refers to the finger and its position with the tangible pascaline. Two separated buttons are associated to each wheel, for clockwise and counter-clockwise rotation.

This difference between pascaline and e·pascaline enables a complementarity between the two. Indeed, the action buttons on the wheels are not always displayed on the e·pascaline, whereas rotation is always possible on the tangible artefact. The buttons are only displayed on the e·pascaline, and are therefore usable, when their activation has a possible mathematical meaning in the field of whole numbers. Only when it is possible to add or remove a unit, a ten or a hundred and get a result in the whole-number interval [0, 999] are the corresponding arrows displayed.

This choice results from the possibilities of the environment, but above all from the didactic analysis. With the tangible pascaline, the calculation 000 – 1 is possible and it produces the display 999. The mechanical explanation given by the students can also be interpreted mathematically in elementary school (one has to think of 1000 – 1 rather than 000 – 1, by imagining that an additional hidden wheel would be initialized at 1 while the others are evidently at 0). Once this point has been elucidated with the students, the use of the e·pascaline makes it possible to focus on the mathematical properties to be learned in elementary school by avoiding the subtraction to 000.

The complementarities in the duo concern additional features, like the initialization of the display. The use of the pascaline requires regular resets to 000. It leads the user to investigate different reset procedures when discovering the machine and to consider that the proper initialization of a procedure is a crucial step in solving problems. With the e·pascaline, reset is made possible by a single action on a 'reset' button. Thus, two complementarities are generated: with the pascaline, the reset is explicit and leads to study the properties of the artefact; with the e·pascaline, the reset is automatic to avoid errors and to free up time to solve other mathematical problems.

Other tools and functionalities have also been added, like the counter of clicks (Maschietto and Soury-Lavergne 2013). Additional features of the e·pascaline create complementarities and make the e·pascaline more than an imitation of the pascaline. It provides an environment that enhances the mathematical experience of students compared with using the pascaline alone. The pascaline itself also has specific features (like sounds, the haptic feedback and its mechanical reality) that are not present in the e·pascaline.

The e·pascaline is included in a collection of e-books, designed with the *Cabri Elem* technology (Laborde 2016). They are resources directly usable by teachers. The added value of the digital part of the duo, in addition to the complementarities, redundancies and antagonisms, also results from the didactic situations that organize student activity. We also designed feedback and constraints that characterize the didactic situations, according to the principles of the theory of didactic situations and adidactic analysis in terms of milieu, variables, procedures and feedback (Brousseau 1997). *Cabri Elem* e-books organize adidactical milieu for the learning of numbers and principles of base-10 place value notation.

The design of different kinds of feedback (Mackrell et al. 2013) and additional features thanks to the digital environment, is one main way to create complementarities between a digital artefact and the tangible one: the possibility to navigate freely from one page to another; a random generation of problem situations within the chosen didactic variable values; a free reloading of a new task; the possibility to do the task as many times as desired and to request evaluation from the system. They also offer tools for the teacher: the possibility to check the list of problems, to monitor the students' results locally, to choose some values and to get feedback about their choices.

Learning Arithmetic with the Duo of Artefacts

We have carried out numerous observations of French students using the duo pascaline and e·pascaline, mainly at CP and CE1 level,[1] with various objectives and contexts: classes with the usual teacher using the duo or one of the two artefacts, classes with intervention of the researchers, classes visiting the research lab or the MMI[2] in Lyon (Soury-Lavergne and Maschietto 2015). The purpose of these observations was to gather information from the users that would nurture the design of the duo over the course of the projects. We did not conduct experiments allowing for the individualized record of student procedures (the e·pascaline e-books do not offer this possibility). However, our observations highlighted the stability of student procedures. We were able to observe procedures and theoretically to anticipate certain phenomena. Finally, even some unanticipated phenomena appeared very significant and were taken into account in the development of the duo and the e-books.

Didactical Situations in the e-books Collection: Adding Two Numbers

Based on the principles of base-10, place-value notation (Houdement and Tempier 2019), we have distinguished two main conceptions (in the sense of Balacheff 2017) of numbers by 6-year-olds. A first way to understand numbers is to link collections, quantities and numerical writings. According to this conception, the digit-based writing of a number, like '17', is seen as a whole, a label which is the symbol associated to a quantity or a position: the number 17 refers to seventeen objects or seventeen clicks on the pascaline (its cardinality). It is also a position in a list, after 16 and before 18 (its ordinality). The main problems solved within this conception are measuring a quantity or marking a position. Procedures are, for instance, counting one by one or finding the result of a sum by over-counting a collection (of objects or numbers) and, with the pascaline, procedure by adjustment or by iteration.

Another conception includes the properties of base-10, place-value notation, which is not only a provider of labels for the numbers but also a means to know some of their properties (for instance the writing of number 17 indicates that 17 is one ten and seven units). According to this conception, the position and value of the digits can be combined to indicate the quantity. It becomes possible to operate directly on the writing of the numbers to solve problems. One can count the tens and make a conversion to get the quantity; one can get the result of a sum by operating on the digits. With the pascaline, procedure by decomposition is associated with this way of understanding numbers.

[1] 6- and 7-year-old students, the first two levels of French compulsory schooling.

[2] The MMI is an exhibition and mediation center for mathematics and computer science in Lyon.

The operational part of these two conceptions can be observed, when using the pascaline or the e·pascaline, thanks to the differences between gestures and actions characterizing the iteration procedure and the decomposition procedure. Moreover, the didactic situation organizes the conditions to provoke the evolution from one to the other. In order to calculate sums with the pascaline, both iteration and decomposition procedures are possible. Once the first term is displayed on the pascaline with one of the procedures, iterative addition consists in incrementing the units wheel by a number of clicks equal to the second term.

The decomposition procedure consists of separating the second term into units, tens and hundreds, and incrementing each of the wheels by the corresponding number. At the end of the process, the number displayed on the pascaline corresponds to the sum. For each of the procedures, the actions to be carried out for the second term are not the same as for the first term. A user manipulates a numeral representation of the first term and can visualize and control it through the position of its digits. The second term is never displayed on the pascaline. When turning the wheels to increment the second term, it cannot be visualized on the machine. The second term needs to be enacted on the machine and the action controlled by counting the clicks.

The e·pascaline in the 'Adding with e·pascaline' e-book works similarly to the pascaline. But it has a critical additional constraint that consists in blocking the iteration of the unit wheel beyond 9 notches. Such feedback makes the decomposition procedure, combined with the use of place value base-10 notation, unavoidable for successful computation (Fig. 4).

The choices of didactic variables values from one page of the e-book to another concern the following: the locking of the units wheel which is not active on first page, but active on the following ones and the size of the numbers to be added (sum up to 30 then sum up to 69), making the iteration procedure more and more costly then impossible. The two variables of the didactic situation seek to provoke an evolution from the iteration procedure to the decomposition one and, consequently, an evolution towards a conception that includes the base-10 number system.

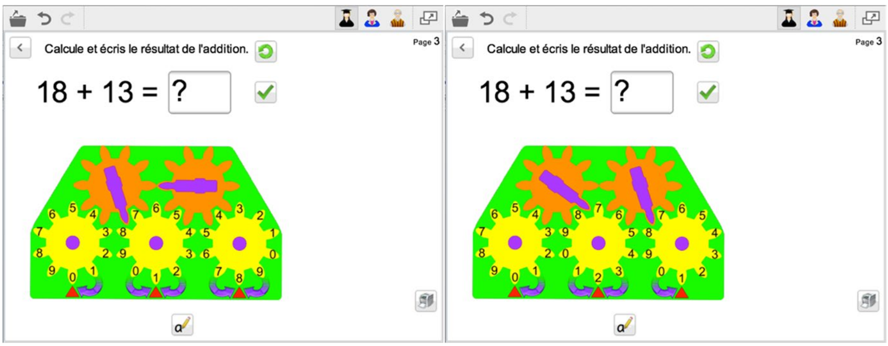

Fig. 4 In the addition e-book, when calculating the sum 18 + 13, after 8 + 3 clicks on the units wheel the arrow to the right of the units wheel disappears

Complementarity and Antagonism in the Duo: Effect of a Feedback on Pupils' Procedures

Our observations in all K–3 classes have shown that pupils develop a wide range of procedures to add numbers when they discover the pascaline. Some students (and even some adults during training sessions) wrote the two terms side by side on the wheels of the pascaline and waited for the machine to produce the result as with a calculator (Soury-Lavergne and Maschietto 2013). Some of them calculated the sum mentally and wrote it on the machine. The iteration procedure appeared only after the teacher had intervened and discussed the two previous strategies. Moreover, pupils did not hesitate to use the iteration procedure even with large numbers, close to or greater than 100. Feedback, which blocks the action on the units wheel, is the feature that should provoke the evolution of the procedures. This feature contributes to complementarity of the digital artefact in the duo. To study the effect of the e·pascaline e-book on adding procedures, we observed two CP classes (Soury-Lavergne and Maschietto 2015) with pupils who already knew the pascaline (these two classes also were involved in an experiment relating to the appropriation of the duo by the teachers, discussed in the next sub-section).

In Cleo's class, the iteration procedure with the pascaline appeared after the teacher intervened to require the use of the units wheel only. Then, with numbers greater than 10 but for sums less than 30, the iteration procedure did not generate enough errors to lead pupils to look for another strategy, even when the teacher encouraged them to do so. When they used the e·pascaline, the pupils were stopped by the disappearance of the arrow on the units wheel. They asked to use the pascaline to perform the calculation and then wrote the result on the e·pascaline (the result is considered to be valid).

This episode precisely illustrates the difficulty of diagnosing students' procedures, who in this case did not perform the calculation with the e·pascaline. Indeed, when the student enters 13 + 18 on the e·pascaline with the decomposition procedure, or when they enter 31 directly, they perform exactly the same actions at the interface, i.e. 3 clicks on the tens wheel and 1 click on the units wheel. The e·pascaline cannot differentiate between these two strategies, and therefore cannot provide an adapted feedback. After that, the teacher prevented them from using the tangible pascaline, which was easy since they are two distinct artefacts. However, the students were unable to mobilize another strategy to perform the calculations, showing the difficulty in conceptualizing the number through its writing in units and tens.

In Stina's class, as in Cleo's class, the pupils were unable to complete the calculations with the e·pascaline when the units wheel was blocked after a certain amount of clicks. Stina thought initially it was a bug and stopped the pupils' work. Therefore, the implementation of feedback on the e·pascaline was not satisfactory. It should be more explicit, with a message or, more interestingly, a pop-up peg stopping the wheel, in line with the mechanical model underlining the duo.

During the next class, Stina brought back the problem to the students and asked them how they could explain the fact that the units wheel was stuck. Pupils came up with several explanations: it is not possible to turn the units wheel more than once, the wheel has been turned too much or a limit such as ten or twenty has been exceeded. It shows that they clearly identified the constraint, but not the knowledge at stake, i.e. digit-based writing. They could not get the solution. Hence, the teacher proposed an alternative problem, by asking them to decompose the addends in different ways. Different decompositions were proposed (such as 20 + 3; 10 + 10 + 3; 13 + 10; 10 + 10 + 2 + 1, etc. for the number 23) and most students were able to mobilize the decomposition strategy and to compare the effectiveness of the two strategies.

Thus, in Stina's class, blocking the units wheel did create a problem-solving situation for the pupils. This situation would not have existed without the digital artefact. However, without the teacher's intervention and the use of additive decompositions, the students would not have been able to construct the alternative procedure targeted by the situation.

These first studies have confirmed the great resistance of pupils to abandon iterative strategies in favour of decomposition strategies, revealing a conceptualization of number that does not yet incorporate the principles of number decimal writing.

Teaching Arithmetic with a Duo of Artefacts

We have conducted a study to assess the extent to which the duo could be useful and usable by teachers. We have recruited 8 voluntary teachers, who were not involved in the design of the e·pascaline and we followed their work with the duo during four months (Maschietto and Soury-Lavergne 2017). With this experiment, our research hypothesis was that the complementarities of the two artefacts in the duo could be perceived by teachers and made use of in the classroom with their students. One of the questions was related to the articulation of the use of both artefacts. Our methodology involved some direct observations in the classroom. But the main data were collected through teachers' reports, pupils' productions, photographs and interviews of the teachers on a regular basis.

Seven teachers created various didactic configurations to organize, in their classrooms, their students' access to the duo's two artefacts (one teacher left the protocol for medical reason). They combined the equipment at their disposal, including 30 pascalines per school and computers, the spatial configuration of the classes, the presence of additional adults to organize their class sessions in order to have the students work collectively, in groups, in pairs, simultaneously or successively. They created or acquired additional artefacts to orchestrate the use of the pascaline and e·pascaline. For example, one equipped herself with a wifi mouse so that the students could act on the projected e·pascaline from their seats. The use of a video projector with the e·pascaline, while directly using the tangible pascaline, was one of the didactic configurations corresponding to the simultaneous use of the two artefacts of the duo (5 out of the 7 teachers).

Teachers combined the use of the pascaline and the e·pascaline in their sessions. The combination was temporal, with a simultaneous or successive use of the two artefacts. There was also a combination of individual or collective use. This produced different possible didactic configurations with the duo of artefacts, such as: succession of individual use of pascaline then e·pascaline by one pupil, succession of individual use of the pascaline by one pupil then collective use of the projected e·pascaline by the class, or simultaneous use of the pascaline by one's pupils and the e·pascaline by another working together. As the experiment progressed, the teachers refined their choice for one or another configuration according to the available equipment, the given task and their didactic objective. But, more importantly, they became aware of the different possibilities of control over the pupils' procedures.

The teachers' choice of configurations reveals that they took into account the antagonism and the complementarity between the two artefacts. The example of Cleo's class about the addition e-book is important to understand the role played by the two artefacts. The pupils were confronted with the impossibility of adding two numbers by using the iterative procedure with the e·pascaline. Cleo helped them by providing the pascaline. By doing so, she modified the didactic situation and allowed the pupils to solve the problem without having to change their procedure. Thus, the duo of artefacts became a system of instruments in the teacher's hand. Depending on the long-term evolution of the pupils (unobserved), it can be a way to avoid learning, if pupils never go through the decomposition procedure, or a way momentarily to alleviate the difficulty and keep pupils involved in the learning situation.

Another example is given by Stina. Her students had to solve a new problem which consisted in writing a given number with a minimum of clicks on the e·pascaline. To do this, they had to count the clicks on the e·pascaline wheels, while controlling the procedure of writing a number. Even though the e·pascaline had a counter of clicks, the pupils had difficulty co-ordinating the two different processes. Confronted with the students' lack of success, Stina had the students work in pairs with the tangible pascaline. In the pair, one student had to write the number on the machine while the other listened and counted the clicks. The sound of the pascaline was a key feature to help the pupils. After that, they successfully solved the problem with the e·pascaline.

These two examples illustrate how teachers were able to exploit the complementarity and antagonism of the two artefacts, using one to assist the use of the other. The pascaline was used to assist students in the instrumentation of the e·pascaline, allowing a wider range of procedures. The e·pascaline assisted the learning by providing didactic constraints and different kinds of feedback. The role of the teacher was to manage the use from one artefact and the other. These examples show the teachers appropriation of the duo of artefacts.

Conclusion About the Duo Pascaline and e·pascaline

I have proposed to consider the duo formed by the pascaline and the e·pascaline as an exemplar of duos of artefacts that achieve a concrete application of the theoretical principles of instrumental approach and theory of didactic situations. Although our study did not allow us to identify the precise learning trajectories of the students, nor the interweaving of schemes in the passage from one artefact to the other, it did provide evidence that both students and teachers' instrumental genesis transformed the duo in a system of instruments to learn and to teach mathematics.

It enabled us to understand better how complementarities, redundancies and antagonisms between two artefacts play their role in the learning and teaching situations. Complementarity results from the specific features of each artefact. The pascaline produces both sound and haptic feedback that can be used by the pupils to control their procedure. It allows a wider range of procedures than the e·pascaline. On the other hand, the e·pascaline offers richer feedback and stronger constraints to create mathematical problems and didactic situations, although implementation still needs to be improved. Indeed, one of the reasons for using a digital artefact in a duo is pragmatic, resulting from the facility to create complementarities by adding feedback. Antagonism, by blocking the iterative procedure, plays a clear role in the didactic situation. In fact, the same feature may be both a complementarity, because it creates the problem to be solved, and an antagonism, which obliges the adaptation of the procedures.

The experiment conducted to follow the use of the duo by seven teachers gave positive results regarding the development of instrumental geneses for five of them. In particular, the instrumental orchestrations observed revealed new configurations combining the two artefacts and the substitution of an artefact in order to modify the student's learning situation. The duo has made it possible to question the appropriation of resources by primary school teachers and to provide a start, rather positively, to the initial question: is a duo of artefacts a means of encouraging the appropriation of technologies by teachers? Further work needs to be carried out to provide a definitive answer.

Duo of Tangible and Digital Artefacts: A Possible Framework and Methodology for Mathematics Education Research

With the pascaline and e·pascaline, I have presented how concrete problems that could be solved with a duo of artefacts are those of designing didactic situations integrating digital technology that can be disseminated, and actually used and appropriated by teachers and students in the field. For researchers, it may be also a fruitful methodology grounded in didactic principles. It combines the viewpoint on resources and technologies from the instrumental approach (Verillon and Rabardel 1995), a conception of learning inherited from the theory of didactical situations (Brousseau 1997) where mathematical knowledge is the property of equilibrium of a student-milieu system, and current knowledge about manipulation in learning (Abrahamson 2019).

The design of the duo to provoke joint instrumental geneses has highlighted the role of complementarities, redundancies and antagonisms between the two artefacts. Starting from the case of the design of a digital artefact from a given tangible artefact, the design emphasized didactic analysis as a tool for choosing the features of the digital one. Also, the design process can just as easily begin with the digital environment and continue with the creation of the tangible artefact (as in the work in Voltolini 2018). Creating or selecting a digital and a tangible artefact is a pragmatic method to ensure the existence of complementarities and antagonisms between the two.

In particular, one reason for using a digital artefact in the duo is the possibility of creating complementarities with feedback, especially feedback about the problem-solving strategy. But, manipulating a digital interface is not enacting manipulatives. Some dimensions are absent and must be preserved for the students, which is possible with the tangible artefact. In the end, what is important is the didactic situation: it creates the need to use both artefacts and to integrate them into a system of instruments for solving problems.

The artefact duo is also a response to the question of transformation of practices in the field. Introducing a digital artefact in association with another already existing in schools is a way to help teachers perceive how digital technology is useful in teaching mathematics. It also helps to ensure its usability, because it is compatible with the usual classroom equipment and institutional demands. In addition, the two artefacts provide teachers with a fairly flexible set of configurations and many possibilities for adaptation. Duo of artefacts may prove to be a hands-on tool to design learning situations that take advantage of both physical and digital manipulatives for students, as well as for teachers.

Declarations

Conflict of Interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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