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The Geometry of Movement: Encounters with Spatial Inscriptions for Making and Exploring Mathematical Figures

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Abstract
In this article, we examine spatial inscriptions marked in real or rendered spaces, rather than on two-dimensional surfaces, conceptualize spatial inscriptions from an inclusive materialist perspective and consider realizations of spatial inscriptions that are possible with emerging technologies (e.g. 3D pens, immersive virtual reality). We then describe two cases of immersive environments that allowed learners to make and interact with spatial inscriptions. Next, we analyze how movements of participant–environment–inscription assemblages realized diagrams. Our analysis highlights how varying scale and changing perspective can become resources for doing mathematical work with spatial inscriptions.

Keywords Immersive environments · Virtual reality · Dynamic geometry

Mathematical work can be categorized by the quality, volume and materiality of the inscriptions with which communities of mathematicians generate and disseminate mathematical knowledge (Artemeva and Fox 2011; de Freitas 2016; Greiffenhagen 2014). We take *inscription* as an undefined term (Panciera 2012). Among others in mathematics, inscriptions include writing, symbols, pictures, graphs or diagrams (Roth, 2005). From Archimedes sketching in sand to children touching screens (Ng and Sinclair 2015), acts of inscribing have called to mind inscribable surfaces (de Freitas 2016; Dimmel and Bock 2019; Ng et al. 2018). But emergent technologies (e.g. 3D pens; virtual or augmented reality) are disrupting this status quo by transforming space into a canvas for making inscriptions (Dimmel and Bock 2019; Ng and Sinclair 2018).

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In the study detailed in this article, we explored two questions that concern *spatial inscriptions* – namely those marked in real or rendered spaces, rather than on two-dimensional surfaces (Dimmel and Bock 2019). How might spatially inscribed realizations of mathematical figures be encountered by learners? How might such encounters affect the learning and teaching of mathematics?

These broad questions reflect the emergent nature of our subject of inquiry. We aim to describe encounters with spatial inscriptions to the goal of generating hypotheses about how they partake in mathematical activity. We focus on *(im)material* spatial inscriptions that are rendered in immersive environments. Examples of such environments have been fixtures in research settings for decades (Lai et al. 2016; Kaufmann et al. 2000; Robinett and Rolland 1992; Steinicke and Hinrichs 2006; Sutherland 1968), but the advent of affordable, reliable equipment for engineering and rendering such environments was more recent (Dibbern et al. 2018).

We investigate these research questions through two cases of immersive environments. One was a spatial canvas for inscribing points, lines and spheres; we refer to this as the *line & sphere* environment. It was designed to be a space where immersed participants could draw and interact with spatial diagrams, including diagrams that were apparently unbounded – i.e. ones that appeared to extend indefinitely. The other environment was a $10 \times 10 \times 10$ lattice in which *edges* or *faces* could be inscribed by different pointing gestures; we refer to this environment as *LatticeLand*. It was designed to be a spatial analogue of the Geoboard – i.e. a three-dimensional canvas for exploring lattice polyhedra.

We analyzed mathematical *encounters* that occurred in each environment, namely loci of activity during which *assemblages* of agents, tools, and representations work in concert to vitalize mathematical concepts (Dimmel and Pandiscio 2020; Ferrara and Ferrari 2017). Our analysis highlights how immersed participants positioned their bodies near, within and among spatial inscriptions in order to explore diagrams from novel perspectives. We consider how body-based explorations of spatial inscriptions create new opportunities both for learning and for teaching.

Surfaces, Styluses and Encounters with Mathematical Figures

Inscriptions are entangled with surfaces: "Whether one be scratching away on slate, parchment, paper, or screen, mathematical activity often entails a material-semiotic activity that engages various kinds of two-dimensional surfaces" (de Freitas 2016, p. 193). Our use of the term *entanglement* is grounded in an inclusive materialist perspective in which agency is shared among subjects, objects and environments (de Freitas 2016; de Freitas and Sinclair 2014; Ferrara and Ferrari 2017). Inclusive materialism challenges us to consider the physical experiences of doing mathematics and how those experiences are intertwined with the qualities of surfaces and styluses. De Freitas examined the physicality of chalkboards in particular:

> Might the specific haptic, visual, and sonic aspects of the encounter – for instance, the speed with which the hand is able to move with the chalk, the fuzziness of any white line that is drawn, the sound and heat that are produced – be bound or fused with the kind of mathematics that is practiced? (2016, p. 195)

One kind of mathematics that is practiced with chalkboards is performative (Artemeva and Fox 2011), namely the mathematics of lectures and colloquia, where mature arguments are synthesized by experts. Speed, sound, scale, contrast, the ability to easily erase and rewrite – these affordances are well-adapted for showcasing mathematics as a disciplined practice. Chalkboards are material contexts for generating and presenting refined mathematics (Greiffenhagen 2014).

Napkins, as in the proverbial back-of-the-napkin sketch, represent a different kind of surface for generating inscriptions. They are convenient, disposable surfaces that accommodate the extemporaneous entry of mathematics into otherwise non-mathematical activities – e.g., drinking coffee, having dinner. Compared with the speed, heat, and sound of blackboard markings, napkin markings are finer, slower, softer, and quieter. Their smaller surfaces are positioned beneath (rather than alongside) those huddled around them and they can be quickly passed from person to person. Over napkins, mathematical ideas can flow in and out of a conversation that might span several hours and survey a range of topics.

In comparing chalkboards with napkins, we are not prescribing what mathematics can be done on each surface – chalkboards can be used for informal mathematics as readily as napkins can be used for sharing polished proofs. Rather, we are attempting to illustrate how surfaces and their qualities – e.g. size, texture, re-writeability – actively shape the kinds of inscriptions that are produced with them. Thus, a napkin/pen pair (small, non-erasable surface; slow, fine-point marks) generate inscriptions differently from a blackboard/chalk one (large, erasable surface; fast, coarse strokes), and both participate in the doing of mathematics.

Inscribable surfaces are more than inert, passive receptacles for the signs that are generated by thinking agents. Surfaces and styluses together with the agents who wield them and the inscriptions they collectively generate are examples of *assemblages*: open networks of human and non-human components (e.g. tools, representations, concepts) that belie any centralizing control (Ferrara and Ferrari 2017, p. 23). We are concerned here with analyzing mathematical encounters realized through movements of participant–tool–inscription assemblages in *inscribable spatial canvases*. We use the term *encounter* to underscore the fact that agency is shared and distributed throughout the assemblage (Dimmel and Pandiscio 2020).

Material and (Im)Material Spatial Inscriptions

Material spatial inscriptions physically extend into real space. They are inscriptions that blend the surface-mediated practices of writing/drawing/diagramming with the space-mediated practices of sculpting/carving/assembling. One technique for inscribing space is to use soap and string to create bubble manifolds. 3D pens are emerging as tools that enable spatial drawing, where it is possible to make, grasp, move, and feel corporeal representations of mathematical figures (Ng and Ferrara 2020; Ng and Sinclair 2018; Ng et al. 2018; Palatnik 2019).

Spatial drawing activates the motor skills that are used to draw diagrams on surfaces. They create the possibility of leaving “a material record of one’s gestural history” (Ng et al. 2018, p. 566) and provide a means for materializing unidimensional inscriptions that could, for example, offer “a *physical instantiation* of a tangent to a curve, in that the student could feel the idea of local linearity by sense of touch" (p.568). This physicalization of spatial movements makes possible new kinds of mathematical encounters that will evolve in complexity as extrusion technologies become more refined.

Material spatial inscriptions are also constrained by their physicality. Their need to be anchored and balanced determines what figures can be inscribed. Furthermore, material spatial inscriptions are not easily transformed: They can be picked up, moved through space, rotated or revolved, but they are limited in their capacity to be continuously varied.[1]

We use the term *(im)material* to refer to spatial inscriptions that are rendered by light-projection technologies (e.g. holograms, inter-actable lasers)[2] including virtual or augmented reality displays.[3] Virtual reality refers to visual and audio immersion within a digitally rendered world, while augmented reality refers to mixing digitally rendered objects, as an additional layer of information, into one's field of view (Dede et al. 2017). Though there are differences between virtual reality and augmented reality when it comes to designing user experiences, we focus here on their shared capacity of rendering spatial inscriptions in a person's field of view.

Gesture-tracking technologies make it possible to 'touch' (im)material spatial inscriptions, in the sense that they can be manipulated by making movements (e.g. pinching, grasping, pointing) with one's hands or body (Dimmel and Bock 2019). But the experiences of *feeling* mathematical structures described above are not yet possible.[4] They are thus biased against people with impaired vision. Finally, (im)material spatial inscriptions leave no durable, physical records. There is no *thing* to show, store, or share that is independent of a rendering device. They are, thus, less accessible than their material counterparts.

But even given these limitations, (im)material spatial inscriptions warrant study. They are neither subject to laws that constrain the behavior of massive objects nor impeded by things in the world (Hart et al. 2017a, b; Kaufmann 2011).They combine the spatial presence of real things with the transformability of dynamic diagrams. They allow for representations of mathematical figures to be extended in three spatial dimensions (Kaufmann 2011), explored at various scales, and viewed from different perspectives (Dimmel and Bock 2019).

Below, we analyze encounters with spatial inscriptions that were situated within two cases of inscribable spatial canvases.[5] In one case, individual participants explored the

[1] Hinges, telescoping, bendable/stretchable materials, disassembly/reassembly, gears, levers, or engines allow physical things to be varied in size/shape, but these capacities for transformation are still constrained by their materiality.

[2] Researchers have demonstrated the proof-of-concept that tangible holographic images can be projected into real space (e.g., Ochiai et al. 2016). We set such inscriptions aside because the technologies needed to display and interact with them are not yet commercially available.

[3] Projecting spatial inscriptions depends on physically inscribing computer disks, but for users immersed in the rendered space such inscriptions are experienced spatially.

[4] Haptic clothing (e.g. gloves, full-body suits) can simulate some sense of touch, but these technologies are still in the proof-of-concept phase.

[5] The development of the environments was supported by grants from the Maine Space Grant Consortium, the Spencer Foundation, and the Office of Vice President for Research at the University of Maine. The views expressed in this article are the authors' and do not necessarily reflect the views of the Maine Space Grant Consortium, the Spencer Foundation, or the University of Maine.

Table 1 Background information about participants

Pseudonym	Class Year	Major	Gender
Heather*	2nd	Bio./pre-med.	F
Seth	2nd	Elem. education	M
Mike	3rd	Civil eng.	M
Sally	2nd	Child development	F
Sean	2nd	Rec. & tourism	M
Conrad	1st	Electrical eng.	M
Brad*	1st	Economics	M
Doug	2nd	Construction eng.	M
Alan	2nd	Computer science	M

*indicates participants whose experiences are not part of the analysis (see below)

immersive environment while conversing with a guide (the first author). In the other, a focus group of secondary mathematics teachers explored the environment together, each taking turns as the immersed participant. Across the cases, we analyze how immersed participants moved their bodies, used their gaze, and transformed the spatial canvases by making inscriptions.

Case One: The *Line & Sphere* Environment

The *line & sphere environment* was a spatial canvas for inscribing points, lines, and spheres. In the environment, spatial inscriptions were created through pinching, stretching, and spinning gestures immersed participants made by moving rendered tools with their hands. The environment was designed to be a first approximation of a movement-based, inscribable, spatial canvas for realizing plane-and-sphere constructions – i.e. the spatial analogue of compass and straight-edge constructions (Franklin 1919; Dimmel and Bock 2019). The aim of the study reported on here was to observe how immersed participants worked with the tools to inscribe diagrams spatially.

Nine undergraduates at a public university in the northeastern U.S. explored the *line & sphere* environment during the Spring of 2019. Participants were recruited from across fields of study and class years. Table 1 comprises demographic information and pseudonyms for participants.

Immersion in the environment was mediated by a wireless, HTC Vive head-mounted display. Participants interacted with the environment by making gestures that were tracked via a sensor[6] affixed to the front of the headset. The sensor tracked participants' hand movements in real space and rendered hands within the environment. These rendered hands were visible in the first-person record of participants' movements within the environment (records of participant encounters in the environments are described and illustrated in the first sub-section below).

[6] This is a *Leap Motion stereoscopic* infrared sensor (Yang et al. 2019). See Dimmel and Bock (2019) for more details about the design and development of the gesture-based user interface.

The first author was a guide for each participant. Participants explored the environment independently of other participants and did not see the experiences of other participants in the environment. We conceptualized the guide as a participant-observer who was partially immersed in the research setting (Hufnagel 2015; DeWalt and DeWalt 2011). The guide watched participants' activities within the environment via a monitor that displayed a two-dimensional projection of their three-dimensional, first person views.

The guide answered questions, described tools, and prompted participants to explain or elaborate on observations they made about the experience of being in the environment. The guide also alerted participants if they approached real-world obstacles (e.g. walls, tables). This active role for the guide both oriented participants and supported the emergence of (researcher)–participant–tool–inscription assemblages. In addition to the guide, a second researcher observed immersed participants' real-world movements, while a third researcher provided technical support.

The unstructured explorations lasted 15–30 min. Participants were asked to investigate the tools and to describe what they were thinking/doing. The guide answered questions about how the tools functioned, but gave no direction about what to do with the tools, save one prompt: "How could you use the tools to make a mathematical picture?" This prompt was used at the guide's discretion, once the immersed participants had learned how each tool functioned.

Activity was documented by two simultaneous and synchronized recordings. Participants' movements in real space were recorded via a fixed camera. Participants' movements within the *line & sphere* environment were captured via recording their first-person view.[7] Seven of the nine participants' experiences within the environment were documented in this way.[8] Video records of participants' activity were reviewed to identify episodes where movements of participant–tool–inscription assemblages were salient. Participants' encounters with lines and spheres emerged from this review. Our analysis focused on how inscriptions partook in movements as representations of figures were created and transformed.

Tools for Inscribing the Spatial Canvas

Figure 1 shows the environment just after Sally donned the headset. This is a flattened image of her first-person view. A black sky meets a grey floor at a horizon, and the tool shelf is visible near the center of the frame.

We use panels of images to enrich our descriptions of how participants moved during encounters with spatial inscriptions. In each panel, the left frame provides an observer view of a participant's actual movements in real space. The right frame captures what the participant saw in the environment. Importantly, the change in perspective from observer (left frame) to participant (right frame) can also reverse the left–right orientation of the images, depending on how a participant was oriented toward the camera. For example, in Fig. 2, the left frame shows Sally grabbing a tool with her right hand; from an observer perspective, this is the hand that appears on the

[7] The first-person camera shows a two-dimensional projection of the participants' view of the environment.

[8] Data from the other two participants was not useable because of camera failure (first-person camera did not record) or technical difficulties with the environment (hands failed to render).

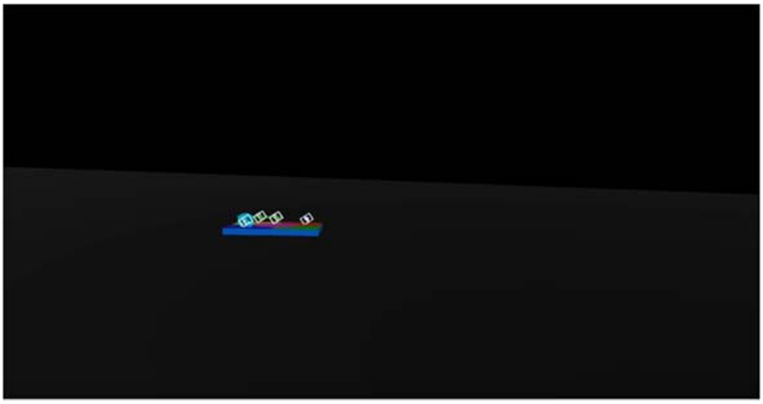

Fig. 1 Initial, first-person view of the tool shelf in the line & sphere environment. The tools are docked in white, wire-frame cubes that are angled toward the shelf. From the left, there is a blue wheel (spatial compass), a green line, a green point and (on the far right) an eraser

left side of the left frame. But the right frame shows the situation from the participant's perspective, and so we see the tool in Sally's right hand on the right side of the frame. Throughout the analysis, we inserted a white arrow icon at the top of the boundary between the *observer view* and *participant view* to flag instances where these images are mirror-reversed.

Figure 2 shows Sally pinching and slightly turning the spatial compass tool (the blue wheel) using her right hand. In the panels, Sally's left hand is fixed in a pinching gesture. She used her left hand to pinch and hold the shelf as she removed the wheel with her right hand. From the angle of Sally's gaze,[9] we can see how Sally was engaged by visually examining the wheel. Her head was turned toward and angled down to her right hand (top panel); she moved her right hand slightly up and then down again (middle panels) and then opened her hand to release her grasp of the wheel (bottom panel). The wheel had a material presence for Sally that shaped how she used her fingers to hold/release it and how she swayed her hand to move it.

Analysis of Encounters with Spatial Lines: "It looks like it goes on forever"

To instantiate lines, one grabbed a green segment from the shelf (back left of Fig. 3). Lines were realized as 200m x 10mm rectangles. The orientation of the rectangles was continually varied so that the face of the rectangle was orthogonal to the direction of an immersed participant's gaze (Anderson 2008). This ensured that the lines were always visible. With these dimensions, the lines in the environment were apparently unbounded – i.e.,they were sufficiently long, given the resolution of the head mounted display, for two parallel lines to appear to converge.

When participants instantiated lines, one of their first movements was to sweep the line with their gaze. Each of the seven participants engaged in this gaze-sweep movement. The three panels in Fig. 4 show an example of this movement-based encounter for Conrad.

[9] Black arrows are overlaid to highlight the general direction of an immersed participant's gaze.

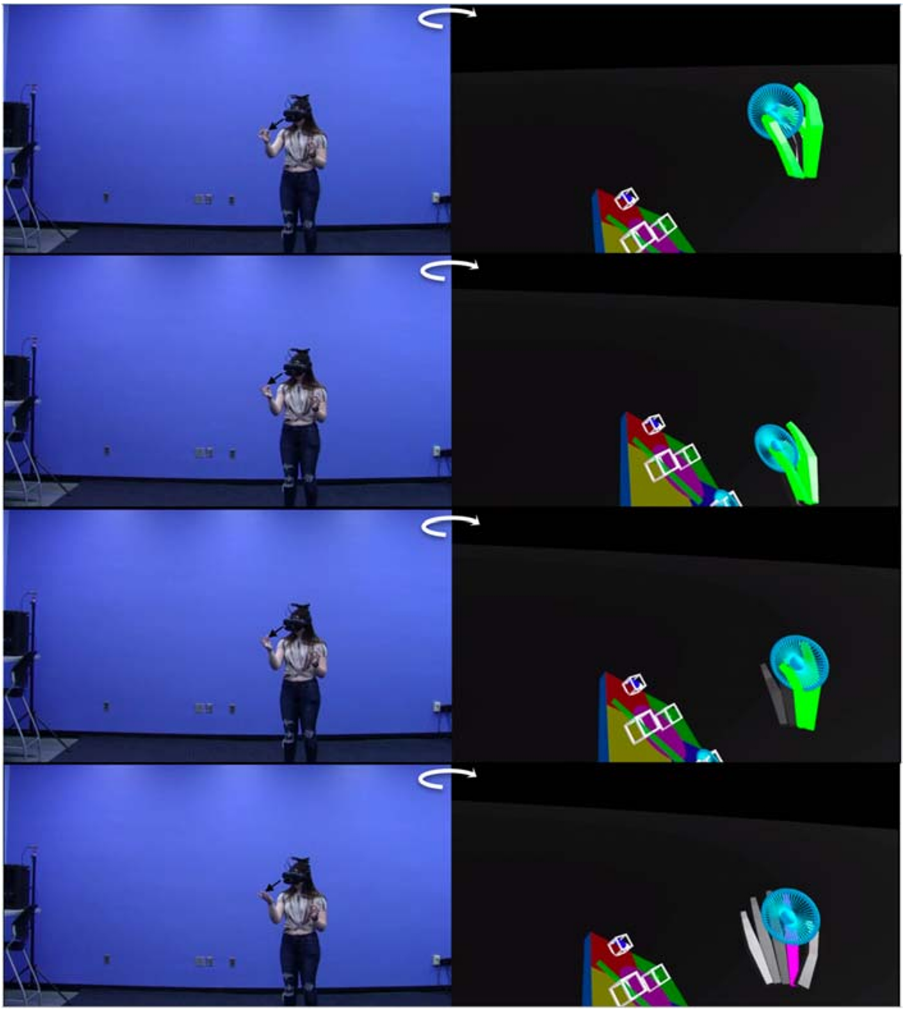

Fig. 2 Grasping and releasing a tool: the white arrow icon at the top of each panel indicates that the participant view (right frame) is left/right reversed when compared with the observer view (left frame)

The top panel of Fig. 4 shows Conrad holding the line in his right hand, not quite an arm's length from his torso.The line is roughly horizontal/parallel to the plane of the floor. He began his sweep of the line by looking left, turning his head to see the end of the line apparently disappear in the distance. Panels 2 and 3 (Fig. 4) show Conrad continuing to sweep the apparently unbounded length of the line. As he moved his gaze over the line, Conrad turned not only his head but also his shoulders, torso and waist, as evident in the bottom panel of Fig. 4.

The apparently unbounded line prompted Conrad to turn his head and body from side-to-side, sweeping his gaze in a manner that approached a half-turn. Conrad appeared to be looking for the ends of the line when he turned his head to the left and then back to the right. The apparent unboundedness of the line thus partook in the

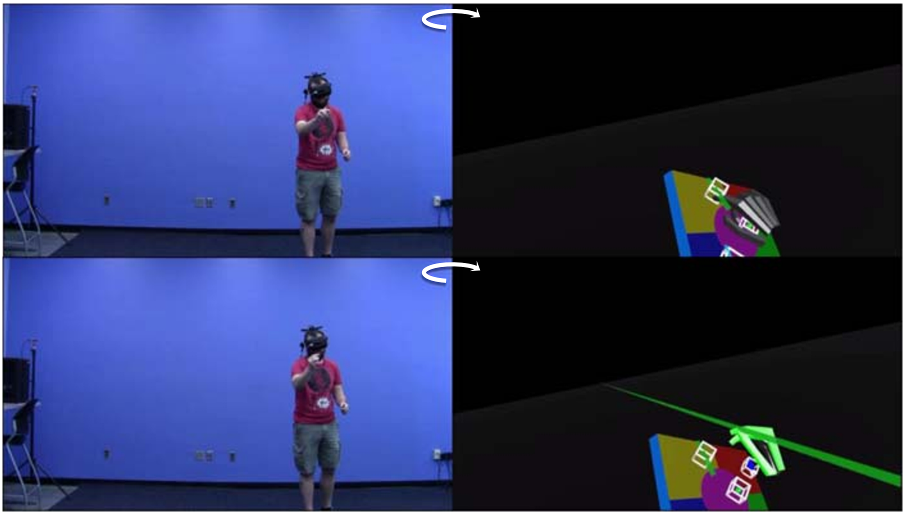

Fig. 3 Conrad reaching for the line tool (top panel), pinching to grab it/instantiate a line, and then moving the line by moving his hand (bottom panel)

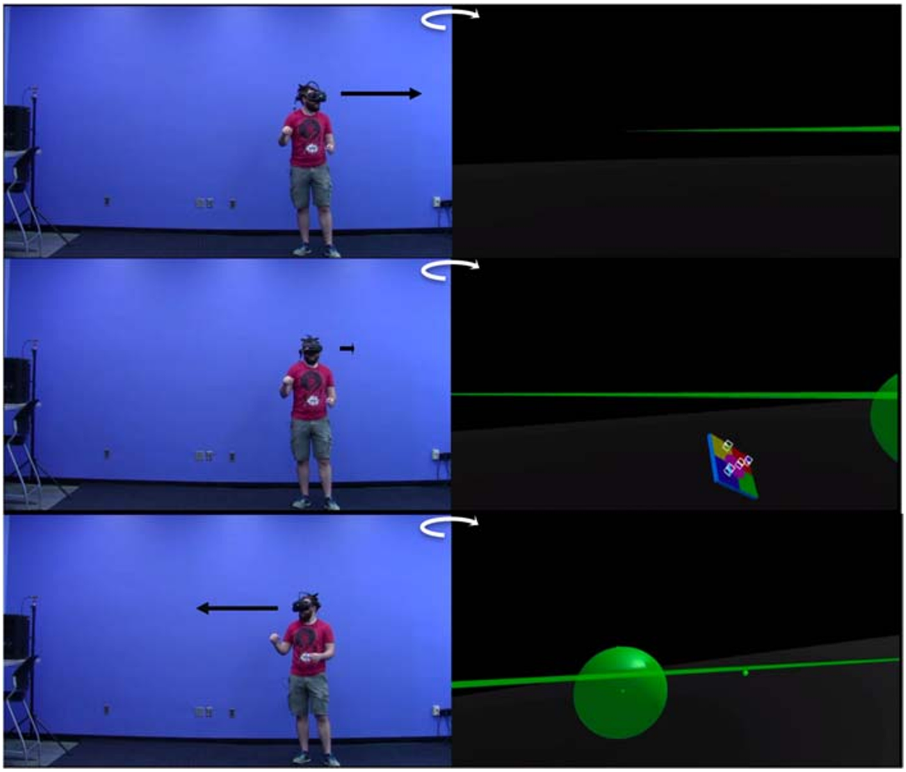

Fig. 4 Using gaze to track an apparently unbounded line, Conrad looks to his left (top panel) to see the line apparently disappear in the distance; he then turns (middle panel) to his right (bottom panel) to see the other end of the line disappear into the plane he is standing on

gaze/sweep movement by drawing Conrad's attention – turning his head from side to side was a means to apprehend the dimensions of the object he was now holding in his hand.

The panels in Fig. 4 illustrate how spatial realizations of lines can be explored from perspectives that are unavailable when diagrams are inscribed on surfaces. The top panel in Fig. 4 shows Conrad *looking along* the line, with his gaze oriented toward the invisible end of the line. This looking along approaches what Gerofsky (2011) described as a *character* viewpoint, where one uses movements of one's body to become a representation of a mathematical figure.

Sean also adopted a character viewpoint. He brought the line to be almost collinear with his gaze as he attempted to intersect the line with a sphere and its center. Figure 5 shows Sean moving the line with his right hand. In the top panel, his hand is slightly below the line of his gaze, and we can see the two points he was targeting in the distance. One is a point on the surface of the sphere (green dot with blue arc, to the left of the other green dot). The other point is the center of the sphere, which is the center of the spatial compass (the blue wheel).

In the middle panel, Sean had raised his hand to be more closely aligned with his gaze. The rendering of his right hand occluded the view of the point that was on the surface of the sphere. Sean released his grasp of the line to set it in place. In the bottom panel, Sean had changed the position of his head, appearing now to be gazing directly

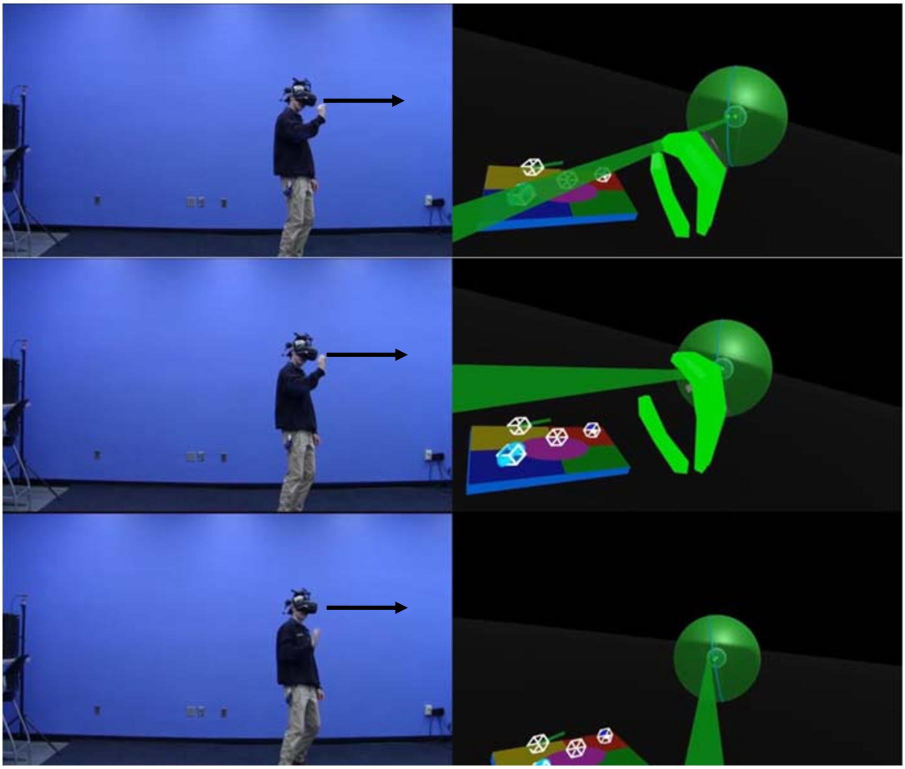

Fig. 5 Sean adopting a character viewpoint to position a line so that it appeared to intersect two points

along the line, as if he were standing in/on it. From this perspective, the line no longer appeared to be co-incident with the points.

The top panel of Fig. 6 shows Sean adjusting the line from this new perspective. When he set the line where he wanted, Sean stepped toward the sphere to investigate how close the line came to passing through both points (Fig. 6, middle panel); “almost”, Sean said. The guide asked Sean what he had done. Sean explained, “So now it’s going through the sphere and it’s going through the little blue spiral in the middle there”.

Sean then turned around to gaze at the other end of the line (Fig. 6, bottom panel). Here is another instance of the line/half-turn movement that we identified above: Sean moved his feet, trunk, torso, shoulders, and head in near unison, like a cylinder turning in space. As he turned, he said, “it looks like it extends in the dark – [looking down at the ground] in the depths [returning his gaze to end of the line], like all the way out there”.

Apart from being instantiated and moved, lines could be made perpendicular to the floor. Five of the seven participants used the gesture shown in Fig. 7 to create a line that was perpendicular to the floor. Once they had created the apparently unbounded, perpendicular line, each participant stepped away and looked up at the line extending off into the distance.

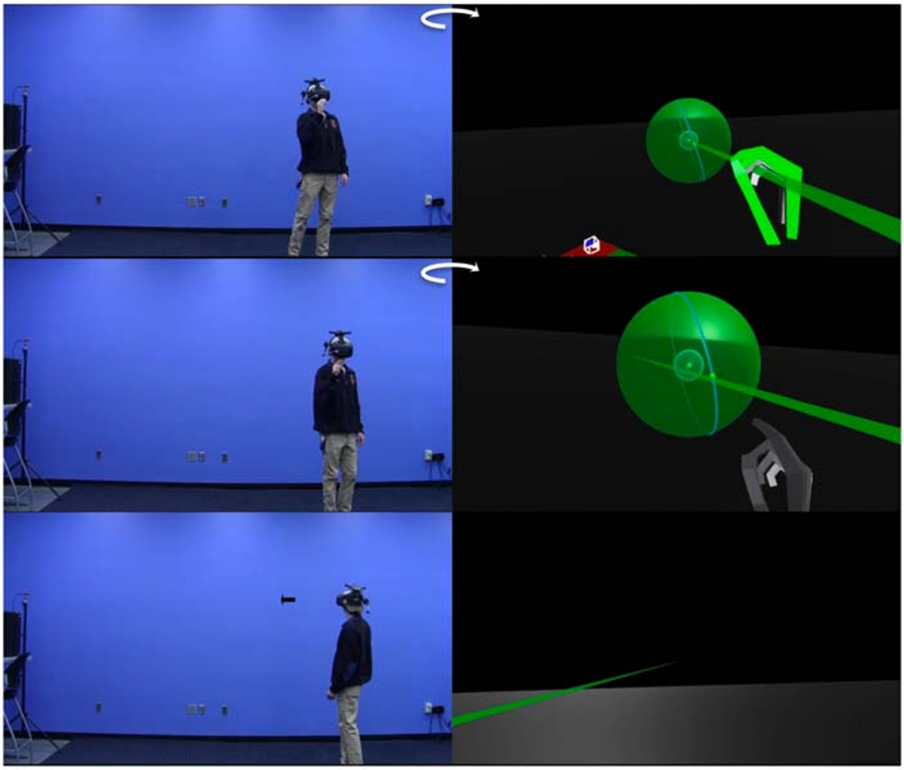

Fig. 6 Sean adjusting the position of the line so that it intersected two points (the top two panels) and then looked off into the distance

As she gazed at the line, Sally remarked, "It looks like it goes on forever". After he had created the line, Seth said something nearly identical: "It goes on forever". Figure 8 shows a montage of participants gazing up at the unbounded line in the moments after they had created it.

Looking up immediately followed each instance where participants snapped a line to be perpendicular to the floor. In real space, their bodies were positioned as if straining to see something spectacular, like a satellite or a shooting star. Through their bodies, they appeared to be in awe. As with the sweeping movement described earlier, gazing up at the line was a bodily response to noticing a change in the environment. The line that had been at an arbitrary angle (Fig. 7, second panel) was now standing straight up. But also, as before, the spatial representation partook in an encounter where a line could be experienced as something one *looks along* as opposed to something one *looks at*.

Euclid described lines as *breadthless length*. Geometry textbooks describe lines as having no width and extending infinitely in two directions, yet represent them as small, finite strokes (Dimmel and Herbst 2015; see Fig. 9). The property that lines are infinite is relegated to thought experiments or analogies.

The encounters with apparently unbounded spatial lines show something more: Students leaned back/looked up or else moved their heads from side to side. Nathan et al. (2014) argued that linking geometric concepts to specific movements of one's body (e.g. using one's torso and arm-span to experience the triangle inequality)

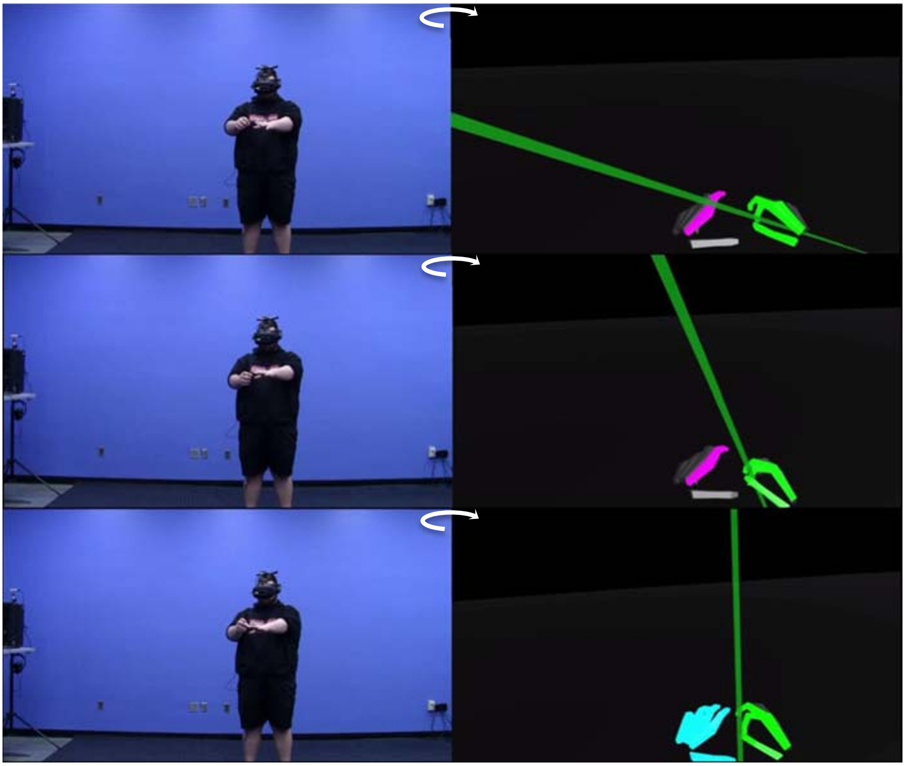

Fig. 7 Seth, snapping a line to be perpendicular to the floor, grabbed the line with one hand while holding out the other one, palm down

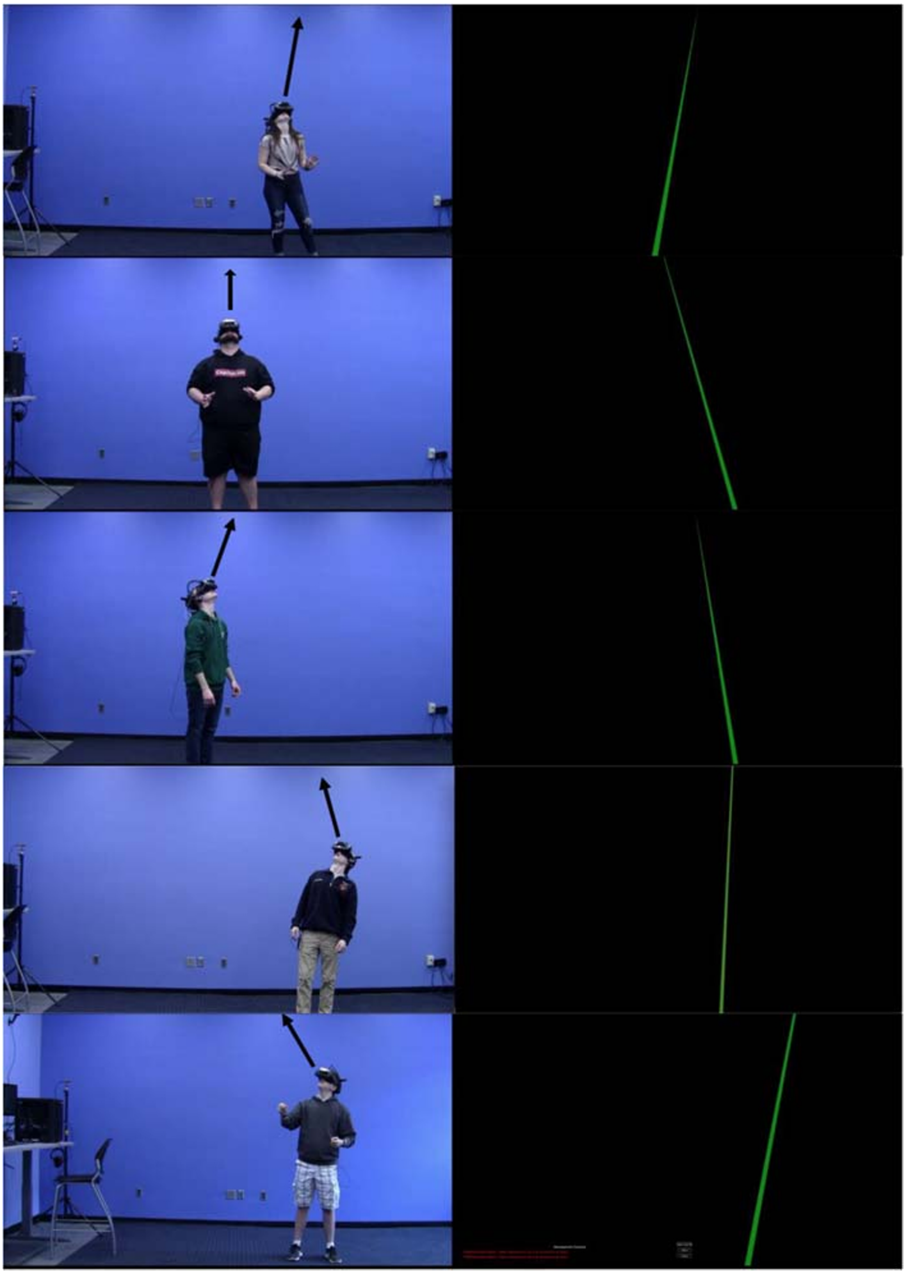

Fig. 8 Gazing up at an unbounded line

Fig. 9 A line

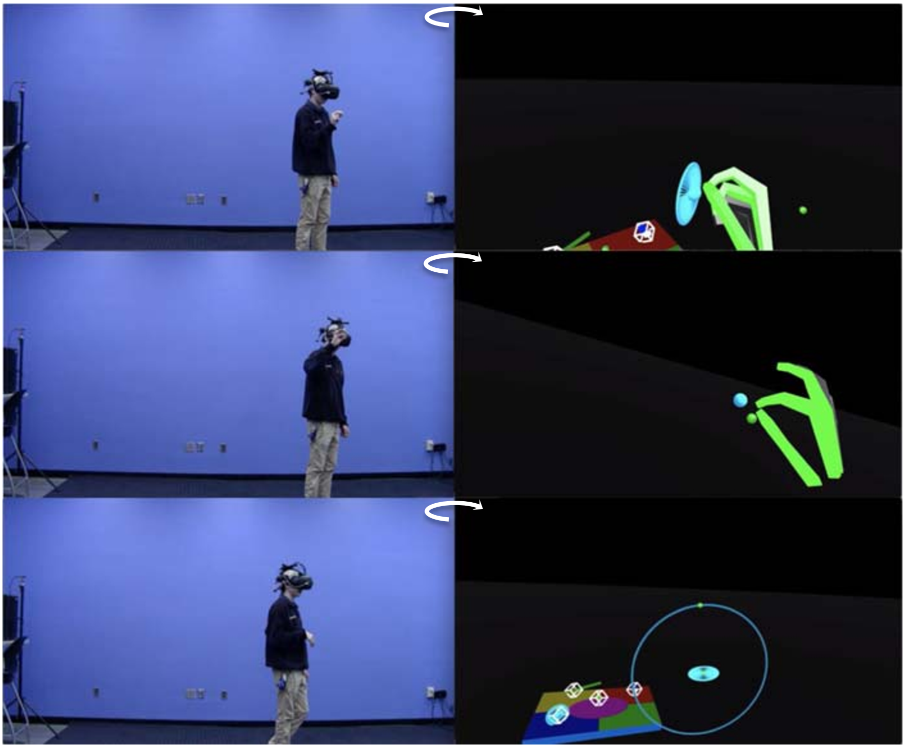

Fig. 10 Preparing to generate a sphere

provides grounding for mathematical proof.[10] The gaze-tracking and full-body half-turns that were apparent across participant–tool–inscription assemblages show how inscribable spatial canvases can be designed/configured to foster a correspondence between mathematical concepts and how we move our bodies in space (Abrahamson and Sánchez-García 2016).

Encounters with Spatial Inscriptions that Represent Spheres: Can I Go inside?

The spatial compass created occasions for participants to use their bodies to explore the size of a spatial diagram and the perspectives from which it can be viewed. Figures 10 and 11 show Sean using the spatial compass to inscribe a sphere in space.

The spatial compass acts on two points – the center of the sphere and a point on its surface. In the top panel of Fig. 10, Sean brought the spatial compass toward a point. When he touched the tool to the point (Fig. 10, second panel from top), a small blue sphere was spawned. The small blue sphere serves as a dock for a second point that is used to set the circumference of the sphere (Fig. 10, bottom panel). To inscribe a sphere in space, one spins the wheel (see Fig. 11).

The point at the center of the circumference becomes the center of the sphere. The logic of the tool is that a circle is rotated through space to trace the surface of a

[10] We acknowledge an anonymous reviewer who noted that physical instantiations of the triangle inequality are at least as old as the theorem itself.

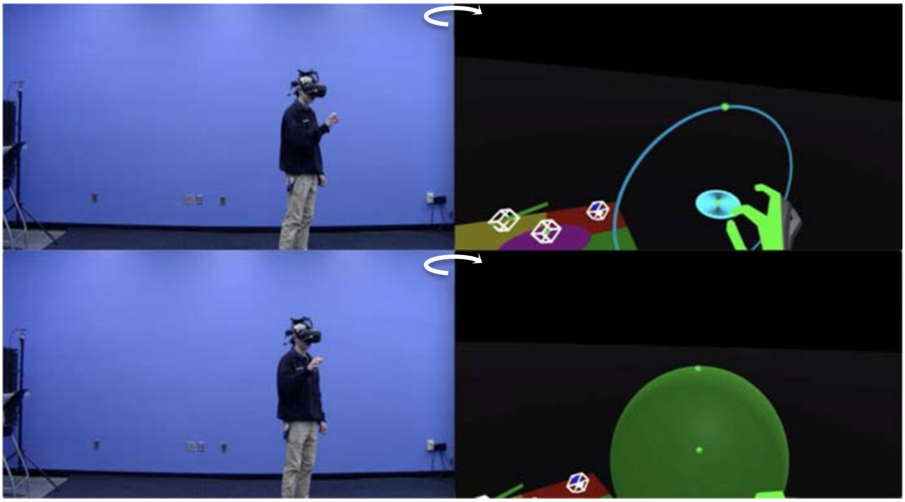

Fig. 11 Spinning the wheel to generate a sphere

sphere.The axis of rotation is orthogonal to the disk of the wheel (see Fig. 12). We designed the spatial compass around half-turns of complete circles (as opposed to complete turns of half-circles) to highlight the dimensional connection between circles and spheres.

Participants could scale spheres by moving either (or both) of its defining points. Spheres could be translated by pinching both of its defining points and moving them in space while preserving their distance. Figure 13 shows Alan increasing the length of the radius of a sphere until he could fit inside it.

In the top panel of Fig. 13, Alan's hands are approximately 25 cm apart, and the sphere is roughly the size of a beach ball. In the next panel, he stretched apart his hands (by pushing his left hand away from his body) to vary the size of the sphere. By the bottom panel, Alan had finished stretching and released his grasp of the segment to fix its length at approximately 70 cm; he was enveloped by the sphere. As he stood on the inside of the sphere, he used his hand to map the boundary of its surface, curving his fingers to mirror its curved interior surface (Fig. 13, bottom panel).

Every participant varied the sizes of spheres. Seth made a sphere large enough for him to stand inside it and then shrunk the sphere to be as small as possible (Fig. 14). As he did so, he said: "It kind of disappears when you put the points on top of each other". He also said that he "really liked" being able to change the size of the sphere. When

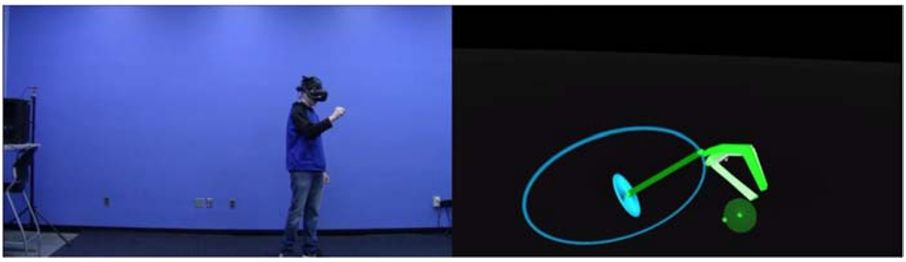

Fig. 12 The circumference of the sphere was rotated orthogonally to the wheel

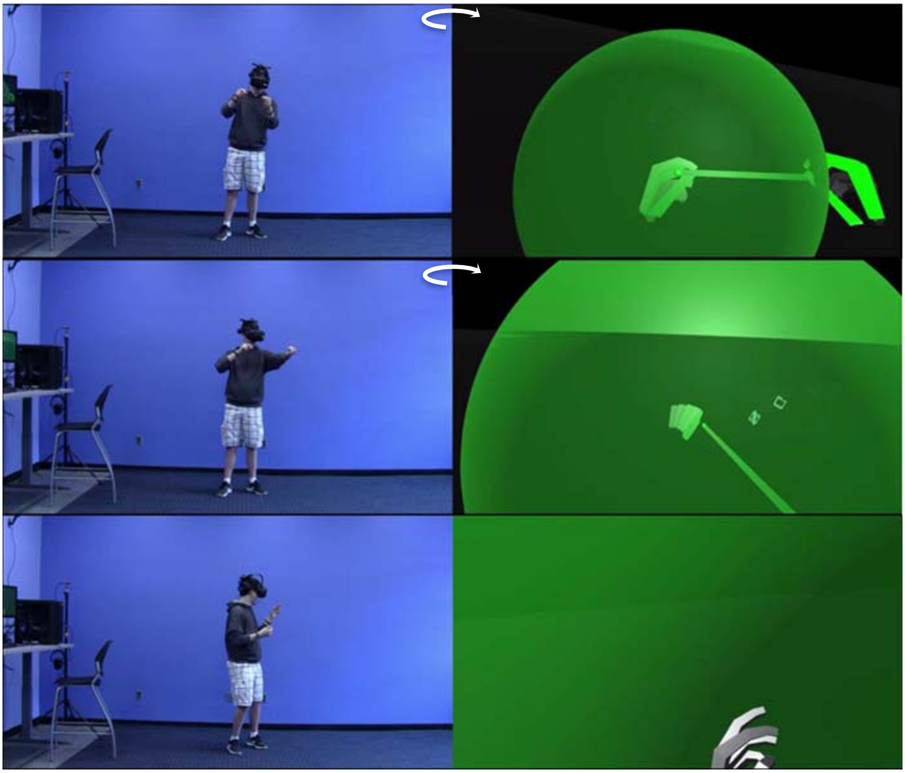

Fig. 13 Alan increasing the radius of the sphere by stretching apart his hands

asked what he liked about it, he said that it was "fun", and that it was "easier to understand that, like, that's a sphere, than seeing it on paper".

All but one participant explored spheres from their interior. Figure 15 shows different participants exploring spheres from the inside.

As with *looking along* a line, manipulating the size of a sphere and exploring it *from within* appeared to be natural actions of participant–tool–inscription assemblages. In both cases, participants interacted with inscriptions that varied through a wider range of scales and that they viewed from different perspectives from what is typical with two-dimensional inscriptions. We return to the scale and perspective themes raised in this case in the discussion section.

Case Two: Lattice Polytopes in *LatticeLand*

A complement to the *line &sphere* environment is *LatticeLand*, a 10 × 10 × 10 grid that is a spatial analogue of the Geoboard (Gattegno 1971). The unit distance (along each of the three intersecting axes) between each point was 0.25 m. Figure 16 shows an image of a high school mathematics teacher immersed in *LatticeLand*. Together, the left (observer) and right (first-person viewpoint) panels help to capture not only what the immersed participant saw, but also how she positioned her body in the rendered space.

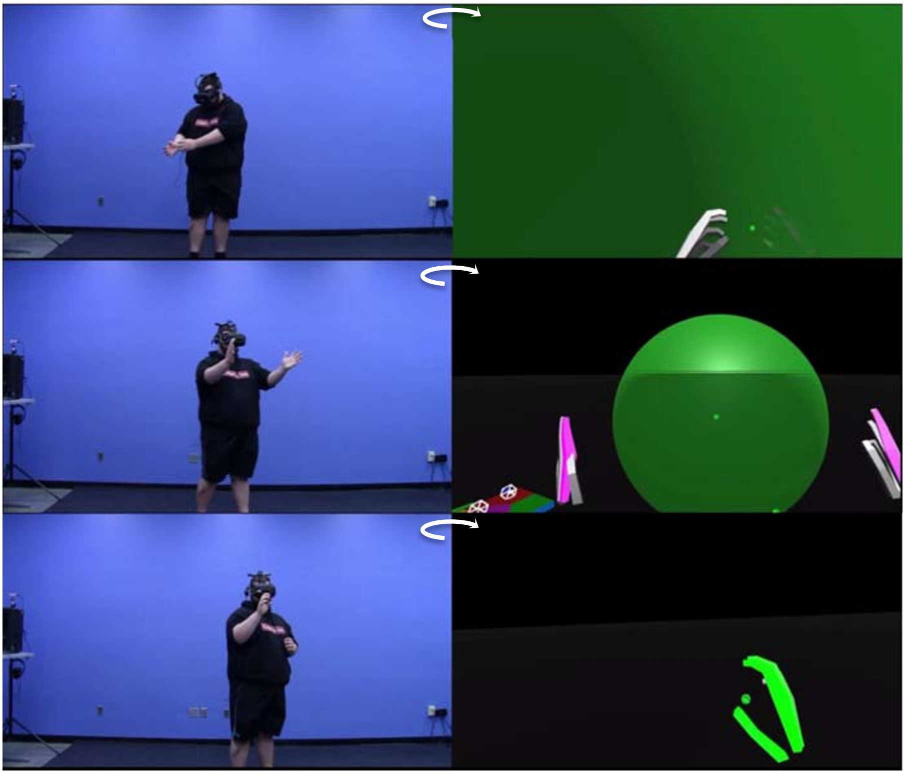

Fig. 14 Seth shrinking a sphere as large as a room down to a point, by using his hands to move a point on the surface of the sphere, in order for it to coincide with the center of the sphere

The left and right panels capture some sense of the depth of the lattice, through the arrays of points that are shown in perspective. A guide for using gestures to inscribe the lattice was displayed in the environment: pointing gestures allowed one to inscribe segments by connecting the points in the lattice; a backward ‘L’ gesture (with thumb and index finger) allowed one to create polygonal faces by pointing to a circuit of points (a polygon is generated once the circuit is completed); a hand-swipe gesture, where one passes the center of one’s palm through an inscription, erased an edge or face of an inscribed figure. Shawna is pointing to inscribe segments in Fig. 16. Figure 17 shows a detail of the gesture guide.

Participants and Method

In June, 2018, we convened a focus group (Kitzinger and Barbour 1999; Krueger and Casey 2000) of four secondary mathematics teachers to explore *LatticeLand*. Shawna, Bob, Dylan and Kevin (pseudonyms) took turns as the immersed participant while their peers observed what he or she did via a monitor that displayed the immersed, first-person view.

The focus group was documented using procedures similar to those that were described for case one. Both observer and first-person views of activity within

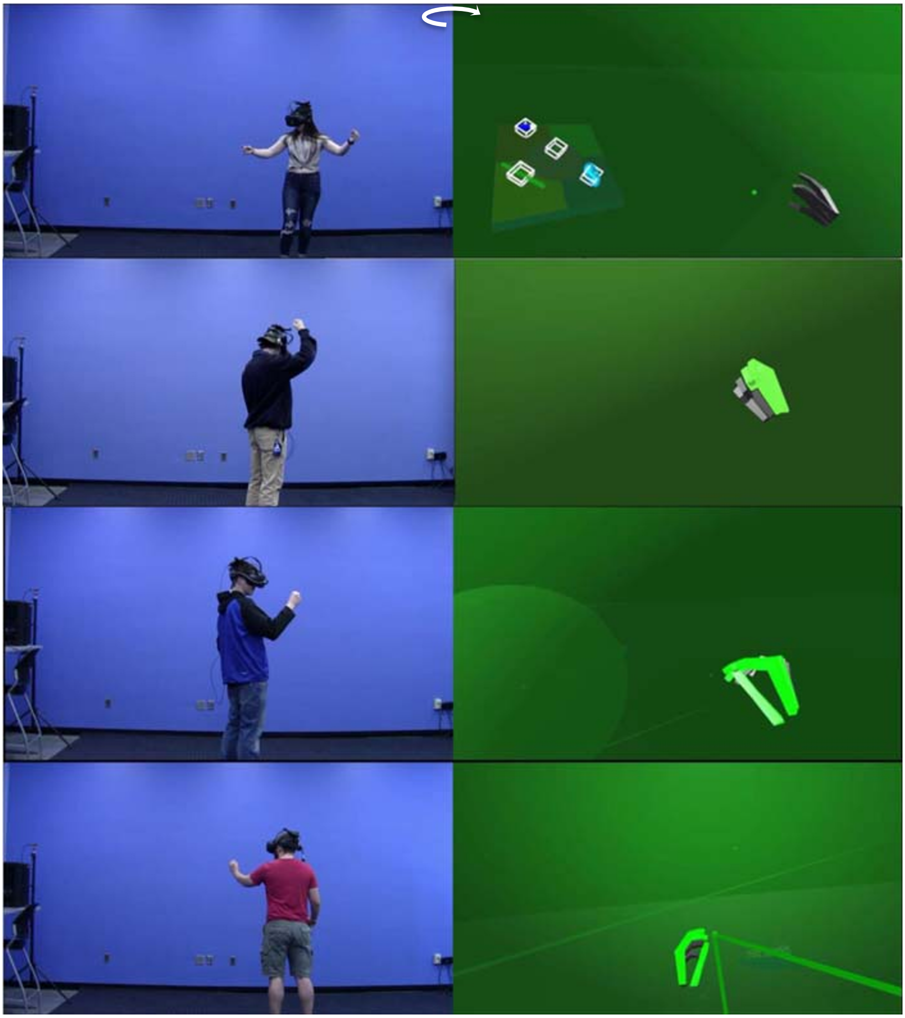

Fig. 15 Exploring the inside of a sphere

LatticeLand were recorded. The left panel of Fig. 16 shows a mixed-reality view (BR 2017; Sheftel and Williams 2019) of Shawna immersed in the environment. To record

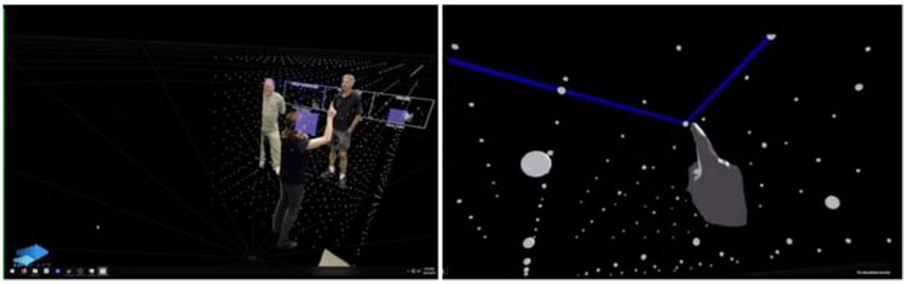

Fig. 16 Shawna (foreground, wearing a headset) immersed in LatticeLand

Fig. 17 Gesture-based operators in LatticeLand: **a** how to point to inscribe segments; **b** how to use a backwards "L" gesture to inscribe polygonal faces; **c** how to use a hand swipe to erase

this view, an in-world camera (inside the environment) was location synched to a wall-mounted physical camera and green screens were used to replace the real background with the rendered background (Fig. 18).

Below, we analyze two episodes of participant–tool–inscription assemblages within *LatticeLand*. The aim of the analysis is to describe how participants experienced a large-scale, inscribable spatial canvas.[11] In particular, we examine what immersed participants said, the figures they inscribed in the lattice and how they moved their bodies relative to the lattice as they worked.

Episode 1: Inscribing a Cube and its Internal Diagonals

The focus group began with unstructured exploration. Its purpose was to document participants' initial encounters with the spatial lattice. The first episode we describe involves Kevin.

Kevin inscribed a cube with his body positioned outside of the lattice. Figure 19 shows Kevin in the foreground with the lattice in the background. He reached into the lattice to inscribe the edges of the cube. When asked what he was making, Kevin said: "I'm just making a basic cube…just to experience walking around it and getting different perspectives".

He commented on this as he worked, "I'm just experimenting with reaching under, reaching around, as compared to walking around, just to see the difference". Shawna asked him about this, "I saw you reach up under it. Was that weird to do in 3D space? Without feeling it, like you almost went through it." Shawna was pressing Kevin to describe the experience of passing through a spatially extended, yet (im)material, inscription. Kevin did not respond to Shawna. He had moved on to inscribing the internal diagonals of the cube. In Dylan's words, Kevin was "mesmerized" by what he was creating. The three panels in Fig. 20 show Kevin inscribing the internal diagonals of a cube.

After Kevin had inscribed one of the internal diagonals, he asked, "Now, if we were to draw another diagonal, essentially to intersect with this one, is that going to work?" As he spoke, he inscribed a second internal diagonal (Fig. 20, middle panel). He then inscribed a third diagonal (Fig. 20, bottom panel) and visually investigated what he had inscribed by moving his head around the sides of the cube. He remarked, "I like [that] it gives you a real good visual representation of those diagonals inside the cube". Here is another instance where the (im)materiality of spatial inscriptions appeared to be significant.

[11] *LatticeLand* filled an approximate volume of 11.5m^3, about the size of small, low-ceilinged room.

Fig. 18 A view of Shawna gesturing in real space as she inscribed a segment by pointing with her index finger

The diagonals were spatially extended, in the sense that they appeared to pass through a measurable volume, but did not occupy any of the space in which they appeared to be contained. Thus, it was possible for two or three or four diagonals to intersect each other at what appeared, visually, to be *exactly the same point.* This is clearly different from what is possible with physical models (e.g. straws and paperclips), but also goes beyond what has been possible to date with screen-based DGEs. Environments such as GeoGebra 3D can display two-dimensional projections of

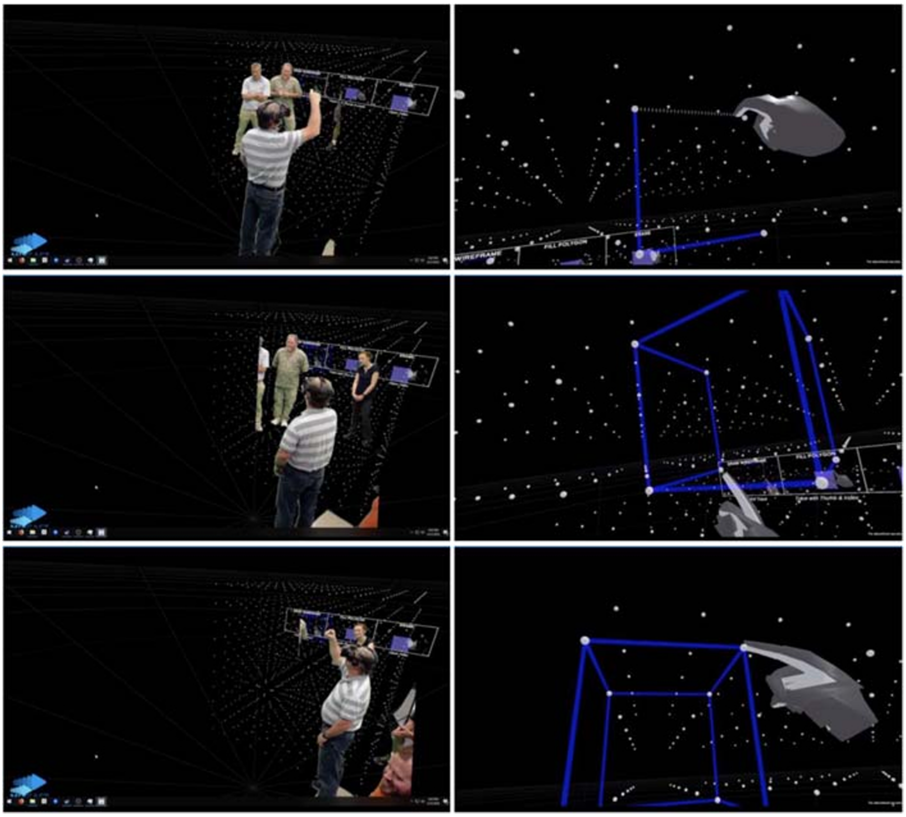

Fig. 19 Kevin inscribing a cube in the lattice,with Dylan (back left, top panel), Bob (next to Dylan), and Shawna (right, middle panel) watching Kevin working on a monitor

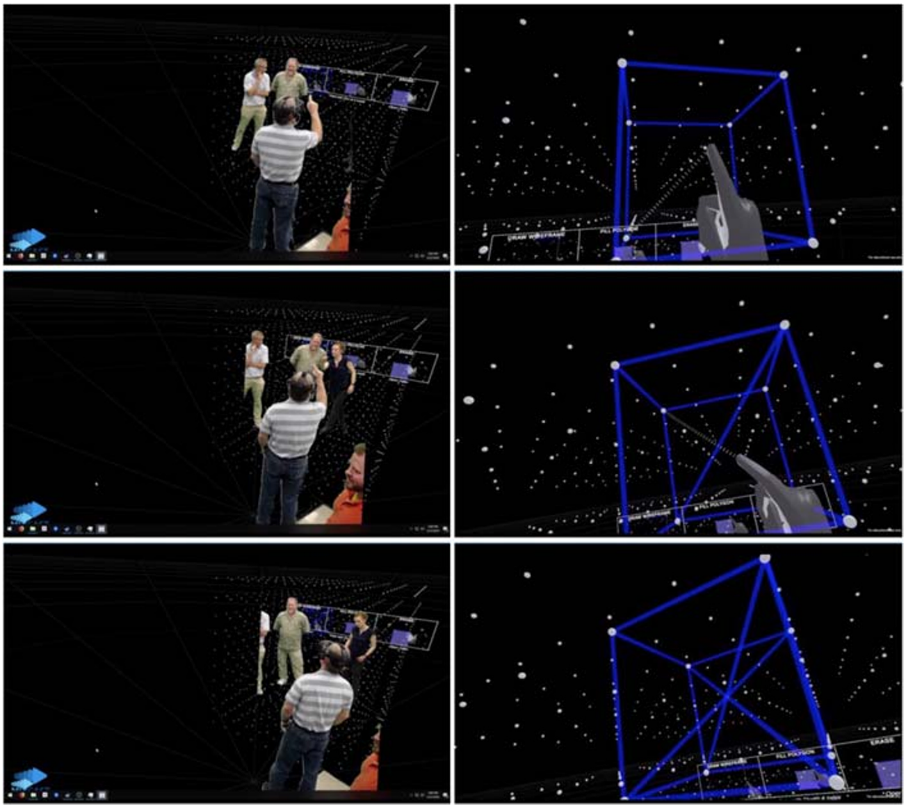

Fig. 20 Reaching into a cube to inscribe its internal diagonals

the internal diagonals of a cube, but the third dimension is compressed, so the viewer still needs to know how to *see* the spatiality of the figure, as mediated by a perspective drawing (Mithalal and Balacheff 2019). By contrast, the interior of Kevin's cube was directly accessible as a spatial object.

Episode 2: Enclosing oneself in a Box

Shawna inscribed polygonal faces in the lattice. As she worked, she commented, "You have to think very differently. I'm used to doing this on a white board in my classroom, and this is very different. The 3D changes how I think about the shapes." The first author asked what she was doing:

R: What are you trying to do right now?
Shawna: Make [a box] big enough to fit inside of.
R: Why do you want to do that?
Shawna: I don't know. Just to see what it looks like.

The panels in Fig. 21 show Shawna inscribing the box. As she worked, she moved in a circle around the inside of the box she was creating. She inscribed four sides of the box, working in a clockwise direction (Fig. 21, first three panels). Once she completed the sides, she leaned back, reached up and inscribed the top of the box, creating a roof over her head (Fig. 21, bottom panel).

As Shawna inscribed the box, she positioned her body among the points, adopting a *from within* perspective to engage the lattice. This was possible because the points were spatial without occupying space. Nevertheless, the box she created conveyed a material presence, as shown in Fig. 22. Shawna pulled her arms in to her torso, as if she were squeezing to fit inside something.

Kevin and Shawna were drawn to the interiors of figures in *LatticeLand*. Kevin inscribed the internal diagonals of a cube, where multiple (im)material segments intersected at one point. Shawna created a figure that allowed her to experience the volume of a prism from inside-out. In both episodes, the interior spaces of polyhedrons – flattened and beyond reach when polyhedra are inscribed on two-dimensional

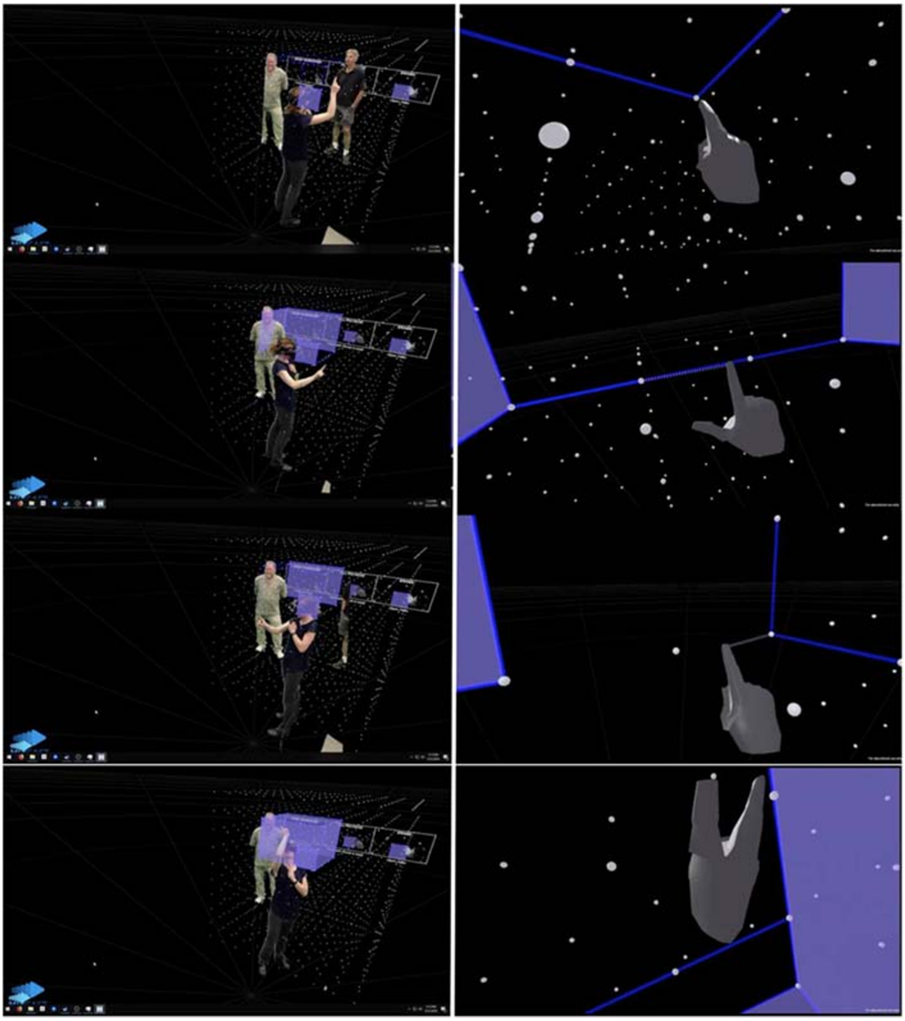

Fig. 21 Shawna encloses herself in a box

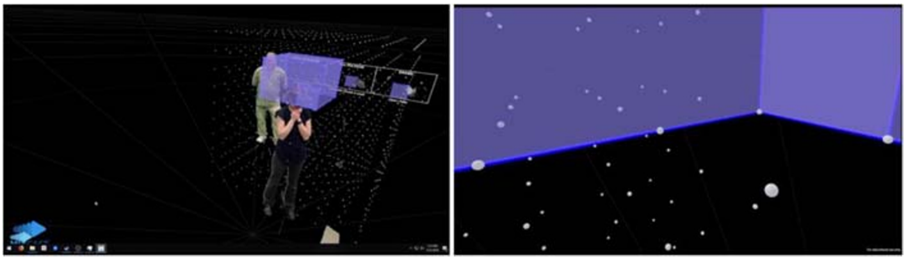

Fig. 22 Shawna pulling her arms in close to her torso as she stood inside the box

surfaces – became settings for mathematical activity. These examples show how the lattice partook in the generation of mathematical figures.

Discussion

Mathematical Encounters in Inscribable Spatial Canvases

This study was motivated by two questions: How might spatially inscribed realizations of mathematical figures be encountered by learners? How might such encounters affect the learning and teaching of mathematics? To respond to these questions, we analyzed the movements of participant–tool–inscription assemblages within the *line & sphere* and *LatticeLand* environments. We found that participants were drawn to explore spatial inscriptions at scales and from perspectives that are not usually available when diagrams are inscribed on two-dimensional surfaces.

In *LatticeLand,* Shawna created a roughly .3 m^3 box that was large enough to surround her. She remarked that the experience of drawing the box around herself required her to think differently about the box compared with when she would draw a two-dimensional representation of a three-dimensional shape on a whiteboard. In the *line & sphere* environment, participants looked along lines and from within spheres. The encounters with large-scale spheres and lines reported here were driven by shared experiences of curiosity that were entangled with particular spatial inscriptions. The similar experiences[12] of different participants interacting with the line and sphere inscriptions illustrates the distributed agency at the core of inclusive materialism. The students were not merely using the line and sphere tools, but, rather, the student–tool–inscription assemblages were acting together to spur explorations.

How did the spatial canvases shape inscription events? De Freitas (2016) conceptualizes inscribable surfaces as sensory boundaries that "partake in the making of mathematics" (p. 198). These surface-based boundaries are activated by hand movements. The episodes analyzed above show people moving not only their hands, but their entire bodies in relation to how they perceived spatial inscriptions – e.g. a half-turn of one's head, torso, and legs to look along a line; pulling in one's shoulders to feel a volume. In inscribable spaces, the boundary between agent and canvas is not just the tip

[12] The similarities are significant because, in the *line & sphere* environment, participants explored the environment independently; they had no prior knowledge of the environment, nor any knowledge of other participants' experiences in the environment during their encounters.

of a finger that holds the chalk or touches a screen – it is a person's entire body. It is an active boundary that moves and is reconfigured as people position themselves among spatial inscriptions.

How might encounters with spatial inscriptions affect the teaching and learning of mathematics? Our analysis provides evidence for two hypotheses. The first concerns perspective. Diagrams record events that are re-animated through exploration (de Freitas 2012). We become different characters (i.e. mathematical objects) in the story of a diagram as we examine (with gaze, with hands) its different parts. These changes in perspective are instances of *virtual voicing*, wherein "an authorial narrator shifts to a figural or character-bound narrator[and …] speaks directly [as that character]" (p. 30).

One changes perspective when viewing a two-dimensional diagram by *virtually* becoming its different parts. Our analysis suggests a complementary phenomenon: moving one's body to position one's physical self *actually* to see as parts of spatial diagrams (if personified as humans) would see. The *looking along* and *from within* perspectives provide two examples of how immersed learners spatially became diagrams by moving their bodies to explore, as characters, features of diagrams they would typically only have had access to as observers.

That participants adopted character viewpoints to engage diagrams spatially suggests one hypothesis for teaching and learning: *Movement and perspective-based descriptions of geometric figures and relationships could help learners vitalize spatial geometry concepts.* For example, a *regular* polyhedron could be defined in terms of how it looks from the inside out. An immersed learner would see an equal number of congruent polygonal faces meeting at each vertex of the polyhedron. *Convexity* and *concavity* could be experienced by positioning oneself at a point on the surface of a polyhedron and scanning with one's gaze to see what other points on the surface are visible and whether one's gaze goes outside the polyhedron in order to see them.

A second hypothesis concerns scale. Large-scale settings can be transformed into rich contexts for mathematical activities (e.g. Dimmel and Milewski 2019; Herbst and Boileau 2018; Kelton and Ma 2018; Ma 2017; Nemirovsky et al. 2012). But, in the material world, large-scale representations can be difficult or impossible to manipulate. By contrast, access to (im)material spatial inscriptions led participants to create and explore large-scale diagrams (e.g. spheres as big as a room; boxes big enough to stand in[13]). Since issues of scale are fundamental to spatial thinking (Herbst et al. 2017; Thurston 1994; Whiteley et al. 2015), our analysis suggests a second hypothesis: *Variable-scale spatial inscriptions could spur new ways of exploring and apprehending geometric figures.*

Conclusion

With the commercial availability of spatial displays (Dimmel and Bock 2019), the race is on to research their educational potential. Comparative studies of VR, AR and book-based teaching methods have shown, for example, that solid geometry can be more effectively learned through interactions with VR or AR than with books (Demitriadou

[13] The suitability of an immersive canvas for realizing large-scale figures is constrained by the size/configuration of its physical spatial host.

et al. 2020). Psychometric studies have shown that AR environments can significantly increase the skill necessary to create, measure and compare three-dimensional figures (İbili et al. 2020). These are two instances of what will soon be hundreds of subject-specific, experimental studies that assess the possibilities for device-mediated learning.

But studies that compare VR and AR with other technologies of instruction will only tell part of the story. Our analysis of encounters with spatial inscriptions serves as an example of research that can help to tell the rest of the story. The hypotheses concerning scale and perspective distilled from our analysis set our work apart from studies that would constrain how spatial inscriptions can partake in learning by comparing them against small, two-dimensional inscriptions. Instead, we need to consider how transformable spatial inscriptions can create new opportunities for encountering, exploring and even becoming mathematical figures.

For example, the spatial compass is an example of a tool that can only be realized by (im)material spatial inscriptions. It is possible to fabricate physical objects that are spheres, but material processes cannot easily recreate the logic of tracing a curve through space to generate a surface. Extruders could be rigged around a circular frame to inscribe a sphere by rotating it in space. And soap films, which have been shown invaluable to investigating the mathematics of surface minimization (Almgren and Taylor 1976), are examples of two-dimensional surfaces that are generated by moving one-dimensional curves through space. But it would be difficult, with physical things, to recreate the control that one has over the scale of and position of (im)material, spatially inscribed spheres.

Our work shows what is possible with off-the-shelf hardware and software that was developed in an academic research lab. Yet, even from these basic examples, it is clear that (im)material spatial inscriptions have the potential to reconfigure how mathematics can be experienced. We see this work as a step toward realizing a mathematics of movement, where spatial canvases can be inscribed by co-ordinating actions among groups of bodies immersed in the same rendered world.

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Compliance with Ethical Standards

Conflict of Interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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