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ORIGINAL ARTICLE

Fundamental properties of fuzzy rough sets based on triangular norms and fuzzy implications: the properties characterized by fuzzy neighborhood and fuzzy topology

Zhaohao Wang[1]

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Abstract

Fuzzy rough set models are useful tools for dealing with fuzzy and real-valued data. They have been used in many real-world applications. In this paper, we investigate the fuzzy rough set model based on triangular norms and fuzzy implications. First, we extend some results in the published literature by removing the condition that is the continuity of triangular norms, and obtain more general conclusion about fuzzy upper approximation operators. Then, for the fuzzy neighborhood and the fuzzy lower approximation operator based on fuzzy implications, we investigate their characterization with each other. Finally, we establish the relationships between fuzzy rough sets and fuzzy topology. In this work, researches on the properties of fuzzy rough sets based on triangular norms which need not be continuous provide generalization results for fuzzy rough set theory from viewpoint of mathematics.

Keywords Fuzzy set · Rough set · Fuzzy neighborhood · Fuzzy topology · Fuzzy implication

Introduction

Rough set theory [1] has been widely used in many fields of applied sciences, such as machine learning, data mining and so on. It is a useful tool to deal with imperfect and inconsistent information in data analysis [2, 3]. However, a disadvantage of rough set theory is that it is designed to handle discrete data [4]. In fact, symbolic, fuzzy, and real-valued attributes exist, e.g., in medical analysis and fault diagnosis [5–7]. In order to deal with these complex datasets, a generalized rough set model needs to be established. Fuzzy set theory [8] is a generalization of classical set theory for dealing with vague concepts and graded indiscernibility. Fuzzy set theory and rough set theory complement each other, so researchers presented the hybrid model, i.e., the fuzzy rough set model, by integrating them, which can be an alternative for dealing with fuzzy and real-valued data. In the big data era, the datasets collected from real-life applications have high dimension and may have hundreds of features. Removing redundant features is necessary for the knowledge discovery method. Fuzzy rough set theory becomes a useful tool for feature selection of high dimensional data [9]. For example, Zhao et al. [10] applied fuzzy rough set theory to design a feature selection strategy for hierarchical classification of high dimensional data.

In 1990, Dubois and Prade [11] introduced the fuzzy rough set model that follows from replacing the equivalence relation in the Pawlak rough set model by the fuzzy similarity relation in fuzzy set theory. Fuzzy implications have been applied in many fields, such as, approximate reasoning [12], fuzzy control [13], and fuzzy mathematical morphology [14]. To extend application of the fuzzy rough set model, fuzzy implications were introduced into fuzzy rough set theory. Morsi and Yakout [15] presented a more general approach to extend the fuzzy rough set model, and they provided the concept of (I, T)-fuzzy rough sets, which are defined by a triangular norm T and the R-implication I based on T. Then Radzikowska and Kerre [16] discussed three classes of (I, T)-fuzzy rough sets taking into account three well-known classes of fuzzy implications, that are the S-implication, R-implication, and QL-implication. Cock et al. [17] and Wu et al. [18] explored (I, T)-fuzzy rough sets based on a general triangular norm T and a general fuzzy implication I. These studies found significant interest of researchers in

✉ Zhaohao Wang
nysywzh@163.com

1 School of Mathematics and Computer Science, Shanxi Normal University, Taiyu Road, Taiyuan, 030000 Shanxi, R.P. China

fuzzy rough set theory. Recently, a great diversity of research on this topic has appeared [19–28]. For example, Hu et al. [20, 21] proposed interval-valued fuzzy rough sets based on interval-valued fuzzy logical operators; moreover, from the viewpoint of lattice theory, the authors [19, 23–26] conducted further research into this topic.

Fuzzy rough sets have been applied to optimization of knowledge engineering algorithms. The main and successful application is in attribute reduction [29–31]. The benefit of the attribute reduction mainly focuses on that fuzzy rough feature extraction preserves the semantics of the selected features. Moreover, the model of fuzzy rough sets can also be used for general data mining operations, like clustering or classification in the case of uncertain input domains [32]. Obviously, the discussion of mathematical theory of fuzzy rough sets is beneficial to improve their application effect. In addition, in [33, 34], the authors pointed out that fuzzy rough sets based on t-norms and fuzzy implications satisfy closure property under some conditions, which indicates that fuzzy rough set model based on t-norms and fuzzy implications has good mathematical properties. This paper focuses on the fuzzy rough set model based on t-norms and fuzzy implications, and investigate the fundamental properties of this model. In practical application of fuzzy rough set model, we usually face two issues: which t-norm and fuzzy implication should one choose? What properties should they satisfy? The results of this paper may provide support for solving these issues.

It is well known that the properties of the lower and upper approximation operators can be determined by the properties of neighborhood operator; for example, in a crisp rough set model based on a neighborhood, if the neighborhood is reflexive, then the upper approximation operator induced by it satisfies property of extension [37]. Thus, it is fundamental to study the connection between the fuzzy neighborhood operator and the fuzzy rough set model induced by the fuzzy neighborhood. In [18, 35, 36], the relationships between fuzzy neighborhood operators and fuzzy upper approximation operators based on a triangular norm was established. However, there are no corresponding results for the fuzzy neighborhood operator and the fuzzy lower approximation operator based on a fuzzy implication. In this paper, we give the connection between the fuzzy neighborhood operator and the fuzzy lower approximation operator based on a fuzzy implication. In addition, the results in [18, 35, 36] limited the triangular norm to be continuous. In this work, we remove the limitation of continuity of the triangular norm, and we obtain the more general result which reveals that the T-transitivity of fuzzy neighborhood operators only relates to the upper approximation of the singleton sets. Finally, by means of the above results, we discuss the relationships between fuzzy topology and fuzzy rough sets based on t-norms and fuzzy implications. The main contributions of this paper can be described as follows:

– The result (see Theorem 1) in [18, 35, 36] is extended by removing the condition that T is continuous. Thus, the more general conclusions about fuzzy upper approximation operators based on t-norms are established.
– This paper concludes that the properties of fuzzy neighborhood operators can be characterized by a fuzzy lower approximation operator based on a fuzzy implication, and vice versa.
– The relationships between fuzzy topology structures and fuzzy rough sets based on t-norms and fuzzy implications are further investigated.

The rest of this paper is organized as follows. Some related notions and results are reviewed in "Notations, definitions and basic concepts". In "Properties of fuzzy rough sets with respect to fuzzy neighborhood operators", we discuss the fundamental properties of the fuzzy lower and upper approximation operators based on the triangular norm and the fuzzy implication. "The relationships between the fuzzy topology and the fuzzy lower (or upper) approximation operator" establishes the relationships between the fuzzy rough set and the fuzzy topology. Finally, "Conclusion" concludes this paper.

Notations, definitions and basic concepts

In this section, we review some basic concepts and notations used in this paper.

Let U be a nonempty universe of discourse. In this paper, we denote $\mathcal{F}(U)$ as a family of fuzzy sets on U.

Fuzzy logical operators

A triangular norm is used to define the operation on two fuzzy sets. Its definition is as follows [38]:

Definition 1 ([38]) A mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ is called a triangular norm (t-norm for short) if for $a, b, c \in [0, 1]$, it satisfies the following conditions:

(T1) $T(a, 1) = a$;
(T2) $b \le c \Rightarrow T(a, b) \le T(a, c)$;
(T3) $T(a, b) = T(b, a)$;
(T4) $T(a, T(b, c)) = T(T(a, b), c)$.

Briefly, a t-norm is an increasing, associative and commutative mapping from $[0, 1] \times [0, 1]$ to $[0, 1]$ that satisfies the boundary condition: $\forall a \in [0, 1], T(a, 1) = a$.

Clearly, a t-norm can be viewed as a real function of two variables. Thus, according to the theory of mathematical analysis, we can give the definition of left-continuous t-norm, that is, a t-norm is said to be left-continuous if it is left-continuous in each component [39]. This definition can be reformulated as follows.

Definition 2 A t-norm T is said to be left-continuous, if T satisfies the following condition:

$$\forall \emptyset \neq A \subseteq [0,1] \text{ and } b \in [0,1],\quad T(\sup A, b) = \sup_{a\in A} T(a,b). \tag{1}$$

In fact, since T is commutative, it is easy to verify that if T is left-continuous, then T also satisfies the condition: $\forall A \subseteq [0,1]$ and $a \in [0,1]$, $T(a, \sup A) = \sup_{b\in A} T(a,b)$.

A fuzzy implication is seen as the extension of an implication in binary classical logic to the multi-valued domain. However, there are different definitions for a fuzzy implication (see [39, 40]). In this paper, we choose the most common definition of fuzzy implication defined by Kitainik in [41].

Definition 3 ([41]) A mapping $I : [0,1]\times[0,1] \to [0,1]$ is called a fuzzy implication if it satisfies the following conditions:

(I1) $\forall x_1, x_2, y \in [0,1]$, if $x_1 \le x_2$, then $I(x_1, y) \ge I(x_2, y)$,

(I2) $\forall y_1, y_2, x \in [0,1]$, if $y_1 \le y_2$, then $I(x, y_1) \le I(x, y_2)$,

(I3) $I(0,0) = I(1,1) = 1$ and $I(1,0) = 0$.

We can easily deduce that I satisfies the following properties: $\forall a \in [0,1]$, $I(0,a) = I(a,1) = 1$. Clearly, $I(0,1) = 1$.

Definition 4 ([39]) A mapping $I : [0,1]\times[0,1] \to [0,1]$ is called an R-implication if there exists a t-norm T such that $\forall a, b \in [0,1]$,

$$I(a,b) = \sup\{c \in [0,1] : T(a,c) \le b\}. \tag{2}$$

Fuzzy neighborhood operators

We know that a fuzzy subset A of U is a mapping from U to $[0,1]$, that is, $A : U \to [0,1]$. Specially, a classical subset $K \subseteq U$ can be seen as the fuzzy subset whose membership function is identical to its characteristic function, and this fuzzy subset corresponding to K is still denoted as K throughout this paper. For $A, B \in \mathcal{F}(U)$, Zadeh [8] gave the definitions of containment, equality, union and intersection between fuzzy sets which are as follows:

- Containment; $A \subseteq B \Leftrightarrow \forall x \in U$, $A(x) \le B(x)$.
- Equality; $A = B \Leftrightarrow \forall x \in U$, $A(x) = B(x)$.
- Union; $\forall x \in U$, $(A \cup B)(x) = \max\{A(x), B(x)\}$.
- Intersection; $\forall x \in U$, $(A \cap B)(x) = \min\{A(x), B(x)\}$.

In [42], D'eer et al. proposed the definition of a fuzzy neighborhood operator as follows:

Definition 5 ([42]) A fuzzy neighborhood operator is a mapping $N : U \to \mathcal{F}(U)$.

In Definition 5, for $x \in U$, $N(x) \in \mathcal{F}(U)$ is a fuzzy set on U, and it is called the fuzzy neighborhood of x. For $y \in U$, $N(x)(y)$ is the membership degree of y in the neighborhood $N(x)$.

Let N be a fuzzy neighborhood operator on U. Similarly, the concepts of a fuzzy reflexive (symmetric, transitive, resp.) neighborhood operator are given as follows.

Definition 6 ([42]) Let N be a fuzzy neighborhood operator and T be a t-norm. N is called

- a serial fuzzy neighborhood operator, if $\forall x \in U$, $\sup_{y\in U} N(x)(y) = 1$;
- a reflexive fuzzy neighborhood operator, if $\forall x \in U$, $N(x)(x) = 1$;
- a symmetric fuzzy neighborhood operator, if $\forall x, y \in U$, $N(x)(y) = N(y)(x)$;
- a T-transitive fuzzy neighborhood operator, if $\forall x, y, z \in U$, $T(N(x)(y), N(y)(z)) \le N(x)(z)$.

In [33, 42], by means of the fuzzy neighborhood, D'eer et al. constructed the fuzzy rough set model based on a t-norm and a fuzzy implication as follows.

Definition 7 [33] Let N be a fuzzy neighborhood operator on U, T a t-norm and I a fuzzy implication. Then the fuzzy approximation operators $(\underline{apr}_N^I, \overline{apr}_N^T)$ with respect to N are defined by the following formulas:

$\forall A \in \mathcal{F}(U)$ and $\forall x \in U$,

$$\left(\underline{apr}_N^I(A)\right)(x) = \inf_{y\in U} I(N(x)(y), A(y)),$$

$$\left(\overline{apr}_N^T(A)\right)(x) = \sup_{y\in U} T(N(x)(y), A(y)).$$

Properties of fuzzy rough sets with respect to fuzzy neighborhood operators

In this section, we use the fuzzy lower and upper approximation operators to characterize the fuzzy neighborhood operators, and the converse issues are also discussed.

Properties of fuzzy upper approximation operators based on a *t*-norm

In this section, we extend a result (see Theorem 1) in [18, 35, 36] by removing the condition that T is continuous, and obtain more general conclusion about fuzzy upper approximation operators (see Theorem 2).

Remark 1 In this paper, given $\alpha \in [0, 1]$, we denote the **constant fuzzy set** of U by $\hat{\alpha}$, that is, $\forall x \in U, \hat{\alpha}(x) = \alpha$.

At first, we give the following properties of fuzzy upper approximation operators based on a t-norm. In this result, we also do not limit the continuity of t-norm.

Proposition 1 *Let N be a fuzzy neighborhood operator on U and T a t-norm. Then, the following assertions hold:*
(1) $\overline{apr}^T_N(\emptyset) = \emptyset$.
(2) $\forall \alpha \in [0, 1]$, $\overline{apr}^T_N(\hat{\alpha}) \subseteq \hat{\alpha}$.

Proof (1) By the monotonicity and boundary conditions of T, we derive that $\forall a \in [0, 1]$, $T(a, 0) \le T(1, 0) = 0$, that is, $T(a, 0) = 0$. Therefore,

$$\forall x \in U, \quad \left(\overline{apr}^T_N(\emptyset)\right)(x) = \sup_{y \in U} T(N(x)(y), \emptyset(y)) = \sup_{y \in U} T(N(x)(y), 0) = 0 = \emptyset(x).$$

In other words, $\overline{apr}^T_N(\emptyset) = \emptyset$.

(2) Let $\alpha \in [0, 1]$. By $\forall a \in U$, $T(a, 1) = a$, we obtain that, for all $x \in U$,

$$\left(\overline{apr}^T_N(\hat{\alpha})\right)(x) = \sup_{y \in U} T(N(x)(y), \alpha) \le \sup_{y \in U} T(1, \alpha) = \sup_{y \in U} \alpha = \alpha = \hat{\alpha}(x).$$

Therefore, we have that $\overline{apr}^T_N(\hat{\alpha}) \subseteq \hat{\alpha}$. □

Remark 2 In this paper, $\forall x \in U$, we denote the fuzzy set μ_x of U as:

$$\forall y \in U, \quad \mu_x(y) = \begin{cases} 1, & y = x; \\ 0, & y \ne x. \end{cases} \tag{3}$$

Theorem 1 [18, 35, 36] *Let N be a fuzzy neighborhood operator on U and T a continuous t-norm. Then, the following statements hold:*
(1) N is serial $\Longleftrightarrow \overline{apr}^T_N(U) = U$,
$\Longleftrightarrow \overline{apr}^T_N(\hat{\alpha}) = \hat{\alpha}, \forall \alpha \in [0, 1]$.
(2) N is reflexive $\Longleftrightarrow \forall A \in \mathcal{F}(U)$, $A \subseteq \overline{apr}^T_N(A)$.
(3) N is symmetric $\Longleftrightarrow \forall x, y \in U$, $\left(\overline{apr}^T_N(\mu_x)\right)(y) \subseteq \left(\overline{apr}^T_N(\mu_y)\right)(x)$.
(4) *N is T*-transitive $\Longleftrightarrow \forall A \in \mathcal{F}(U), \overline{apr}^T_N(\overline{apr}^T_N(A)) \subseteq \overline{apr}^T_N(A)$.

Theorem 1 holds under the condition that T is a continuous t-norm. In this section, we delete this condition and establish the corresponding results of Theorem 1.

The following example shows that if T is not continuous, then N is serial does not necessarily imply $\forall \alpha \in [0, 1]$, $\overline{apr}^T_N(\hat{\alpha}) = \hat{\alpha}$. That is to say, Theorem 1 (1) is not true when T is not continuous.

Example 1 Let $U = \{1, 2, 3, \dots\}$. $\forall k, m \in U$, we take that $N(k)(m) = \frac{m-1}{m}$. Clearly, N is a fuzzy neighborhood operator on U. Since $\forall k \in U, \sup_{m \in U} N(k)(m) = \sup_{m \in U} \frac{m-1}{m} = 1$, N is serial. We choose the drastic t-norm T_0 as follows:

$$\forall a, b \in [0, 1], \quad T_0(a, b) = \begin{cases} b, & \text{if } a = 1; \\ a, & \text{if } b = 1; \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

In what follows, we take $\alpha = 0.3 \in [0, 1]$. Thus, for $k \in U$, we have

$$\left(\overline{apr}^{T_0}_N(\hat{\alpha})\right)(k) = \sup_{m \in U} T_0\left(N(k)(m), \hat{\alpha}(m)\right) = \sup_{m \in U} T_0\left(\frac{m-1}{m}, 0.3\right) = \sup_{m \in U} 0 = 0 \ne 0.3 = \hat{\alpha}(k).$$

This indicates that if T is not continuous, then N is serial does not necessarily imply $\forall \alpha \in [0, 1]$, $\overline{apr}^T_N(\hat{\alpha}) = \hat{\alpha}$.

The following example indicates that N is T-transitive does not necessarily imply $\forall A \in \mathcal{F}(U), \overline{apr}^T_N(\overline{apr}^T_N(A)) \subseteq \overline{apr}^T_N(A)$. That is to say, Theorem 1 (4) is not true when T is not continuous.

Example 2 Let $U = \{1, 2, 3, \dots\}$. The fuzzy neighborhood N is given as follows:

$$N(1)(m) = \begin{cases} 0.3, & \text{if } m = 2; \\ 0.2, & \text{otherwise.} \end{cases}$$

$$\forall n \in U \text{ and } n \ge 2, \ \forall m \in U, \ N(n)(m) = \frac{m}{2m+1}.$$

In [38], Klement et al. provided a not left-continuous t-norm as follows:

$$\forall a, b \in [0, 1], \quad T_1(a, b) = \begin{cases} 0, & \text{if } (a, b) \in (0, 0.5)^2; \\ \min\{a, b\}, & \text{otherwise.} \end{cases} \tag{5}$$

In addition, $\forall n, m, k \in U$, it is easy to see that

$$0 < N(n)(m) < 0.5, \text{ and } 0 < N(m)(k) < 0.5.$$

Thus, we have

$$T_1\big(N(n)(m), N(m)(k)\big) = 0 \le N(n)(k).$$

That is to say, N is a T_1-transitive fuzzy neighborhood. We take the fuzzy set A whose membership function is given by the formula $\forall m \in U$, $A(m) = \frac{m}{m+6}$. Next, it is easy to compute that

$$\left(\overline{apr}_N^{T_1}(A)\right)(n) = \begin{cases} 0.2, & \text{if } n = 1; \\ 0.5, & \text{otherwise.} \end{cases}$$

$$\left(\overline{apr}_N^{T_1}\left(\overline{apr}_N^{T_1}(A)\right)\right)(1) = \sup_{m\in U} T\left(N(1)(m), \left(\overline{apr}_N^{T_1}(A)\right)(m)\right) = 0.3.$$

Thus, $(\overline{apr}_N^{T_1}(\overline{apr}_N^{T_1}(A)))(1) > (\overline{apr}_N^{T_1}(A))(1)$. This implies $\overline{apr}_N^{T_1}\left(\overline{apr}_N^{T_1}(A)\right) \nsubseteq \overline{apr}_N^{T_1}(A)$, which shows that N is T-transitive does not necessarily imply $\forall A \in \mathcal{F}(U)$, $\overline{apr}_N^T(\overline{apr}_N^T(A)) \subseteq \overline{apr}_N^T(A)$.

Next, we remove the limitation of continuity of t-norm in Theorem 1, and give the corresponding results. First, we can obtain the following results.

Lemma 1 *Let N be a fuzzy neighborhood operator on U and T a t-norm. If $\forall x \in U$, $\overline{apr}_N^T(\overline{apr}_N^T(\mu_x)) \subseteq \overline{apr}_N^T(\mu_x)$, then N is T-transitive.*

Proof For $z \in U$, we know that $\overline{apr}_N^T(\overline{apr}_N^T(\mu_z)) \subseteq \overline{apr}_N^T(\mu_z)$. Thus, $\forall x \in U$, we have

$$\left(\overline{apr}_N^T\left(\overline{apr}_N^T(\mu_z)\right)\right)(x) \le \left(\overline{apr}_N^T(\mu_z)\right)(x). \tag{6}$$

In addition, from the properties of t-norm, $\forall a \in [0, 1]$, $T(a, 0) = 0$, $T(a, 1) = a$, and Eq. (3), we have that $\forall x \in U$,

$$\begin{aligned}\left(\overline{apr}_N^T(\mu_z)\right)(x) &= \sup_{w\in U} T\left(N(x)(w), \mu_z(w)\right) \\ &= T\left(N(x)(z), \mu_z(z)\right) \\ &= T\left(N(x)(z), 1\right) = N(x)(z),\end{aligned}$$

and

$$\begin{aligned}&\left(\overline{apr}_N^T\left(\overline{apr}_N^T(\mu_z)\right)\right)(x) \\ &= \sup_{u\in U} T\left(N(x)(u), \left(\overline{apr}_N^T(\mu_z)\right)(u)\right) \\ &= \sup_{u\in U} T\left(N(x)(u), \sup_{w\in U} T\left(N(u)(w), \mu_z(w)\right)\right) \\ &= \sup_{u\in U} T\left(N(x)(u), T\left(N(u)(z), \mu_z(z)\right)\right) \\ &= \sup_{u\in U} T\left(N(x)(u), T\left(N(u)(z), 1\right)\right) \\ &= \sup_{u\in U} T\left(N(x)(u), N(u)(z)\right).\end{aligned}$$

Therefore, by Eq. (6), we obtain that $\sup_{u\in U} T(N(x)(u), N(u)(z)) \le N(x)(z)$. Clearly, for $y \in U$,

$$T\left(N(x)(y), N(y)(z)\right) \le \sup_{u\in U} T\left(N(x)(u), N(u)(z)\right).$$

This implies that $T\Big(N(x)(y), N(y)(z)\Big) \le N(x)(z)$. We have proved that $\forall x, y, z \in U$, $T\Big(N(x)(y), N(y)(z)\Big) \le N(x)(z)$. Consequently, N is T-transitive. □

Proposition 2 *Let N be a fuzzy neighborhood operator on U and T a t-norm. Then, the following statements hold:*

(1) If $\forall \alpha \in [0, 1]$, $\overline{apr}_N^T(\hat{\alpha}) = \hat{\alpha}$, then N is serial.

(2) If $\forall A \in \mathcal{F}(U)$, $\overline{apr}_N^T(\overline{apr}_N^T(A)) \subseteq \overline{apr}_N^T(A)$, then N is T-transitive.

Proof (1) Assuming that $\forall \alpha \in [0, 1]$, $\overline{apr}_N^T(\hat{\alpha}) = \hat{\alpha}$, we have $\overline{apr}_N^T(\hat{1}) = \hat{1}$. By Definition 7, this implies that $\forall x \in U$, $\left(\overline{apr}_N^T(\hat{1})\right)(x) = \sup_{y\in U} T(N(x)(y), \hat{1}(y)) = \sup_{y\in U} T(N(x)(y), 1) = \sup_{y\in U} N(x)(y) = 1 = \hat{1}(x)$, that is, $\forall x \in U$, $\sup_{y\in U} N(x)(y) = 1$. Therefore, N is serial.

(2) By Lemma 1, it is obvious. □

Remark 3 In [33], D'eer et al. provided a result: that is, if N is reflexive, then $\forall A \in \mathcal{F}(U)$, $\underline{apr}_N^I(A) \subseteq A \subseteq \overline{apr}_N^T(A)$. In this paper, we only use the upper approximation to characterize the reflexivity of a fuzzy neighborhood operator.

Theorem 2 *Let N be a fuzzy neighborhood operator on U and T a t-norm. Then, the following statements hold:*

(1) N is serial $\Longleftrightarrow \overline{apr}_N^T(U) = U$.

(2) n is reflexive $\Longleftrightarrow \forall A \in \mathcal{F}(U)$, $A \subseteq \overline{apr}_N^T(A)$.

(3) N is symmetric $\Longleftrightarrow \forall x, y \in U$, $\Big(\overline{apr}_N^T(\mu_x)\Big)(y) \subseteq \Big(\overline{apr}_N^T(\mu_y)\Big)(x)$.

(4) N is T-transitive $\Longleftrightarrow \forall x \in U$, $\overline{apr}_N^T(\overline{apr}_N^T(\mu_x)) \subseteq \overline{apr}_N^T(\mu_x)$.

Proof (1) By $\forall a \in [0, 1]$, $T(a, 1) = a$, we conclude that

$$\begin{aligned}\overline{apr}_N^T(U) = U &\Leftrightarrow \forall x \in U,\ \left(\overline{apr}_N^T(U)\right)(x) = 1 \\ &\Leftrightarrow \forall x \in U,\ \sup_{y\in U} T(N(x)(y), U(y)) = 1 \\ &\Leftrightarrow \forall x \in U,\ \sup_{y\in U} T(N(x)(y), 1) = 1 \\ &\Leftrightarrow \forall x \in U,\ \sup_{y\in U} N(x)(y) = 1 \\ &\Leftrightarrow N \text{ is serial.}\end{aligned}$$

This completes the proof.

(2) The necessity is obvious from Remark 3.

Conversely, by $\forall a \in [0, 1]$, $T(a, 0) = 0$ and $T(a, 1) = a$, we have that, for all $x \in U$,

$$\begin{aligned}\overline{apr}_N^T(\mu_x)(x) &= \sup_{y\in U} T(N(x)(y), \mu_x(y)) \\ &= T(N(x)(x), \mu_x(x)) \\ &= T(N(x)(x), 1) = N(x)(x).\end{aligned}$$

Since $\mu_x \subseteq \overline{apr}_N^T(\mu_x)$, it follows that $1 = \mu_x(x) \leq (\overline{apr}_N^T(\mu_x))(x) = N(x)(x)$. This implies that $N(x)(x) = 1$. We have proved that $\forall x \in U$, $N(x)(x) = 1$. Therefore, N is reflexive. This completes the proof of the sufficiency.

(3) It is straightforward from Definition 7 and the definition of symmetry.

(4) The sufficiency is obvious from Lemma 1.

Conversely, let $x \in U$. By $\forall a \in [0, 1]$, $T(a, 0) = 0$, $T(a, 1) = a$ and Eq. (3), we have that, for all $y \in U$,

$$\begin{aligned}
&\left(\overline{apr}_N^T\left(\overline{apr}_N^T(\mu_x)\right)\right)(y)\\
&= \sup_{z\in U} T\left(N(y)(z), \left(\overline{apr}_N^T(\mu_x)\right)(z)\right)\\
&= \sup_{z\in U} T\left(N(y)(z), \sup_{w\in U} T(N(z)(w), \mu_x(w))\right)\\
&= \sup_{z\in U} T\left(N(y)(z), T(N(z)(x), \mu_x(x))\right)\\
&= \sup_{z\in U} T\left(N(y)(z), T(N(z)(x), 1)\right)\\
&= \sup_{z\in U} T\left(N(y)(z), N(z)(x)\right).
\end{aligned}$$

Since N is T-transitive, it follows that, for all $y \in U$,

$$\begin{aligned}
&\left(\overline{apr}_N^T\left(\overline{apr}_N^T(\mu_x)\right)\right)(y)\\
&= \sup_{z\in U} T\left(N(y)(z), N(z)(x)\right) \leq \sup_{z\in U} N(y)(x) = N(y)(x).
\end{aligned} \tag{7}$$

On the other hand, by $\forall a \in [0, 1]$, $T(a, 0) = 0$, $T(a, 1) = a$ and Eq. (3), we know that, for all $y \in U$,

$$\begin{aligned}
\overline{apr}_N^T(\mu_x)(y) &= \sup_{z\in U} T(N(y)(z), \mu_x(z))\\
&= T(N(y)(x), \mu_x(x)) = T(N(y)(x), 1) = N(y)(x).
\end{aligned}$$

Therefore, by Eq. (7), we have proved that, for all $y \in U$, $\left(\overline{apr}_N^T\left(\overline{apr}_N^T(\mu_x)\right)\right)(y) \leq \overline{apr}_N^T(\mu_x)(y)$. Hence $\forall x \in U$, $\overline{apr}_N^T(\overline{apr}_N^T(\mu_x)) \subseteq \overline{apr}_N^T(\mu_x)$. This completes the proof of the necessity. □

By combining Theorem 2 (2) and Proposition 1 (2), we can provide the following result.

Corollary 1 *Let N be a fuzzy neighborhood operator on U and T a t-norm. If N is reflexive, then $\forall \alpha \in [0, 1]$, $\overline{apr}_N^T(\hat{\alpha}) = \hat{\alpha}$.*

Properties of fuzzy lower approximation operators based on the Gödel implication I_M

In this section, we discuss the relationships between the fuzzy lower approximation operators based on a fuzzy implication and the fuzzy neighborhood operators. However, it is difficult to investigate the fuzzy lower approximation operator based on a general fuzzy implication. Thus, we discuss mainly the characterization of the fuzzy lower approximation operator based on the Gödel implication.

Remark 4 The popular left-continuous t-norm is the standard min operator and it is defined by

$$T_M(a, b) = \min\{a, b\}, \quad \text{where } a, b \in [0, 1]. \tag{8}$$

The R-implication I_M based on T_M is also well known and is called the Gödel implication [39]. It is defined as follows:

$$\forall x, y \in [0, 1], \quad I_M(x, y) = \begin{cases} y, & \text{if } x > y;\\ 1, & \text{if } x \leq y. \end{cases} \tag{9}$$

First, we give the properties of the fuzzy lower approximation based on I_M.

Proposition 3 *Let N be a fuzzy neighborhood operator on U and I_M the Gödel implication. Then, the following statements hold:*

(1) $\underline{apr}_N^{I_M}(U) = U$.

(2) *If N is serial, then* $\underline{apr}_N^{I_M}(\emptyset) = \emptyset$.

(3) $\forall \alpha \in [0, 1]$, $\underline{apr}_N^{I_M}(\hat{\alpha}) \supseteq \hat{\alpha}$.

Proof (1) By Remark 4, we can derive $\forall a \in [0, 1]$, $I_M(a, 1) = 1$. Thus, for all $x \in U$, we have that

$$\begin{aligned}
\left(\underline{apr}_N^{I_M}(U)\right)(x) &= \inf_{y\in U} I_M(N(x)(y), U(y))\\
&= \inf_{y\in U} I_M(N(x)(y), 1) = 1 = U(x).
\end{aligned}$$

Thus, $\underline{apr}_N^{I_M}(U) = U$.

(2) Since N is serial, we have that $\forall x \in U$, $\exists y$ such that $N(x)(y) \neq 0$. This implies that $\forall x \in U$, $\exists y$ such that $I_M(N(x)(y), 0) = 0$. Hence, $\forall x \in U$, we conclude that

$$\begin{aligned}
\left(\underline{apr}_N^{I_M}(\emptyset)\right)(x) &= \inf_{y\in U} I_M(N(x)(y), \emptyset(y))\\
&= \inf_{y\in U} I_M(N(x)(y), 0) = 0 = \emptyset(x).
\end{aligned}$$

Thus, $\underline{apr}_N^{I_M}(\emptyset) = \emptyset$.

(3) Let $\alpha \in [0, 1]$. Since $\forall a \in [0, 1]$, $I_M(1, a) = a$, and I_M is the monotone decreasing function with respect to the first variable, it follows that $\forall x \in U$, we have that

$$\begin{aligned}
\left(\underline{apr}_N^{I_M}(\hat{\alpha})\right)(x) &= \inf_{y\in U} I_M(N(x)(y), \alpha) \geq \inf_{y\in U} I_M(1, \alpha)\\
&= \inf_{y\in U} \alpha = \alpha = \hat{\alpha}(x).
\end{aligned}$$

This implies that $\underline{apr}_N^{I_M}(\hat{\alpha}) \supseteq \hat{\alpha}$. □

For $\beta \in [0, 1]$ and $x \in U$, we define the fuzzy set β_x of U as follows:

$$\forall y \in U, \quad \beta_x(y) = \begin{cases} \beta, & \text{if } y = x; \\ 1, & \text{otherwise.} \end{cases} \tag{10}$$

Lemma 2 *Let N be a fuzzy neighborhood operator on U, T_M the standard min operator and I_M the Gödel implication. If $\forall \beta \in [0, 1]$ and $x \in U$, $\underline{apr}_N^{I_M}(\underline{apr}_N^{I_M}(\beta_x)) \supseteq \underline{apr}_N^{I_M}(\beta_x)$, then N is T_M-transitive.*

Proof Suppose that N is not T_M-transitive, then there exist $x_0, y_0, z_0 \in U$ such that $T_M\big(N(x_0)(y_0), N(y_0)(z_0)\big) > N(x_0)(z_0)$. We choose $\beta = \frac{T_M\left(N(x_0)(y_0), N(y_0)(z_0)\right) + N(x_0)(z_0)}{2}$. Clearly, $N(x_0)(z_0) < \beta < T_M(N(x_0)(y_0), N(y_0)(z_0))$. By Remark 4, we can derive $\forall a \in [0, 1]$, $I_M(a, 1) = 1$. This implies that

$$\begin{aligned}
&\left(\underline{apr}_N^{I_M}\left(\underline{apr}_N^{I_M}(\beta_{z_0})\right)\right)(x_0) \\
&= \inf_{u \in U} I_M\left(N(x_0)(u), \left(\underline{apr}_N^{I_M}(\beta_{z_0})\right)(u)\right) \\
&= \inf_{u \in U} I_M\left(N(x_0)(u), \inf_{w \in U} I_M\left(N(u)(w), \beta_{z_0}(w)\right)\right) \\
&= \inf_{u \in U} I_M\left(N(x_0)(u), I_M(N(u)\left(z_0\right), \beta_{z_0}(z_0))\right) \\
&= \inf_{u \in U} I_M\left(N(x_0)(u), I_M(N(u)(z_0), \beta)\right) \\
&\le I_M\left(N(x_0)(y_0), I_M\left(N(y_0)(z_0), \beta\right)\right).
\end{aligned}$$

and by $N(x_0)(z_0) < \beta$, we have

$$\begin{aligned}
\left(\underline{apr}_N^{I_M}(\beta_{z_0})\right)(x_0) &= \inf_{w \in U} I_M\left(N(x_0)(w), \beta_z(w)\right) \\
&= I_M\left(N(x_0)(z_0), \beta_{z_0}(z_0)\right) = I_M\left(N(x_0)(z_0), \beta\right) = 1.
\end{aligned}$$

That is, $\left(\underline{apr}_N^{I_M}\left(\underline{apr}_N^{I_M}(\beta_{z_0})\right)\right)(x_0) \le I_M(N(x_0)(y_0), T(N(y_0)(z_0), \beta))$ and $\left(\underline{apr}_N^{I_M}(\beta_{z_0})\right)(x_0) = 1$. By

$$\beta < T_M(N(x_0)(y_0), N(y_0)(z_0)),$$

we can get that $\beta < N(x_0)(y_0)$ and $\beta < N(y_0)(z_0)$. This implies that

$$\begin{aligned}
&\left(\underline{apr}_N^{I_M}\left(\underline{apr}_N^{I_M}(\beta_{z_0})\right)\right)(x_0) \\
&\quad \le I_M\left(N(x_0)(y_0), T(N(y_0)(z_0), \beta)\right) = \beta < 1 \\
&\quad = \left(\underline{apr}_N^{I_M}(\beta_{z_0})\right)(x_0).
\end{aligned}$$

That is to say, we conclude that $\left(\underline{apr}_N^{I_M}\left(\underline{apr}_N^{I_M}(\beta_{z_0})\right)\right)(x_0) < \left(\underline{apr}_N^{I_M}(\beta_{z_0})\right)(x_0)$. This contradicts the condition: $\forall \beta \in [0, 1]$ and $x \in U$, $\underline{apr}_N^{I_M}(\underline{apr}_N^{I_M}(\beta_x)) \supseteq \underline{apr}_N^{I_M}(\beta_x)$. Consequently, N is T_M-transitive. □

Theorem 3 *Let N be a fuzzy neighborhood operator on U and I_M the Gödel implication. Then, the following statements hold:*

(1) *N is serial $\Longleftrightarrow \forall \alpha \in [0, 1]$, $\underline{apr}_N^{I_M}(\hat{\alpha}) = \hat{\alpha}$.*
(2) *N is reflexive $\Longleftrightarrow \forall A \in \mathcal{F}(U)$, $\underline{apr}_N^{I_M}(A) \subseteq A$.*
(3) *N is symmetric $\Longleftrightarrow \forall \beta \in [0, 1]$ and $x, y \in U$, $\left(\underline{apr}_N^{I_M}(\beta_x)\right)(y) = \left(\underline{apr}_N^{I_M}(\beta_y)\right)(x)$.*
(4) *N is T_M-transitive $\Longleftrightarrow \forall A \in \mathcal{F}(U)$, $\underline{apr}_N^{I_M}(\underline{apr}_N^{I_M}(A)) \supseteq \underline{apr}_N^{I_M}(A)$, where T_M is the standard min operator.*

Proof (1) For $\alpha \in [0, 1]$, if $\alpha = 1$, then by the definition of lower approximation and Remark 4, we can obtain that $\forall x \in U$, $\left(\underline{apr}_N^{I_M}(\hat{\alpha})\right)(x) = \inf_{y \in U} I_M(N(x)(y), 1) = 1 = \hat{\alpha}(x)$, that is, $\underline{apr}_N^{I_M}(\hat{\alpha}) = \hat{\alpha}$. Next, let $\alpha < 1$. Since N is serial, we know that $\forall x \in U$, $\sup_{y \in U} N(x)(y) = 1$. Thus, for $x \in U$, there exists $y_0 \in U$ such that $N(x)(y_0) > \alpha$.

$$\begin{aligned}
\left(\underline{apr}_N^{I_M}(\hat{\alpha})\right)(x) &= \inf_{y \in U} I_M(N(x)(y), \alpha) \\
&\le I_M(N(x)(y_0), \alpha) = \alpha = \hat{\alpha}(x).
\end{aligned}$$

That is to say, $\underline{apr}_N^{I_M}(\hat{\alpha}) \subseteq \hat{\alpha}$. On the other hand, by Proposition 3 (3), we know that $\underline{apr}_N^{I_M}(\hat{\alpha}) \supseteq \hat{\alpha}$. In summary, $\underline{apr}_N^{I_M}(\hat{\alpha}) = \hat{\alpha}$. This completes the proof of the necessity.

Conversely, suppose that N is not serial. Then, there exists $x_0 \in U$ such that $\sup_{y \in U} N(x_0)(y) \neq 1$. That is to say, $\sup_{y \in U} N(x_0)(y) < 1$. We take

$$\alpha_0 = \frac{1 + \sup_{y \in U} N(x_0)(y)}{2} > 0.$$

Clearly, $\sup_{y \in U} N(x_0)(y) < \alpha_0 < 1$. Thus, $\forall y \in U$, $N(x_0)(y) < \alpha_0$. This implies that $\forall y \in U$, $I_M(N(x_0)(y), \alpha_0) = 1$. It follows that $\left(\underline{apr}_N^{I_M}(\hat{\alpha_0})\right)(x_0) = 1 \neq \alpha_0 = \hat{\alpha_0}(x_0)$. Thus, $\underline{apr}_N^{I_M}(\hat{\alpha_0}) \neq \hat{\alpha_0}$, which contradicts the condition: $\forall \alpha \in [0, 1]$, $\underline{apr}_N^{I_M}(\hat{\alpha}) = \hat{\alpha}$. Consequently, N is serial.

(2) Since N is reflexive, we have that $\forall x \in U$, $N(x)(x) = 1$. Thus, $\forall A \in \mathcal{F}(U)$ and $x \in U$,

$$\begin{aligned}
\left(\underline{apr}_N^{I_M}(A)\right)(x) &= \inf_{y \in U} I_M\left(N(x)(y), A(y)\right) \\
&\le I_M(N(x)(x), A(x)) \\
&= I_M(1, A(x)) = A(x).
\end{aligned}$$

This implies that $\forall A \in \mathcal{F}(U)$, $\underline{apr}_N^{I_M}(A) \subseteq A$.

Conversely, suppose that N is not reflexive, then there exists $x_0 \in U$ such that $N(x_0)(x_0) \neq 1$. That is to say, $N(x_0)(x_0) < 1$. We choose $\beta = \frac{1+N(x_0)(x_0)}{2}$. It is clear that

$N(x_0)(x_0) < \beta < 1$. By Eq. (10), we can construct the fuzzy subset β_{x_0} of U. By $\forall a \in [0,1]$, $I_M(a,1)=1$, we can compute that

$$\left(\underline{apr}_N^{I_M}(\beta_{x_0})\right)(x_0) = \inf_{y\in U} I_M\left(N(x)(y), \beta_{x_0}(y)\right) = I_M\left(N(x_0)(x_0), \beta_{x_0}(x_0)\right) = I_M\left(N(x_0)(x_0), \beta\right) = 1 > \beta = \beta_{x_0}(x_0).$$

This implies that $\underline{apr}_N^{I_M}(\beta_{x_0}) \nsubseteq \beta_{x_0}$, which contradicts with the condition: $\forall A \in \mathcal{F}(U)$, $\underline{apr}_N^{I_M}(A) \subseteq A$. This completes the proof of the sufficiency.

(3) Let $\beta \in [0,1]$ and $x, y \in U$. By $\forall a \in [0,1]$, $I_M(a,1)=1$, we have that

$$\left(\underline{apr}_N^{I_M}(\beta_x)\right)(y) = \inf_{z\in U} I_M(N(y)(z), \beta_x(z)) = I_M(N(y)(x), \beta), \tag{11}$$

and

$$\left(\underline{apr}_N^{I_M}(\beta_y)\right)(x) = \inf_{z\in U} I_M\left(N(y)(z), \beta_x(z)\right) = I_M(N(x)(y), \beta)\,. \tag{12}$$

Since N is symmetric, it follows that $N(x)(y) = N(y)(x)$. Thus, by Eqs. (11) and (12), we conclude $\left(\underline{apr}_N^{I_M}(\beta_x)\right)(y) = \left(\underline{apr}_N^{I_M}(\beta_y)\right)(x)$.

Conversely, suppose that N is not symmetric, then there exist $x, y \in U$ such that $N(x)(y) \neq N(y)(x)$. Without loss of generality, we assume $N(x)(y) < N(y)(x)$. We choose $\beta = \frac{N(x)(y)+N(y)(x)}{2}$. Clearly, $N(x)(y) < \beta < N(y)(x)$. By Remark 4 and Eqs. (11) and (12), we can derive

$$\left(\underline{apr}_N^{I_M}(\beta_y)\right)(x) = I_M(N(x)(y), \beta) = 1 \neq \beta = I_M(N(y)(x), \beta) = \left(\underline{apr}_N^{I_M}(\beta_x)\right)(y).$$

This is a contradiction to the condition: $\forall \beta \in [0,1]$ and $x, y \in U$, $\left(\underline{apr}_N^{I_M}(\beta_x)\right)(y) = \left(\underline{apr}_N^{I_M}(\beta_y)\right)(x)$. This completes the proof.

(4) The sufficiency is obvious from Lemma 2.

Conversely, we shall prove that $\forall A \in \mathcal{F}(U)$, $\underline{apr}_N^{I_M}\left(\underline{apr}_N^{I_M}(A)\right) \supseteq \underline{apr}_N^{I_M}(A)$. We only need to prove that $\forall x \in U$, $\left(\underline{apr}_N^{I_M}\left(\underline{apr}_N^{I_M}(A)\right)\right)(x) \geq \left(\underline{apr}_N^{I_M}(A)\right)(x)$. Suppose that there exists $x_0 \in U$ such that $\left(\underline{apr}_N^{I_M}\left(\underline{apr}_N^{I_M}(A)\right)\right)(x_0) < \left(\underline{apr}_N^{I_M}(A)\right)(x_0)$, that is to say,

$$\inf_{y\in U} I_M\left(N(x_0)(y), \inf_{z\in U} I_M(N(y)(z), A(z))\right) < \inf_{z\in U} I_M(N(x_0)(z), A(z)).$$

Then, there exists $y_0 \in U$ such that

$$I_M\left(N(x_0)(y_0), \inf_{z\in U} I_M(N(y_0)(z), A(z))\right) < \inf_{z\in U} I_M(N(x_0)(z), A(z)).$$

It is clear that $I_M\left(N(x_0)(y_0), \inf_{z\in U} I_M(N(y_0)(z), A(z))\right) < 1$. By the definition of I_M, we can obtain that

$$N(x_0)(y_0) > \inf_{z\in U} I_M(N(y_0)(z), A(z)), \quad \text{and}$$

$$I_M\left(N(x_0)(y_0), \inf_{z\in U} I_M(N(y_0)(z), A(z))\right) = \inf_{z\in U} I_M(N(y_0)(z), A(z)) < \inf_{z\in U} I_M(N(x)(z), A(z)),$$

that is, $\inf_{z\in U} I_M(N(y_0)(z), A(z)) < \inf_{z\in U} I_M(N(x)(z), A(z))$. This implies that there exists $z_1 \in U$ such that

$$I_M(N(y_0)(z_1), A(z_1)) < \inf_{z\in U} I_M(N(x)(z), A(z)). \tag{13}$$

In addition, according to $N(x_0)(y_0) > \inf_{z\in U} I_M(N(y_0)(z), A(z))$, we can conclude that there exists $z_2 \in U$ such that

$$N(x_0)(y_0) > I_M(N(y_0)(z_2), A(z_2)). \tag{14}$$

By Eqs. (13) and (14), we can conclude that $I_M(N(y_0)(z_1), A(z_1)) < 1$ and $I_M(N(y_0)(z_2), A(z_2) < 1$. It follows from the definition of I_M that

$$N(y_0)(z_1) > A(z_1), \tag{15}$$

$$I_M(N(y_0)(z_1), A(z_1)) = A(z_1); \text{ and} \tag{16}$$

$$N(y_0)(z_2) > A(z_2), \tag{17}$$

$$I_M(N(y_0)(z_2), A(z_2)) = A(z_2). \tag{18}$$

Without loss of generality, we assume that $A(z_2) \geq A(z_1)$. Thus, by Eqs. (13) and (16), we have

$$A(z_1) < \inf_{z\in U} I_M(N(x)(z), A(z)); \tag{19}$$

and by Eqs. (14) and (18), we have

$$N(x_0)(y_0) > A(z_2) \geq A(z_1). \tag{20}$$

By Eqs. (15) and (20), this implies that $T_M(N(x_0)(y_0), N(y_0)(z_1)) > A(z_1)$. Since N is T_M-transitive, we have

$$T_M(N(x_0)(y_0), N(y_0)(z_1)) \leq N(x_0)(z_1).$$

Thus, $N(x_0)(z_1) > A(z_1)$. This implies that $I_M(N(x_0)(z_1), A(z_1)) = A(z_1)$. Therefore, we obtain

$$\inf_{z\in U} I_M(N(x)(z), A(z)) \leq I_M(N(x_0)(z_1), A(z_1)) = A(z_1),$$

which is a contradiction with Eq. (19). Consequently, $\underline{apr}_N^{I_M}\big(\underline{apr}_N^{I_M}(A)\big) \supseteq \underline{apr}_N^{I_M}(A)$. This completes the proof of the necessity. □

In this section, the notations $\widehat{\alpha}$, μ_x and β_x represent three types of fuzzy sets. They are similar and easily confused notations. For easy understanding, we summarize them in Table 1.

The relationships between the fuzzy topology and the fuzzy lower (or upper) approximation operator

Topology is an important topic of mathematics. It is applied in many fields. We know that closure operator and topology can determine each other. Thus, closure operators (characterized by closure axioms [43]), are an important tool to investigate topology theory. In 1976, Lowen extended this concept into fuzzy set theory, and established the concept of fuzzy closure operators [44]. Since the lower and upper approximation operators in the rough set theory is strongly similar to the closure operator and interior operator in topology theory. Hence there are many researches on the connections between topological structures and rough sets [36, 45–48]. Wiweger [46] discussed the relationships between Pawlak rough sets and topological spaces, and he showed that an upper approximation is a closure operator while a lower approximation is an interior operator. Then, the connections between crisp rough set model and crisp topology were explored [49, 50]. Subsequently, Qin and Pei [51] discussed topological structures of fuzzy rough sets by the interior operator and closure operator. Mi et al. [35] and Wu [36] gave different collections of independent axiomatic sets to characterize various types of fuzzy rough approximations based on t-norm. In this section, by means of the results of Section 3, we further investigate the relationships between fuzzy topology structures and fuzzy rough sets based on t-norms and fuzzy implications.

Fuzzy upper approximation operators and fuzzy closure operators

In this subsection, we shall investigate the connections of fuzzy upper approximation operators and fuzzy closure operators. In [36], Wu introduced the closure axioms of a fuzzy topology on U as follows:

Definition 8 [36] Let U be a nonempty set. A mapping $cl : \mathcal{F}(U) \to \mathcal{F}(U)$ is called a fuzzy closure operator if it satisfies following axioms:

(C1) $\forall A \in \mathcal{F}(U)$, $A \subseteq cl(A)$;
(C2) $\forall A, B \in \mathcal{F}(U)$, $cl(A \cup B) = cl(A) \cup cl(B)$;
(C3) $\forall A \in \mathcal{F}(U)$, $cl(cl(A)) = cl(A)$;
(C4) $\forall \alpha \in [0, 1]$, $cl(\hat{\alpha}) = \hat{\alpha}$.

In [33], D'eer et al. showed that $\overline{apr}_N^T$ satisfies the property: $\forall A, B \in \mathcal{F}(U)$, $\overline{apr}_N^T(A\cup B) = \overline{apr}_N^T(A)\cup\overline{apr}_N^T(B)$, and they proved that the following result hold:

- Let N be a fuzzy neighborhood operator on U, T a t-norm and I a R-implication based on T. If T is left-continuous, and N is reflexive and T-Euclidean, or if T is a left-continuous and N is T-similarity relation, then $\forall A \in \mathcal{F}(U)$, $\overline{apr}_N^T(\overline{apr}_N^T(A)) = \overline{apr}_N^T(A)$ and $\underline{apr}_N^I(\underline{apr}_N^I(A)) = \underline{apr}_N^I(A)$.

Note that N is referred to as T-similarity relation if N is reflexive, symmetric and T-transitive. In fact, the limitation of symmetry in above result is not necessary. Next, we provide a more general condition under which $\forall A \in \mathcal{F}(U)$, $\overline{apr}_N^T(\overline{apr}_N^T(A)) = \overline{apr}_N^T(A)$.

Proposition 4 *Let N be a fuzzy neighborhood operator on U and T a t-norm. If T is left-continuous, and N is reflexive and T-transitive, then $\forall A \in \mathcal{F}(U)$, $\overline{apr}_N^T\big(\overline{apr}_N^T(A)\big) = \overline{apr}_N^T(A)$.*

Proof Let $A \in \mathcal{F}(U)$. Since N is reflexive, it follows from Theorem 2 (2) that $\overline{apr}_N^T\big(\overline{apr}_N^T(A)\big) \supseteq \overline{apr}_N^T(A)$. On the other hand, since N is T-transitive and T is left-continuous,

Table 1 The summary of notations of three fuzzy sets

Notation	$\widehat{\alpha}$	μ_x	β_x
Definition	$\widehat{\alpha}(x) = \alpha$	If $y = x$, $\mu_x(y) = 1$ If $y \neq x$, $\mu_x(y) = 0$	If $y = x$, $\beta_x(y) = \beta$ If $y \neq x$, $\beta_x(y) = 1$

we have that $\forall x \in U$,

$$\Big(\overline{apr}_N^T(\overline{apr}_N^T(A))\Big)(x) = \sup_{y\in U} T\Big(N(x)(y), (\overline{apr}_N^T(A))(y)\Big)$$
$$= \sup_{y\in U} T\Big(N(x)(y), \sup_{z\in U} T\big(N(y)(z), A(z)\big)\Big)$$
$$= \sup_{y\in U}\sup_{z\in U} T\Big(N(x)(y), T\big(N(y)(z), A(z)\big)\Big)$$
$$= \sup_{y\in U}\sup_{z\in U} T\Big(T\big(N(x)(y), N(y)(z)\big), A(z)\Big)$$
$$\leq \sup_{y\in U}\sup_{z\in U} T\Big(N(x)(z), A(z)\Big)$$
$$= \sup_{z\in U} T\Big(N(x)(z), A(z)\Big) = \Big(\overline{apr}_N^T(A)\Big)(x).$$

We have proved that $\forall x \in U$, $\Big(\overline{apr}_N^T(\overline{apr}_N^T(A))\Big)(x) \leq \Big(\overline{apr}_N^T(A)\Big)(x)$. That is to say, $\overline{apr}_N^T(\overline{apr}_N^T(A)) \subseteq \overline{apr}_N^T(A)$. In summary, $\overline{apr}_N^T(\overline{apr}_N^T(A)) = \overline{apr}_N^T(A)$. □

By combining Theorem 2 (2), Corollary 1 and Proposition 4, we can establish the following conclusion.

Theorem 4 *Let N be a fuzzy neighborhood operator on U and T a t-norm. If T is left-continuous, and N is reflexive and T-transitive, then $\overline{apr}_N^T$ is a fuzzy closure operator.*

Fuzzy lower approximations and fuzzy interior operators

In this subsection, we shall investigate the connections of fuzzy lower approximation operators and fuzzy interior operators. In [36], Wu introduced the interior axioms of a fuzzy topology on U as follows:

Definition 9 [36] Let U be a nonempty set. A mapping $int : \mathcal{F}(U) \to \mathcal{F}(U)$ is called a fuzzy interior operator if it satisfies following axioms:

(I1) $int(A) \subseteq A, \forall A \in \mathcal{F}(U)$;
(I2) $int(A \cap B) = int(A) \cap int(B), \forall A, B \in \mathcal{F}(U)$;
(I3) $int(int(A)) = int(A), \forall A \in \mathcal{F}(U)$;
(I4) $int(\hat{\alpha}) = \hat{\alpha}, \forall \alpha \in [0, 1]$.

We first provide the following conclusion.

Proposition 5 *Let N be a fuzzy neighborhood operator on U, T_M the standard min operator and I_M the Gödel implication. If N is reflexive and T_M-transitive, then $\forall A \in \mathcal{F}(U)$, $\underline{apr}_N^{I_M}\big(\underline{apr}_N^{I_M}(A)\big) = \underline{apr}_N^{I_M}(A)$.*

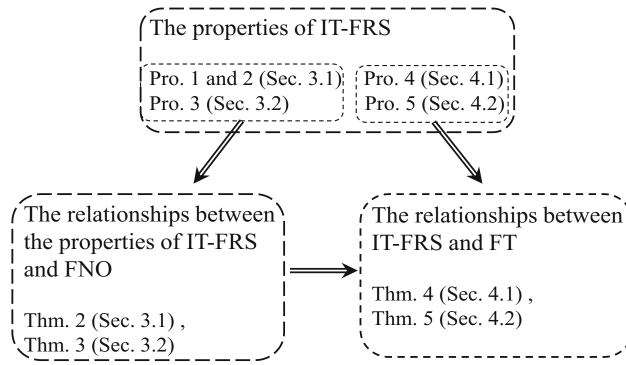

Fig. 1 The connections among the proposed conclusions

Proof It is straightforward from (2) and (4) of Theorem 3. □

By combining Theorem 3 and Proposition 5, we can obtain the following conclusion.

Theorem 5 *Let N be a fuzzy neighborhood operator on U, T_M the standard min operator and I_M the Gödel implication. If N is reflexive and T_M-transitive, then $\underline{apr}_N^{I_M}$ is a fuzzy interior operator.*

Conclusion

This paper proposed some fundamental results on fuzzy neighborhood operators (FNO), fuzzy topology (FT) and fuzzy rough sets based on t-norm and fuzzy implication (IT-FRS). These conclusions were summarized by Fig. 1. In Fig. 1, we can see that the paper first gave the properties of IT-FRS. Then, using these properties, IT-FRS was characterized by FNO and vice versa. Finally, combining the results of Sect. 3, topological properties of IT-FRS were discussed. In summary, this work provided theoretic foundation for fuzzy rough sets based on arbitrary t-norms and fuzzy implications. In our future work, we will take into account the following issue: Whether are these results of Sect. 3.2 true or not when we replace the fuzzy implication I_M by a general fuzzy implication I. In addition, we will research on the applications of the IT-FRS model in real problems.

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Data availability No data was used for the research described in the article.

Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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