



Hesitant Fermatean fuzzy Bonferroni mean operators for multi-attribute decision-making

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Abstract

Hesitant Fermatean fuzzy sets (HFFS) can characterize the membership degree (MD) and non-membership degree (NMD) of hesitant fuzzy elements in a broader range, which offers superior fuzzy data processing capabilities for addressing complex uncertainty issues. In this research, first, we present the definition of the hesitant Fermatean fuzzy Bonferroni mean operator (HFFBM). Further, with the basic operations of HFFS in Einstein t-norms, the definition and derivation process of the hesitant Fermatean fuzzy Einstein Bonferroni mean operator (HFFEEM) are given. In addition, considering how weights affect decision-making outcomes, the hesitant Fermatean fuzzy weighted Bonferroni mean (HFFWBM) operator and the hesitant Fermatean fuzzy Einstein weighted Bonferroni mean operator (HFFEEM) are developed. Then, the properties of the operators are discussed. Based on HFFWBM and HFFEEM operator, a new multi-attribute decision-making (MADM) approach is provided. Finally, we apply the proposed decision-making approach to the case of a depression diagnostic evaluation for three depressed patients. The three patients' diagnosis results confirmed the proposed method's validity and rationality. Through a series of comparative experiments and analyses, the proposed MADM method is an efficient solution for decision-making issues in the hesitant Fermatean fuzzy environment.

Keywords Hesitant Fermatean fuzzy set · Einstein operation · Bonferroni mean · Aggregation operator · Decision-making

Introduction

The fuzzy sets (FS) theory [1] has been utilized in several fields to address the uncertainty problem, since its introduction by Zadeh in 1965. Atanassov extended FS to

intuitionistic fuzzy sets (IFS) [2], which employ membership and non-membership degree to describe the fuzziness of things. To accommodate an increasingly complex social context, FS theory has been extensively researched [3, 4]. Nowadays, fuzzy set theory and its extensions are widely utilized in many fields [5, 6]. Multi-criteria decision-making (MCDM) in uncertain environments is a popular field of study. Alali et al. [7] applied the TODIM method to the portfolio allocation process. Tolga et al. [8] proposed a finite interval Type-2 Gaussian fuzzy number and then applied its extended TODIM method to the economic evaluation of medical device selection. Tolga et al. [9] evaluated the technical types of three vertical farm alternatives using Weighted Euclidean Distance-Based Approximation (WEDBA) and Measuring Attractiveness by a Categorical-Based Evaluation Technique (MACBETH). Deveci et al. [10] developed an integrated decision model based on the fuzzy Dombi norms-based Logarithmic Methodology of Additive Weights (LMAW) and Evaluation based on the Distance from the Average Solution (EDAS). Then, they applied it to a measurement scheme for a goods mobility application.

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Decision-makers may be hesitant to assessments when making decisions in uncertain complex situations, such as mental health assessments, making it difficult to reach a consensus on the assessments. As an extension of FS, Torra et al. [11] introduced the notion of hesitant fuzzy sets (HFS) in 2009. The MD of HFS is represented by a set of probable values. This feature of HFS is appropriate for describing hesitant information. Consequently, HFS has been extensively investigated and developed [12–14]. In 2012, Zhu and Xu extended HFS with dual-hesitant fuzzy sets (DHFS) [15]. DHFS employs hesitant MD and NMD functions to characterize hesitancy, which goes a step further in addressing uncertain hesitant complicated situations. Consequently, after DHFS was proposed, it garnered considerable interest from researchers. Chen and Peng et al. [16] analyzed the correlation coefficient of DHFS and studied it by MADM method. Yu and Li et al. [17] applied DHFS to the supplier selection issues and presented a dual-hesitant fuzzy multi-attribute group decision-making (MAGDM) method. Singh [18] introduced the DHFS distance and similarity measurement techniques. Then, they established the ranking by the distance between the ideal solution and alternatives to solve MADM problem. The DHFS constraints ensure that the total of all possible MD and NMD is in the interval 0 and 1, which carries some disadvantages. Liang et al. [19] developed the hesitant Pythagorean fuzzy sets (HPFS) in 2017. At the same time, Wei et al. [20] developed the dual-hesitant Pythagorean fuzzy sets (DHPFS). As a combination of HFS and Pythagorean fuzzy sets [21, 22], the constraint of HPFS is $0 \leq \max\{MD\}^2 + \max\{NMD\}^2 \leq 1$. In addition, Liang and Xu [19] also applied the HPFS to MCDM and presented a new extension of TOPSIS method. The proposal of HPFS has attracted the extensive attention of researchers. And HPFS has undergone significant development in recent years. Wu et al. [23] presented a MAGDM flexibility enhancement method based on information fusion HPFS. Akram et al. [24] presented a comprehensive ELECTRE-I method for risk assessment based on HPFS. Geetha et al. [25] extended the ELECTRE-III method under HPFS. Then, they used the method to the problem of plastic recycling.

In fuzzy environments, the key of MADM is information aggregation. The HFS information aggregation offers distinct benefits in uncertain MADM issues. In general, The HFS aggregation operations are based on Archimedean t-norms [26, 27]. Combining basic algebraic products and algebraic sums of Archimedean t-norms is the most commonly used fuzzy aggregation operation approach. As is commonly known, Algebraic t-norms and Einstein t-norms are typical examples of the strictly Archimedean t-norms class [28–30]. Einstein's product and sum yield the same smooth approximation as algebraic product and sum. Therefore, it is a suitable replacement for algebraic t-norms. Zhou

and Li [31] developed the hesitant fuzzy geometric aggregation operator, which is based on Einstein t-norms. Zhao and Xu [32] proposed some Einstein aggregation operators under dual-hesitant fuzzy environments. Farid et al. [33] improved the basic fuzzy algorithm and proposed the Einstein interactive geometric aggregation operator on q-rung orthogonal pair fuzzy sets. Anusha and Sireesha [34] developed an aggregation operator for interval-valued intuitionistic fuzzy sets, which combined with the Heronian mean operator based on Einstein t-norms. It is known that traditional aggregation operators have significant limitations. Traditional aggregation operators, for instance, cannot eliminate the influence of extreme values, and each element of aggregation operations is independent of the other. Real complex situations must take the above restrictions into account. As a mean operator, Bonferroni mean (BM) [35] can effectively tackle abovementioned problems. Researchers have paid considerable attention to BM due to its capacity to characterize the interrelationships between input data. In 2009, Yager [36] formulated this operator. Then, Beliakov et al. [37] researched systematically a family of extended BM combinatorial aggregation functions in an effort to expand the limitation that BM can only handle exact numbers. Zhu and Xu [38] addressed the details of the hesitant fuzzy BM operator in their study of the hesitant fuzzy aggregation operator. In addition, they explained the concept of the hesitant Bonferroni element as bonding satisfaction.

Fermatean hesitant fuzzy sets [39, 40] and hesitant Fermatean fuzzy sets [41] are the generalization of Fermatean fuzzy sets (FFS) [42] and DHFS. The condition of HFFS is $0 \leq \max\{MD\}^3 + \max\{NMD\}^3 \leq 1$. Therefore, HFFS is more advantageous than FFS and DHFS in addressing uncertainty issues. In decision-making in complex fuzzy environments, the evaluation of attributes is sometimes hesitant. Meanwhile, specific correlations occur between multiple attributes. As mentioned, fuzzy information aggregation is an effective approach, particularly based on HFFS. Consequently, this paper primarily proposes several new BM aggregation operators based on HFFS. Then, a new MADM method is presented by the proposed aggregation operators. Moreover, the diagnostic evaluation of depression is a complex issue. On the one hand, the description of depression symptoms by patient may be ambiguous and hesitant. On the other hand, there is a correlation between each symptom diagnostic evaluation by doctors. In this paper, three depressed patients were diagnosed and analyzed using the proposed MADM method in the context of HFFS. The experimental results and comparative analysis demonstrate that our proposed method is reasonable and effective, and has obvious advantages. The main contributions of this paper are as follows:

- (1) This paper first extends the HFFS aggregation operation based on algebraic t-norms to the BM operator. Then, the hesitant Fermatean fuzzy Bonferroni mean (HFFBM) operator and hesitant Fermatean fuzzy weighted Bonferroni mean (HFFWBM) operator are proposed.
- (2) This paper further proposes the fundamental operations and theorems of HFFS under Einstein t-norms. The HFFS aggregation operation based on Einstein operation rules is then extended to the BM operator. Hesitant Fermatean fuzzy Einstein Bonferroni mean (HFFEEM) operator and hesitant Fermatean fuzzy Einstein weighted Bonferroni mean (HFFEWBEM) operator are given next.
- (3) A novel MADM method based on the HFFWBM and HFFEWBEM operators is presented. The method is then applied to a case of depression diagnosis. By evaluating the depression diagnosis of three depressed patients, the effectiveness of the proposed method is demonstrated. Finally, compared to the existing decision-making methods, the proposed method has more advantages and more reasonable.

The structure of this paper is as follows: the section “[Preliminaries](#)” describes several fundamental notions, including the basic concepts of HFFS, the Einstein product, and Einstein sum and Bonferroni mean. The form and derivation of the hesitant Fermatean fuzzy BM operators and the hesitant Fermatean fuzzy weighted BM operators are introduced in the section “[Hesitant Fermatean fuzzy aggregation operator](#)”. The section “[A New method for MADM based on HFFWBM and HFFEWBEM aggregation operator](#)” presents a novel MADM method based on the HFFWBM and HFFEWBEM operators. The section “[Case study and comparative analysis](#)” verifies the rationality and effectiveness of the proposed method through a case study of three patients with depression diagnosis and evaluation. In addition, the advantages of the proposed method are illustrated using a series of comparisons. In the section “[Conclusion](#)”, a conclusion is presented.

Preliminaries

This section begins with a review of t-norm and t-conorm, which are crucial to fuzzy operations. The concept of DHFS is then reviewed. DHFS extends IFS from the perspective of hesitation. Similarly, HFFS is an extension of FFS from a hesitant perspective. Consequently, we then review the basic operation of DHFS from the perspective of t-norm and t-conorm. In addition, the concept of HFFS is given, and the basic operations of HFFS in terms of t-norm and t-conorm are reviewed. Finally, the concepts of Einstein sum, Einstein product, and Bonferroni mean are reviewed.

t-Norm and t-conorm

Definition 2.1 [43] A function $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a t-norm if it meets the following conditions:

- (1) $T(1, x) = x$, for all x .
- (2) $T(x, y) = T(y, x)$, for all x and y .
- (3) $T(x, T(y, z)) = T(T(x, y), z)$, for all x, y , and z .
- (4) If $x \leq x'$ and $y \leq y'$, then $T(x, y) \leq T(x', y')$.

Definition 2.2 [43] A function $S : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a t-conorm if it meets the following conditions:

- (1) $S(0, x) = x$, for all x .
- (2) $S(x, y) = S(y, x)$, for all x and y .
- (3) $S(x, S(y, z)) = S(S(x, y), z)$, for all x, y , and z .
- (4) If $x \leq x'$ and $y \leq y'$, then $S(x, y) \leq S(x', y')$.

As we all know [44], a strict Archimedean t-norm is represented by its additive generator k as $T(a, b) = k^{-1}(k(a) + k(b))$. Similarly, the dual t-conorm of t-norm, a strict Archimedean t-conorm, is represented by its additive generator l as $S(a, b) = l^{-1}(l(a) + l(b))$ with $l(t) = k(1 - t)$. On the basis of Archimedean t-norm and t-conorm, the sum operation of IFS is then given by Beliakov et al. [45], that is, $\alpha_1 \oplus \alpha_2 = (S(\mu_1, \mu_2), T(v_1, v_2))$, where $\alpha_1 = (\mu_1, v_1)$ and $\alpha_2 = (\mu_2, v_2)$ are two intuitionistic fuzzy numbers. Expanding the sum operation yields the following formula:

$$\alpha_1 \oplus \alpha_2 = (S(\mu_1, \mu_2), T(v_1, v_2)) = (l^{-1}(l(\mu_1) + l(\mu_2)), k^{-1}(k(v_1) + k(v_2))).$$

Next, the concept of DHFS and the basic operations of DHFS from the perspective of Archimedean t-norm and t-conorm are reviewed.

Dual hesitant fuzzy sets

Definition 2.3 [15] Let \mathcal{X} be a finite set, then a DHFS D on \mathcal{X} is described as

$$D = \{ \langle x, h(x), g(x) \rangle \mid x \in \mathcal{X} \},$$

where $h(x)$ and $g(x)$ are two sets of some values in $[0, 1]$, denoting the possible MDs and NMDs of element $x \in \mathcal{X}$ to the set D , respectively, with the condition $0 \leq \theta, \vartheta \leq 1, 0 \leq \theta^+ + \vartheta^+ \leq 1$, where $\theta \in h(x), \vartheta \in g(x), \theta^+ = \max_{\theta \in h(x)} \{\theta\}$, and $\vartheta^+ = \max_{\vartheta \in g(x)} \{\vartheta\}$ for all $x \in \mathcal{X}$. For convenience, the pair $d(x) = (h(x), g(x))$ is called a dual-hesitant fuzzy element (DHFE), simply as $d = (h, g)$.

According to [15], IFS can be viewed as a particular case of DHFS. When all possible MDs and NMDs of elements in

a DHFS only have one value, the DHFS is reduced to an IFS. Next, the basic DHFS operations from the perspective of t-norm and t-conorm are reviewed. In 2016, Wang et al. [46] provided the basic operational laws of DHFS, which were inspired by Beliakov et al. [45] and based on Archimedean t-norm and t-conorm.

Definition 2.4 [46] For any three DHFEs $d = (h, g)$, $d_1 = (h_1, g_1)$ and $d_2 = (h_2, g_2)$, then

- (1) $d^\gamma = \bigcup_{\theta \in h, \vartheta \in g} \{ \{k^{-1}(\gamma k(\theta))\}, \{l^{-1}(\gamma l(\vartheta))\} \}, \gamma > 0$
- (2) $\gamma d = \bigcup_{\theta \in h, \vartheta \in g} \{ \{l^{-1}(\gamma l(\theta))\}, \{k^{-1}(\gamma k(\vartheta))\} \}, \gamma > 0$
- (3) $d_1 \otimes d_2 = \bigcup_{\theta_1 \in h_1, \theta_2 \in h_2, \vartheta_1 \in g_1, \vartheta_2 \in g_2} \{ \{T(\theta_1, \theta_2)\}, \{S(\vartheta_1, \vartheta_2)\} \} = \bigcup_{\theta_1 \in h_1, \theta_2 \in h_2, \vartheta_1 \in g_1, \vartheta_2 \in g_2} \{ \{k^{-1}(k(\theta_1) + k(\theta_2))\}, \{l^{-1}(l(\vartheta_1) + l(\vartheta_2))\} \}$
- (4) $d_1 \oplus d_2 = \bigcup_{\theta_1 \in h_1, \theta_2 \in h_2, \vartheta_1 \in g_1, \vartheta_2 \in g_2} \{ \{S(\theta_1, \theta_2)\}, \{T(\vartheta_1, \vartheta_2)\} \} = \bigcup_{\theta_1 \in h_1, \theta_2 \in h_2, \vartheta_1 \in g_1, \vartheta_2 \in g_2} \{ \{l^{-1}(l(\theta_1) + l(\theta_2))\}, \{k^{-1}(k(\vartheta_1) + k(\vartheta_2))\} \}$.

Let $k(t) = -\log(t)$, $l(t) = k(1 - t) = -\log(1 - t)$. Then, we have the operational laws of DHFS in [15] as follows:

- (1) $d^\gamma = \bigcup_{\theta \in h, \vartheta \in g} \{ \{\theta^\gamma\}, \{1 - (1 - \vartheta)^\gamma\} \}, \gamma > 0$
- (2) $\gamma d = \bigcup_{\theta \in h, \vartheta \in g} \{ \{1 - (1 - \theta)^\gamma\}, \{\vartheta^\gamma\} \}, \gamma > 0$
- (3) $d_1 \otimes d_2 = \bigcup_{\theta_1 \in h_1, \theta_2 \in h_2, \vartheta_1 \in g_1, \vartheta_2 \in g_2} \{ \{\theta_1 \theta_2\}, \{\vartheta_1 + \vartheta_2 - \vartheta_1 \vartheta_2\} \}$
- (4) $d_1 \oplus d_2 = \bigcup_{\theta_1 \in h_1, \theta_2 \in h_2, \vartheta_1 \in g_1, \vartheta_2 \in g_2} \{ \{\theta_1 + \theta_2 - \theta_1 \theta_2\}, \{\vartheta_1 \vartheta_2\} \}$.

Hesitant Fermatean fuzzy sets

IFS can be regarded as DHFS with only one value for MD and NMD for all elements. Likewise, FFS can be viewed as HFFS when each element has a single value for MD and NMD. This section begins with a review of the concept of HFFS. Then, the basic operational laws of HFFS are reviewed from the t-norm and t-conorm perspectives.

Definition 2.5 [41] Let \mathcal{X} be a finite set. The definition of a HFFS on \mathcal{X} is

$$F = \{ \langle u, (\mu_{\mathcal{F}}(u), \nu_{\mathcal{F}}(u)) \rangle \mid u \in \mathcal{X} \},$$

where $\mu_{\mathcal{F}}(u)$ and $\nu_{\mathcal{F}}(u)$ are two sets, denoting the probable MDs and NMDs of the element $u \in \mathcal{X}$ to the set \mathcal{F} respectively, with the conditions $0 \leq \theta, \vartheta \leq 1, 0 \leq (\theta^+)^3 + (\vartheta^+)^3 \leq 1$, where $\theta \in \mu_{\mathcal{F}}(u), \vartheta \in \nu_{\mathcal{F}}(u)$ $\theta^+ = \max_{\theta \in \mu_{\mathcal{F}}(u)} \{ \theta \}$, and $\vartheta^+ = \max_{\vartheta \in \nu_{\mathcal{F}}(u)} \{ \vartheta \}$ for all $u \in \mathcal{X}$. The pair $\zeta_{\mathcal{F}}(u) = (\mu_{\mathcal{F}}(u), \nu_{\mathcal{F}}(u))$ is represented as the Hesitant Fermatean Fuzzy element (HFFE). For convenience, HFFE is simplified to $\zeta = (\mu, \nu)$.

Definition 2.6 [39] Let $\zeta = (\mu, \nu)$ be an HFFE. The score function $\mathcal{S}_{\mathcal{F}}$ and the accuracy function $\mathcal{P}_{\mathcal{F}}$ of the HFFE are depicted as follows:

$$\mathcal{S}_{\mathcal{F}}(\zeta) = \frac{1}{l_\mu} \sum_{\theta \in \mu} \theta^3 - \frac{1}{l_\nu} \sum_{\vartheta \in \nu} \vartheta^3$$

$$\mathcal{P}_{\mathcal{F}}(\zeta) = \frac{1}{l_\mu} \sum_{\theta \in \mu} \theta^3 + \frac{1}{l_\nu} \sum_{\vartheta \in \nu} \vartheta^3,$$

where l_μ and l_ν are the cardinal number of elements in μ and ν , respectively. Let $\zeta_i = (u_i, v_i)(i = 1, 2)$ be any two HFFEs, then

- (1) if $\mathcal{S}_{\mathcal{F}}(\zeta_1) > \mathcal{S}_{\mathcal{F}}(\zeta_2)$, then $\zeta_1 > \zeta_2$
- (2) if $\mathcal{S}_{\mathcal{F}}(\zeta_1) = \mathcal{S}_{\mathcal{F}}(\zeta_2)$, then
 - (a) if $\mathcal{P}_{\mathcal{F}}(\zeta_1) = \mathcal{P}_{\mathcal{F}}(\zeta_2)$, then $\zeta_1 = \zeta_2$
 - (b) if $\mathcal{P}_{\mathcal{F}}(\zeta_1) > \mathcal{P}_{\mathcal{F}}(\zeta_2)$, then $\zeta_1 > \zeta_2$.

As the basic operational laws of IFS based on Archimedean t-norm and t-conorm given by Beliakov et al. [45], the basic operational laws of FFS can also be proposed. For example, the sum operation of Fermatean fuzzy numbers can be written as $\alpha_1 \oplus \alpha_2 = \left(\sqrt[3]{S(\mu_1^3, \mu_2^3)}, \sqrt[3]{T(\nu_1^3, \nu_2^3)} \right)$, where $\alpha_1 = (\mu_1, \nu_1)$ and $\alpha_2 = (\mu_2, \nu_2)$ are two Fermatean fuzzy numbers. Expanding the sum operation yields the following formula:

$$\alpha_1 \oplus \alpha_2 = \left(\sqrt[3]{S(\mu_1^3, \mu_2^3)}, \sqrt[3]{T(\nu_1^3, \nu_2^3)} \right) = \left(\sqrt[3]{l^{-1}(l(\mu_1^3) + l(\mu_2^3))}, \sqrt[3]{k^{-1}(k(\nu_1^3) + k(\nu_2^3))} \right).$$

Let $k(t) = -\log(t)$, then $l(t) = k(1 - t) = -\log(1 - t)$. Then, we have the sum operation of FFS in [42] is shown as follows:

$$\alpha_1 \oplus \alpha_2 = \left(\sqrt[3]{S(\mu_1^3, \mu_2^3)}, \sqrt[3]{T(\nu_1^3, \nu_2^3)} \right) = \left(\sqrt[3]{l^{-1}(l(\mu_1^3) + l(\mu_2^3))}, \sqrt[3]{k^{-1}(k(\nu_1^3) + k(\nu_2^3))} \right) = \left(\sqrt[3]{\mu_1^3 + \mu_2^3 - \mu_1^3 \mu_2^3}, \nu_1 \nu_2 \right).$$

Extending the operational laws of FFS based on Archimedean t-norm and t-conorm to HFFS, we can derive the operational laws of HFFS based on Archimedean t-norm and t-conorm. For any three HFFEs $\zeta = (\mu, \nu), \zeta_1 = (\mu_1, \nu_1)$ and $\zeta_2 = (\mu_2, \nu_2)$, then the operational laws are shown as follows:

$$(1) \quad \zeta^\gamma = \bigcup_{\theta \in \mu, \vartheta \in \nu} \left\{ \left\{ \sqrt[k-1]{\gamma k(\theta^3)} \right\}, \left\{ \sqrt[l-1]{\gamma l(\vartheta^3)} \right\} \right\}, \gamma > 0;$$

$$(2) \quad \gamma \zeta = \bigcup_{\theta \in \mu, \vartheta \in \nu} \left\{ \left\{ \sqrt[l-1]{\gamma l(\theta^3)} \right\}, \left\{ \sqrt[k-1]{\gamma k(\vartheta^3)} \right\} \right\}, \gamma > 0;$$

$$(3) \quad \begin{aligned} \zeta_1 \otimes \zeta_2 &= \bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \left\{ \sqrt[3]{T(\theta_1^3, \theta_2^3)} \right\}, \left\{ \sqrt[3]{S(\vartheta_1^3, \vartheta_2^3)} \right\} \right\} \\ &= \bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \left\{ \sqrt[k-1]{k(\theta_1^3) + k(\theta_2^3)} \right\}, \right. \\ &\quad \left. \left\{ \sqrt[l-1]{l(\vartheta_1^3) + l(\vartheta_2^3)} \right\} \right\}; \end{aligned}$$

$$(4) \quad \begin{aligned} \zeta_1 \oplus \zeta_2 &= \bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \left\{ \sqrt[3]{S(\theta_1^3, \theta_2^3)} \right\}, \left\{ \sqrt[3]{T(\vartheta_1^3, \vartheta_2^3)} \right\} \right\} \\ &= \bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \left\{ \sqrt[l-1]{l(\theta_1^3) + l(\theta_2^3)} \right\}, \right. \\ &\quad \left. \left\{ \sqrt[k-1]{k(\vartheta_1^3) + k(\vartheta_2^3)} \right\} \right\}. \end{aligned}$$

Let $k(t) = -\log(t)$, then $l(t) = k(1 - t) = -\log(1 - t)$, and then, the operational laws of HFFS in [41] can be derived as follows.

Definition 2.7 [41] For any three HFFEs, $\zeta = (\mu, \nu)$, $\zeta_1 = (\mu_1, \nu_1)$ and $\zeta_2 = (\mu_2, \nu_2)$, we have the following basic operations:

- (1) $\zeta^c = \bigcup_{\theta \in \mu, \vartheta \in \nu} \{ \{\vartheta\}, \{\theta\} \}$, if $\nu \neq \emptyset, \mu \neq \emptyset$
- (2) $\zeta^\gamma = \bigcup_{\theta \in \mu, \vartheta \in \nu} \left\{ \{\theta^\gamma\}, \left\{ \sqrt[3]{1 - (1 - \vartheta^3)^\gamma} \right\} \right\}, \gamma > 0$
- (3) $\gamma \zeta = \bigcup_{\theta \in \mu, \vartheta \in \nu} \left\{ \left\{ \sqrt[3]{1 - (1 - \theta^3)^\gamma} \right\}, \{\vartheta^\gamma\} \right\}, \gamma > 0$
- (4) $\zeta_1 \otimes \zeta_2 = \bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \{\theta_1 \theta_2\}, \left\{ \sqrt[3]{\vartheta_1^3 + \vartheta_2^3 - \vartheta_1^3 \vartheta_2^3} \right\} \right\}$
- (5) $\zeta_1 \oplus \zeta_2 = \bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \left\{ \sqrt[3]{\theta_1^3 + \theta_2^3 - \theta_1^3 \theta_2^3} \right\}, \{\vartheta_1 \vartheta_2\} \right\}$.

The operations of fuzzy sets are crucial in fuzzy sets theory. Archimedean t-norm and t-conorm contain fuzzy operations and satisfy the requirements of conjunction and disjunction operator, respectively. The t-norm and t-conorm are, in fact, the most general families of binary functions that map the unit square to the unit interval [28]. Einstein product and Einstein sum, also called Einstein t-norm and t-conorm, is a commonly used fuzzy logic operation.

Definition 2.8 [28, 29] The Einstein product \mathcal{T}_E and Einstein sum \mathcal{S}_E are related by the De Morgan duality $\mathcal{S}_E(s, t) = 1 - \mathcal{T}_E(1 - s, 1 - t)$. For any $(s, t) \in [0, 1]^2$

$$\mathcal{T}_E(s, t) = \frac{st}{1 + (1 - s)(1 - t)}$$

$$\mathcal{S}_E(s, t) = \frac{s + t}{1 + st}.$$

Then, $\mathcal{S}_E(s, t)$ is called the dual t-conorm of \mathcal{T}_E . Analogously, $\mathcal{T}_E(s, t)$ is called the dual t-norm of \mathcal{S}_E .

Bonferroni mean

BM is an aggregation operation that considers aggregation elements independently [47]. Yager [36] interprets BM as a product of each argument with the mean of the other arguments. The BM operator can also be seen as an aggregation of the correlation of any element with other elements. The definition of BM is shown below.

Definition 2.9 [35, 48] Let $u_i \geq 0, u_j \geq 0 (i, j = 1, 2, \dots, n)$ be from a same set of nonnegative numbers. $p, q \geq 0$. Then, the Bonferroni mean is defined as

$$BM^{p,q}(u_1, u_2, \dots, u_n) = \left(\frac{1}{n(n-1)} \sum_{i,j=1; i \neq j}^n (u_i^p \cdot u_j^q) \right)^{\frac{1}{p+q}}.$$

Hesitant Fermatean fuzzy aggregation operator

In this section, four hesitant Fermatean fuzzy aggregation operators are introduced. First and foremost, the definition and derivation of the HFFBM operator are presented. In addition, in conjunction with the fundamental operations of HFFS in Einstein t-norms, the HFFEBM operator is defined and derived. We also develop the HFFWBM operator and the HFFEWBM operator in consideration of the impact of weights on decision-making results. The characteristics of the HFFEBM and HFFEWBM operators are then discussed.

HFFBM and HFFWBM operators

This subsection introduces the HFFBM aggregation operator based on the algebraic operations of HFFEs. And then, the HFFWBM aggregation operator is given. Before the operators have shown, we list some HFFS theorems of [41], and then, the more detailed proofs are given.

Theorem 3.1 For any three HFFEs $\zeta_1 = (\mu_1, \nu_1)$ and $\zeta_2 = (\mu_2, \nu_2), \gamma \geq 0$, then we have

- (1) $\zeta_1 \oplus \zeta_2 = \zeta_2 \oplus \zeta_1$
- (2) $\zeta_1 \otimes \zeta_2 = \zeta_2 \otimes \zeta_1$
- (3) $\gamma(\zeta_1 \oplus \zeta_2) = \gamma\zeta_1 \oplus \gamma\zeta_2$
- (4) $(\zeta_1 \otimes \zeta_2)^\gamma = \zeta_1^\gamma \otimes \zeta_2^\gamma$.

Theorem 3.2 Let $p, q > 0$, $\zeta_i = (\mu_i, \nu_i)\zeta_j = (\mu_j, \nu_j)(i, j = 1, 2, \dots, n)$ be from a same HFFS. $\delta_{i,j} = (\rho_{i,j}, \varrho_{i,j}) = (\zeta_i^p \otimes \zeta_j^q) \oplus (\zeta_i^q \otimes \zeta_j^p)$ is a collection of HFFBEs with $i \neq j$. Then, $\text{HFFBM}^{p,q}(\zeta_1, \zeta_2, \dots, \zeta_n) \in \text{HFFE}$ and

$\text{HFFBM}^{p,q}(\zeta_1, \zeta_2, \dots, \zeta_n)$

$$= \left(\frac{1}{n(n-1)} \left(\bigoplus_{i,j=1; i \neq j}^n \delta_{i,j} \right) \right)^{\frac{1}{p+q}} \bigcup_{\theta_{i,j} \in \rho_{i,j}, \vartheta_{i,j} \in \varrho_{i,j}} \left\{ \left[\sqrt[3]{1 - \prod_{i,j=1; i \neq j}^n \left((1 - A_i^p A_j^q)(1 - A_i^q A_j^p) \right)^{\frac{1}{n(n-1)}}} \right]^{\frac{1}{p+q}}, \right. \\ \left. \left[\sqrt[3]{1 - \left(1 - \prod_{i,j=1; i \neq j}^n \left((1 - B_i^p B_j^q)(1 - B_i^q B_j^p) \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}}} \right] \right\} \tag{3}$$

The proof of Theorem 3.1 can be found in the Appendix.

HFFBM operator

According to Definitions 2.7 and 2.9, we extend the hesitant Fermatean fuzzy aggregation operations to the BM operator as follows.

Definition 3.1 Let $\zeta_i = (\mu_i, \nu_i)\zeta_j = (\mu_j, \nu_j)(i, j = 1, 2, \dots, n)$ be from a same collection of HFFEs, $p, q > 0$. For any i, j and $i \neq j$, then

$$\text{HFFBM}^{p,q}(\zeta_1, \zeta_2, \dots, \zeta_n) = \left(\frac{1}{n(n-1)} \left(\bigoplus_{i,j=1; i \neq j}^n \zeta_i^p \otimes \zeta_j^q \right) \right)^{\frac{1}{p+q}} \tag{1}$$

is called hesitant Fermatean fuzzy Bonferroni mean operator.

Lemma 3.1 Let $p, q > 0$, $\zeta_i = (\mu_i, \nu_i), \zeta_j = (\mu_j, \nu_j)(i \neq j)$ be from a same HFFS. Then

$$\left(\zeta_i^p \otimes \zeta_j^q \right) \oplus \left(\zeta_i^q \otimes \zeta_j^p \right) = \bigcup_{\theta_i \in \mu_i, \theta_j \in \mu_j, \vartheta_i \in \nu_i, \vartheta_j \in \nu_j} \left\{ \left[\sqrt[3]{1 - \left(1 - A_i^p A_j^q \right) \left(1 - A_i^q A_j^p \right)} \right], \left[\sqrt[3]{\left(1 - B_i^p B_j^q \right) \left(1 - B_i^q B_j^p \right)} \right] \right\} \tag{2}$$

is hesitant Fermatean fuzzy Bonferroni elements (HFFBEs), where

$$A_i = \theta_i^3, A_j = \theta_j^3, B_i = 1 - \vartheta_i^3, B_j = 1 - \vartheta_j^3.$$

The proof of Lemma 3.1 can be found in the Appendix.

where

$$A_i = \theta_i^3, A_j = \theta_j^3, B_i = 1 - \vartheta_i^3, B_j = 1 - \vartheta_j^3.$$

The proof of Theorem 3.1 can be found in the Appendix.

Property 3.1 (Monotonicity) Let $\{\zeta_{\varphi_1}, \zeta_{\varphi_2}, \dots, \zeta_{\varphi_n}\}$ and $\{\zeta_{\psi_1}, \zeta_{\psi_2}, \dots, \zeta_{\psi_n}\}$ be two sets of HFFEs, $p, q > 0$. $\zeta_{\varphi_i} = (\mu_{\varphi_i}, \nu_{\varphi_i}), \zeta_{\psi_i} = (\mu_{\psi_i}, \nu_{\psi_i})$. For any $i, \theta_{\varphi_i} \in \mu_{\varphi_i}, \vartheta_{\varphi_i} \in \nu_{\varphi_i}$ and $\theta_{\psi_i} \in \mu_{\psi_i}, \vartheta_{\psi_i} \in \nu_{\psi_i}$, if $\theta_{\varphi_i} \leq \theta_{\psi_i}, \vartheta_{\varphi_i} \geq \vartheta_{\psi_i}$, then

$$\text{HFFBM}^{p,q}(\zeta_{\varphi_1}, \zeta_{\varphi_2}, \dots, \zeta_{\varphi_n}) \leq \text{HFFBM}^{p,q}(\zeta_{\psi_1}, \zeta_{\psi_2}, \dots, \zeta_{\psi_n}).$$

Property 3.2 (Idempotency) Let $\zeta_i = (\mu_i, \nu_i)(i = 1, 2, \dots, n)$ from an HFFS. $p, q > 0$. Assume that all ζ_i are equal and $\zeta_i = \zeta(\mu, \nu)$, then $\text{HFFBM}^{p,q}(\zeta_1, \zeta_2, \dots, \zeta_n) = \zeta$.

Property 3.3 (Boundedness) Let $\zeta_i = (\mu_i, \nu_i)(i = 1, 2, \dots, n)$ from an HFFS

$$\zeta_i^+ = \bigcup_{\theta_i \in \mu_i, \vartheta_i \in \nu_i} \{\max\{\theta_i\}, \min\{\vartheta_i\}\}$$

$$\zeta_i^- = \bigcup_{\theta_i \in \mu_i, \vartheta_i \in \nu_i} \{\min\{\theta_i\}, \max\{\vartheta_i\}\},$$

$\theta_i^+ \in \mu_i^+, \theta_i^- \in \mu_i^-, \vartheta_i^+ \in \nu_i^+, \vartheta_i^- \in \nu_i^-$, then we have

$$\zeta_i^- \leq \text{HFFBM}^{p,q}(\zeta_1, \zeta_2, \dots, \zeta_n) \leq \zeta_i^+.$$

Example 1 Assume that $\zeta_1 = (\{0.7\}, \{0.1, 0.2\}), \zeta_2 = (\{0.6, 0.9\}, \{0.5\}), \zeta_3 = (\{0.8\}, \{0.4\})$ are three HFFEs, then we use the HFFBM operator to aggregate these three

HFFE. Let $p = 1, q = 2$, then the aggregated HFFE is as follows:

$$\text{HFFBM}^{1,2}(\{(0.7), \{0.1, 0.2\}\}, \{(0.6, 0.9), \{0.5\}\}, \{(0.8), \{0.4\}\}) = \left(\left\{ \begin{array}{l} 0.710, 0.750, 0.726, 0.760, 0.744, 0.773, 0.755, 0.781, \\ 0.766, 0.789, 0.775, 0.796, 0.786, 0.804, 0.793, 0.810 \\ 0.383, 0.383, 0.385, 0.385, 0.384, 0.384, 0.386, 0.387, \\ 0.387, 0.387, 0.389, 0.390, 0.388, 0.389, 0.390, 0.391 \end{array} \right\} \right).$$

Then, we calculate the score value of this aggregation result based on Definition 2.2, and the result is 0.400.

HFFWBM operator

Considering the influence of weights on aggregation operations, we extend the weighted hesitant Fermatean fuzzy aggregation operation to BM operator. According to Definitions 2.7 and 2.9, the weighted extended BM operator is shown below.

Definition 3.2 Let $\zeta_i = (\mu_i, \nu_i)\zeta_j = (\mu_j, \nu_j)(i, j = 1, 2, \dots, n)$ be from a same collection of HFFEs, $p, q > 0$. For any i, j and $i \neq j$. The weight vector ω is $(\omega_1, \omega_2, \dots, \omega_n)^T$, with $\sum_{i=1}^n \omega_i = 1$ and $\omega_i \in [0, 1]$, then

$$\text{HFFWBM}_{\omega}^{p,q}(\zeta_1, \zeta_2, \dots, \zeta_n) = \left(\frac{1}{n(n-1)} \bigoplus_{i,j=1;i \neq j}^n ((\omega_i \zeta_i)^p \otimes (\omega_j \zeta_j)^q) \right)^{\frac{1}{p+q}} \quad (4)$$

is called hesitant Fermatean fuzzy weighted Bonferroni mean operator.

Lemma 3.2 Let $p, q > 0, \zeta_i = (\mu_i, \nu_i), \zeta_j = (\mu_j, \nu_j)(i \neq j)$ be from a same HFFS. The weight vector ω is $(\omega_1, \omega_2, \dots, \omega_n)^T$, with $\sum_{i=1}^n \omega_i = 1$ and $\omega_i \in [0, 1]$, then

$$\begin{aligned} & ((\omega_i \zeta_i)^p \otimes (\omega_j \zeta_j)^q) \oplus ((\omega_i \zeta_i)^q \otimes (\omega_j \zeta_j)^p) \\ &= \bigcup_{\theta_i \in \mu_i, \theta_j \in \mu_j, \vartheta_i \in \nu_i, \vartheta_j \in \nu_j} \left\{ \left\{ \sqrt[3]{1 - (1 - A_i^p A_j^q)(1 - A_i^q A_j^p)} \right\}, \left\{ \sqrt[3]{(1 - B_i^p B_j^q)(1 - B_i^q B_j^p)} \right\} \right\} \end{aligned} \quad (5)$$

is called hesitant Fermatean fuzzy weighted Bonferroni elements (HFFWBEs), where

$$A_i = 1 - (1 - \theta_i^3)^{\omega_i}, A_j = 1 - (1 - \theta_j^3)^{\omega_j}, B_i = 1 - (\vartheta_i^3)^{\omega_i}, B_j = 1 - (\vartheta_j^3)^{\omega_j}.$$

The proof of Lemma 3.2 can be found in the Appendix.

Theorem 3.3 Let $p, q > 0, \zeta_i = (\mu_i, \nu_i)\zeta_j = (\mu_j, \nu_j)(i, j = 1, 2, \dots, n)$ be from a same HFFS. The weight vector ω is $(\omega_1, \omega_2, \dots, \omega_n)^T$, with $\sum_{i=1}^n \omega_i = 1$ and $\omega_i \in [0, 1]$. $\delta_{i,j} = (\rho_{i,j}, \varrho_{i,j}) = ((\omega_i \zeta_i)^p \otimes (\omega_j \zeta_j)^q) \oplus ((\omega_i \zeta_i)^q \otimes (\omega_j \zeta_j)^p)$ is a collection of HFFWBEs with $i \neq j$. Then, $\text{HFFWBM}_{\omega}^{p,q}(d_1, d_2, \dots, d_n) \in \text{HFFEs}$ and

$$\text{HFFWBM}_{\omega}^{p,q}(\zeta_1, \zeta_2, \dots, \zeta_n) = \left(\frac{1}{n(n-1)} \bigoplus_{i,j=1;i \neq j}^n \delta_{i,j} \right)^{\frac{1}{p+q}} \bigcup_{\theta_i, j \in \rho_i, j, \vartheta_i, j \in \varrho_i, j} \left\{ \left\{ \sqrt[3]{1 - \prod_{i,j=1;i \neq j}^n \left((1 - A_i^p A_j^q)(1 - A_i^q A_j^p) \right)^{\frac{1}{n(n-1)}}} \right\}, \left\{ \sqrt[3]{1 - \prod_{i,j=1;i \neq j}^n \left((1 - B_i^p B_j^q)(1 - B_i^q B_j^p) \right)^{\frac{1}{n(n-1)}}} \right\} \right\} \quad (6)$$

where

$$A_i = 1 - (1 - \theta_i^3)^{\omega_i}, A_j = 1 - (1 - \theta_j^3)^{\omega_j}, B_i = 1 - (\vartheta_i^3)^{\omega_i}, B_j = 1 - (\vartheta_j^3)^{\omega_j}.$$

Similar to the proof of the HFFBM^{p,q} in Lemma 3.2, we can obtain the HFFWBM^{p,q}, so omitted in this section.

Example 2 Here, we use the same HFFEs as in Example 1 with $p = 1$ and $q = 2$. $\omega = (0.3, 0.2, 0.5)^T$ is the weighted vector of the three HFFEs. Then, the aggregation result is as follows:

$$\text{HFFWBM}_{\omega}^{1,2}(\{(0.7), \{0.1, 0.2\}\}, \{(0.6, 0.9), \{0.5\}\}, \{(0.8), \{0.4\}\}) = \left(\left\{ \begin{array}{l} 0.533, 0.548, 0.540, 0.554, 0.569, 0.578, 0.573, 0.582, \\ 0.567, 0.577, 0.571, 0.580, 0.591, 0.598, 0.594, 0.601 \\ 0.686, 0.687, 0.691, 0.692, 0.693, 0.694, 0.698, 0.699, \\ 0.703, 0.704, 0.708, 0.709, 0.710, 0.711, 0.716, 0.716 \end{array} \right\} \right).$$

Then, we calculate the score value of this aggregation result based on Definition 2.2, and the result is -0.157 .

HFFE and HFFEWBM operators

This subsection first gives the basic operations of HFFE based on Einstein t-norm and t-conorm of Definition 2.8. Further, this subsection develops the HFFE and HFFEWBM Operator. Then, the idea and derivation process of HFFEWBM are given. In particular, let additive generator $k(t) = \log\left(\frac{2-t}{t}\right)$ of Archimedean t-norm, then we have the basic Einstein operations of HFFEs.

Definition 3.3 For any three HFFEs $\zeta = (\mu, \nu), \zeta_1 = (\mu_1, \nu_1), \zeta_2 = (\mu_2, \nu_2)$, the basic Einstein operations of HFFEs are as follows:

$$(1) \zeta^\gamma = \bigcup_{\theta \in \mu, \vartheta \in \nu} \left\{ \left\{ \sqrt[3]{\frac{2(\theta^3)^\gamma}{(2-\theta^3)^\gamma + (\theta^3)^\gamma}} \right\}, \left\{ \sqrt[3]{\frac{(1+\vartheta^3)^\gamma - (1-\vartheta^3)^\gamma}{(1+\vartheta^3)^\gamma + (1-\vartheta^3)^\gamma}} \right\} \right\}, \gamma > 0$$

$$\begin{aligned}
 (2) \quad & \gamma \zeta = \bigcup_{\theta \in \mu, \vartheta \in \nu} \left\{ \left\{ \sqrt[\gamma]{\frac{(1+\theta^3)^\gamma - (1-\theta^3)^\gamma}{(1+\theta^3)^\gamma + (1-\theta^3)^\gamma}}, \left\{ \sqrt[\gamma]{\frac{2(\vartheta^3)^\gamma}{(2-\vartheta^3)^\gamma + (\vartheta^3)^\gamma}} \right\} \right\}, \right. \\
 & \left. \gamma > 0 \right. \\
 (3) \quad & \zeta_1 \otimes \zeta_2 = \bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \left\{ \frac{\theta_1 \theta_2}{\sqrt[3]{1+(1-\theta_1^3)(1-\theta_2^3)}}, \left\{ \sqrt[3]{\frac{\vartheta_1^3 + \vartheta_2^3}{1+\vartheta_1^3 \vartheta_2^3}} \right\} \right\} \right. \\
 (4) \quad & \left. \zeta_1 \oplus \zeta_2 = \bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \left\{ \sqrt[3]{\frac{\theta_1^3 + \theta_2^3}{1+\theta_1^3 \theta_2^3}}, \left\{ \frac{\vartheta_1 \vartheta_2}{\sqrt[3]{1+(1-\vartheta_1^3)(1-\vartheta_2^3)}} \right\} \right\} \right.
 \end{aligned}$$

Theorem 3.4 Let $\zeta_1 = (\mu_1, \nu_1), \zeta_2 = (\mu_2, \nu_2)$, and $\zeta_3 = (\mu_3, \nu_3)$ be three HFFEs with $\gamma \geq 0$. Then, we have the associativity and commutativity of HFFEs based on Definition 3.1 are as follows:

- (1) $\zeta_1 \oplus \zeta_2 = \zeta_2 \oplus \zeta_1$
- (2) $\zeta_1 \otimes \zeta_2 = \zeta_2 \otimes \zeta_1$
- (3) $\gamma(\zeta_1 \oplus \zeta_2) = \gamma \zeta_1 \oplus \gamma \zeta_2$
- (4) $(\zeta_1 \otimes \zeta_2)^\gamma = \zeta_1^\gamma \otimes \zeta_2^\gamma$
- (5) $\zeta_1 \oplus (\zeta_2 \otimes \zeta_3) = (\zeta_1 \oplus \zeta_2) \otimes \zeta_3$
- (6) $\zeta_1 \otimes (\zeta_2 \oplus \zeta_3) = (\zeta_1 \otimes \zeta_2) \oplus \zeta_3$.

The proof of Theorem 3.3 can be found in the Appendix.

HFFEBM operator

According to Definition 3.3 and Definition 2.9, we extend the hesitant Fermatean fuzzy aggregation operations based on Einstein operations to the BM operator as follows.

Definition 3.4 Let $\zeta_i = (\mu_i, \nu_i) \zeta_j = (\mu_j, \nu_j) (i, j = 1, 2, \dots, n)$ be from a same collection of HFFEs. $p, q > 0$. For any $i, j, i \neq j$, then

$$\begin{aligned}
 & \text{HFFEBM}^{p,q}(\zeta_1, \zeta_2, \dots, \zeta_n) \\
 & = \left(\frac{1}{n(n-1)} \bigoplus_{i,j=1; i \neq j}^n \left(\zeta_i^p \otimes \zeta_j^q \right) \right)^{\frac{1}{p+q}} \tag{7}
 \end{aligned}$$

is called hesitant Fermatean fuzzy Einstein Bonferroni mean operator.

Lemma 3.3 Let $p, q > 0, \zeta_i = (\mu_i, \nu_i), \zeta_j = (\mu_j, \nu_j) (i \neq j)$ be from a same HFFS. Then

$$\begin{aligned}
 & (\zeta_i^p \otimes \zeta_j^q) \oplus (\zeta_i^q \otimes \zeta_j^p) \\
 & = \bigcup_{\theta_i \in \mu_i, \theta_j \in \mu_j, \vartheta_i \in \nu_i, \vartheta_j \in \nu_j} \left(\begin{array}{c} \left(\frac{(B_i^p B_j^q + 3A_i^p A_j^q)(B_i^q B_j^p + 3A_i^q A_j^p)}{-(B_i^p B_j^q - A_i^p A_j^q)(B_i^q B_j^p - A_i^q A_j^p)} \right)^{\frac{1}{3}} \\ \left(\frac{(B_i^p B_j^q + 3A_i^p A_j^q)(B_i^q B_j^p + 3A_i^q A_j^p)}{+(B_i^p B_j^q - A_i^p A_j^q)(B_i^q B_j^p - A_i^q A_j^p)} \right)^{\frac{1}{3}} \\ \left(\frac{2(M_i^p M_j^q - N_i^p N_j^q)(M_i^q M_j^p - N_i^q N_j^p)}{(M_i^p M_j^q + 3N_i^p N_j^q)(M_i^q M_j^p + 3N_i^q N_j^p)} \right)^{\frac{1}{3}} \\ \left(\frac{(M_i^p M_j^q - N_i^p N_j^q)(M_i^q M_j^p - N_i^q N_j^p)}{+(M_i^p M_j^q - N_i^p N_j^q)(M_i^q M_j^p - N_i^q N_j^p)} \right)^{\frac{1}{3}} \end{array} \right) \tag{8}
 \end{aligned}$$

is called hesitant Fermatean fuzzy Einstein Bonferroni elements (HFFEBEs), where

$$A_i = \theta_i^3, B_i = 2 - \theta_i^3, M_i = 1 + \vartheta_i^3, N_i = 1 - \vartheta_i^3.$$

The proof of Lemma 3.3 can be found in the Appendix.

Theorem 3.5 Let $p, q > 0, \zeta_i = (\mu_i, \nu_i) \zeta_j = (\mu_j, \nu_j) (i, j = 1, 2, \dots, n)$ be from a same HFFS. $\delta_{i,j} = (\rho_{i,j}, \varrho_{i,j}) = (\zeta_i^p \otimes \zeta_j^q) \oplus (\zeta_i^q \otimes \zeta_j^p)$ is a collection of HFFEBEs with $i \neq j$. Then, $\text{HFFEBM}^{p,q}(\zeta_1, \zeta_2, \dots, \zeta_n) \in$ HFFE and

$$\begin{aligned}
 \text{HFFEBM}^{p,q}(\zeta_1, \zeta_2, \dots, \zeta_n) &= \left(\frac{1}{n(n-1)} \left(\bigoplus_{i,j=1; i \neq j}^n \delta_{i,j} \right) \right)^{\frac{1}{p+q}} \\
 &= \bigcup_{\theta_i, j \in \rho_{i,j}, \vartheta_i, j \in \varrho_{i,j}} \left\{ \sqrt[n]{ \left(1 - \prod_{i,j=1; i \neq j}^n \left(1 - \frac{\left((B_i^p B_j^q + 3A_i^p A_j^q)(B_i^q B_j^p + 3A_i^q A_j^p) - (B_i^p B_j^q - A_i^p A_j^q)(B_i^q B_j^p - A_i^q A_j^p) \right)}{\left((B_i^p B_j^q + 3A_i^p A_j^q)(B_i^q B_j^p + 3A_i^q A_j^p) + (B_i^p B_j^q - A_i^p A_j^q)(B_i^q B_j^p - A_i^q A_j^p) \right)} \right)^{\frac{1}{n(n-1)}}} \right)^{\frac{1}{p+q}} \right\} \\
 &\quad \bigcup_{\theta_i, j \in \mu_i, \vartheta_i, j \in \nu_i} \left\{ \sqrt[n]{ \left(1 - \prod_{i,j=1; i \neq j}^n \left(1 - \frac{2(M_i^p M_j^q - N_i^p N_j^q)(M_i^q M_j^p - N_i^q N_j^p)}{\left((M_i^p M_j^q + 3N_i^p N_j^q)(M_i^q M_j^p + 3N_i^q N_j^p) + (M_i^p M_j^q - N_i^p N_j^q)(M_i^q M_j^p - N_i^q N_j^p) \right)} \right)^{\frac{1}{n(n-1)}}} \right)^{\frac{1}{p+q}} \right\} \tag{9}
 \end{aligned}$$

where

$$A_i = \theta_i^3, B_i = 2 - \theta_i^3, M_i = 1 + \vartheta_i^3, N_i = 1 - \vartheta_i^3.$$

Similar to the proof of the HFFBM^{p,q} in Lemma 3.2, we can obtain the HFFEBM^{p,q} by Lemma 3.3 and Theorem 3.4, so omitted in this section.

Property 3.4 (Monotonicity) Let {ζ_{φ₁}, ζ_{φ₂}, ..., ζ_{φ_n}} and {ζ_{ψ₁}, ζ_{ψ₂}, ..., ζ_{ψ_n}} be two sets of HFFEs, p, q > 0. HFFEs, ζ_{φ_i} = (μ_{φ_i}, ν_{φ_i}), ζ_{ψ_i} = (μ_{ψ_i}, ν_{ψ_i}). For any i, θ_{φ_i} ∈ μ_{φ_i}, θ_{ψ_i} ∈ ν_{ψ_i} and θ_{ψ_i} ∈ μ_{ψ_i}, θ_{φ_i} ∈ ν_{φ_i}, if θ_{φ_i} ≤ θ_{ψ_i}, θ_{ψ_i} ≥ θ_{φ_i}, then

$$\text{HFFEBM}^{p,q}(\zeta_{\varphi_1}, \zeta_{\varphi_2}, \dots, \zeta_{\varphi_n}) \leq \text{HFFEBM}^{p,q}(\zeta_{\psi_1}, \zeta_{\psi_2}, \dots, \zeta_{\psi_n}).$$

Property 3.5 (Idempotency) Let ζ_i = (μ_i, ν_i) (i = 1, 2, ..., n) be from an HFFS. p, q > 0. Assume that all ζ_i are equal and ζ_i = ζ(μ, ν), then HFFEBM^{p,q}(ζ₁, ζ₂, ..., ζ_n) = ζ.

Property 3.6 (Boundedness) Let ζ_i = (μ_i, ν_i) (i = 1, 2, ..., n) be from an HFFS, and

$$\zeta_i^+ = \bigcup_{\theta_i \in \mu_i, \vartheta_i \in \nu_i} \{\max\{\theta_i\}, \min\{\vartheta_i\}\}$$

$$\zeta_i^- = \bigcup_{\theta_i \in \mu_i, \vartheta_i \in \nu_i} \{\min\{\theta_i\}, \max\{\vartheta_i\}\},$$

θ_i⁺ ∈ μ_i⁺, θ_i⁻ ∈ μ_i⁻, θ_i⁺ ∈ ν_i⁺, θ_i⁻ ∈ ν_i⁻, then we have

$$\zeta_i^- \leq \text{HFFEBM}^{p,q}(\zeta_1, \zeta_2, \dots, \zeta_n) \leq \zeta_i^+.$$

Example 3 Here, we use the same HFFEs as in Example 1 with p = 1 and q = 2, then the aggregate HFFE is as follows:

$$\text{HFFEBM}^{1,2}(\{(0.7), \{0.1, 0.2\}\}, \{(0.6, 0.9), \{0.5\}\}, \{(0.8), \{0.4\}\})$$

$$= \left(\left\{ \begin{aligned} &0.714, 0.758, 0.731, 0.767, 0.752, 0.780, 0.762, 0.787, \\ &0.775, 0.797, 0.783, 0.802, 0.793, 0.811, 0.799, 0.815 \end{aligned} \right\}, \left\{ \begin{aligned} &0.383, 0.383, 0.385, 0.386, 0.384, 0.385, 0.386, 0.387, \\ &0.387, 0.387, 0.389, 0.390, 0.388, 0.389, 0.390, 0.391 \end{aligned} \right\} \right).$$

Then, we calculate the score value of this aggregation result based on Definition 2.2, and the result is 0.413.

HFFEWBM operator

Without considering the weights, we obtain the aggregation operator as shown in Theorem 3.3, while, in general, the weights is a crucial factor in the dual-hesitant fuzzy decision-making. The weights refer to the weight of each HFFE, and different weights will lead to different final decision results. We extend the weighted hesitant Fermatean fuzzy aggregation operation based on Einstein operations to BM operator. According to Definitions 3.3 and 2.9, the HFFEWBM operator is shown below.

Definition 3.5 Let $\zeta_i = (\mu_i, \nu_i)\zeta_j = (\mu_j, \nu_j)$ ($i, j = 1, 2, \dots, n$) be from a same set of HFFEs. $p, q > 0$. And, for any $i, j, i \neq j$. The weight vector ω is $(\omega_1, \omega_2, \dots, \omega_n)^T$, with $\sum_{i=1}^n \omega_i = 1$ and $\omega_i \in [0, 1]$. Then

$$\begin{aligned} & \text{HFFEWBM}_{\omega}^{p,q}(\zeta_1, \zeta_2, \dots, \zeta_n) \\ &= \left(\frac{1}{n(n-1)} \bigoplus_{i,j=1; i \neq j}^n (\omega_i \zeta_i)^p \otimes (\omega_j \zeta_j)^q \right)^{\frac{1}{p+q}} \end{aligned} \tag{10}$$

is called hesitant Fermatean fuzzy Einstein weighted Bonferroni mean operator.

Lemma 3.4 Let $p, q > 0, \zeta_i = (\mu_i, \nu_i), \zeta_j = (\mu_j, \nu_j)$ ($i \neq j$) be from a same HFFS. The weight vector ω is $(\omega_1, \omega_2, \dots, \omega_n)^T$, with $\sum_{i=1}^n \omega_i = 1$ and $\omega_i \in [0, 1]$. Then

$$\begin{aligned} & ((\omega_i \zeta_i)^p \otimes (\omega_j \zeta_j)^q) \oplus ((\omega_i \zeta_i)^q \otimes (\omega_j \zeta_j)^p) \\ &= \bigcup_{\theta_i \in \mu_i, \theta_j \in \mu_j, \vartheta_i \in \nu_i, \vartheta_j \in \nu_j} \left\{ \begin{array}{l} \left[\begin{array}{l} (B_i^p B_j^q + 3A_i^p A_j^q)(B_i^q B_j^p + 3A_i^q A_j^p) \\ -(B_i^p B_j^q - A_i^p A_j^q)(B_i^q B_j^p - A_i^q A_j^p) \\ (B_i^p B_j^q + 3A_i^p A_j^q)(B_i^q B_j^p + 3A_i^q A_j^p) \\ +(B_i^p B_j^q - A_i^p A_j^q)(B_i^q B_j^p - A_i^q A_j^p) \end{array} \right] \\ \left[\begin{array}{l} 2(M_i^p M_j^q - N_i^p N_j^q)(M_i^q M_j^p - N_i^q N_j^p) \\ (M_i^p M_j^q + 3N_i^p N_j^q)(M_i^q M_j^p + 3N_i^q N_j^p) \\ +(M_i^p M_j^q - N_i^p N_j^q)(M_i^q M_j^p - N_i^q N_j^p) \end{array} \right] \end{array} \right\} \end{aligned} \tag{11}$$

is called Hesitant Fermatean Fuzzy Einstein weighted Bonferroni Elements (HFFEWBEs), where

$$\begin{aligned} A_i &= (1 + \theta_i^3)^{\omega_i} - (1 - \theta_i^3)^{\omega_i}, B_i = (1 + \theta_i^3)^{\omega_i} + 3(1 - \theta_i^3)^{\omega_i}, \\ M_i &= (2 - \vartheta_i^3)^{\omega_i} + 3(\vartheta_i^3)^{\omega_i}, N_i = (2\vartheta_i^3)^{\omega_i} - 3(\vartheta_i^3)^{\omega_i}. \end{aligned}$$

The proof of Lemma 3.4 can be found in the Appendix.

Theorem 3.6 Let $p, q > 0, \zeta_i = (\mu_i, \nu_i)\zeta_j = (\mu_j, \nu_j)$ ($i, j = 1, 2, \dots, n$) be from a same HFFS. The weight vector ω is $(\omega_1, \omega_2, \dots, \omega_n)^T$, with $\sum_{i=1}^n \omega_i = 1$ and $\omega_i \in [0, 1]$. $\delta_{i,j} = (\rho_{i,j}, \varrho_{i,j}) = ((\omega_i \zeta_i)^p \otimes (\omega_j \zeta_j)^q) \oplus ((\omega_i \zeta_i)^q \otimes (\omega_j \zeta_j)^p)$ is a collection of HFFEWBEs with $i \neq j$. Then, $\text{HFFEWBM}_{\omega}^{p,q}(d_1, d_2, \dots, d_n) \in \text{HFFEs}$ and

$$\begin{aligned}
 \text{HFFEWBM}_{\omega}^{p,q}(\zeta_1, \zeta_2, \dots, \zeta_n) &= \left(\frac{1}{n(n-1)} \left(\bigoplus_{i,j=1; i \neq j}^n \delta_{i,j} \right) \right)^{\frac{1}{p+q}} \\
 &= \bigcup_{\theta_i, j \in \rho_i, j, \vartheta_i, j \in \Omega_i, j} \left\{ \left[\sqrt{ \left(1 - \prod_{i,j=1; i \neq j}^n \left(1 - \frac{\left(B_i^p B_j^q + 3A_i^p A_j^q \right) \left(B_i^q B_j^p + 3A_i^q A_j^p \right) - \left(B_i^p B_j^q - A_i^p A_j^q \right) \left(B_i^q B_j^p - A_i^q A_j^p \right)}{\left(B_i^p B_j^q + 3A_i^p A_j^q \right) \left(B_i^q B_j^p + 3A_i^q A_j^p \right) + \left(B_i^p B_j^q - A_i^p A_j^q \right) \left(B_i^q B_j^p - A_i^q A_j^p \right)} \right)^{\frac{1}{n(n-1)}} \right]^{\frac{1}{p+q}} \right\} \\
 &\quad \left\{ \left[\sqrt{ \left(1 - \prod_{i,j=1; i \neq j}^n \left(1 - \frac{2 \left(M_i^p M_j^q - N_i^p N_j^q \right) \left(M_i^q M_j^p - N_i^q N_j^p \right)}{\left(M_i^p M_j^q + 3N_i^p N_j^q \right) \left(M_i^q M_j^p + 3N_i^q N_j^p \right) + \left(M_i^p M_j^q - N_i^p N_j^q \right) \left(M_i^q M_j^p - N_i^q N_j^p \right)} \right)^{\frac{1}{n(n-1)}} \right]^{\frac{1}{p+q}} \right\} \quad (12)
 \end{aligned}$$

where

$$\begin{aligned}
 A_i &= (1 + \theta_i^3)^{\omega_i} - (1 - \theta_i^3)^{\omega_i}, \quad B_i = (1 + \theta_i^3)^{\omega_i} + 3(1 - \theta_i^3)^{\omega_i}, \\
 M_i &= (2 - \vartheta_i^3)^{\omega_i} + 3(\vartheta_i^3)^{\omega_i}, \quad N_i = (2\vartheta_i^3)^{\omega_i} - 3(\vartheta_i^3)^{\omega_i}.
 \end{aligned}$$

Similar to the proof of the HFFEBM^{p,q} in Lemma 3.3, we can obtain the HFFEWBM^{p,q} by Lemma 3.4 and Theorem 3.4, so omitted in this section.

Example 4 Here, we use the same HFFEs as in Example 2 with $p = 1$ and $q = 2$. $\omega = (0.3, 0.2, 0.5)^T$ is the weighted vector of the three HFFEs. Then, the aggregation result is as follows:

$$\begin{aligned}
 &\text{HFFEWBM}_{\omega}^{1,2}(\{(0.7), \{0.1, 0.2\}\}, \{(0.6, 0.9), \{0.5\}\}, \{(0.8), \{0.4\}\}) \\
 &= \left(\left\{ \begin{array}{l} 0.522, 0.532, 0.527, 0.536, 0.553, 0.559, 0.556, 0.561, \\ 0.547, 0.554, 0.550, 0.557, 0.569, 0.574, 0.571, 0.576 \\ 0.722, 0.723, 0.728, 0.728, 0.730, 0.730, 0.735, 0.736, \\ 0.739, 0.740, 0.745, 0.746, 0.748, 0.749, 0.754, 0.755 \end{array} \right\} \right)
 \end{aligned}$$

Then, we calculate the score value of this aggregation result based on Definition 2.2, and the result is -0.233 .

A new method for MADM based on HFFWBM and HFFEWBM aggregation operator

This section will examine MADM under the hesitant Fermatean fuzzy environment. In this environment, each alternative is aggregated using HFFWBM or HFFEWBM operator and then ranked by the score function. Assume that there are m alternatives and n attributes. $A = \{A_1, A_2, \dots, A_m\}$ be a discrete collection of alternatives,

and $C = \{C_1, C_2, \dots, C_n\}$ be a collection of attributes. The hesitant MD and NMD under each attribute of each alternative are given by the decision-makers. i.e., they can be considered as HFFEs $\zeta_{ij} = (\mu_{ij}, \nu_{ij})$. Then, we get the hesitant Fermatean fuzzy decision matrix (HFF-DM) $D = (\zeta_{ij})_{m \times n}$ and is as follows:

$$D_{m \times n} = \begin{bmatrix} \zeta_{11} & \zeta_{12} & \dots & \zeta_{1n} \\ \zeta_{21} & \zeta_{22} & & \zeta_{2n} \\ \vdots & & \ddots & \vdots \\ \zeta_{m1} & \zeta_{m2} & \dots & \zeta_{mn} \end{bmatrix}. \quad (13)$$

Next, we use the HFFWBM and HFFEWBM operator to handle this MADM problem. The specific algorithm is described as follows.

Step 1. The HFF-DM $D = (\zeta_{ij})_{m \times n}$ is generated by evaluating the decision-makers under each attribute of the alternatives. Here, the evaluation of the cost attributes needs to be converted into a benefits evaluation by the complementary operation in Definition 2.7. The conversion is as follows:

$$\zeta'_{ij} = \begin{cases} \zeta_{ij} & \text{for benefit attribute } C_j, \\ \zeta_{ij}^C & \text{for cost attribute } C_j \end{cases} \quad (14)$$

where $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$. ζ_{ij}^C is the complement of ζ_{ij} . After converted, we have the HFF-DM D' from D as follows:

$$D'_{m \times n} = \begin{bmatrix} \zeta'_{11} & \zeta'_{12} & \dots & \zeta'_{1n} \\ \zeta'_{21} & \zeta'_{22} & \dots & \zeta'_{2n} \\ \vdots & & \ddots & \vdots \\ \zeta'_{m1} & \zeta'_{m2} & \dots & \zeta'_{mn} \end{bmatrix}. \quad (15)$$

Step 2. If weights are considered, the HFFEWBM^{p,q} and HFFWBM^{p,q} operators are used to aggregate the multi-attribute evaluations of the m alternatives in the decision matrix D' into one evaluation value. Alternatively, using the HFFEBM and HFFBM operator to aggregate attributes without considering the weight of each attribute. Using Theorem 3.3 or 3.6 is as follows:

$$\tilde{\zeta}_i = \text{HFFWBM}_{\omega}^{p,q}(\zeta'_{i1}, \zeta'_{i2}, \dots, \zeta'_{in}) \quad (16)$$

$$\tilde{\zeta}_i = \text{HFFEWBM}_{\omega}^{p,q}(\zeta'_{i1}, \zeta'_{i2}, \dots, \zeta'_{in}), \quad (17)$$

where $i = 1, 2, \dots, m$ and ω_i indicate the weight of i th attribute.

Step 3. The score and accuracy values are calculated for each evaluation by the score function and accuracy function in Definition 2.6.

Step 4. The m alternatives are ranked by the comparison rule of the score function and the accuracy function. This rank-order solves the MADM under the hesitant Fermatean fuzzy environment.

Case study and comparative analysis

To demonstrate the viability and efficacy of our proposed method, this section illustrates a case of depression diagnosis. First, the proposed method based on the HFFWBM and HFFEWBM operators is used to diagnose depression for three patients. Second, we analyzed how parameters in the HFFWBM and HFFEWBM operators affect depression diagnosis. The analysis indicates that the parameters can be regarded as the risk preference of the patient or the doctor. Finally, a comparison experiment among the existing MADM methods and the proposed method is conducted.

Case study

Depression is a prevalent chronic disease that impacts both physical and mental health. Studies indicate that depression is a chronic disease with a low mortality rate and a high rate of disability. As social pressure has increased over the past 3 decades, depression has been one of the top three causes of nonfatal health impairment and the incidence of depression has increased yearly. Estimated 173 million adults in China suffer from mental health issues, with 4.30 million

having severe mental health issues [49]. The global ranking of disability-adjusted life years (DALY) for China rose from 15th in 1990 to 10th in 2017 as the burden of depression increased [50]. According to studies, the risk of death due to depression doubles [51]. Consequently, the diagnosis and treatment of depression is a significant concern in China. Individual diagnosis of depression is based on the Diagnostic and Statistical Manual of Mental Disorders, Fifth Edition (DSM-5) [52], and the Chinese Classification and Diagnostic Criteria for Mental Disorders, Third Edition (CCMD-3) [53], which require consideration of four criteria: symptom criteria, course criteria, exclusion criteria, and severity criteria. The main symptoms of depression are loss of interest, unhappiness, decreased energy, fatigue, trauma, low self-esteem, self-blame or guilt, difficulty associating or thinking, sleep disturbances, significant weight loss, less sexual desire, and severe cases, suicide [54]. The existing assessment criteria for depression are based on the depression score scale. The Hamilton Depression Scale (HAMD) and the nine-item Patient Health Questionnaire Depression Scale (PHQ-9) are typically used to determine severity [55–58].

The current method of diagnosis is an initial assessment by the doctor using the HAMD or PHQ-9 based on the patient's clinical depressive symptoms within 2 weeks. However, both the score on the Depression Score Scale for patients and the evaluation of the depressive status value of doctors contain uncertainty and ambiguity. Diagnostic evaluation for existing depressive symptoms is complicated by the fuzziness and uncertainty of symptoms under precise values, which makes diagnosis difficult. Additionally, due to the ambiguity of depressive symptoms, it is difficult for patients to describe their specific symptoms clearly, causing doctors to hesitate in diagnosing them.

Fuzzy sets, a powerful tool for dealing with uncertainty and fuzziness, can also be helpful in the diagnosis of depression. Considering the possibility of correlations between depressive symptoms, we employ the proposed hesitant Fermatean fuzzy BM operators and the MADM method to diagnose depression. The HAMD and PHQ-9 classify individuals with depression according to three types of symptoms: mild depression (D_1), moderate depression (D_2), and major depression (D_3). In this case, we use it as a diagnostic option set $D = \{D_1, D_2, D_3\}$. According to [54], the core symptoms of major depressive disorder are anxiety state (C_1), sleep disturbance (C_2), decreased interest and fatigue (C_3), cognitive dysfunction (C_4), and sexual dysfunction (C_5). Then, the diagnostic set of depressive symptoms is $C = \{C_1, C_2, C_3, C_4, C_5\}$. The weight vector for the five core symptoms is $\omega = (0.2, 0.15, 0.2, 0.3, 0.15)$. Three patients were then diagnosed. Suppose the depression diagnostic evaluation tables for patient P_A , patient P_B , and patient

Table 1 Depression diagnostic evaluation for P_A

	C_1	C_2	C_3	C_4	C_5
D_1	{{0.3,0.5}, {0.6}}	{{0.6}, {0.4,0.6}}	{{0.4}, {0.2}}	{{0.5}, {0.4}}	{{0.3}, {0.6}}
D_2	{{0.8}, {0.2}}	{{0.6,0.7}, {0.1}}	{{0.7}, {0.3,0.4}}	{{0.3}, {0.2}}	{{0.6}, {0.1}}
D_3	{{0.2}, {0.3}}	{{0.3}, {0.5}}	{{0.5,0.8}, {0.2}}	{{0.3}, {0.7}}	{{0.5}, {0.4,0.5}}

Table 2 Depression diagnostic evaluation for P_B

	C_1	C_2	C_3	C_4	C_5
D_1	{{0.7}, {0.2}}	{{0.8}, {0.4}}	{{0.4}, {0.5, 0.6}}	{{0.5}, {0.2}}	{{0.6, 0.4}, {0.1}}
D_2	{{0.3}, {0.8}}	{{0.5}, {0.2}}	{{0.5, 0.8}, {0.7}}	{{0.5}, {0.5}}	{{0.2}, {0.5, 0.7}}
D_3	{{0.1, 0.3}, {0.4}}	{{0.7}, {0.6}}	{{0.2}, {0.5}}	{{0.6}, {0.6}}	{{0.3}, {0.4, 0.5}}

Table 3 Depression diagnostic evaluation for P_C

	C_1	C_2	C_3	C_4	C_5
D_1	{{0.6}, {0.7}}	{{0.3,0.6}, {0.2}}	{{0.7}, {0.7,0.8}}	{{0.2}, {0.4}}	{{0.1}, {0.3,0.5}}
D_2	{{0.6,0.7}, {0.4}}	{{0.5}, {0.3,0.4}}	{{0.2}, {0.8}}	{{0.6}, {0.5}}	{{0.3}, {0.3}}
D_3	{{0.9}, {0.1,0.2}}	{{0.6,0.7}, {0.4}}	{{0.5}, {0.4}}	{{0.8}, {0.2}}	{{0.5}, {0.1}}

Table 4 Comprehensive diagnostic score based on HFFWBM aggregation operator

	D_1	D_2	D_3
P_A	-0.5989	-0.3117	-0.5898
P_B	-0.4139	-0.6806	-0.6499
P_C	-0.6241	-0.6119	-0.3305

Table 5 Comprehensive diagnostic score based on HFFEWM aggregation operator

	D_1	D_2	D_3
P_A	-0.6715	-0.4309	-0.6706
P_B	-0.5151	-0.7382	-0.7221
P_C	-0.6949	-0.6909	-0.4438

P_C are shown in Tables 1, 2, 3. The diagnostic process is shown below.

Step 1. The symptoms of depression can be regarded as negative attributes (cost attributes). The final diagnosis ranking result is mainly symptom severity, so there is no need to convert.

Setp 2. Aggregate each depression type for each patient using the HFFWBM operator by Eq. (16) and the HFFEWM operator by Eq. (17). The weight vector of depression symptoms is $\omega = (0.2, 0.15, 0.2, 0.3, 0.15)$. The aggregation parameters are set to $p = 1$ and $q = 1$. Then, the aggregation results represent the comprehensive diagnostic evaluation of each patient under each depression type.

Step 3. Score function in Definition 2.6 is used to calculate the diagnostic evaluation score of each patient under each depression type. The diagnostic evaluation scores by HFFWBM are shown in Table 4, and the diagnostic evaluation scores by HFFEWM are shown in Table 5.

Table 6 Depression diagnosis for three patients

	Patients	Diagnosis result
HFFWBM ($p = 1, q = 1$)	P_A	$D_2 > D_3 > D_1$
	P_B	$D_1 > D_3 > D_2$
	P_C	$D_3 > D_2 > D_1$
HFFEWM ($p = 1, q = 1$)	P_A	$D_2 > D_3 > D_1$
	P_B	$D_1 > D_3 > D_2$
	P_C	$D_3 > D_2 > D_1$

Step 4. According to the comprehensive diagnostic evaluation score of each patient under each depression type obtained in step 3, the depression type of each patient is sorted. The results are shown in Table 6. From the diagnostic evaluation results, we know that the diagnosis of P_A is moderate depression, the diagnosis of P_B is mild depression, and the diagnosis of P_C is major depression.

Influence of the parameter on the diagnosis result

BM operation has two parameters, p and q , which can be regarded as a parameter vector. Variations in parameters make aggregation operators more flexible and result in distinct outcomes. Let the three depressed patients in the sections “Case study” be an example. The diagnosis results vary based on the parameter vector, as shown in Tables 7, 8, 9, 10, 11, 12. We can find that the parameter vector impacts the diagnosis ranking results, but the diagnosis results remain the same. P_A is still diagnosed with moderate depression. P_B has mild depression, while P_C is diagnosed with major

depression. In addition, the score increases as the parameter increases, and the score decreases as the parameter decreases. Therefore, the parameter vector can be regarded as the risk preference of the doctors or patients.

Comparative analysis with other method

This section discusses the comparative analysis of the proposed method and existing approaches to verify the reliability of the proposed MADM method. Kirişci [39] presented a MADM method based on the Fermatean hesitant fuzzy

Table 7 The influence of parameter vector changes of HFFWBM operator on the depression diagnosis for P_A

Parameters	D_1	D_2	D_3	Diagnosis result
$p = 1, q = 1$	-0.5989	-0.3118	-0.5898	$D_2 \succ D_3 \succ D_1$
$p = 1, q = 2$	-0.5766	-0.3001	-0.5684	$D_2 \succ D_3 \succ D_1$
$p = 1, q = 3$	-0.5500	-0.2871	-0.5397	$D_2 \succ D_3 \succ D_1$
$p = 2, q = 3$	-0.5520	-0.2893	-0.5498	$D_2 \succ D_3 \succ D_1$
$p = 3, q = 3$	-0.5413	-0.2857	-0.5439	$D_2 \succ D_1 \succ D_3$
$p = 3, q = 4$	-0.5270	-0.2797	-0.5308	$D_2 \succ D_1 \succ D_3$
$p = 3, q = 5$	-0.5128	-0.2728	-0.5152	$D_2 \succ D_1 \succ D_3$
$p = 4, q = 5$	-0.5050	-0.2715	-0.5134	$D_2 \succ D_1 \succ D_3$
$p = 5, q = 5$	-0.4960	-0.2685	-0.5082	$D_2 \succ D_1 \succ D_3$

Table 8 The influence of parameter vector changes of HFFEWBM operator on the depression diagnosis for P_A

Parameters	D_1	D_2	D_3	Diagnosis result
$p = 1, q = 1$	-0.6715	-0.4309	-0.6706	$D_2 \succ D_3 \succ D_1$
$p = 1, q = 2$	-0.6449	-0.4168	-0.6479	$D_2 \succ D_1 \succ D_3$
$p = 1, q = 3$	-0.6171	-0.4000	-0.6194	$D_2 \succ D_1 \succ D_3$
$p = 2, q = 3$	-0.6162	-0.4019	-0.6275	$D_2 \succ D_1 \succ D_3$
$p = 3, q = 3$	-0.6044	-0.3974	-0.6214	$D_2 \succ D_1 \succ D_3$
$p = 3, q = 4$	-0.5910	-0.3898	-0.6092	$D_2 \succ D_1 \succ D_3$
$p = 3, q = 5$	-0.5790	-0.3807	-0.5957	$D_2 \succ D_1 \succ D_3$
$p = 4, q = 5$	-0.5995	-0.3801	-0.5996	$D_2 \succ D_1 \succ D_3$
$p = 5, q = 5$	-0.5923	-0.3771	-0.6207	$D_2 \succ D_1 \succ D_3$

Table 9 The influence of parameter vector changes of HFFWBM operator on the depression diagnosis for P_B

Parameters	D_1	D_2	D_3	Diagnosis result
$p = 1, q = 1$	-0.4139	-0.6806	-0.6498	$D_1 \succ D_3 \succ D_2$
$p = 1, q = 2$	-0.3921	-0.6506	-0.6367	$D_1 \succ D_3 \succ D_2$
$p = 1, q = 3$	-0.3677	-0.6161	-0.6247	$D_1 \succ D_2 \succ D_3$
$p = 2, q = 3$	-0.3694	-0.6188	-0.6217	$D_1 \succ D_2 \succ D_3$
$p = 3, q = 3$	-0.3598	-0.6063	-0.6163	$D_1 \succ D_2 \succ D_3$
$p = 3, q = 4$	-0.3473	-0.5900	-0.6106	$D_1 \succ D_2 \succ D_3$
$p = 3, q = 5$	-0.3346	-0.5738	-0.6052	$D_1 \succ D_2 \succ D_3$
$p = 4, q = 5$	-0.3286	-0.5670	-0.6023	$D_1 \succ D_2 \succ D_3$
$p = 5, q = 5$	-0.3212	-0.5584	-0.5991	$D_1 \succ D_2 \succ D_3$

Table 10 The influence of parameter vector changes of HFFEWBM operator on the depression diagnosis for P_B

Parameters	D_1	D_2	D_3	Diagnosis result
$p = 1, q = 1$	-0.5151	-0.7382	-0.7221	$D1 > D3 > D2$
$p = 1, q = 2$	-0.4861	-0.7103	-0.7093	$D1 > D3 > D2$
$p = 1, q = 3$	-0.4555	-0.6807	-0.6979	$D1 > D2 > D3$
$p = 2, q = 3$	-0.4554	-0.6813	-0.6951	$D1 > D2 > D3$
$p = 3, q = 3$	-0.4449	-0.6702	-0.6901	$D1 > D2 > D3$
$p = 3, q = 4$	-0.4319	-0.6570	-0.6849	$D1 > D2 > D3$
$p = 3, q = 5$	-0.4187	-0.6445	-0.6800	$D1 > D2 > D3$
$p = 4, q = 5$	-0.4146	-0.6402	-0.6774	$D1 > D2 > D3$
$p = 5, q = 5$	-0.4087	-0.6397	-0.6737	$D1 > D2 > D3$

Table 11 The influence of parameter vector changes of HFFWBM operator on the depression diagnosis for P_C

Parameters	D_1	D_2	D_3	Diagnosis result
$p = 1, q = 1$	-0.6241	-0.6119	-0.3305	$D3 > D2 > D1$
$p = 1, q = 2$	-0.5959	-0.5929	-0.2990	$D3 > D2 > D1$
$p = 1, q = 3$	-0.5671	-0.5761	-0.2666	$D3 > D1 > D2$
$p = 2, q = 3$	-0.5650	-0.5717	-0.2645	$D3 > D1 > D2$
$p = 3, q = 3$	-0.5528	-0.5638	-0.2503	$D3 > D1 > D2$
$p = 3, q = 4$	-0.5388	-0.5563	-0.2343	$D3 > D1 > D2$
$p = 3, q = 5$	-0.5254	-0.5499	-0.2192	$D3 > D1 > D2$
$p = 4, q = 5$	-0.5183	-0.5457	-0.2107	$D3 > D1 > D2$
$p = 5, q = 5$	-0.5103	-0.5417	-0.2013	$D3 > D1 > D2$

Table 12 The influence of parameter vector changes of HFFEWBM operator on the depression diagnosis for P_C

Parameters	D_1	D_2	D_3	Diagnosis result
$p = 1, q = 1$	-0.6949	-0.6909	-0.4438	$D3 > D2 > D1$
$p = 1, q = 2$	-0.6666	-0.6732	-0.4071	$D3 > D1 > D2$
$p = 1, q = 3$	-0.6391	-0.6581	-0.3702	$D3 > D1 > D2$
$p = 2, q = 3$	-0.6375	-0.6549	-0.3658	$D3 > D1 > D2$
$p = 3, q = 3$	-0.6271	-0.6485	-0.3504	$D3 > D1 > D2$
$p = 3, q = 4$	-0.6151	-0.6421	-0.3338	$D3 > D1 > D2$
$p = 3, q = 5$	-0.6037	-0.6363	-0.3184	$D3 > D1 > D2$
$p = 4, q = 5$	-0.5985	-0.6332	-0.3103	$D3 > D1 > D2$
$p = 5, q = 5$	-0.5944	-0.6402	-0.3016	$D3 > D1 > D2$

weighted average (FHFWA) operator and the Fermatean hesitant fuzzy weighted geometric (FHFWG) operator. Grag [59] proposed an MADM method that is based on weighted hesitant Pythagorean fuzzy Maclaurin symmetric mean operator (WHPFMSM). Considering HFFS as an extension of HPFS, we compare the proposed method to the approach of [59]. Furthermore, Hadi et al. [60] examined in detail the Fermatean fuzzy Hamacher hybrid average (FFHHA) operator and the Fermatean fuzzy Hamacher hybrid geometric (FFHHG) operator and proposed a MADM approach. Since FFS can be considered a particular case of HFFS, we can compare them analytically with [60] approach. HFF-DMs can be converted into Fermatean fuzzy decision matrices

under certain conditions. Therefore, the maximum, minimum, and mean values of HFFE MDs and NMDs are considered for use as the MD and NMD of Fermatean fuzzy numbers. After this conversion, a comparison analysis is conducted. Hamacher operations [61] as a generalized t-norm degenerate to Einstein t-norm and t-conorm when the parameter is equal to 2. We only investigate the scenario in which the Hamacher parameter equals 2. Finally, the depression diagnostic methods of three patients are compared. The weight vector $\omega = (0.2, 0.15, 0.2, 0.3, 0.15)$ is still utilized. The diagnostic results of different diagnostic methods for the three patients are listed in Tables 13, 14, 15.

Table 13 Results of comparative analysis of different diagnostic methods for P_A

Methods	Operators	Conditions	D_1	D_2	D_3	Diagnosis result
The proposed method	HFFWBM	$p = 1, q = 1$	-0.5989	-0.3117	-0.5898	$D_2 \succ D_3 \succ D_1$
	HFFEWBM	$p = 1, q = 1$	-0.6715	-0.4309	-0.6706	$D_2 \succ D_3 \succ D_1$
The method of Hadi et al. [60]	FFHWA	Maximum	0.0262	0.3010	0.1021	$D_2 \succ D_3 \succ D_1$
		Minimum	0.0261	0.2769	0.0070	$D_2 \succ D_1 \succ D_3$
		Average	0.0232	0.2875	0.0373	$D_2 \succ D_3 \succ D_1$
	FFHHWG	Maximum	-0.0588	0.1871	-0.0827	$D_2 \succ D_1 \succ D_3$
		Minimum	-0.0518	0.1767	-0.0837	$D_2 \succ D_1 \succ D_3$
		Average	-0.0511	0.1826	-0.0822	$D_2 \succ D_1 \succ D_3$
The method of Garg [59]	WHPFMSM	$k = 1$	0.0421	0.3714	0.0318	$D_2 \succ D_1 \succ D_3$
	WHPFMSM	$k = 2$	-0.6779	-0.4197	-0.6768	$D_2 \succ D_3 \succ D_1$
The method of Kirişci [39]	FHFVA		0.0330	0.2692	0.0423	$D_2 \succ D_3 \succ D_1$
	FHFVG		-0.0398	0.1427	-0.1126	$D_2 \succ D_1 \succ D_3$

Table 14 Results of comparative analysis of different diagnostic methods for P_B

Methods	Operators	Conditions	D_1	D_2	D_3	Diagnosis result
The proposed method	HFFWBM	$p = 1, q = 1$	-0.4139	-0.6806	-0.6499	$D_1 \succ D_3 \succ D_2$
	HFFEWBM	$p = 1, q = 1$	-0.5151	-0.7382	-0.7221	$D_1 \succ D_3 \succ D_2$
The method of Hadi et al. [60]	FFHWA	Maximum	0.2556	0.0413	-0.0009	$D_1 \succ D_2 \succ D_3$
		Minimum	0.2309	-0.0335	0.0116	$D_1 \succ D_3 \succ D_2$
		Average	0.2402	-0.0133	0.0041	$D_1 \succ D_3 \succ D_2$
	FFHHWG	Maximum	0.1347	-0.2177	-0.0979	$D_1 \succ D_3 \succ D_2$
		Minimum	0.1127	-0.1930	-0.1121	$D_1 \succ D_3 \succ D_2$
		Average	0.1257	-0.2004	-0.1029	$D_1 \succ D_3 \succ D_2$
The method of Garg [59]	WHPFMSM	$k = 1$	None	None	None	None
	WHPFMSM	$k = 2$	None	None	None	None
The method of Kirişci [39]	FHFVA		0.2205	-0.0058	0.0014	$D_1 \succ D_3 \succ D_2$
	FHFVG		0.1141	-0.1958	-0.1077	$D_1 \succ D_3 \succ D_2$

Comparative analysis experiments demonstrated that the diagnostic results for P_A using the existing decision-making method were the same as the proposed method. Similarly, the diagnostic results for P_B and P_C were identical. Although there was inconsistency in the diagnostic ranking results for the three patients, this did not affect their final diagnoses. P_A had moderate depression, P_B had mild depression, and P_C had major depression. Based on the results of a comparison of three patients, the proposed method is effective and feasible. Next, we discuss the advantages of the proposed method.

Advantage analysis

The proposed method is in the hesitant Fermatean fuzzy environment, which can describe a larger fuzzy information space

and handle uncertainty. In addition, we extended the hesitant Fermatean fuzzy aggregation operation by combining with the Bonferroni mean operator, which has two advantages. First, the decision result is guaranteed less affected by extreme values. Second, the hesitant Fermatean fuzzy BM operator can capture the relationship between two HFFEs.

Describing larger fuzzy information space

HFFS can describe a larger fuzzy information space than DHFS and HPFS. Garg [59] proposes a multi-attribute decision-making approach based on hesitant Pythagorean fuzzy sets. According to Table 13, this approach can diagnose depression for P_A , but not for P_B and P_C shown in Tables 14 and 15. Because, in the depression diagnosis evaluation table of P_B shown in Table 2, the decreased interest and fatigue

Table 15 Results of comparative analysis of different diagnostic methods for P_C

Methods	Operators	Conditions	D_1	D_2	D_3	Diagnosis result
The proposed method	HFFWBM	$p = 1, q = 1$	-0.6241	-0.6119	-0.3305	$D_3 > D_2 > D_1$
	HFFEWM	$p = 1, q = 1$	-0.6949	-0.6909	-0.4438	$D_3 > D_2 > D_1$
The method of Hadi et al. [60]	FFHWA	Maximum	0.0650	0.0600	0.4063	$D_3 > D_1 > D_2$
		Minimum	0.0611	0.0444	0.3895	$D_3 > D_1 > D_2$
		Average	0.0562	0.0503	0.3966	$D_3 > D_1 > D_2$
	FFHHWG	Maximum	-0.1941	-0.1105	0.2591	$D_3 > D_2 > D_1$
		Minimum	-0.1441	-0.1106	0.2350	$D_3 > D_2 > D_1$
		Average	-0.1636	-0.1100	0.2475	$D_3 > D_2 > D_1$
The method of Garg [59]	WHPFMSM	$k = 1$	None	None	None	None
	WHPFMSM	$k = 2$	None	None	None	None
The method of Kirişci [39]	FHFVA		0.0492	0.0542	0.4260	$D_3 > D_2 > D_1$
	FHFVG		-0.1717	-0.1087	0.2790	$D_3 > D_2 > D_1$

(C_3) of moderate depression (D_2) are exceeded the information representation range of HPFS. As shown in Table 3, the decreased interest and fatigue (C_3) for mild depression (D_1) on the depression diagnostic evaluation table of P_B also exceeded the information representation range of the HPFS.

The Hadi et al.'s [60] MADM method is based on FFS. In contrast to HFFS, this method does not account for hesitation situations. For complex diagnostic situations, such as diagnosing depression, the processing of FFS is limited. In the comparative analysis, the MDs and NMDs of HFFE are degenerated into Fermatean fuzzy numbers according to their maximum, minimum, and mean values. This way of dealing with fuzzy numbers loses the hesitant fuzzy information. Therefore, the MADM method based on the HFFS is superior to the MADM method based on the FFS when considering complex hesitation situations.

Reducing the impact of extreme values

The influence of extreme values may result in unreasonable result. However, the hesitant Fermatean fuzzy BM operator can effectively reduce the impact of extreme values. Kirişci's [39] MADM method is based on the FHFVA and FHFVG, which is sensitive to extreme values and then renders the final decision unreasonable. Next, we illustrate with Example 5.

Example 5 As shown in Table 2, we performed a diagnostic evaluation for depression on P_B . Particularly, the diagnostic evaluation of cognitive dysfunction (C_4) with mild depression (D_1) in Table 2 was changed to $\{\{0.001\}, \{0.999\}\}$. The new depression diagnostic for P_B is shown in Table 16, and the final diagnosis results are shown in Table 17.

As seen in Table 17, the diagnosis for P_B by our proposed method is still mild depression. However, the diagnosis for P_B by Kirişci's method may be mild depression or major depression, which is unreasonable.

Flexibility and reasonability

Depression diagnosis is a complex issue. Whether the patient describes the symptoms or the doctor diagnoses the symptoms, there is ambiguity, hesitation, and correlation. The method of [60] requires the diagnosis for a patient by doctor must be a definite Fermatean fuzzy number and ignores the correlation between symptoms. The method of [39] is hesitancy, because it is based on the Fermatean hesitation fuzzy set, but its aggregation operator does not account for the correlation between individual symptoms. Not only does the proposed method consider the correlation between attributes, but can also describe hesitation in a larger fuzzy space.

Comparative experiments verify that the proposed method based on the HFFWBM and HFFEWM operators is reasonable and effective. Further research shows that the proposed method has obvious advantages. First, the propose MADM method based on HFFS, which can describe a larger fuzzy information space. Second, the BM operators of HFFE are given, which allowing the proposed method to consider the correlation between multiple attributes. Finally, due to the advantages of the BM operator, the proposed method can reduce the impact of extreme values. The characteristics of the proposed method and existing MADM methods are shown in Table 18.

Table 16 The new depression diagnostic evaluation for P_B

	C_1	C_2	C_3	C_4	C_5
D_1	{{0.7}, {0.2}}	{{0.8}, {0.4}}	{{0.4}, {0.5, 0.6}}	{{0.001}, {0.999}}	{{0.6, 0.4}, {0.1}}
D_2	{{0.3}, {0.8}}	{{0.5}, {0.2}}	{{0.5, 0.8}, {0.7}}	{{0.5}, {0.5}}	{{0.2}, {0.5, 0.7}}
D_3	{{0.1, 0.3}, {0.4}}	{{0.7}, {0.6}}	{{0.2}, {0.5}}	{{0.6}, {0.6}}	{{0.3}, {0.4, 0.5}}

Table 17 The depression diagnostic results for P_B in Example 5

Method	Operator	D_1	D_2	D_3	Diagnosis result
The proposed method	HFFWBM ($p = 1, q = 1$)	- 0.5851	- 0.6806	- 0.6498	$D_1 > D_3 > D_2$
	HFFEWM ($p = 1, q = 1$)	- 0.6639	- 0.7382	- 0.7221	$D_1 > D_3 > D_2$
The method of Kirişci [39]	FHFWA	0.1415	- 0.0058	0.0014	$D_1 > D_3 > D_2$
	FHFWG	- 0.8328	- 0.1958	- 0.1077	$D_3 > D_2 > D_1$

Table 18 Comparison of characteristics of different methods

	Whether to consider the correlation between aggregated attributes	Whether it is hesitant fuzzy information	Describe a larger fuzzy information space	Low impact on extreme values
The method of Hadi et al. [60]	No	No	Yes	No
The method of Kirişci [39]	No	Yes	Yes	No
The method of Garg [59]	Yes	Yes	No	Yes
The Proposed method	Yes	Yes	Yes	Yes

Conclusion

HFFS is an extension of DHFS and FFS and is more flexible in addressing uncertainty issues. On the one hand, HFFS can represent a larger fuzzy information space than DHFS. On the other hand, different from FFS, HFFS can handle hesitant information. Two types of aggregation operators based on HFFS are developed in this paper. This paper extends the HFFS aggregation operation to the BM operation and proposes the HFFBM and HFFWBM operators. And the HFFEWM and HFFEWM operators are then proposed based on Einstein’s t-norm and t-conorm. The proposed operator has the BM property, which allows it to consider the interrelationships between aggregate elements. In addition, a novel MADM approach based on HFFWBM and HFFEWM is also proposed. Finally, depression diagnosis is a complex issue. Sometimes, it may be difficult for patients to describe symptoms and their relationships accurately. Similarly, doctors may make hesitant symptom diagnoses based on fuzzy patient descriptions. Consequently, a diagnostic scheme based on HFFS is an effective solution. This

paper uses the proposed method for diagnosing depression in three patients as a case study. It analyzes the influence of HFFWBM and HFFEWM parameters on the diagnosis results. The final results demonstrate the effectiveness and rationality of the proposed operator and MADM method. Experiment also shows that the proposed method is better than [39, 59, 60], concerning the representation of fuzzy information space, the influence of extreme values, and flexibility and rationality.

For future work, the proposed method can be extended to q-rung orthopair hesitant fuzzy sets [62] and complex q-rung orthopair hesitant fuzzy sets [63]. The proposed method can also be combined with other aggregation operations, such as MSM [59], power averaging [46], and Choquet Integral operator [64]. Moreover, group decision-making is crucial to decision-making issues. Finally, in reality, there are often some uncertain decision-making problems that are ambiguous and hesitant. These decision-making problems include not only the diagnosis of depression but also anxiety disorders, bipolar disorder, and other mental illness problems. Consequently, new decision-making schemes based on more

extended hesitant fuzzy sets can be used to research such issues. The proposed method is hoped to provide a viable solution for these decision-making problems.

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Data availability Enquiries about data availability should be directed to the authors.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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Appendix

The Proof of Theorem 3.1

Proof (1) From operation (5) in Definition 2.7, we have

$$\zeta_1 \oplus \zeta_2 = \bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \left\{ \sqrt[3]{\theta_1^3 + \theta_2^3 - \theta_1^3 \theta_2^3} \right\}, \{\vartheta_1 \vartheta_2\} \right\}$$

$$\zeta_2 \oplus \zeta_1 = \bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \left\{ \sqrt[3]{\theta_2^3 + \theta_1^3 - \theta_2^3 \theta_1^3} \right\}, \{\vartheta_2 \vartheta_1\} \right\}.$$

Obviously, for any $\theta_1 \in \mu_1, \theta_2 \in \mu_2$ and $\vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2$,

$$\sqrt[3]{\theta_1^3 + \theta_2^3 - \theta_1^3 \theta_2^3} = \sqrt[3]{\theta_2^3 + \theta_1^3 - \theta_2^3 \theta_1^3}$$

$$\vartheta_1 \vartheta_2 = \vartheta_2 \vartheta_1.$$

Then, $\zeta_1 \oplus \zeta_2 = \zeta_2 \oplus \zeta_1$.

(2) From operation (4) in Definition 2.7, we have

$$\zeta_1 \otimes \zeta_2 = \bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \{\theta_1 \theta_2\}, \left\{ \sqrt[3]{\vartheta_1^3 + \vartheta_2^3 - \vartheta_1^3 \vartheta_2^3} \right\} \right\}$$

$$\zeta_2 \otimes \zeta_1 = \bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \{\theta_2 \theta_1\}, \left\{ \sqrt[3]{\vartheta_2^3 + \vartheta_1^3 - \vartheta_2^3 \vartheta_1^3} \right\} \right\}.$$

Obviously, for any $\theta_1 \in \mu_1, \theta_2 \in \mu_2$ and $\vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2$
 $\theta_1 \theta_2 = \theta_2 \theta_1$

$$\sqrt[3]{\vartheta_1^3 + \vartheta_2^3 - \vartheta_1^3 \vartheta_2^3} = \sqrt[3]{\vartheta_2^3 + \vartheta_1^3 - \vartheta_2^3 \vartheta_1^3}.$$

Then, $\zeta_1 \otimes \zeta_2 = \zeta_2 \otimes \zeta_1$.

(3) According to the operations (3) and (5) of Definition 2.7

$$\begin{aligned} \gamma(\zeta_1 \oplus \zeta_2) &= \gamma \left(\bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \left\{ \sqrt[3]{\theta_1^3 + \theta_2^3 - \theta_1^3 \theta_2^3} \right\}, \{\vartheta_1 \vartheta_2\} \right\} \right) \\ &= \bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \left\{ \sqrt[3]{1 - (1 - (\theta_1^3 + \theta_2^3 - \theta_1^3 \theta_2^3))^\gamma} \right\}, \{(\vartheta_1 \vartheta_2)^\gamma\} \right\} \\ &= \bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \left\{ \sqrt[3]{1 - (1 - \theta_1^3 - \theta_2^3 + \theta_1^3 \theta_2^3)^\gamma} \right\}, \{(\vartheta_1 \vartheta_2)^\gamma\} \right\} \\ &= \bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \left\{ \sqrt[3]{1 - ((1 - \theta_1^3)(1 - \theta_2^3))^\gamma} \right\}, \{(\vartheta_1 \vartheta_2)^\gamma\} \right\}. \end{aligned}$$

Likewise according to operations (3) and (5) of Definition 3.3

$$\begin{aligned} \gamma \zeta_1 \oplus \gamma \zeta_2 &= \bigcup_{\theta_1 \in \mu_1, \vartheta_1 \in \nu_1} \left\{ \left\{ \sqrt[3]{1 - (1 - \theta_1^3)^\gamma}, \{\vartheta_1^\gamma\} \right\} \oplus \bigcup_{\theta_2 \in \mu_2, \vartheta_2 \in \nu_2} \left\{ \left\{ \sqrt[3]{1 - (1 - \theta_2^3)^\gamma}, \{\vartheta_2^\gamma\} \right\} \right. \right. \\ &= \bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \left\{ \sqrt[3]{(1 - (1 - \theta_1^3)^\gamma) + (1 - (1 - \theta_2^3)^\gamma)}, \{\vartheta_1^\gamma \vartheta_2^\gamma\} \right\} \right. \\ &= \bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \left\{ \sqrt[3]{1 - (1 - \theta_1^3)^\gamma (1 - \theta_2^3)^\gamma}, \{\vartheta_1^\gamma \vartheta_2^\gamma\} \right\} \right\}. \end{aligned}$$

Then, we can obtain $\gamma(\zeta_1 \oplus \zeta_2) = \gamma \zeta_1 \oplus \gamma \zeta_2$.

(4) According to the operations (2) and (4) of Definition 2.7

$$\begin{aligned} (\zeta_1 \otimes \zeta_2)^\gamma &= \left(\bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \{\theta_1 \theta_2\}, \left\{ \sqrt[3]{\vartheta_1^3 + \vartheta_2^3 - \vartheta_1^3 \vartheta_2^3} \right\} \right\} \right)^\gamma \\ &= \bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \{(\theta_1 \theta_2)^\gamma\}, \left\{ \sqrt[3]{1 - (1 - (\vartheta_1^3 + \vartheta_2^3 - \vartheta_1^3 \vartheta_2^3))^\gamma} \right\} \right\} \\ &= \bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \{(\theta_1 \theta_2)^\gamma\}, \left\{ \sqrt[3]{1 - ((1 - \vartheta_1^3)(1 - \vartheta_2^3))^\gamma} \right\} \right\}. \end{aligned}$$

Likewise according to operations (2) and (4) of Definition 2.7

$$\begin{aligned} \zeta_1^\gamma \otimes \zeta_2^\gamma &= \bigcup_{\theta_1 \in \mu_1, \vartheta_1 \in \nu_1} \left\{ \{\theta_1^\gamma\}, \left\{ \sqrt[3]{1 - (1 - \vartheta_1^3)^\gamma} \right\} \right\} \otimes \bigcup_{\theta_2 \in \mu_2, \vartheta_2 \in \nu_2} \left\{ \{\theta_2^\gamma\}, \left\{ \sqrt[3]{1 - (1 - \vartheta_2^3)^\gamma} \right\} \right\} \\ &= \bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \{\theta_1^\gamma \theta_2^\gamma\}, \left\{ \sqrt[3]{(1 - (1 - \vartheta_1^3)^\gamma) + (1 - (1 - \vartheta_2^3)^\gamma)} \right\} \right\} \\ &= \bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \{\theta_1^\gamma \theta_2^\gamma\}, \left\{ \sqrt[3]{1 - (1 - \vartheta_1^3)^\gamma (1 - \vartheta_2^3)^\gamma} \right\} \right\}. \end{aligned}$$

Then, we can obtain $(\zeta_1 \otimes \zeta_2)^\gamma = \zeta_1^\gamma \otimes \zeta_2^\gamma$. The proof of Theorem 3.1 is completed.

The Proof of Lemma 3.1

Proof From the operation (2) in Definition 2.7, we have

$$\zeta_i^p = \bigcup_{\theta_i \in \mu_i, \vartheta_i \in \nu_i} \left\{ \left\{ \theta_i^p \right\}, \left\{ \sqrt[3]{1 - (1 - \vartheta_i^3)^p} \right\} \right\}$$

$$\zeta_j^q = \bigcup_{\theta_j \in \mu_j, \vartheta_j \in \nu_j} \left\{ \left\{ \theta_j^q \right\}, \left\{ \sqrt[3]{1 - (1 - \vartheta_j^3)^q} \right\} \right\};$$

then from the operation (4) in Definition 2.7, we have

$$\zeta_i^p \otimes \zeta_j^q = \bigcup_{\theta_i \in \mu_i, \theta_j \in \mu_j, \vartheta_i \in \nu_i, \vartheta_j \in \nu_j} \left\{ \left\{ \theta_i^p \theta_j^q \right\}, \left\{ \sqrt[3]{1 - (1 - \vartheta_i^3)^p (1 - \vartheta_j^3)^q} \right\} \right\}.$$

Similarly, we can obtain $\zeta_i^q \otimes \zeta_j^p$. Then, from the operation (5) from Definition 2.7, we have

$$\left(\zeta_i^p \otimes \zeta_j^q \right) \oplus \left(\zeta_i^q \otimes \zeta_j^p \right) = \bigcup_{\theta_i \in \mu_i, \theta_j \in \mu_j, \vartheta_i \in \nu_i, \vartheta_j \in \nu_j} \left\{ \left\{ \sqrt[3]{1 - (1 - A_i^p A_j^q)(1 - A_i^q A_j^p)} \right\}, \left\{ \sqrt[3]{(1 - B_i^p B_j^q)(1 - B_i^q B_j^p)} \right\} \right\}.$$

The proof of Lemma 3.1 is completed.

The Proof of Theorem 3.2

Proof From the operation law (3) of Theorem 3.1, we have

$$\frac{1}{n(n-1)} \left(\bigoplus_{i,j=1; i \neq j}^n \delta_{i,j} \right) = \bigoplus_{i,j=1; i \neq j}^n \left(\frac{1}{n(n-1)} \delta_{i,j} \right).$$

From Lemma 3.1, we can obtain

$$\begin{aligned} \frac{1}{n(n-1)} \delta_{i,j} &= \frac{1}{n(n-1)} \left(\zeta_i^p \otimes \zeta_j^q \right) \oplus \left(\zeta_i^q \otimes \zeta_j^p \right) \\ &= \frac{1}{n(n-1)} \left(\bigcup_{\theta_i \in \mu_i, \theta_j \in \mu_j, \vartheta_i \in \nu_i, \vartheta_j \in \nu_j} \left\{ \left\{ \sqrt[3]{1 - (1 - A_i^p A_j^q)(1 - A_i^q A_j^p)} \right\}, \left\{ \sqrt[3]{(1 - B_i^p B_j^q)(1 - B_i^q B_j^p)} \right\} \right\} \right) \\ &= \bigcup_{\theta_i \in \mu_i, \theta_j \in \mu_j, \vartheta_i \in \nu_i, \vartheta_j \in \nu_j} \left\{ \left\{ \sqrt[3]{1 - \left((1 - A_i^p A_j^q)(1 - A_i^q A_j^p) \right)^{\frac{1}{n(n-1)}}} \right\}, \left\{ \sqrt[3]{\left((1 - B_i^p B_j^q)(1 - B_i^q B_j^p) \right)^{\frac{1}{n(n-1)}}} \right\} \right\}. \end{aligned}$$

$$\left(\zeta_i^p \otimes \zeta_j^q \right) \oplus \left(\zeta_i^q \otimes \zeta_j^p \right) = \bigcup_{\theta_i \in \mu_i, \theta_j \in \mu_j, \vartheta_i \in \nu_i, \vartheta_j \in \nu_j} \left\{ \left\{ \sqrt[3]{1 - (1 - (\theta_i^3)^p (\theta_j^3)^q)(1 - (\theta_i^3)^q (\theta_j^3)^p)} \right\}, \left\{ \sqrt[3]{(1 - (1 - \vartheta_i^3)^p (1 - \vartheta_j^3)^q)(1 - (1 - \vartheta_i^3)^q (1 - \vartheta_j^3)^p)} \right\} \right\}.$$

Let

$$A_i = \theta_i^3, A_j = \theta_j^3, B_i = 1 - \vartheta_i^3, B_j = 1 - \vartheta_j^3;$$

then we have

where

$$A_i = \theta_i^3, A_j = \theta_j^3, B_i = 1 - \vartheta_i^3, B_j = 1 - \vartheta_j^3.$$

Then, from the operation (5) in Definition 2.7, we have

$$\bigoplus_{i,j=1; i \neq j}^n \left(\frac{1}{n(n-1)} \delta_{i,j} \right) = \bigoplus_{i,j=1; i \neq j}^n \left(\bigcup_{\theta_i \in \mu_i, \theta_j \in \mu_j, \vartheta_i \in \nu_i, \vartheta_j \in \nu_j} \left\{ \left\{ \sqrt[3]{1 - \left((1 - A_i^p A_j^q)(1 - A_i^q A_j^p) \right)^{\frac{1}{n(n-1)}}} \right\}, \left\{ \sqrt[3]{\left((1 - B_i^p B_j^q)(1 - B_i^q B_j^p) \right)^{\frac{1}{n(n-1)}}} \right\} \right\} \right)$$

Let $C_{i,j} = \left((1 - A_i^p A_j^q)(1 - A_i^q A_j^p) \right)^{\frac{1}{n(n-1)}}$, $D_{i,j} = \left((1 - B_i^p B_j^q)(1 - B_i^q B_j^p) \right)^{\frac{1}{n(n-1)}}$. Then

$$\begin{aligned} \bigoplus_{i,j=1;i \neq j}^n \left(\frac{1}{n(n-1)} \delta_{i,j} \right) &= \bigoplus_{i,j=1;i \neq j}^n \left(\bigcup_{\theta_i \in \mu_i, \theta_j \in \mu_j, \vartheta_i \in \nu_i, \vartheta_j \in \nu_j} \left\{ \left\{ \sqrt[3]{1 - C_{i,j}} \right\}, \left\{ \sqrt[3]{D_{i,j}} \right\} \right\} \right) \\ &= \bigcup_{\theta_i \in \mu_i, \theta_j \in \mu_j, \vartheta_i \in \nu_i, \vartheta_j \in \nu_j} \left\{ \left\{ \sqrt[3]{1 - \prod_{i,j=1;i \neq j}^n C_{i,j}} \right\}, \left\{ \sqrt[3]{\prod_{i,j=1;i \neq j}^n D_{i,j}} \right\} \right\}. \end{aligned}$$

Finally, according to the operation (2) in Definition 2.7, we can obtain the following.

The proof of Theorem 3.2 is completed

$$\begin{aligned} \left(\bigoplus_{i,j=1;i \neq j}^n \left(\frac{1}{n(n-1)} \delta_{i,j} \right) \right)^{\frac{1}{p+q}} &= \bigcup_{\theta_i \in \mu_i, \theta_j \in \mu_j, \vartheta_i \in \nu_i, \vartheta_j \in \nu_j} \left\{ \left\{ \sqrt[3]{\left(1 - \prod_{i,j=1;i \neq j}^n C_{i,j} \right)^{\frac{1}{p+q}}}, \right. \right. \\ &\quad \left. \left. \sqrt[3]{1 - \left(1 - \prod_{i,j=1;i \neq j}^n D_{i,j} \right)^{\frac{1}{p+q}}} \right\} \right\} \\ &= \bigcup_{\theta_i \in \mu_i, \theta_j \in \mu_j, \vartheta_i \in \nu_i, \vartheta_j \in \nu_j} \left\{ \left\{ \sqrt[3]{\left(1 - \prod_{i,j=1;i \neq j}^n \left((1 - A_i^p A_j^q) (1 - A_i^q A_j^p) \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}}}, \right. \right. \\ &\quad \left. \left. \sqrt[3]{1 - \left(1 - \prod_{i,j=1;i \neq j}^n \left((1 - B_i^p B_j^q) (1 - B_i^q B_j^p) \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}}} \right\} \right\}. \end{aligned}$$

The Proof of Lemma 3.2

Proof From operations (2) and (3) in Definition 2.7, we have

$$\begin{aligned} (\omega_i \zeta_i)^p &= \bigcup_{\theta_i \in \mu_i, \vartheta_i \in \nu_i} \left\{ \left\{ \sqrt[3]{(1 - (1 - \theta_i^3) \omega_i)^p}, \left\{ \sqrt[3]{1 - (1 - (\vartheta_i^3) \omega_i)^p} \right\} \right\} \right\} \\ (\omega_j \zeta_j)^q &= \bigcup_{\theta_j \in \mu_j, \vartheta_j \in \nu_j} \left\{ \left\{ \sqrt[3]{(1 - (1 - \theta_j^3) \omega_j)^q}, \left\{ \sqrt[3]{1 - (1 - (\vartheta_j^3) \omega_j)^q} \right\} \right\} \right\}. \end{aligned}$$

From the operation (4) in Definition 2.7, we have

$$(\omega_i \zeta_i)^p \otimes (\omega_j \zeta_j)^q = \bigcup_{\theta_i \in \mu_i, \theta_j \in \mu_j, \vartheta_i \in \nu_i, \vartheta_j \in \nu_j} \left\{ \left\{ \sqrt[3]{(1 - (1 - \theta_i^3)^{\omega_i})^p (1 - (1 - \theta_j^3)^{\omega_j})^q} \right\}, \left\{ \sqrt[3]{1 - (1 - (\vartheta_i^3)^{\omega_i})^p (1 - (\vartheta_j^3)^{\omega_j})^q} \right\} \right\}.$$

Let

$$A_i = 1 - (1 - \theta_i^3)^{\omega_i}, \quad A_j = 1 - (1 - \theta_j^3)^{\omega_j}, \\ B_i = 1 - (\vartheta_i^3)^{\omega_i}, \quad B_j = 1 - (\vartheta_j^3)^{\omega_j}.$$

Obviously, for any $\theta_1 \in \mu_1, \theta_2 \in \mu_2$ and $\vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2$

Then, we have

$$(\omega_i \zeta_i)^p \otimes (\omega_j \zeta_j)^q = \bigcup_{\theta_i \in \mu_i, \theta_j \in \mu_j, \vartheta_i \in \nu_i, \vartheta_j \in \nu_j} \left\{ \left\{ \sqrt[3]{A_i^p A_j^q} \right\}, \left\{ \sqrt[3]{1 - B_i^p B_j^q} \right\} \right\}.$$

$$\sqrt[3]{\frac{\theta_1^3 + \theta_2^3}{1 + \theta_1^3 \theta_2^3}} = \sqrt[3]{\frac{\theta_2^3 + \theta_1^3}{1 + \theta_2^3 \theta_1^3}} \\ \frac{\vartheta_1 \vartheta_2}{\sqrt[3]{1 + (1 - \vartheta_1^3)(1 - \vartheta_2^3)}} = \frac{\vartheta_2 \vartheta_1}{\sqrt[3]{1 + (1 - \vartheta_2^3)(1 - \vartheta_1^3)}}.$$

Similarly, we can obtain $(\omega_i \zeta_i)^q \otimes (\omega_j \zeta_j)^p$. Then, from the operation (5) and Definition 2.7, we have

$$((\omega_i \zeta_i)^p \otimes (\omega_j \zeta_j)^q) \oplus ((\omega_i \zeta_i)^q \otimes (\omega_j \zeta_j)^p) = \bigcup_{\theta_i \in \mu_i, \theta_j \in \mu_j, \vartheta_i \in \nu_i, \vartheta_j \in \nu_j} \left\{ \left\{ \sqrt[3]{1 - (1 - A_i^p A_j^q)(1 - A_i^q A_j^p)} \right\}, \left\{ \sqrt[3]{(1 - B_i^p B_j^q)(1 - B_i^q B_j^p)} \right\} \right\}.$$

The proof of Lemma 3.2 is completed.

Then, $\zeta_1 \oplus \zeta_2 = \zeta_2 \oplus \zeta_1$.

(2) From the operation (3) in Definition 3.3, we have

The Proof of Theorem 3.4

Proof (1) From the operation (4) in Definition 3.3, we have

$$\zeta_1 \oplus \zeta_2 = \bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \left\{ \sqrt[3]{\frac{\theta_1^3 + \theta_2^3}{1 + \theta_1^3 \theta_2^3}} \right\}, \left\{ \frac{\vartheta_1 \vartheta_2}{\sqrt[3]{1 + (1 - \vartheta_1^3)(1 - \vartheta_2^3)}} \right\} \right\} \\ \zeta_2 \oplus \zeta_1 = \bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \left\{ \sqrt[3]{\frac{\theta_2^3 + \theta_1^3}{1 + \theta_2^3 \theta_1^3}} \right\}, \left\{ \frac{\vartheta_2 \vartheta_1}{\sqrt[3]{1 + (1 - \vartheta_2^3)(1 - \vartheta_1^3)}} \right\} \right\}.$$

$$\zeta_1 \otimes \zeta_2 = \bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \left\{ \frac{\theta_1 \theta_2}{\sqrt[3]{1 + (1 - \theta_1^3)(1 - \theta_2^3)}} \right\}, \left\{ \sqrt[3]{\frac{\vartheta_1^3 + \vartheta_2^3}{1 + \vartheta_1^3 \vartheta_2^3}} \right\} \right\}$$

$$\zeta_2 \otimes \zeta_1 = \bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \left\{ \frac{\theta_2 \theta_1}{\sqrt[3]{1 + (1 - \theta_2^3)(1 - \theta_1^3)}} \right\}, \left\{ \sqrt[3]{\frac{\vartheta_2^3 + \vartheta_1^3}{1 + \vartheta_2^3 \vartheta_1^3}} \right\} \right\}.$$

Obviously, for any $\theta_1 \in \mu_1$, $\theta_2 \in \mu_2$ and $\vartheta_1 \in \nu_1$, $\vartheta_2 \in \nu_2$

$$\frac{\theta_1 \theta_2}{\sqrt[3]{1 + (1 - \theta_1^3)(1 - \theta_2^3)}} = \frac{\theta_2 \theta_1}{\sqrt[3]{1 + (1 - \theta_2^3)(1 - \theta_1^3)}} \quad \sqrt[3]{\frac{\vartheta_1^3 + \vartheta_2^3}{1 + \vartheta_1^3 \vartheta_2^3}} = \sqrt[3]{\frac{\vartheta_2^3 + \vartheta_1^3}{1 + \vartheta_2^3 \vartheta_1^3}}.$$

Then, $\zeta_1 \otimes \zeta_2 = \zeta_2 \otimes \zeta_1$.

(3) According to the operations (2) and (4) of Definition

3.3

$$\gamma(\zeta_1 \oplus \zeta_2) = \gamma \left(\bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \left\{ \sqrt[3]{\frac{\theta_1^3 + \theta_2^3}{1 + \theta_1^3 \theta_2^3}} \right\}, \left\{ \frac{\vartheta_1 \vartheta_2}{\sqrt[3]{1 + (1 - \vartheta_1^3)(1 - \vartheta_2^3)}} \right\} \right\} \right)$$

$$= \bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \left[\sqrt[3]{\frac{\left(1 + \frac{\theta_1^3 + \theta_2^3}{1 + \theta_1^3 \theta_2^3}\right)^\gamma - \left(1 - \frac{\theta_1^3 + \theta_2^3}{1 + \theta_1^3 \theta_2^3}\right)^\gamma}{\left(1 + \frac{\theta_1^3 + \theta_2^3}{1 + \theta_1^3 \theta_2^3}\right)^\gamma + \left(1 - \frac{\theta_1^3 + \theta_2^3}{1 + \theta_1^3 \theta_2^3}\right)^\gamma}} \right], \left[\sqrt[3]{\frac{2 \left(\frac{\vartheta_1^3 \vartheta_2^3}{1 + (1 - \vartheta_1^3)(1 - \vartheta_2^3)}\right)^\gamma}{\left(2 - \frac{\vartheta_1^3 \vartheta_2^3}{1 + (1 - \vartheta_1^3)(1 - \vartheta_2^3)}\right)^\gamma + \left(\frac{\vartheta_1^3 \vartheta_2^3}{1 + (1 - \vartheta_1^3)(1 - \vartheta_2^3)}\right)^\gamma}} \right] \right\}$$

$$\bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \left[\sqrt[3]{\frac{\left(1 + \theta_1^3 + \theta_2^3 + \theta_1^3 \theta_2^3\right)^\gamma - \left(1 - \theta_1^3 - \theta_2^3 + \theta_1^3 \theta_2^3\right)^\gamma}{\left(1 + \theta_1^3 + \theta_2^3 + \theta_1^3 \theta_2^3\right)^\gamma + \left(1 - \theta_1^3 - \theta_2^3 + \theta_1^3 \theta_2^3\right)^\gamma}} \right], \left[\sqrt[3]{\frac{2(\vartheta_1^3 \vartheta_2^3)^\gamma}{(2^2 - 2\vartheta_1^3 - 2\vartheta_2^3 + \vartheta_1^3 \vartheta_2^3)^\gamma + (\vartheta_1^3 \vartheta_2^3)^\gamma}} \right] \right\}$$

$$\bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \left[\sqrt[3]{\frac{\left(1 + \theta_1^3\right)^\gamma \left(1 + \theta_2^3\right)^\gamma - \left(1 - \theta_1^3\right)^\gamma \left(1 - \theta_2^3\right)^\gamma}{\left(1 + \theta_1^3\right)^\gamma \left(1 + \theta_2^3\right)^\gamma + \left(1 - \theta_1^3\right)^\gamma \left(1 - \theta_2^3\right)^\gamma}} \right], \left[\sqrt[3]{\frac{2(\vartheta_1^3 \vartheta_2^3)^\gamma}{(2 - \vartheta_1^3)^\gamma (2 - \vartheta_2^3)^\gamma + (\vartheta_1^3 \vartheta_2^3)^\gamma}} \right] \right\}.$$

Let

$$A_1 = (1 + \theta_1^3)^\gamma, A_2 = (1 + \theta_2^3)^\gamma, B_1 = (1 - \theta_1^3)^\gamma, B_2 = (1 - \theta_2^3)^\gamma, \\ M_1 = (\vartheta_1^3)^\gamma, M_2 = (\vartheta_2^3)^\gamma, N_1 = (2 - \vartheta_1^3)^\gamma, N_2 = (2 - \vartheta_2^3)^\gamma.$$

Then, we have

$$\gamma(\zeta_1 \oplus \zeta_2) = \bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \left\{ \sqrt[3]{\frac{A_1 A_2 - B_1 B_2}{A_1 A_2 + B_1 B_2}} \right\}, \left\{ \sqrt[3]{\frac{2M_1 M_2}{N_1 N_2 + M_1 M_2}} \right\} \right\}.$$

Likewise, according to operations (2) and (4) of Definition 3.3

$$\begin{aligned} \gamma \zeta_1 \oplus \gamma \zeta_2 &= \bigcup_{\theta_1 \in \mu_1, \vartheta_1 \in \nu_1} \left\{ \left\{ \sqrt[3]{\frac{(1 + \theta_1^3)^\gamma - (1 - \theta_1^3)^\gamma}{(1 + \theta_1^3)^\gamma + (1 - \theta_1^3)^\gamma}} \right\}, \left\{ \sqrt[3]{\frac{2(\vartheta_1^3)^\gamma}{(2 - \vartheta_1^3)^\gamma + (\vartheta_1^3)^\gamma}} \right\} \right\} \\ &\oplus \bigcup_{\theta_2 \in \mu_2, \vartheta_2 \in \nu_2} \left\{ \left\{ \sqrt[3]{\frac{(1 + \theta_2^3)^\gamma - (1 - \theta_2^3)^\gamma}{(1 + \theta_2^3)^\gamma + (1 - \theta_2^3)^\gamma}} \right\}, \left\{ \sqrt[3]{\frac{2(\vartheta_2^3)^\gamma}{(2 - \vartheta_2^3)^\gamma + (\vartheta_2^3)^\gamma}} \right\} \right\} \\ &= \bigcup_{\theta_1 \in \mu_1, \vartheta_1 \in \nu_1} \left\{ \left\{ \sqrt[3]{\frac{A_1 - B_1}{A_1 + B_1}} \right\}, \left\{ \sqrt[3]{\frac{2M_1}{N_1 + M_1}} \right\} \right\} \oplus \bigcup_{\theta_2 \in \mu_2, \vartheta_2 \in \nu_2} \left\{ \left\{ \sqrt[3]{\frac{A_2 - B_2}{A_2 + B_2}} \right\}, \left\{ \sqrt[3]{\frac{2M_2}{N_2 + M_2}} \right\} \right\} \\ &= \bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \left\{ \sqrt[3]{\frac{\frac{A_1 - B_1}{A_1 + B_1} + \frac{A_2 - B_2}{A_2 + B_2}}{1 + \frac{A_1 - B_1}{A_1 + B_1} \cdot \frac{A_2 - B_2}{A_2 + B_2}}} \right\}, \left\{ \sqrt[3]{\frac{\frac{2M_1}{N_1 + M_1} \cdot \frac{2M_2}{N_2 + M_2}}{1 + \frac{N_1 - M_1}{N_1 + M_1} \cdot \frac{N_2 - M_2}{N_2 + M_2}}} \right\} \right\} \\ &= \bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \left\{ \sqrt[3]{\frac{(A_1 - B_1)(A_2 + B_2) + (A_1 + B_1)(A_2 - B_2)}{(A_1 + B_1)(A_2 + B_2) - (A_1 - B_1)(A_2 - B_2)}} \right\}, \left\{ \sqrt[3]{\frac{2^2 M_1 M_2}{(N_1 + M_1)(N_2 + M_2) + (N_1 - M_1)(N_2 - M_2)}} \right\} \right\} \\ &= \bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \left\{ \sqrt[3]{\frac{A_1 A_2 - B_1 B_2}{A_1 A_2 + B_1 B_2}} \right\}, \left\{ \sqrt[3]{\frac{2M_1 M_2}{N_1 N_2 + M_1 M_2}} \right\} \right\}. \end{aligned}$$

Then, we can obtain $\gamma(\zeta_1 \oplus \zeta_2) = \gamma \zeta_1 \oplus \gamma \zeta_2$.

(4) From the operations (1) and (3) in Definition 3.3

$$\begin{aligned}
(\zeta_1 \otimes \zeta_2)^\gamma &= \left(\bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \left\{ \frac{\theta_1 \theta_2}{\sqrt[3]{1 + (1 - \theta_1^3)(1 - \theta_2^3)}} \right\}, \left\{ \sqrt[3]{\frac{\vartheta_1^3 + \vartheta_2^3}{1 + \vartheta_1^3 \vartheta_2^3}} \right\} \right\} \right)^\gamma \\
&= \bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \left[\begin{array}{c} \sqrt[3]{\frac{2 \left(\frac{\theta_1^3 \theta_2^3}{1 + (1 - \theta_1^3)(1 - \theta_2^3)} \right)^\gamma}{\left(2 - \frac{\theta_1^3 \theta_2^3}{1 + (1 - \theta_1^3)(1 - \theta_2^3)} \right)^\gamma} + \left(\frac{\theta_1^3 \theta_2^3}{1 + (1 - \theta_1^3)(1 - \theta_2^3)} \right)^\gamma}} \right]} \right\} \\
&= \bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \left[\begin{array}{c} \sqrt[3]{\frac{\left(1 + \frac{\vartheta_1^3 + \vartheta_2^3}{1 + \vartheta_1^3 \vartheta_2^3} \right)^\gamma - \left(1 - \frac{\vartheta_1^3 + \vartheta_2^3}{1 + \vartheta_1^3 \vartheta_2^3} \right)^\gamma}{\left(1 + \frac{\vartheta_1^3 + \vartheta_2^3}{1 + \vartheta_1^3 \vartheta_2^3} \right)^\gamma + \left(1 - \frac{\vartheta_1^3 + \vartheta_2^3}{1 + \vartheta_1^3 \vartheta_2^3} \right)^\gamma}} \right]} \right\} \\
&= \bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \left[\begin{array}{c} \sqrt[3]{\frac{2(\theta_1^3 \theta_2^3)^\gamma}{(2^2 - 2\theta_1^3 - 2\theta_2^3 + \theta_1^3 \theta_2^3)^\gamma + (\theta_1^3 \theta_2^3)^\gamma}} \\ \sqrt[3]{\frac{\left(1 + \vartheta_1^3 + \vartheta_2^3 + \vartheta_1^3 \vartheta_2^3 \right)^\gamma - \left(1 - \vartheta_1^3 - \vartheta_2^3 + \vartheta_1^3 \vartheta_2^3 \right)^\gamma}{\left(1 + \vartheta_1^3 + \vartheta_2^3 + \vartheta_1^3 \vartheta_2^3 \right)^\gamma + \left(1 - \vartheta_1^3 - \vartheta_2^3 + \vartheta_1^3 \vartheta_2^3 \right)^\gamma}} \right]} \right\} \\
&= \bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \left[\begin{array}{c} \sqrt[3]{\frac{2(\theta_1^3 \theta_2^3)^\gamma}{(2 - \theta_1^3)^\gamma (2 - \theta_2^3)^\gamma + (\theta_1^3 \theta_2^3)^\gamma}} \\ \sqrt[3]{\frac{\left(1 + \vartheta_1^3 \right)^\gamma \left(1 + \vartheta_2^3 \right)^\gamma - \left(1 - \vartheta_1^3 \right)^\gamma \left(1 - \vartheta_2^3 \right)^\gamma}{\left(1 + \vartheta_1^3 \right)^\gamma \left(1 + \vartheta_2^3 \right)^\gamma + \left(1 - \vartheta_1^3 \right)^\gamma \left(1 - \vartheta_2^3 \right)^\gamma}} \right]} \right\}.
\end{aligned}
\right.$$

Let

$$\begin{aligned}
A_1 &= (\theta_1^3)^\gamma, \quad A_2 = (\theta_2^3)^\gamma, \quad B_1 = (2 - \theta_1^3)^\gamma, \quad B_2 = (2 - \theta_2^3)^\gamma, \\
M_1 &= (1 + \vartheta_1^3)^\gamma, \quad M_2 = (1 + \vartheta_2^3)^\gamma, \quad N_1 = (1 - \vartheta_1^3)^\gamma, \quad N_2 = (1 - \vartheta_2^3)^\gamma.
\end{aligned}$$

Then, we have

$$\begin{aligned}
(\zeta_1 \otimes \zeta_2)^\gamma &= \bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \left\{ \sqrt[3]{\frac{2A_1 A_2}{B_1 B_2 + A_1 A_2}} \right\}, \left\{ \sqrt[3]{\frac{M_1 M_2 - N_1 N_2}{M_1 M_2 + N_1 N_2}} \right\} \right\}.
\end{aligned}$$

Likewise, according to operations (1) and (3) in Definition 3.3

$$\begin{aligned}
 \zeta_1^\gamma \otimes \zeta_2^\gamma &= \bigcup_{\theta_1 \in \mu_1, \vartheta_1 \in \nu_1} \left\{ \left\{ \sqrt[3]{\frac{2(\theta_1^3)^\gamma}{(2 - \theta_1^3)^\gamma + (\theta_1^3)^\gamma}} \right\}, \left\{ \sqrt[3]{\frac{(1 + \vartheta_1^3)^\gamma - (1 - \vartheta_1^3)^\gamma}{(1 + \vartheta_1^3)^\gamma + (1 - \vartheta_1^3)^\gamma}} \right\} \right\} \\
 &\otimes \bigcup_{\theta_2 \in \mu_2, \vartheta_2 \in \nu_2} \left\{ \left\{ \sqrt[3]{\frac{2(\theta_2^3)^\gamma}{(2 - \theta_2^3)^\gamma + (\theta_2^3)^\gamma}} \right\}, \left\{ \sqrt[3]{\frac{(1 + \vartheta_2^3)^\gamma - (1 - \vartheta_2^3)^\gamma}{(1 + \vartheta_2^3)^\gamma + (1 - \vartheta_2^3)^\gamma}} \right\} \right\} \\
 &= \bigcup_{\theta_1 \in \mu_1, \vartheta_1 \in \nu_1} \left\{ \left\{ \sqrt[3]{\frac{2A_1}{B_1 + A_1}} \right\}, \left\{ \sqrt[3]{\frac{M_1 - N_1}{M_1 + N_1}} \right\} \right\} \otimes \bigcup_{\theta_2 \in \mu_2, \vartheta_2 \in \nu_2} \left\{ \left\{ \sqrt[3]{\frac{2A_2}{B_2 + A_2}} \right\}, \left\{ \sqrt[3]{\frac{M_2 - N_2}{M_2 + N_2}} \right\} \right\} \\
 &= \bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \left\{ \sqrt[3]{\frac{\frac{2A_1}{B_1 + A_1} \cdot \frac{2A_2}{B_2 + A_2}}{1 + \frac{B_1 - A_1}{B_1 + A_1} \cdot \frac{B_2 - A_2}{B_2 + A_2}}} \right\}, \right. \\
 &\quad \left. \left\{ \sqrt[3]{\frac{\frac{M_1 - N_1}{M_1 + N_1} + \frac{M_2 - N_2}{M_2 + N_2}}{1 + \frac{M_1 - N_1}{M_1 + N_1} \cdot \frac{M_2 - N_2}{M_2 + N_2}}} \right\} \right\} \\
 &= \bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \left\{ \sqrt[3]{\frac{2^2 A_1 A_2}{(B_1 + A_1)(B_2 + A_2) + (B_1 - A_1)(B_2 - A_2)}} \right\}, \right. \\
 &\quad \left. \left\{ \sqrt[3]{\frac{(M_1 - N_1)(M_2 + N_2) + (M_1 + N_1)(M_2 - N_2)}{(M_1 + N_1)(M_2 + N_2) - (M_1 - N_1)(M_2 - N_2)}} \right\} \right\} \\
 &= \bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \left\{ \sqrt[3]{\frac{2A_1 A_2}{B_1 B_2 + A_1 A_2}} \right\}, \left\{ \sqrt[3]{\frac{M_1 M_2 - N_1 N_2}{M_1 M_2 + N_1 N_2}} \right\} \right\}.
 \end{aligned}$$

Then, we can obtain $(\zeta_1 \otimes \zeta_2)^\gamma = \zeta_1^\gamma \otimes \zeta_2^\gamma$.

(5) From the operation (4) in Definition 3.3, we have

$$\begin{aligned}
 \zeta_1 \oplus (\zeta_2 \oplus \zeta_3) &= \zeta_1 \oplus \left(\bigcup_{\theta_2 \in \mu_2, \theta_3 \in \mu_3, \vartheta_2 \in \nu_2, \vartheta_3 \in \nu_3} \left\{ \left\{ \sqrt[3]{\frac{\theta_2^3 + \theta_3^3}{1 + \theta_2^3 \theta_3^3}} \right\}, \left\{ \frac{\vartheta_2 \vartheta_3}{\sqrt[3]{1 + (1 - \vartheta_2^3)(1 - \vartheta_3^3)}} \right\} \right\} \right) \\
 &= \bigcup_{\substack{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \theta_3 \in \mu_3, \\ \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2, \vartheta_3 \in \nu_3}} \left\{ \left\{ \sqrt[3]{\frac{\theta_1^3 + \frac{\theta_2^3 + \theta_3^3}{1 + \theta_2^3 \theta_3^3}}{1 + \theta_1^3 \frac{\theta_2^3 + \theta_3^3}{1 + \theta_2^3 \theta_3^3}}} \right\}, \right. \\
 &\quad \left. \left\{ \frac{\vartheta_1 \frac{\vartheta_2 \vartheta_3}{\sqrt[3]{1 + (1 - \vartheta_2^3)(1 - \vartheta_3^3)}}}{\sqrt[3]{1 + (1 - \vartheta_1^3) \left(1 - \frac{\vartheta_2^3 \vartheta_3^3}{1 + (1 - \vartheta_2^3)(1 - \vartheta_3^3)} \right)}} \right\} \right\} \\
 &= \bigcup_{\substack{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \theta_3 \in \mu_3, \\ \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2, \vartheta_3 \in \nu_3}} \left\{ \left\{ \sqrt[3]{\frac{\theta_1^3 + \theta_2^3 + \theta_3^3 + \theta_1^3 \theta_2^3 \theta_3^3}{1 + \theta_2^3 \theta_3^3 + \theta_1^3 \theta_2^3 + \theta_1^3 \theta_3^3}} \right\}, \right. \\
 &\quad \left. \left\{ \frac{\vartheta_1 \vartheta_2 \vartheta_3}{\sqrt[3]{3 - \vartheta_1^3 - \vartheta_2^3 - \vartheta_3^3 + (1 - \vartheta_1^3)(1 - \vartheta_2^3)(1 - \vartheta_3^3) + \vartheta_1^3 \vartheta_2^3 \vartheta_3^3}} \right\} \right\}.
 \end{aligned}$$

Likewise, we have

$$\begin{aligned}
 (\zeta_1 \oplus \zeta_2) \oplus \zeta_3 &= \left(\bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \left\{ \sqrt[3]{\frac{\theta_1^3 + \theta_2^3}{1 + \theta_1^3 \theta_2^3}} \right\}, \left\{ \frac{\vartheta_1 \vartheta_2}{\sqrt[3]{1 + (1 - \vartheta_1^3)(1 - \vartheta_2^3)}} \right\} \right\} \right) \oplus \zeta_3 \\
 &= \bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \theta_3 \in \mu_3, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2, \vartheta_3 \in \nu_3} \left\{ \left[\begin{array}{c} \left\{ \sqrt[3]{\frac{\theta_3^3 + \frac{\theta_1^3 + \theta_2^3}{1 + \theta_1^3 \theta_2^3}}{1 + \theta_3^3 \frac{\theta_1^3 + \theta_2^3}{1 + \theta_1^3 \theta_2^3}}} \right\} \\ \frac{\vartheta_3 \sqrt[3]{\frac{\vartheta_1 \vartheta_2}{1 + (1 - \vartheta_1^3)(1 - \vartheta_2^3)}}}{\sqrt[3]{1 + (1 - \vartheta_3^3) \left(1 - \frac{\vartheta_1^3 \vartheta_2^3}{1 + (1 - \vartheta_1^3)(1 - \vartheta_2^3)} \right)}} \end{array} \right] \right\} \\
 &= \bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \theta_3 \in \mu_3, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2, \vartheta_3 \in \nu_3} \left\{ \left[\begin{array}{c} \left\{ \sqrt[3]{\frac{\theta_1^3 + \theta_2^3 + \theta_3^3 + \theta_1^3 \theta_2^3 \theta_3^3}{1 + \theta_2^3 \theta_3^3 + \theta_1^3 \theta_2^3 + \theta_1^3 \theta_3^3}} \right\} \\ \frac{\vartheta_1 \vartheta_2 \vartheta_3}{\sqrt[3]{3 - \vartheta_1^3 - \vartheta_2^3 - \vartheta_3^3 + (1 - \vartheta_1^3)(1 - \vartheta_2^3)(1 - \vartheta_3^3) + \vartheta_1^3 \vartheta_2^3 \vartheta_3^3}} \end{array} \right] \right\}.
 \end{aligned}$$

Then, we can obtain $\zeta_1 \oplus (\zeta_2 \oplus \zeta_3) = (\zeta_1 \oplus \zeta_2) \oplus \zeta_3$

(6) From the operation (3) in Definition 3.3, we have

$$\begin{aligned}
 \zeta_1 \otimes (\zeta_2 \otimes \zeta_3) &= \zeta_1 \otimes \left(\bigcup_{\theta_2 \in \mu_2, \theta_3 \in \mu_3, \vartheta_2 \in \nu_2, \vartheta_3 \in \nu_3} \left\{ \left\{ \frac{\theta_2 \theta_3}{\sqrt[3]{1 + (1 - \theta_2^3)(1 - \theta_3^3)}} \right\}, \left\{ \sqrt[3]{\frac{\vartheta_2^3 + \vartheta_3^3}{1 + \vartheta_2^3 \vartheta_3^3}} \right\} \right\} \right) \\
 &= \bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \theta_3 \in \mu_3, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2, \vartheta_3 \in \nu_3} \left\{ \left[\begin{array}{c} \frac{\theta_1 \frac{\theta_2 \theta_3}{\sqrt[3]{1 + (1 - \theta_2^3)(1 - \theta_3^3)}}}{\sqrt[3]{1 + (1 - \theta_1^3) \left(1 - \frac{\theta_2^3 \theta_3^3}{1 + (1 - \theta_2^3)(1 - \theta_3^3)} \right)}} \\ \left\{ \sqrt[3]{\frac{\vartheta_1^3 + \frac{\vartheta_2^3 + \vartheta_3^3}{1 + \vartheta_2^3 \vartheta_3^3}}{1 + \vartheta_1^3 \frac{\vartheta_2^3 + \vartheta_3^3}{1 + \vartheta_2^3 \vartheta_3^3}}} \right\} \end{array} \right] \right\} \\
 &= \bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \theta_3 \in \mu_3, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2, \vartheta_3 \in \nu_3} \left\{ \left[\begin{array}{c} \frac{\theta_1 \theta_2 \theta_3}{\sqrt[3]{3 - \theta_1^3 - \theta_2^3 - \theta_3^3 + (1 - \theta_1^3)(1 - \theta_2^3)(1 - \theta_3^3) + \theta_1^3 \theta_2^3 \theta_3^3}} \\ \left\{ \sqrt[3]{\frac{\vartheta_1^3 + \vartheta_2^3 + \vartheta_3^3 + \vartheta_1^3 \vartheta_2^3 \vartheta_3^3}{1 + \vartheta_2^3 \vartheta_3^3 + \vartheta_1^3 \vartheta_2^3 + \vartheta_1^3 \vartheta_3^3}} \right\} \end{array} \right] \right\}.
 \end{aligned}$$

Likewise, we have

$$\begin{aligned}
 (\zeta_1 \otimes \zeta_2) \otimes \zeta_3 &= \bigcup_{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2} \left\{ \left\{ \frac{\theta_1 \theta_2}{\sqrt[3]{1 + (1 - \theta_1^3)(1 - \theta_2^3)}} \right\}, \left\{ \sqrt[3]{\frac{\vartheta_1^3 + \vartheta_2^3}{1 + \vartheta_1^3 \vartheta_2^3}} \right\} \right\} \otimes \zeta_3 \\
 &= \bigcup_{\substack{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \theta_3 \in \mu_3, \\ \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2, \vartheta_3 \in \nu_3}} \left\{ \left\{ \frac{\theta_3 \sqrt[3]{\frac{\theta_1 \theta_2}{1 + (1 - \theta_1^3)(1 - \theta_2^3)}}}}{\sqrt[3]{1 + (1 - \theta_3^3) \left(1 - \frac{\theta_1^3 \theta_2^3}{1 + (1 - \theta_1^3)(1 - \theta_2^3)}\right)}} \right\}, \left\{ \sqrt[3]{\frac{\vartheta_3^3 + \frac{\vartheta_1^3 + \vartheta_2^3}{1 + \vartheta_1^3 \vartheta_2^3}}{1 + \vartheta_3^3 \frac{\vartheta_1^3 + \vartheta_2^3}{1 + \vartheta_1^3 \vartheta_2^3}}} \right\} \right\} \\
 &= \bigcup_{\substack{\theta_1 \in \mu_1, \theta_2 \in \mu_2, \theta_3 \in \mu_3, \\ \vartheta_1 \in \nu_1, \vartheta_2 \in \nu_2, \vartheta_3 \in \nu_3}} \left\{ \left\{ \frac{\theta_1 \theta_2 \theta_3}{\sqrt[3]{(1 - \theta_1^3)(1 - \theta_2^3)(1 - \theta_3^3) + \theta_1^3 \theta_2^3 \theta_3^3}} \right\}, \left\{ \sqrt[3]{\frac{\vartheta_1^3 + \vartheta_2^3 + \vartheta_3^3 + \vartheta_1^3 \vartheta_2^3 \vartheta_3^3}{1 + \vartheta_2^3 \vartheta_3^3 + \vartheta_1^3 \vartheta_2^3 + \vartheta_1^3 \vartheta_3^3}} \right\} \right\}.
 \end{aligned}$$

Then, we can obtain $\zeta_1 \otimes (\zeta_2 \otimes \zeta_3) = (\zeta_1 \otimes \zeta_2) \otimes \zeta_3$. The proof of Theorem 3.4 is completed.

$$\zeta_i^p = \bigcup_{\theta_i \in \mu_i, \vartheta_i \in \nu_i} \left\{ \left\{ \sqrt[3]{\frac{2A_i^p}{B_i^p + A_i^p}} \right\}, \left\{ \sqrt[3]{\frac{M_i^p - N_i^p}{M_i^p + N_i^p}} \right\} \right\}$$

The Proof of Lemma 3.3

Proof From the operation (1) in Definition 3.3, we have

$$\zeta_j^q = \bigcup_{\theta_j \in \mu_j, \vartheta_j \in \nu_j} \left\{ \left\{ \sqrt[3]{\frac{2A_j^q}{B_j^q + A_j^q}} \right\}, \left\{ \sqrt[3]{\frac{M_j^q - N_j^q}{M_j^q + N_j^q}} \right\} \right\}.$$

Then, from the operation (3) in Definition 3.3, we have

$$\zeta_i^p \otimes \zeta_j^q = \bigcup_{\theta_i \in \mu_i, \theta_j \in \mu_j, \vartheta_i \in \nu_i, \vartheta_j \in \nu_j} \left\{ \left\{ \sqrt[3]{\frac{2A_i^p A_j^q}{B_i^p B_j^q + A_i^p A_j^q}} \right\}, \left\{ \sqrt[3]{\frac{M_i^p M_j^q - N_i^p N_j^q}{M_i^p M_j^q + N_i^p N_j^q}} \right\} \right\}.$$

Similarly, we can obtain $\zeta_i^q \otimes \zeta_j^p$. From the operation (4) in Definition 3.3, we can obtain Eq. (8). The proof of Lemma 3.3 is completed.

$$\begin{aligned}
 \zeta_i^p &= \bigcup_{\theta_i \in \mu_i, \vartheta_i \in \nu_i} \left\{ \left\{ \sqrt[3]{\frac{2(\theta_i^3)^p}{(2 - \theta_i^3)^p + (\theta_i^3)^p}} \right\}, \left\{ \sqrt[3]{\frac{(1 + \vartheta_i^3)^p - (1 - \vartheta_i^3)^p}{(1 + \vartheta_i^3)^p + (1 - \vartheta_i^3)^p}} \right\} \right\} \\
 \zeta_j^q &= \bigcup_{\theta_j \in \mu_j, \vartheta_j \in \nu_j} \left\{ \left\{ \sqrt[3]{\frac{2(\theta_j^3)^q}{(2 - \theta_j^3)^q + (\theta_j^3)^q}} \right\}, \left\{ \sqrt[3]{\frac{(1 + \vartheta_j^3)^q - (1 - \vartheta_j^3)^q}{(1 + \vartheta_j^3)^q + (1 - \vartheta_j^3)^q}} \right\} \right\}.
 \end{aligned}$$

The Proof of Lemma 3.4

Proof From the operation (2) in Definition 3.3, we have

Let $A_i = \theta_i^3$, $B_i = 2 - \theta_i^3$, $M_i = 1 + \vartheta_i^3$, $N_i = 1 - \vartheta_i^3$, then ζ_i^p and ζ_j^q can be replaced as follows:

$$\omega_i \zeta_i = \bigcup_{\theta_i \in \mu_i, \vartheta_i \in \nu_i} \left\{ \left\{ \sqrt[3]{\frac{(1 + \theta_i^3)^{\omega_i} - (1 - \theta_i^3)^{\omega_i}}{(1 + \theta_i^3)^{\omega_i} + (1 - \theta_i^3)^{\omega_i}}} \right\}, \left\{ \sqrt[3]{\frac{2(\vartheta_i^3)^{\omega_i}}{(2 - \vartheta_i^3)^{\omega_i} + (\vartheta_i^3)^{\omega_i}}} \right\} \right\}$$

$$\omega_j \zeta_j = \bigcup_{\theta_j \in \mu_j, \vartheta_j \in \nu_j} \left\{ \left\{ \sqrt[3]{\frac{(1 + \theta_j^3)^{\omega_j} - (1 - \theta_j^3)^{\omega_j}}{(1 + \theta_j^3)^{\omega_j} + (1 - \theta_j^3)^{\omega_j}}} \right\}, \left\{ \sqrt[3]{\frac{2(\vartheta_j^3)^{\omega_j}}{(2 - \vartheta_j^3)^{\omega_j} + (\vartheta_j^3)^{\omega_j}}} \right\} \right\}.$$

Then, from the operation (1) in Definition 3.3, we have

$$(\omega_i \zeta_i)^P = \bigcup_{\theta_i \in \mu_i, \vartheta_i \in \nu_i} \left\{ \left\{ \sqrt[3]{\frac{2((1 + \theta_i^3)^{\omega_i} - (1 - \theta_i^3)^{\omega_i})^P}{((1 + \theta_i^3)^{\omega_i} + 3(1 - \theta_i^3)^{\omega_i})^P + ((1 + \theta_i^3)^{\omega_i} - (1 - \theta_i^3)^{\omega_i})^P}} \right\}, \left\{ \sqrt[3]{\frac{((2 - \vartheta_i^3)^{\omega_i} + 3(\vartheta_i^3)^{\omega_i})^P - ((2 - \vartheta_i^3)^{\omega_i} - 3(\vartheta_i^3)^{\omega_i})^P}{((2 - \vartheta_i^3)^{\omega_i} + 3(\vartheta_i^3)^{\omega_i})^P + ((2 - \vartheta_i^3)^{\omega_i} - 3(\vartheta_i^3)^{\omega_i})^P}} \right\} \right\}$$

$$(\omega_i \zeta_i)^Q = \bigcup_{\theta_j \in \mu_j, \vartheta_j \in \nu_j} \left\{ \left\{ \sqrt[3]{\frac{2((1 + \theta_j^3)^{\omega_j} - (1 - \theta_j^3)^{\omega_j})^Q}{((1 + \theta_j^3)^{\omega_j} + 3(1 - \theta_j^3)^{\omega_j})^Q + ((1 + \theta_j^3)^{\omega_j} - (1 - \theta_j^3)^{\omega_j})^Q}} \right\}, \left\{ \sqrt[3]{\frac{((2 - \vartheta_j^3)^{\omega_j} + 3(\vartheta_j^3)^{\omega_j})^Q - ((2 - \vartheta_j^3)^{\omega_j} - 3(\vartheta_j^3)^{\omega_j})^Q}{((2 - \vartheta_j^3)^{\omega_j} + 3(\vartheta_j^3)^{\omega_j})^Q + ((2 - \vartheta_j^3)^{\omega_j} - 3(\vartheta_j^3)^{\omega_j})^Q}} \right\} \right\}.$$

Let

$$A_i = (1 + \theta_i^3)^{\omega_i} - (1 - \theta_i^3)^{\omega_i}, \quad B_i = (1 + \theta_i^3)^{\omega_i} + 3(1 - \theta_i^3)^{\omega_i},$$

$$M_i = (2 - \vartheta_i^3)^{\omega_i} + 3(\vartheta_i^3)^{\omega_i}, \quad N_i = (2\vartheta_i^3)^{\omega_i} - 3(\vartheta_i^3)^{\omega_i};$$

then $(\omega_i \zeta_i)^P$ and $(\omega_j \zeta_j)^Q$ can be replaced as follows:

$$(\omega_i \zeta_i)^P = \bigcup_{\theta_i \in \mu_i, \vartheta_i \in \nu_i} \left\{ \left\{ \sqrt[3]{\frac{2A_i^P}{B_i^P + A_i^P}} \right\}, \left\{ \sqrt[3]{\frac{M_i^P - N_i^P}{M_i^P + N_i^P}} \right\} \right\}$$

$$(\omega_j \zeta_j)^Q = \bigcup_{\theta_j \in \mu_j, \vartheta_j \in \nu_j} \left\{ \left\{ \sqrt[3]{\frac{2A_j^Q}{B_j^Q + A_j^Q}} \right\}, \left\{ \sqrt[3]{\frac{M_j^Q - N_j^Q}{M_j^Q + N_j^Q}} \right\} \right\}.$$

From the operation (3) in Definition 3.3, we have

$$(\omega_i \zeta_i)^P \otimes (\omega_j \zeta_j)^Q = \bigcup_{\theta_i \in \mu_i, \theta_j \in \mu_j, \vartheta_i \in \nu_i, \vartheta_j \in \nu_j} \left\{ \left\{ \sqrt[3]{\frac{2A_i^P A_j^Q}{B_i^P B_j^Q + A_i^P A_j^Q}} \right\}, \left\{ \sqrt[3]{\frac{M_i^P M_j^Q - N_i^P N_j^Q}{M_i^P M_j^Q + N_i^P N_j^Q}} \right\} \right\}.$$

Similarly, we can obtain $(\omega_i \zeta_i)^Q \otimes (\omega_j \zeta_j)^P$. Then, from the operation (4) in Definition 3.3, we can obtain Eq. (11). The proof of Lemma 3.4 is completed.

The code of HFFWBM and HFFEWM

Python code of HFFWBM and HFFEWM

```
def HFFWBM(alter, qrung, weight=None, p=1, q=1):
    def bonferroni_element(d1, d2, p_, q_):
        s1 = fns.algeb_multiply(d1.algeb_power(p_), d2.algeb_power(q_))
        s2 = fns.algeb_multiply(d1.algeb_power(q_), d2.algeb_power(p_))
        return fns.algeb_plus(s1, s2)

    dlist = []
    if weight is None:
        dl = alter
        for i, d1 in enumerate(dl):
            for j, d2 in enumerate(dl[:i]):
                dlist.append(bonferroni_element(d1, d2, p, q))
    else:
        dl = []
        assert len(alter) == len(weight), 'The number of alter is not equal to
the number of weight!'
        for i in range(len(alter)):
            dl.append(alter[i].algeb_times(weight[i]))
        for i, d1 in enumerate(dl):
            for j, d2 in enumerate(dl[:i]):
                dlist.append(bonferroni_element(d1, d2, p, q))

    agge = fns.qrungdhfe(qrung, [0], [1])
    for agg in dlist:
        agge = fns.algeb_plus(agg, agge)
    agge = (agge.algeb_times(1 / (len(alter) * (len(alter) - 1)))) .algeb_power(1
/ (p + q))
    return agge

def HFFEWM(alter, qrung, weight=None, p=1, q=1):
    def bonferroni_element(d1, d2, p_, q_):
        s1 = fns.eins_multiply(d1.eins_power(p_), d2.eins_power(q_))
        s2 = fns.eins_multiply(d1.eins_power(q_), d2.eins_power(p_))
        return fns.eins_plus(s1, s2)

    dlist = []
    if weight is None:
        dl = alter
        for i, d1 in enumerate(dl):
            for j, d2 in enumerate(dl[:i]):
                dlist.append(bonferroni_element(d1, d2, p, q))
    else:
        dl = []
        assert len(alter) == len(weight), 'The number of alter is not equal to
the number of weight!'
        for i in range(len(alter)):
            dl.append(alter[i].eins_times(weight[i]))
        for i, d1 in enumerate(dl):
            for j, d2 in enumerate(dl[:i]):
                dlist.append(bonferroni_element(d1, d2, p, q))

    agge = fns.qrungdhfe(qrung, [0], [1])
    for agg in dlist:
        agge = fns.eins_plus(agg, agge)
    agge = (agge.eins_times(1 / (len(alter) * (len(alter) - 1)))) .eins_power(1
/ (p + q))
    return agge
```

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