Complex & Intelligent Systems (2023) 9:3249–3260
https://doi.org/10.1007/s40747-022-00920-5

ORIGINAL ARTICLE

# Dynamic modeling and infinite-dimensional observer-based control for manipulation of flexible beam by a multi-link robot

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Received: 18 August 2022 / Accepted: 6 November 2022 / Published online: 2 December 2022


**Abstract**
This paper concerns an infinite-dimensional observer for manipulation of flexible beam by a rigid arm robot. The complex dynamic of the system is described by distributed parameter model in terms of ordinary differential equations and partial differential equation. A novel infinite-dimensional observer is proposed to estimate the vibration information of the flexible object. In addition, an observer-based independent joint controller is designed to achieve the position control and vibration suppression, which do not need end-point boundary control. The semigroup theory and LaSalle's invariance principle are adopted to prove the asymptotic stability of the robot system. The efficiency of the observers and the proposed control strategy are demonstrated by numerical simulations.

**Keywords** Infinite-dimensional observer · Flexible beam · Independent joint control · Distributed parameter model

## Introduction

In industry, the research of robotic manipulators handling rigid bodies has received extensive attentions [1,2]. However, flexible objects such as solar panels, flexible metal plates and spring components are often used in the automotive, aerospace and medical fields [3,4]. For example, space robotic arms are used to maintain the aircraft and replace failed batteries, and the panels are mostly made of flexible materials. Different from rigid parts, flexible parts have the characteristics of light weight, high flexibility, man–machine interaction and low energy consumption, which also bring vibration. In the field of high accuracy and safety requirements, the vibration is necessary to be solved [5,6]. Therefore, it is of theoretical and practical significance to research the trajectory and vibration control of manipulation for flexible beam during industry operation and assembly tasks.

In the literature, research on manipulation of flexible structures is always based on simplified lumped parameter model [7,8]. In [9,10], the rigid control method has been used in moving flexible object by rigid robot based on assumed modes model. [11] researched two rigid robot manipulating flexible beam by finite element method. However, the simplified model may bring control or observer overflow problem, which may bring instability to the robot system. To avoid the above drawbacks, great attention has been paid to study the control design based upon distributed parameter model [12–14]. Distributed parameter model is an infinite-dimensional model, which refers to the relationship between the system state and the change of space coordinates and time variables. The PDE–ODE model is established for dual-arm coordinated operation of large spatial flexible structures in [15]. [16] studied the position/force control of flexible beams based on PDE model. [17,18] address robust control for flexible system based on distributed parameter system. However, most of the above achievements are about the flexible beam, research about the system of manipulation for flexible object based on distributed parameter model are relatively few, which exists many problems to be solved.

Furthermore, flexible beam operating system is a complex dynamically coupled system, which includes not only the rigid motion of large range, but also the local elastic deformation. The most important problem in this system is the vibration of the flexible structure in motion. Due

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to the requirements of high accuracy and high flexibility, the resulting vibration cannot be ignored [19,20]. At the same time, the characteristics of flexible object to be handled vary with different manipulation tasks. So this makes it impractical to outfit the flexible object with sensors and actuators. Therefore, it is necessary to propose an observer to estimate the vibration states [21]. At present, the results of PDE observers for infinite-dimensional systems are limited. In [22], the PDE observer for flexible single-link robot was proposed to estimate the infinite-dimensional states in task space. Feng et al. [23] designed an exponentially converging observer to estimates the state for the heat system [24] develops PDE observer to estimate the freeway traffic states. In [25], the Luenberger-like observers is proposed for an infinite-dimensional rotating body-beam system. Therefore, the research on infinite dimensional observer design for manipulation of flexible object is necessary and meaningful, and has not been reported yet.

Besides, the boundary control is always used for the control design of flexible system [26–28]. It not only needs joint input actuator to adjust the position of the robot, but also needs end-point input force to suppress the vibration. For the single-link manipulator or multi-link manipulator, boundary control is relatively easy to achieve, and is also effective to suppress the vibration for the manipulator [29,30]. The actuator installed at the end of the manipulator will not vary with the change of the task when the manipulator systems perform different operations. However, different from the manipulator system, the flexible object operated by the manipulator is the actuated mechanism, the actuator must be reinstalled when the operation object is changed, it is not appropriate to install the actuator at the end of the flexible object system [31]. In this paper, we only use the independent joint input to control the system, which do not need the end-point input force. The vibration can be suppressed by adding the root vibration observer signals of flexible object into the joint input controller. The contributions are summarized below:

1. The complex dynamic is expressed as original infinite-dimensional model without any simplification or discretization, which is very effective to reduce the vibration of the system.
2. An infinite-dimensional observer is designed for the manipulation of flexible object by a rigid arm manipulator, it prevents the installation of sensors on flexible objects.
3. The independent joint observer-based controller is proposed based on distributed parameter model. The stability is proved by LaSalle's Invariance Principle.

This paper is structured as follows. "System description" section describes the system dynamic. The infinite-dimensional observer is designed in "Infinite-dimensional observer" section. "Observer-based controller design" section proposes the observer-based controller and the asymptotic stability. The "Simulation" section shows validation results by simulation, and the conclusion is given in "Conclusion" section.

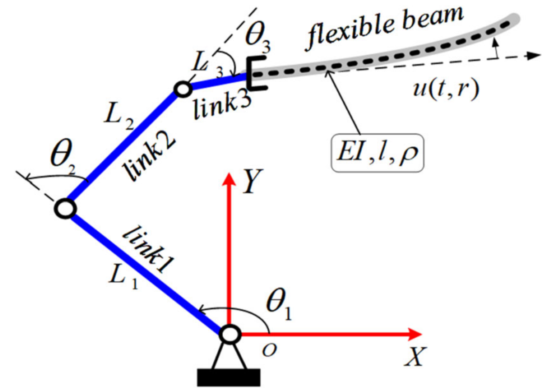



**Fig. 1** The structure diagram of manipulator operation

## System description

### Distributed parameter model

In the following, we consider a three-link robot moving a flexible object as shown in Fig. 1. XOY is the inertial coordinate. The flexible object is supposed to be an Euler–Bernoulli beam due to its own characteristics. The beam has uniform mass density $\rho K$, length $l$ and uniform flexural rigidity $EI$, $u(t,r)$ stands for the elastic deformation at length $r$ of flexible beam at time $t$. The rigid link $i (i = 1, 2, 3)$ has length $L_i$, mass center length $L_{ig}$, mass $m_i$, and moment of inertia $I_i$. $\theta_i$ denotes the rotation angle of link $i$, and $\tau_i$ is the joint torque at the motor $i$. $S_2 = [S_{2x} \quad S_{2y}]^{\mathrm{T}}$, $S_3 = [S_{3x} \quad S_{3y}]^{\mathrm{T}}$ and $S = [S_x \quad S_y]^{\mathrm{T}}$ express the position of the second link, the third link and the mass center of the flexible object in the reference coordinate system, respectively.

$$S_{2x} = L_1 \cos\theta_1 + L_{2g} \cos(\theta_1 + \theta_2), \tag{1}$$

$$S_{2y} = L_1 \sin\theta_1 + L_{2g} \sin(\theta_1 + \theta_2), \tag{2}$$

$$S_{3x} = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) + L_{3g} \cos(\theta_1 + \theta_2 + \theta_3), \tag{3}$$

$$S_{3y} = L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) + L_{3g} \sin(\theta_1 + \theta_2 + \theta_3), \tag{4}$$

$$\begin{aligned} S_x &= L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3) \\ &\quad + r\cos(\theta_1 + \theta_2 + \theta_3) - u \sin(\theta_1 + \theta_2 + \theta_3), \end{aligned} \tag{5}$$

$$\begin{aligned} S_y &= L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3) \\ &\quad + r\sin(\theta_1 + \theta_2 + \theta_3) - u \cos(\theta_1 + \theta_2 + \theta_3). \end{aligned} \tag{6}$$

Two assumptions are introduced [9]:

**Assumption 1** Assumed the longitudinal deflection of flexible object is ignored, and only transverse deflection and deflection angle are taken into consideration.

**Assumption 2** The flexible beam is grasped rigidly, no deformation occurs between the manipulator and the contacted beam.

***Remark 1*** The superscript "." is denoted as the derivative of time $t$ and superscript "′" is the derivative of length $r$. The subscript "$E$" of $u(t,r)$ express variable $r$ equal to $l$ and subscript "0" represent $r$ equal to "0", that is $u_0 = u(t,r)|_{r=0}$, $u_E = u(t,r)|_{r=l}$

The total kinetic and potential energy are expressed as

$$T = \frac{1}{2}\Big[m_1 L_{1g}^2 \dot{\theta}_1^2 + I_1\dot{\theta}_1^2 + m_2 \dot{S}_2^{\mathrm{T}} \dot{S}_2 + I_2(\dot{\theta}_1 + \dot{\theta}_2)^2 + m_3 \dot{S}_3^{\mathrm{T}} \dot{S}_3 + I_3(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)^2\Big] + \frac{1}{2}\rho K \int_0^l \dot{S}^{\mathrm{T}} \dot{S} \mathrm{d}r, \tag{7}$$

$$U = m_1 g L_{1g} \sin\theta_1 + m_2 g L_1 \sin\theta_1 + m_2 g L_{2g} \sin(\theta_1 + \theta_2) + m_3 g L_1 \sin\theta_1 + m_3 g L_2 \sin(\theta_1 + \theta_2) + m_3 g L_{3g} \sin(\theta_1 + \theta_2 + \theta_3) + \rho K g \int_0^l S_y \mathrm{d}r + \frac{1}{2}\int_0^l EI(u'')^2 \mathrm{d}r. \tag{8}$$

The virtual work of the system is

$$\delta W = \sum_{i=1}^{3} \tau_i \delta\theta_i. \tag{9}$$

By Hamilton's principle, we have

$$\int_{t_0}^{t_1} (\delta T - \delta U + \delta W)\mathrm{d}t \equiv 0. \tag{10}$$

The system dynamic model is expressed as

$$M(\theta)\ddot{\theta} + [C(\theta,\dot{\theta}) + f(\theta,\dot{u})]\dot{\theta} + F\left(\theta, u_0{''}, u_0{'''}\right) + G = \tau, \tag{11}$$

$$\ddot{z} + \frac{EIu^{(4)}}{\rho K} = 0, \tag{12}$$

$$u_0 = 0,\quad u_0' = 0,\quad u_E''' = 0,\quad u_E'' = 0. \tag{13}$$

Parameters of (11) and (12) are as following

$$z(r,t) = (r + L_3)(\theta_1 + \theta_2 + \theta_3) + u(t,r) + \int_0^t [\dot{\theta}_1 L_1 \cos(\theta_2 + \theta_3) + (\dot{\theta}_1 + \dot{\theta}_2) L_2 \cos\theta_3]\mathrm{d}t, \tag{14}$$

$\theta = [\theta_1\ \ \theta_2\ \ \theta_3\ ]^T$, $F\left(\theta, u_0{''}, u_0{'''}\right) = [F_1\ \ F_2\ \ F_3\ ]^T$, $\tau = [\tau_1\ \ \tau_2\ \ \tau_3\ ]^T$, $G = [G_1\ \ G_2\ \ G_3\ ]^T$. $M(\theta)$, $C(\theta,\dot{\theta})$ and $f(\theta,\dot{u})$ are $3\times 3$ matrix; $\theta$, $G$ and $\tau$ are $3\times 1$ vector (see Appendix A). The Eq. (11) also have the following property:

**Property 1** $M(\theta)$ is symmetrical and positive definite, $M(\theta)$, $C(\theta,\dot{\theta})$ and $f(\theta,\dot{u})$ satisfy $\dot{M} - 2(C+f) = -(\dot{M} - 2(C+f))^T$.

## Energy analysis of flexible beam

First, the elastic deformation of Euler–Bernoulli beam is assumed to be much smaller than the beam length ($|u| << l$), so the square velocity $\dot{S}^{\mathrm{T}}\dot{S}$ of flexible beam in (7) is simplified as follows

$$\dot{S}^{\mathrm{T}}\dot{S} = \left[\dot{\theta}_1 L_1 \cos(\theta_2+\theta_3) + (L_3 + r)(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3) + (\dot{\theta}_1 + \dot{\theta}_2) L_2 \cos\theta_3 + \dot{u}\right]^2 + \left[\dot{\theta}_1 L_1 \sin(\theta_2+\theta_3) + (\dot{\theta}_1 + \dot{\theta}_2) L_2 \sin\theta_3\right]^2. \tag{15}$$

The transversal velocity $\dot{S}_t$ is only considered since the vibration of beam is hardly affected by the kinetic energy due to the longitudinal velocity. The transversal component $\dot{S}^{\mathrm{T}}\dot{S}$ of the beam is

$$\dot{S}_t^{\mathrm{T}}\dot{S}_t = [L_1\dot{\theta}_1 \cos(\theta_2+\theta_3) + (L_3 + r)(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3) + (\dot{\theta}_1 + \dot{\theta}_2) L_2 \cos\theta_3 + \dot{u}]^2. \tag{16}$$

Then, the kinetic energy $T_f$ and the potential energy $U_f$ of flexible beam due to the transversal velocity are computed as

$$T_f + U_f = \frac{1}{2}\int_0^l \rho K \dot{S}_t^{\mathrm{T}}\dot{S}_t \mathrm{d}r + \frac{1}{2}\int_0^l EI(u'')^2 \mathrm{d}r. \tag{17}$$

Next, the time derivative of the total flexible energy $T_f + U_f$ can be derived as

$$\begin{aligned}\dot{T}_f + \dot{U}_f &= \rho K \int_0^l [L_1\dot{\theta}_1 \cos(\theta_2+\theta_3) + (\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)(L_3 + r) \\ &\quad + \cos\theta_3 L_2(\dot{\theta}_1 + \dot{\theta}_2) + \dot{u}][-\dot{\theta}_1 L_1 \sin(\theta_2+\theta_3)(\dot{\theta}_2 + \dot{\theta}_3) \\ &\quad + L_1\ddot{\theta}_1 \cos(\theta_2+\theta_3) + (\ddot{\theta}_1 + \ddot{\theta}_2) L_2 \cos\theta_3 \\ &\quad - (\dot{\theta}_1 + \dot{\theta}_2) L_2 \sin\theta_3 \dot{\theta}_3 \\ &\quad + (L_3 + r)(\ddot{\theta}_1 + \ddot{\theta}_2 + \ddot{\theta}_3) + \ddot{u}]\mathrm{d}r + \int_0^l EIu^{(4)}\dot{u}\mathrm{d}r,\end{aligned} \tag{18}$$

where $\ddot{u}$ can be get from the vibration Eq. (6).

Finally, according to the boundary condition (7), we can get

$$\dot{T}_f + \dot{U}_f = [L_1\dot{\theta}_1 \cos(\theta_2+\theta_3) + (\dot{\theta}_1 + \dot{\theta}_2) L_2 \cos\theta_3$$

$$
\begin{aligned}
&+(\dot\theta_1+\dot\theta_2+\dot\theta_3)L_3]EIu_0''' \\
&-EIu_0''(\dot\theta_1+\dot\theta_2+\dot\theta_3) = -q^{\mathrm{T}}\dot\theta,
\end{aligned}
\tag{19}
$$

where $\dot\theta = [\dot\theta_1 \quad \dot\theta_2 \quad \dot\theta_3]^{\mathrm{T}}$, $q$ is defined as

$$
q = \begin{pmatrix} [-\cos(\theta_2+\theta_3)L_1 - \cos\theta_3 L_2 - L_3]EIu_0''' + EIu_0'' \\ (-\cos\theta_3 L_2 - L_3)EIu_0''' + EIu_0'' \\ -EIL_3u_0''' + EIu_0'' \end{pmatrix}.
\tag{20}
$$

## Infinite-dimensional observer

### Observer design

For avoid installing sensors in the flexible beam, a nonlinear infinite-dimensional observer is designed to estimate the root strain and shear force of flexible beam based on Eqs. (11)–(13). Assume that the angular $\theta_i$, angular velocity $\dot\theta_i$ are available for measurement, and the estimates of the angular and the flexible deformation are defined as $\hat\theta_i$ and $\hat u(r,t)$. The estimate errors are defined as $\tilde\theta_i = \theta_i - \hat\theta_i$, $\tilde u(t,r) = u(t,r) - \hat u(t,r)$ and $\tilde z(r,t) = z(r,t) - \hat z(r,t)$. The observer is proposed to satisfy the following relations as $t \to \infty$:

$$
\hat\theta_i \to \theta_i, \quad \dot{\hat\theta}_i \to \dot\theta_i, \quad \hat u(t,r) \to u(t,r), \quad \dot{\hat u}(t,r) \to \dot u(t,r).
\tag{21}
$$

Define the estimate $\hat z(r,t)$ of $z(r,t)$ as

$$
\begin{aligned}
\hat z(r,t) &= (r+L_3)(\hat\theta_1+\hat\theta_2+\hat\theta_3) + \hat u(t,r) \\
&\quad + \int_0^t [\dot{\hat\theta}_1 L_1\cos(\theta_2+\theta_3) + (\dot{\hat\theta}_1+\dot{\hat\theta}_2)L_2\cos\theta_3]\mathrm{d}t.
\end{aligned}
\tag{22}
$$

An observer is designed to reconstruct the states in the domain as follows

$$
\begin{aligned}
&M(\theta)\ddot{\hat\theta} + C(\theta,\dot\theta)\dot{\hat\theta} + f(\dot\theta,\hat u)\dot\theta + F(\theta,\hat u_0'',\hat u_0''') \\
&\quad - k_d(\dot{\hat\theta}-\dot\theta) - k_p(\hat\theta-\theta) + G = \tau,
\end{aligned}
\tag{23}
$$

$$
\ddot{\hat z}(r,t) + \frac{EI}{\rho K}\hat u^{(4)}(t,r) = 0,
\tag{24}
$$

$$
\hat u_0 = 0, \quad \hat u_0' = 0, \quad \hat u_E'' = 0, \quad \hat u_E''' = 0,
\tag{25}
$$

where $k_d = diag(k_{d1}, k_{d2}, k_{d3})$, $k_p = diag(k_{p1}, k_{p2}, k_{p3})$, $k_{di}, k_{pi} \in R^+$ $(i = 1, 2, 3)$. Then, we subtract system model (11)–(13) by (23)–(25) and get the model of estimate errors.

$$
\begin{aligned}
&M_{11}(\theta)\ddot{\tilde\theta}_1 + M_{12}(\theta)\ddot{\tilde\theta}_2 + M_{13}(\theta)\ddot{\tilde\theta}_3 + C_{11}(\theta,\dot\theta)\dot{\tilde\theta}_1 \\
&\quad + C_{12}(\theta,\dot\theta)\dot{\tilde\theta}_2 + C_{13}(\theta,\dot\theta)\dot{\tilde\theta}_3 \\
&\quad - \rho K[L_1\sin(\theta_2+\theta_3)]\int_0^l \dot{\tilde u}\mathrm{d}r\dot\theta_2 - \rho K[\sin(\theta_2+\theta_3)L_1 \\
&\quad + \sin\theta_3 L_2]\int_0^l \dot{\tilde u}\mathrm{d}r\dot\theta_3 \\
&\quad - EI\tilde u_0'' + [\cos(\theta_2+\theta_3)L_1 + \cos\theta_3 L_2 + L_3]EI\tilde u_0''' \\
&\quad + k_{d1}\dot{\tilde\theta}_1 + k_{p1}\tilde\theta_1 = 0,
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
&M_{21}(\theta)\ddot{\tilde\theta}_1 + M_{22}(\theta)\ddot{\tilde\theta}_2 + M_{23}(\theta)\ddot{\tilde\theta}_3 + C_{21}(\theta,\dot\theta)\dot{\tilde\theta}_1 \\
&\quad + C_{22}(\theta,\dot\theta)\dot{\tilde\theta}_2 \\
&\quad + C_{23}(\theta,\dot\theta)\dot{\tilde\theta}_3 \\
&\quad + \rho K\sin(\theta_2+\theta_3)L_1\int_0^l \dot{\tilde u}\mathrm{d}r\dot\theta_1 - \rho K\sin\theta_3 L_2 \\
&\int_0^l \dot{\tilde u}\mathrm{d}r\dot\theta_3 - EI\tilde u_0'' \\
&\quad + (L_2\cos\theta_3 + L_3)EI\tilde u_0''' + k_{d2}\dot{\tilde\theta}_2 + k_{p2}\tilde\theta_2 = 0,
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
&M_{31}(\theta)\ddot{\tilde\theta}_1 + M_{32}(\theta)\ddot{\tilde\theta}_2 + M_{33}(\theta)\ddot{\tilde\theta}_3 + C_{31}(\theta,\dot\theta)\dot{\tilde\theta}_1 \\
&\quad + C_{32}(\theta,\dot\theta)\dot{\tilde\theta}_2 + C_{33}(\theta,\dot\theta)\dot{\tilde\theta}_3 \\
&\quad + \rho K[\sin(\theta_2+\theta_3)L_1 + \sin\theta_3 L_2]\int_0^l \dot{\tilde u}\mathrm{d}r\dot\theta_1 \\
&\quad + \rho K L_2\sin\theta_3 \\
&\int_0^l \dot{\tilde u}\mathrm{d}r\dot\theta_2 \\
&\quad - EI\tilde u_0'' + L_3EI\tilde u_0''' + k_{d3}\dot{\tilde\theta}_3 + k_{p3}\tilde\theta_3 = 0,
\end{aligned}
\tag{28}
$$

$$
\ddot{\tilde z}(r,t) + \frac{EI}{\rho K}\tilde u^{(4)}(t,r) = 0,
\tag{29}
$$

$$
\tilde u_0 = 0, \quad \tilde u_0' = 0, \quad \tilde u_E'' = 0, \quad \tilde u_E''' = 0.
\tag{30}
$$

### The asymptotic stability of the observer

Define a generalized coordinate vector as

$$
\begin{aligned}
w &= [w_1 \quad w_2 \quad w_3 \quad w_4 \quad w_5 \quad w_6 \quad w_7 \quad w_8]^{\mathrm{T}} \\
&= [\tilde\theta_1 \quad \dot{\tilde\theta}_1 \quad \tilde\theta_2 \quad \dot{\tilde\theta}_2 \quad \tilde\theta_3 \quad \dot{\tilde\theta}_3 \quad \tilde u \quad \dot{\tilde u}]^{\mathrm{T}}.
\end{aligned}
\tag{31}
$$

The estimate error model (26)–(30) can be rewritten as

$$
\dot w = Aw + Y(w), \quad w(0) \in H,
\tag{32}
$$

where $A$ is infinite-dimensional linear operator, $Y(w)$ is finite dimensional nonlinear operator.

$$
Aw = [w_2 \quad \varphi_1 \quad w_4 \quad \varphi_2 \quad w_6 \quad \varphi_3 \quad w_8 \quad \varphi_4]^{\mathrm{T}}, \quad \forall w \in D(A),
\tag{33}
$$

$$
Y(w) = [0 \quad Y_1 \quad 0 \quad Y_2 \quad 0 \quad Y_3 \quad 0 \quad Y_4]^{\mathrm{T}},
\tag{34}
$$

where

$$\varphi_1 = (EIw_7''(0) - EIL_3w_7'''(0) - k_{d1}w_2 - k_{p1}w_1)/M_{11},$$
$$\varphi_2 = (EIw_7''(0) - EIL_3w_7'''(0) - k_{d2}w_4 - k_{p2}w_3)/M_{22},$$
$$\varphi_3 = (EIw_7''(0) - EIL_3w_7'''(0) - k_{d3}w_6 - k_{p3}w_5)/M_{33},$$
$$\varphi_4 = -EIw_7^{(4)}/\rho K,$$
$$\begin{aligned}Y_1 = \Big\{&-\widehat{M}_{11}(\theta)\dot{w}_2 - M_{12}(\theta)\dot{w}_4 - M_{13}(\theta)\dot{w}_6 - C_{11}w_2\\ &- C_{12}w_4 - C_{13}w_6\\ &+ [L_1\sin(\theta_2+\theta_3)\dot{\theta}_2 + [\sin(\theta_2+\theta_3)L_2\\ &+ \sin\theta_3 L_2]\dot{\theta}_3]\rho K\int_0^l w_8\mathrm{d}r\\ &- [\cos(\theta_2+\theta_3)L_1 + \cos\theta_3 L_2]EIw_7'''(0)\Big\}/M_{11},\end{aligned}$$
$$\begin{aligned}Y_2 = \{&-\widehat{M}_{22}(\theta)\dot{w}_4 - M_{21}(\theta)\dot{w}_2 - M_{23}(\theta)\dot{w}_6 - C_{21}w_2\\ &- C_{22}w_4 - C_{23}w_6\\ &- [L_1\sin(\theta_2+\theta_3)\dot{\theta}_1 - L_2\sin\theta_3\dot{\theta}_3]\rho K\\ &\int_0^l w_8\mathrm{d}r - L_2\cos\theta_3 EIw_7'''(0)\}/M_{22},\end{aligned}$$
$$\begin{aligned}Y_3 = \{&-\widehat{M}_{33}(\theta)\dot{w}_6 - M_{31}(\theta)\dot{w}_2 - M_{32}(\theta)\dot{w}_4 - C_{31}w_2\\ &- C_{32}w_4 - C_{33}w_6\\ &- [\sin(\theta_2+\theta_3)L_1\dot{\theta}_1 + sin\theta_3L_2\dot{\theta}_1 + \sin\theta_3L_2\dot{\theta}_2]\rho K\\ &\int_0^l w_8\mathrm{d}r\}/M_{33},\end{aligned}$$
$$\begin{aligned}Y_4 = &-L_1\dot{w}_2\cos(\theta_2+\theta_3) - L_2(\dot{w}_2+\dot{w}_4)cos(\theta_3)\\ &+ L_2\dot{w}_6\sin\theta_3\\ &- (L_3+r)(\dot{w}_2+\dot{w}_4+\dot{w}_6) - \dot{w}_4[\dot{\theta}_1L_1\sin(\theta_2+\theta_3)\\ &+ \sin\theta_3L_2\dot{\theta}_3] - \dot{w}_6[\dot{\theta}_1L_1\sin(\theta_2+\theta_3)\\ &+ (\dot{\theta}_1+\dot{\theta}_2)L_2\sin\theta_3],\end{aligned}$$

in which $M_{ij}(\theta)$ $(i, j = 1, 2, 3)$ in (23) contains $\theta$ form as $\widehat{M}_{ij}(\theta)$, otherwise form as $M_{ij}$. The defined Hilbert space is shown below

$$H = R^6 \times H^2(\Omega) \times L^2(\Omega),$$
$$D(A) = R^6 \times H^4(\Omega) \times H^2(\Omega),$$

where

$$\Omega = [0, l],\ L^2(\Omega) = \{f : \Omega \to R | \int_0^l |f|^2\mathrm{d}r < \infty\},$$
$$H^k(\Omega) = \{f : \Omega \to R | f, f', f'', \cdots, f^{(k)} \in L^2(\Omega)\}.$$

Define the energy of estimate error model as follows

$$\begin{aligned}E_a = &\frac{1}{2}\int_0^l [\rho K\dot{\tilde{z}}^2(r,t) + EI\tilde{u}''^2(t,r)]\mathrm{d}r\\ &+\frac{1}{2}\dot{\tilde{\theta}}^{\mathrm{T}}M(\theta)\dot{\tilde{\theta}} + \frac{1}{2}k_p\tilde{\theta}^{\mathrm{T}}\tilde{\theta}.\end{aligned} \tag{35}$$

Then the time derivative of (35) is computed as

$$\begin{aligned}\dot{E}_a = &\int_0^l [\rho K\dot{\tilde{z}}(r,t)\ddot{\tilde{z}}(r,t) + EI\tilde{u}''(t,r)\dot{\tilde{u}}''(t,r)]\mathrm{d}r\\ &+\dot{\tilde{\theta}}^{\mathrm{T}}M(\theta)\ddot{\tilde{\theta}} + \frac{1}{2}\dot{\tilde{\theta}}^{\mathrm{T}}\dot{M(\theta)}\dot{\tilde{\theta}} + k_p\dot{\tilde{\theta}}^{\mathrm{T}}\tilde{\theta}.\end{aligned} \tag{36}$$

Substituting the observer error dynamics (26)–(30) into the above equation, then further obtain(see Appendix B)

$$\dot{E}_a = -k_d\dot{\tilde{\theta}}^{\mathrm{T}}\dot{\tilde{\theta}} \le 0. \tag{37}$$

From Eq. (37), it can be easily shown that operator $A$ is dissipated, and $A$ generates a $C_0$ semigroup in $H$. Since $Y(0) = 0$, $Y(w)$ is differentiable, we can easily get $(\lambda I - A)^{-1}$ is a compact operator for sufficiently large $\lambda > 0$(e.g., for$\lambda > \tilde{\alpha}$) [32,33], then the solutions of (32) locally exist in, moreover the bounded solutions are also precompact in $D \subset H$; Based on the LaSalle's Invariance Principle, it then follows that the solution of the system $w(t) \to \hat{C}$ asymptotically as $t \to \infty$, where $\hat{C} = \{w \in D | \dot{E}(w) = 0\}$, and $\hat{\Omega}$ is the largest invariant set in $\hat{C}$. However, Eq. (37) is not sufficient to prove the stability of the designed observer. To strictly prove the observer is asymptotically stable, we should apply the extended LaSalle's Invariance Principle to show $\dot{E}_a = 0$ implies $w = 0$ and then verify the observer stability.

So from $\dot{E}_a = 0$, yields

$$\dot{\tilde{\theta}}_i = 0,\quad \tilde{\theta}_i = \theta_i - \hat{\theta}_i = \text{const}. \tag{38}$$

Putting (38) into (26)–(30) yields

$$\begin{aligned}&\rho K\int_0^l [-L_1\sin(\theta_2+\theta_3)]\dot{\tilde{u}}\mathrm{d}r\dot{\theta}_2 - \rho K[\sin(\theta_2+\theta_3)L_1\\ &+\sin\theta_3L_2]\int_0^l \dot{\tilde{u}}\mathrm{d}r\dot{\theta}_3\\ &-EI\tilde{u}_0'' + [\cos(\theta_2+\theta_3)L_1 + \cos\theta_3L_2 + L_3]EI\tilde{u}_0'''\\ &+k_{p1}\tilde{\theta}_1 = 0,\end{aligned} \tag{39}$$

$$\begin{aligned}&\rho KL_1\sin(\theta_2+\theta_3)\int_0^l \dot{\tilde{u}}\mathrm{d}r\dot{\theta}_1 - \rho KL_2\sin\theta_3\\ &\int_0^l \dot{\tilde{u}}\mathrm{d}r\dot{\theta}_3 - EI\tilde{u}_0''\\ &+(L_2\cos\theta_3 + L_3)EI\tilde{u}_0''' + k_{p2}\tilde{\theta}_2 = 0,\end{aligned} \tag{40}$$

$$\begin{aligned}&\rho K[\sin(\theta_2+\theta_3)L_1 + \sin\theta_3L_2]\int_0^l \dot{\tilde{u}}\mathrm{d}r\dot{\theta}_1 + \rho KL_2\sin\theta_3\\ &\int_0^l \dot{\tilde{u}}\mathrm{d}r\dot{\theta}_2 - EI\tilde{u}_0'' + L_3EI\tilde{u}_0''' + k_{p3}\tilde{\theta}_3 = 0,\end{aligned} \tag{41}$$

$$\ddot{\tilde{u}}(t,r) + \frac{EI}{\rho K}\tilde{u}^4(t,r) = 0, \tag{42}$$

$$\tilde{u}_0 = 0, \tilde{u}_0' = 0, \tilde{u}_E'' = 0, \tilde{u}_E''' = 0. \tag{43}$$

By solving (42) and (43), the solutions $\tilde{u}(t, r)$ can be get according to the method of separating variables.

$$\tilde{u}(t, r) = \varphi(r)e^{\lambda t}, \tag{44}$$

where $\varphi(r)$ are eigenfunctions for Eq. (44) and $\lambda$ are nonzero complex eigenvalues. Substituting (44) into (42), we have

$$\varphi^4(r) = -\frac{\lambda^2 \rho K}{EI}\varphi(r). \tag{45}$$

The solution of (45) is given by

$$\varphi(r) = \gamma_1 e^{\beta r} + \gamma_2 e^{-\beta r} + \gamma_3 e^{j\beta r} + \gamma_4 e^{-j\beta r}, \tag{46}$$

where $\gamma_i (i = 1, 2, 3, 4) \in R$, $\beta = \sqrt[(4)]{-\lambda^2 \rho K/EI} \in C$; $j$ is the imaginary unit, $j^2 = -1$.

Applying Eq. (46) to the boundary condition (43), we obtain

$$\varphi(0) = \varphi'(0) = \varphi''(l) = \varphi'''(l) = 0. \tag{47}$$

Using (46) and (47), we have

$$\begin{cases} \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 = 0 \\ \gamma_1 - \gamma_2 + \gamma_3 - \gamma_4 = 0 \\ \gamma_1 + \gamma_2 - \gamma_3 - \gamma_4 = 0 \\ \gamma_1 - \gamma_2 - \gamma_3 + \gamma_4 = 0. \end{cases} \tag{48}$$

The solution of Eq. (48) is $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 0$, hence $\varphi(r) = 0$, then we have

$$\tilde{u}(t, r) = \varphi(r)e^{\lambda t} = 0. \tag{49}$$

From Eqs. (39)–(41), we obtain that $\tilde{\theta}_i = 0$ and thus get

$$\hat{\theta}_i \to \theta_i,\ \dot{\hat{\theta}}_i \to \dot{\theta}_i,\ \hat{u}(t, r) \to u(t, r),\ \dot{\hat{u}}(t, r) \to \dot{u}(t, r). \tag{50}$$

Therefore, we proved $w = 0$ from $\dot{E}_a = 0$, the designed observer is asymptotic stability.

## Observer-based controller design

Furthermore, we propose the observer-based controller that requires only the independent joint input, and the controller can ensure the asymptotic stability.

**Theorem 1** *If we set the control law as follows based on the estimate of observer (23)–(25), the closed-loop system is asymptotically stability.*

$$\tau = -k_1 e - k_2 \dot{e} - k_3 \dot{e} F(\theta, \hat{u}_0'', \hat{u}_0''')^{\mathrm{T}} F(\theta, \hat{u}_0'', \hat{u}_0''') + G, \tag{51}$$

*in which* $k_1 = diag(k_{11}, k_{12}, k_{13})$, $k_2 = diag(k_{21}, k_{22}, k_{23})$, $k_3 = diag(k_{31}, k_{32}, k_{33})$ *are* $3 \times 3$ *positive-definite diagonal matrix.*

***Proof*** First, define vector $p$ as

$$\begin{aligned} p &= [\,p_1 \quad p_2 \quad p_3 \quad p_4 \quad p_5 \quad p_6 \quad p_7 \quad p_8 \quad w^{\mathrm{T}}\,]^{\mathrm{T}} \\ &= [\,e_1 \quad \dot{e}_1 \quad e_2 \quad \dot{e}_2 \quad e_3 \quad \dot{e}_3 \quad u \quad \dot{u} \quad w^{\mathrm{T}}\,]^{\mathrm{T}}. \end{aligned} \tag{52}$$

The observer error dynamics (26)–(30), closed-loop system model (11)–(13) and (51) are rewritten in the matrix operator form as

$$\dot{p} = \mathrm{B}p + N(p), \quad p(0) \in H, \tag{53}$$

where $B$ represents an infinite-dimensional linear operator as follows

$$\begin{aligned} \mathrm{B}p &= [\,p_2 \quad s_1 \quad p_4 \quad s_2 \quad p_6 \quad s_3 \quad p_8 \quad s_4 \quad Aw^{\mathrm{T}}]^{\mathrm{T}}, \\ N(p) &= [\,0 \quad N_1 \quad 0 \quad N_2 \quad 0 \quad N_3 \quad 0 \quad N_4 \quad Y(w)^{\mathrm{T}}]^{\mathrm{T}}, \end{aligned}$$

in which

$$\begin{aligned}
s_1 &= (EIp_7''(0) - EIL_3 p_7'''(0) - k_{11}p_1 - k_{21}p_2)/M_{11}, \\
s_2 &= (EIp_7''(0) - EIL_3 p_7'''(0) - k_{12}p_3 - k_{22}p_4)/M_{22}, \\
s_3 &= (EIp_7''(0) - EIL_3 p_7'''(0) - k_{13}p_5 - k_{23}p_6)/M_{33}, \\
s_4 &= -\frac{EI}{\rho K} p_7^{(4)}, \\
N_1 &= \Big\{-\widehat{M}_{11}(\theta)\dot{p}_2 - M_{12}(\theta)\dot{p}_4 - M_{13}(\theta)\dot{p}_6 - C_{11}p_2 \\
&\quad - C_{12}p_4 - C_{13}p_6 \\
&\quad + [\sin(\theta_2+\theta_3)L_1 p_4 + [\sin(\theta_2+\theta_3)L_1 \\
&\quad + \sin\theta_3 L_2]p_6]p_2 \rho K \int_0^l p_8 \mathrm{d}r \\
&\quad - [\cos(\theta_2+\theta_3)L_1 + \cos\theta_3 L_2]EIp_7'''(0) - k_{31}p_2 \hat{F}_1{}^{\mathrm{T}}\hat{F}_1\Big\}/M_{11}, \\
N_2 &= \{-\widehat{M}_{22}(\theta)\dot{p}_4 - M_{21}(\theta)\dot{p}_2 - M_{23}(\theta)\dot{p}_6 - C_{21}p_2 - C_{22}p_4 \\
&\quad - C_{23}p_6 - [L_1\sin(\theta_2+\theta_3)p_2 - L_2\sin\theta_3 p_6]p_4 \rho K \int_0^l p_8 \mathrm{d}r \\
&\quad - L_2\cos\theta_3 EIp_7'''(0) - k_{32}p_4 \hat{F}_2{}^{\mathrm{T}}\hat{F}_2\}/M_{22}, \\
N_3 &= \{-\widehat{M}_{33}(\theta)\dot{p}_6 - M_{31}(\theta)\dot{p}_2 - M_{32}(\theta)\dot{p}_4 - C_{31}p_2 - C_{32}p_4 \\
&\quad - C_{33}p_6 - [\sin(\theta_2+\theta_3)L_1 p_2 + \sin(\theta_3)L_2 p_2 \\
&\quad + L_2\sin\theta_3 p_4]p_6 \rho K \int_0^l p_8 \mathrm{d}r - k_{33}p_6 \hat{F}_3{}^{\mathrm{T}}\hat{F}_3\}/M_{33}, \\
N_4 &= -L_1\dot{p}_2\cos(\theta_2+\theta_3) - L_2(\dot{p}_2+\dot{p}_4)\cos(\theta_3) + L_2\dot{p}_6\sin\theta_3 \\
&\quad - (L_3+r)(\dot{p}_2+\dot{p}_4+\dot{p}_6) - \dot{p}_4[\dot{\theta}_1 L_1\sin(\theta_2+\theta_3) + \sin\theta_3 L_2\dot{\theta}_3]
\end{aligned}$$

**Table 1** Physical parameters of the system

| Parameter | Value | Parameter | Value | Parameter | Value |
|---|---|---|---|---|---|
| $L_1$ | 0.7 m | $L_{1g}$ | 0.35 m | $m_1$ | 8 kg |
| $L_2$ | 0.6 m | $L_{2g}$ | 0.35 m | $m_2$ | 8 kg |
| $L_3$ | 0.2 m | $L_{3g}$ | 0.17 m | $m_3$ | 4 kg |
| $I_i$ | $37 \times 10^{-5}$ kg m | $EI$ | 20 Nm$^2$ | $l$ | 1 m |
| $\rho K$ | 0.5 kg/m | | | | |

$$- \dot{p}_6[\dot{\theta}_1 L_1 \sin(\theta_2+\theta_3) + (\dot{\theta}_1+\dot{\theta}_2)L_2 \sin\theta_3].$$

The state space $H$ and the domain $D(\mathrm{B})$ of the operator $B$ are defined as

$$H = R^6 \times H^2(\Omega) \times L^2(\Omega) \times R^6 \times H^2(\Omega) \times L^2(\Omega),$$
$$D(\mathrm{B}) = R^6 \times H^4(\Omega) \times H^2(\Omega) \times R^6 \times H^4(\Omega) \times H^2(\Omega).$$

Then, choose the energy function of the close-loop system as

$$E = \frac{1}{2}\int_0^l [\rho K \dot{z}^2(r,t) + EI u''^2(t,r)]\mathrm{d}r + \frac{1}{2}\dot{e}^{\mathrm{T}} M(\theta)\dot{e} + \frac{1}{2}e^{\mathrm{T}} k_1 e + E_a. \tag{54}$$

Differentiating (54) with respect to time by using (19) and (37) yields

$$\dot{E} = -q^T\dot{e} + \dot{e}^{\mathrm{T}}[M(\theta)\ddot{e} + \frac{1}{2}\dot{M}(\theta)\dot{e}] + \dot{e}^{\mathrm{T}}k_1 e - \dot{\tilde{\theta}}^{\mathrm{T}} k_d \dot{\tilde{\theta}}. \tag{55}$$

We further have

$$\dot{E} = -q^T\dot{e} + \dot{e}^{\mathrm{T}}[\tau - F\left(\theta, u_0{}'', u_0{}'''\right) - G] - \dot{\tilde{\theta}}^{\mathrm{T}} k_d \dot{\tilde{\theta}}. \tag{56}$$

From (20), we can get $F\left(\theta, u_0{}'', u_0{}'''\right) - q = 0$, then substitute controller (51) into (56) and have

$$\dot{E} = -\dot{e}^{\mathrm{T}}k_2\dot{e} - k_3\dot{e}^{\mathrm{T}}\dot{e}F(\theta,\hat{u}_0'',\hat{u}_0''')^{\mathrm{T}}F(\theta,\hat{u}_0'',\hat{u}_0''') - \dot{\tilde{\theta}}^{\mathrm{T}} k_d \dot{\tilde{\theta}} \le 0. \tag{57}$$

According to (57), we can know that operator $B$ is dissipated. It can be also verified operator $B$ generates a $C_0$-semigroup in $H$ based on Lumer–Phillips theorem that used in "Infinite-dimensional observer" section. Then similar to the application of LaSalle's Invariance Principle in "Infinite-dimensional observer" section, from $\dot{E} = 0$ we can also get

$$e = \dot{e} = u(t,r) = \dot{u}(t,r) = \tilde{\theta} = \dot{\tilde{\theta}} = \tilde{u}(t,r) = \dot{\tilde{u}}(t,r) = 0. \tag{58}$$

**Table 2** Control parameters of the system

| Parameter | Value | Parameter | Value | Parameter | Value |
|---|---|---|---|---|---|
| $k_{11}$ | 50 | $k_{12}$ | 20 | $k_{13}$ | 12 |
| $k_{21}$ | 45 | $k_{22}$ | 20 | $k_{23}$ | 9 |
| $k_{31}$ | 10 | $k_{32}$ | 10 | $k_{33}$ | 10 |
| $k_{d1}$ | 37 | $k_{d2}$ | 10 | $k_{d3}$ | 7 |
| $k_{p1}$ | 22 | $k_{p2}$ | 17 | $k_{p3}$ | 8 |

This means $p = 0$, it can be proved that the solution of $\dot{E} = 0$ is $p = 0$, and therefore, we get the asymptotically stable of the proposed observer-based control. □

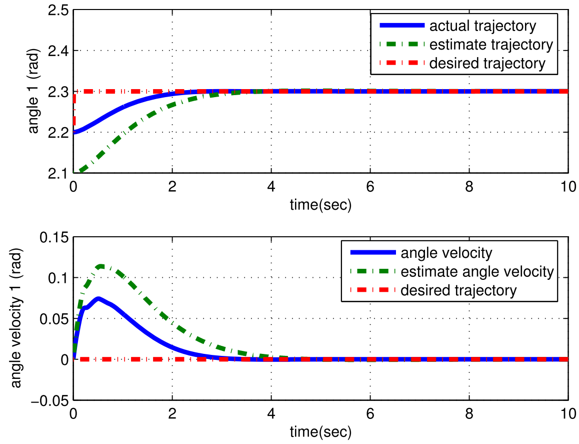
## Simulation

The designed observer and controller strategy are tested with the simulation implemented in MATLAB. System physical parameters are given in Table 1.

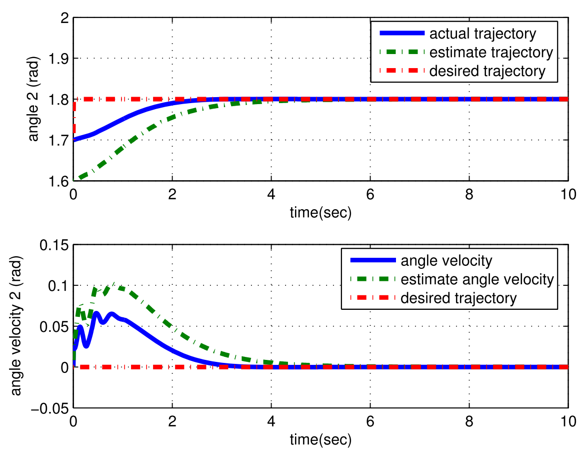
The initial and desired trajectory are as follows:

$$\theta_{10} = 2.2\,\mathrm{rad}, \quad \theta_{20} = 1.7\,\mathrm{rad}, \quad \theta_{30} = 0.2\,\mathrm{rad},$$
$$\theta_{1d} = 2.3\,\mathrm{rad}, \quad \theta_{2d} = 1.8\,\mathrm{rad}, \quad \theta_{3d} = 0.1\,\mathrm{rad}.$$

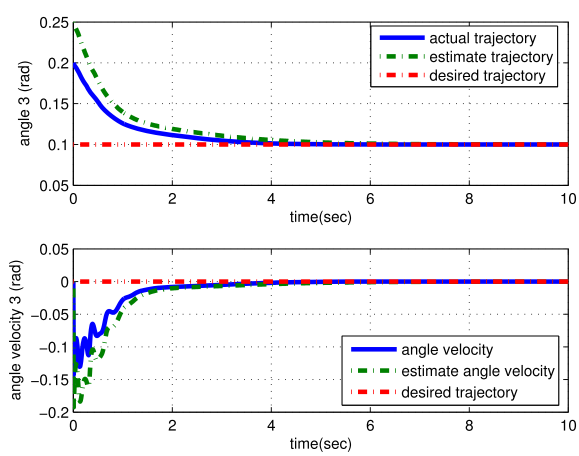
The controller and observer parameters of the system are given in Table 2. The observer-based controller is given to guarantee the asymptotic stability, which make the robot track the desired joint angular and its speed, and suppress the vibration. The simulation result of the proposed controller is provided by Figs. 2, 3, 4, 5, 6, 7 and 8. Note that the simulation time is 10 s. Figure 2 displays the three angles, angular velocities and the corresponding estimates, which can all track the desired trajectory. It is clear that the estimate angles and angle velocities could converge to their true values within 4 s. Figure 3 shows the deflection at link end of the flexible beam and its estimation value, we can conclude that the vibration of the beam is suppressed and the estimate can converge to the real value. Figures 4 and 5 represent the strain force, the shear force and their estimate values, which indicate the vibration informations can be exactly observed. Figure 6 indicates the three-dimensional deformation and its estimation of flexible beam. From Fig. 7, we can see the deformation error




(a) $\theta_1$ , $\hat{\theta}_1$ and $\dot{\theta}_1$, $\dot{\hat{\theta}}_1$




(b) $\theta_2$ , $\hat{\theta}_2$ and $\dot{\theta}_2$, $\dot{\hat{\theta}}_2$




(c) $\theta_3$ , $\hat{\theta}_3$ and $\dot{\theta}_3$, $\dot{\hat{\theta}}_3$

**Fig. 2** Angle $\theta_i$, angle velocity $\dot{\theta}_i$ and estimations $\hat{\theta}_i$, $\dot{\hat{\theta}}_i$

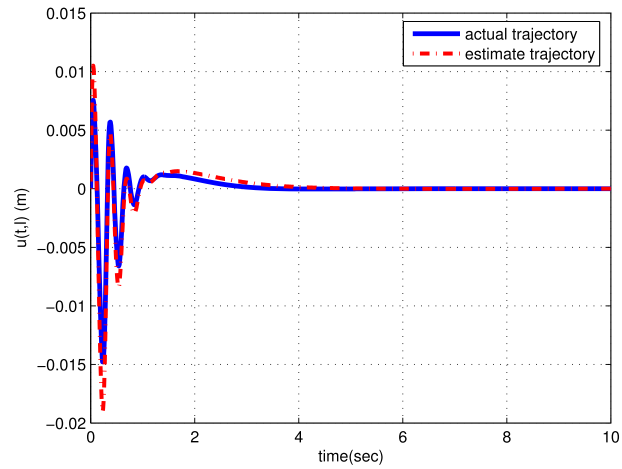



**Fig. 3** End-point deflection $u_E$ and estimation $\hat{u}_E$

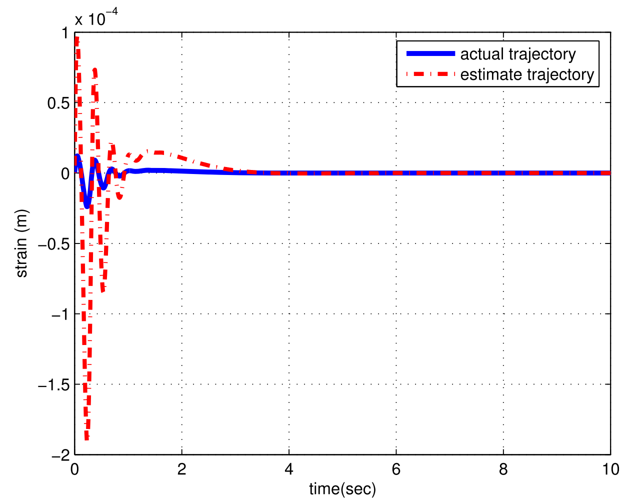



**Fig. 4** Strain force $u_0''$ and estimation $\hat{u}_0''$

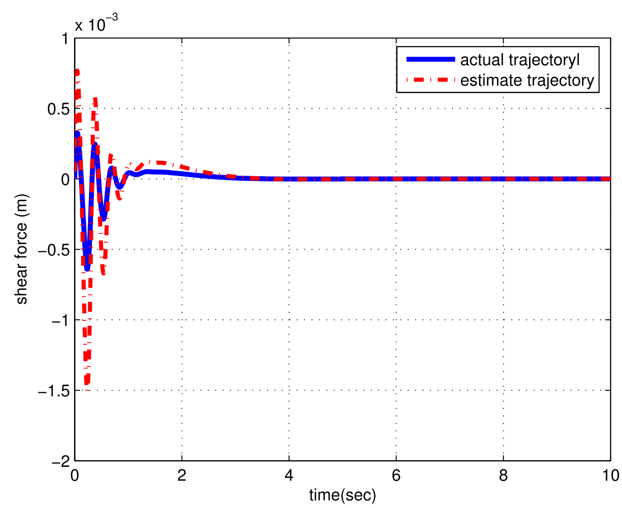



**Fig. 5** Shear force $u_0'''$ and estimation $\hat{u}_0'''$

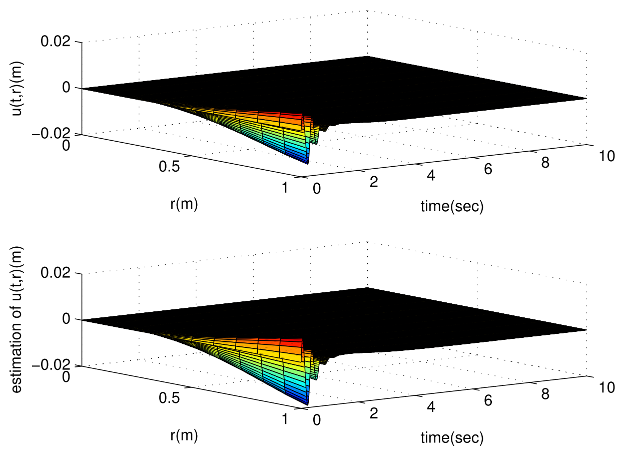



**Fig. 6** Deformation $u(t, r)$ and estimation $\hat{u}(t, r)$ of flexible beam

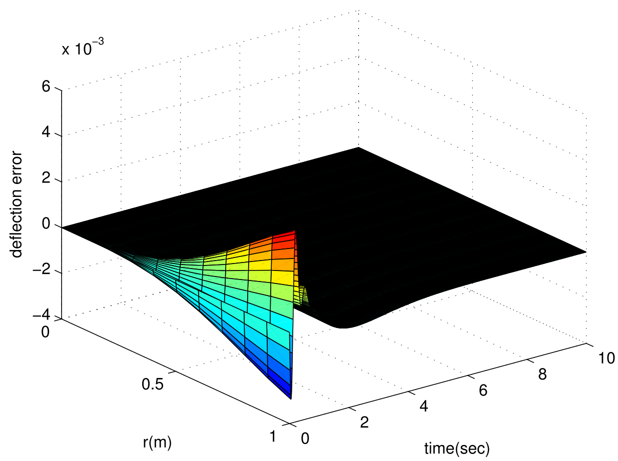



**Fig. 7** Deformation error $\tilde{u}(t, r)$ of flexible beam

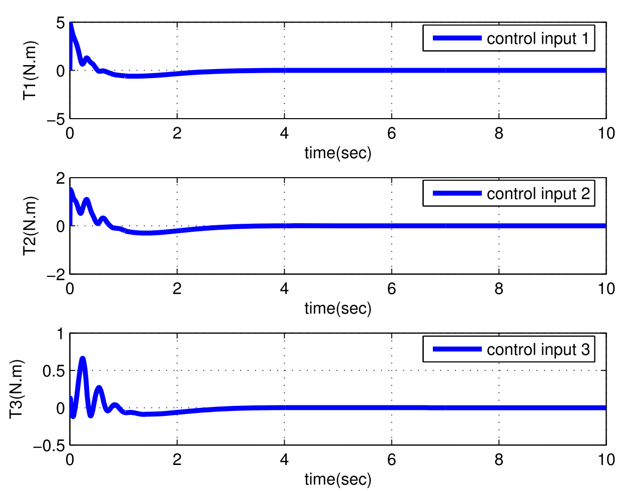



**Fig. 8** Control input $\tau_i (i = 1, 2, 3)$

can be effectively suppressed to asymptotically approach 0, which demonstrates the observer could accurately estimate the vibration deformation states. As shown in Fig. 8, the result is the controller input of the proposed independent joint control. Therefore, the simulations demonstrate the good performance of the proposed infinite-dimensional observer and controller.

## Conclusion

In this paper, a non-linear infinite-dimensional observer is proposed for manipulator operating a flexible beam based on distributed parameter model. We further design an observer-based independent joint control to regulate the angles to follow the desired states and suppress the vibration of the beam simultaneously, which can avoid setting sensors and actuators on the beam. The asymptotic convergence of the observer and the controller is validated through theoretical proof. Numerical simulations have demonstrated the performance of the proposed observer and the control strategy. The next problem to be tackled is to research the cooperative force control of manipulation for flexible object by multi-arm robot based on infinite-dimensional model.

**Acknowledgements** This paper is supported by the National Natural Science Foundation of China (Grant No. 62173047), the Department of Science and Technology of Jilin Province (Grant No. 20210101179JC), the Science and Technology Project of Jilin Provincial Education Department (Grant No. JJKH20200672KJ).

## Declarations

**Conflict of interest** The authors declared no potential conflict of interest with respect to the research, authorship, and/or publication of this article.



## Appendix A: Modeling parameters

The parameters of joint Eq. (11) are as follows.

$$M_{11} = I_1 + I_2 + I_3 + m_1 L_{1g}^2 + m_2 L_{2g}^2 + m_3 L_{3g}^2$$

$$
\begin{aligned}
&+(m_2+m_3)L_1^2+\\
&m_3L_2^2+2L_1(m_2L_{2g}+m_3L_2)\cos\theta_2\\
&+2m_3L_1L_{3g}\cos(\theta_2+\theta_3)\\
&+2m_3L_2L_{3g}\cos\theta_3+\rho Kl[\sin(\theta_2+\theta_3)L_1\\
&+\sin\theta_3L_2]^2,
\end{aligned}\tag{A1}
$$

$$
\begin{aligned}
M_{12}=M_{21}&=I_2+I_3+m_2L_{2g}^2+m_3L_{3g}^2\\
&+m_3L_2^2+(m_2L_{2g}\\
&+m_3L_2)\cos\theta_2L_1+m_3L_1L_{3g}\cos(\theta_2+\theta_3)\\
&+2m_3L_2L_{3g}\cos\theta_3\\
&+\rho KlL_2[\sin(\theta_2+\theta_3)L_1+\sin\theta_3L_2]\sin\theta_3,
\end{aligned}\tag{A2}
$$

$$
\begin{aligned}
M_{13}=M_{31}&=I_3+m_3L_{3g}^2+m_3L_1L_{3g}\cos(\theta_2+\theta_3)\\
&+m_3L_2L_{3g}\cos\theta_3,
\end{aligned}\tag{A3}
$$

$$
\begin{aligned}
M_{22}&=I_2+I_3+m_2L_{2g}^2+m_3L_{3g}^2+m_3L_2^2\\
&+2m_3L_2L_{3g}\cos\theta_3\\
&+\rho K{L_2}^2l\sin\theta_3\sin\theta_3,
\end{aligned}\tag{A4}
$$

$$
M_{23}=M_{32}=I_3+m_3L_{3g}^2+m_3L_2L_{3g}\cos\theta,\tag{A5}
$$

$$
M_{33}=I_3+m_3L_{3g}^2,\tag{A6}
$$

$$
\begin{aligned}
C_{11}&=-(m_2L_{2g}+m_3L_2)L_1\dot\theta_2\sin\theta_2\\
&-m_3L_1L_{3g}\sin(\theta_2+\theta_3)(\dot\theta_2+\dot\theta_3)\\
&-m_3L_2L_{3g}\dot\theta_3\sin\theta_3+\rho Kl[\sin(\theta_2+\theta_3)L_1\\
&+\sin\theta_3L_2][L_1(\dot\theta_2\\
&+\dot\theta_3)\cos(\theta_2+\theta_3)+L_2\dot\theta_3\cos\theta_3],
\end{aligned}\tag{A7}
$$

$$
\begin{aligned}
C_{12}&=-(m_2L_{2g}+m_3L_2)(\dot\theta_1+\dot\theta_2)L_1\sin\theta_2\\
&-m_3(\dot\theta_1+\dot\theta_2+\dot\theta_3)L_1L_{3g}\sin(\theta_2\\
&+\theta_3)-m_3L_2L_{2g}\dot\theta_3\sin\theta_3-(L_3l+\frac{1}{2}l^2)\\
&\times(\dot\theta_1+\dot\theta_2+\dot\theta_3)[L_1\sin(\theta_2\\
&+\theta_3)]+\rho KL_2l\sin\theta_3[L_1(\dot\theta_2+\dot\theta_3)\cos(\theta_2+\theta_3)\\
&+L_2\dot\theta_3\cos\theta_3]-\rho KL_2(\dot\theta_1\\
&+\dot\theta_2)\cos\theta_3[L_1\sin(\theta_2+\theta_3)]\\
&+\rho KL_1L_2l\dot\theta_1\cos(\theta_2+\theta_3)\sin\theta_3,
\end{aligned}\tag{A8}
$$

$$
\begin{aligned}
C_{13}&=-m_3L_1L_{3g}(\dot\theta_1+\dot\theta_2+\dot\theta_3)\sin(\theta_2+\theta_3)\\
&-m_3L_2L_{3g}(\dot\theta_1+\dot\theta_2+\dot\theta_3)\sin\theta_3\\
&-\rho K(L_3l+\frac{1}{2}l^2)(\dot\theta_1+\dot\theta_2+\dot\theta_3)\\
&\times[L_1\sin(\theta_2+\theta_3)+L_2\sin\theta_3],
\end{aligned}\tag{A9}
$$

$$
\begin{aligned}
C_{21}&=(m_2L_{2g}+m_3L_2)L_1\dot\theta_1\sin\theta_2\\
&+m_3L_1L_{3g}\dot\theta_1\sin(\theta_2+\theta_3)\\
&-m_3L_2L_{2g}\dot\theta_3\sin\theta_3+\rho K(L_3l+\frac{1}{2}l^2)(\dot\theta_1+\dot\theta_2+\dot\theta_3)\\
&\times\sin(\theta_2\\
&+\theta_3)+\rho KL_2l\cos\theta_3[L_1(\dot\theta_1+\dot\theta_2+\dot\theta_3)\sin(\theta_2+\theta_3)\\
&+L_2\dot\theta_3\sin\theta_3]\\
&-\rho KL_1L_2l\dot\theta_1\cos(\theta_2+\theta_3)\sin\theta_3,
\end{aligned}\tag{A10}
$$

$$
C_{22}=-(m_3L_{2g}+\rho KL_2\cos\theta_3)L_2\dot\theta_3\sin\theta_3,\tag{A11}
$$

$$
\begin{aligned}
C_{23}&=-m_3L_2L_{3g}(\dot\theta_1+\dot\theta_2+\dot\theta_3)\sin\theta_3\\
&-\rho K(L_3l+\frac{1}{2}l^2)(\dot\theta_1+\dot\theta_2+\dot\theta_3)L_2\sin\theta_3,
\end{aligned}\tag{A12}
$$

$$
\begin{aligned}
C_{31}&=m_3L_1L_{3g}\dot\theta_1\sin(\theta_2+\theta_3)+m_3(\dot\theta_1+\dot\theta_2)\\
&L_2L_{3g}\sin\theta_3\\
&+\rho K(L_3l+\frac{1}{2}l^2)(\dot\theta_1+\dot\theta_2+\dot\theta_3)[\sin(\theta_2+\theta_3)\\
&L_1+\sin\theta_3L_2],
\end{aligned}\tag{A13}
$$

$$
\begin{aligned}
C_{32}&=m_3L_2L_{3g}(\dot\theta_1+\dot\theta_2)\sin\theta_3+\rho K(L_3l\\
&+\frac{1}{2}l^2)(\dot\theta_1+\dot\theta_2+\dot\theta_3)L_2\sin\theta_3,
\end{aligned}\tag{A14}
$$

$$
C_{33}=0,\tag{A15}
$$

$$
F_1=-EIu_0''+(L_1\cos(\theta_2+\theta_3)+L_2\cos\theta_3+L_3)EIu_0''',\tag{A16}
$$

$$
F_2=-EIu_0''+(L_2\cos\theta_3+L_3)EIu_0''',\tag{A17}
$$

$$
F_3=-EIu_0''+L_3EIu_0'''.\tag{A18}
$$

$$
\begin{aligned}
G_1&=g(m_1L_{1g}+m_2L_1+m_3L_1)\cos\theta_1\\
&+g(m_2L_{2g}+m_3L_2)\cos(\theta_1+\theta_2)\\
&+m_3L_{3g}g\cos(\theta_1+\theta_2+\theta_3),
\end{aligned}\tag{A19}
$$

$$
\begin{aligned}
G_2&=g(m_2L_{2g}+m_3L_2)\cos(\theta_1+\theta_2)\\
&+m_3L_{3g}g\cos(\theta_1+\theta_2+\theta_3),
\end{aligned}\tag{A20}
$$

$$
G_3=m_3gL_{3g}\cos(\theta_1+\theta_2+\theta_3),\tag{A21}
$$

$$
f=\begin{pmatrix}
 & -A\int_0^l\dot u\mathrm{d}r & -A-\rho KL_2\sin\theta_3\int_0^l\dot u\mathrm{d}r\\
A\int_0^l\dot u\mathrm{d}r & & -\rho KL_2\sin\theta_3\int_0^l\dot u\mathrm{d}r\\
A+\rho KL_2\sin\theta_3\int_0^l\dot u\mathrm{d}r & \rho KL_2\sin\theta_3\int_0^l\dot u\mathrm{d}r &
\end{pmatrix},\tag{A22}
$$

where $A=\rho KL_1\sin(\theta_2+\theta_3)$.

## Appendix B: Calculation of the derivative of Lyapunov function

Substituting the observer error dynamics (26)–(30) into Eq. (36), then further obtain

$$
\begin{aligned}
\dot E_a&=\rho K\int_0^l\dot{\tilde z}(-\frac{EI}{\rho K}\tilde u^{(4)})\mathrm{d}r+EIu''\dot u'|_0^l-EIu'''\dot u|_0^l\\
&+\int_0^l EIu^{(4)}\dot u\mathrm{d}r\\
&+\dot{\tilde\theta}^{\mathrm T}[M(\theta)\ddot{\tilde\theta}+\frac{1}{2}\dot M(\theta)\dot{\tilde\theta}]+k_p\dot{\tilde\theta}^{\mathrm T}\tilde\theta\\
&=\rho K\int_0^l[\dot{\tilde z}(-\frac{EI}{\rho K}\tilde u^{(4)})+EI\tilde u^{(4)}\dot{\tilde u}]\mathrm{d}r+\dot{\tilde\theta}_1\{EI\tilde u_0''\\
&-[L_1cos(\theta_2+\theta_3)\\
&+L_2\cos\theta_3+L_3]EI\tilde u_0'''-k_{d1}\dot{\tilde\theta}_1-k_{p1}\tilde\theta_1\}\\
&+\dot{\tilde\theta}_2\{EI\tilde u''{}_0\\
&-(L_2\cos\theta_3+L_3)EI\tilde u_0'''-k_{d2}\dot{\tilde\theta}_2-k_{p2}\tilde\theta_2\}\\
&+\dot{\tilde\theta}_3\{EI\tilde u_0''-L_3EI\tilde u_0'''\\
&-k_{d3}\dot{\tilde\theta}_3-k_{p3}\tilde\theta_3\}+k_p\dot{\tilde\theta}^{\mathrm T}\tilde\theta,
\end{aligned}\tag{B1}
$$

where $EIu''\dot u'|_0^l=0$, $EIu'''\dot u|_0^l=0$. Based on the energy dynamic of flexible beam, we can obtain

$$
\begin{aligned}
&\rho K\int_0^l\dot{\tilde z}(-\frac{EI}{\rho K}\tilde u^{(4)})\mathrm{d}r+\int_0^l EI\tilde u^{(4)}\dot{\tilde u}\mathrm{d}r\\
&=\rho K\int_0^l[\dot{\tilde\theta}_1L_1\cos(\theta_2+\theta_3)+(\dot{\tilde\theta}_1+\dot{\tilde\theta}_2)L_2\cos\theta_3\\
&+(L_3+r)(\dot{\tilde\theta}_1+\dot{\tilde\theta}_2+\dot{\tilde\theta}_3)
\end{aligned}
$$

$$
\begin{aligned}
&+ \dot{\tilde{u}}][L_1\ddot{\theta}_1 \cos(\theta_2+\theta_3) - \dot{\tilde{\theta}}_1 \sin(\theta_2+\theta_3)L_1(\dot{\theta}_2+\dot{\theta}_3) \\
&+ L_2(\ddot{\tilde{\theta}}_1 + \ddot{\tilde{\theta}}_2)\cos\theta_3 \\
&- (\dot{\tilde{\theta}}_1 + \dot{\tilde{\theta}}_2)L_2\dot{\theta}_3 \sin\theta_3 + (L_3+r)(\ddot{\tilde{\theta}}_1 + \ddot{\tilde{\theta}}_2 + \ddot{\tilde{\theta}}_3) + \ddot{\tilde{u}}]\mathrm{d}r \\
&+ \int_0^l EI\tilde{u}^{(4)}\dot{\tilde{u}}\mathrm{d}r \\
= {}& \rho K \int_0^l \dot{\tilde{r}}_t^{\mathrm{T}}\ddot{\tilde{r}}_t \mathrm{d}r + \int_0^l EI\tilde{u}^{(4)}\dot{\tilde{u}}\mathrm{d}r. \qquad \text{(B2)}
\end{aligned}
$$

From Eq. (19), we can get

$$
\begin{aligned}
&\rho K \int_0^l \dot{\tilde{z}}\left(-\frac{EI}{\rho K}\tilde{u}^{(4)}\right)\mathrm{d}r + \int_0^l EI\tilde{u}^{(4)}\dot{\tilde{u}}\mathrm{d}r \\
&= [L_1\dot{\tilde{\theta}}_1\cos(\theta_2+\theta_3) + (\dot{\tilde{\theta}}_1+\dot{\tilde{\theta}}_2)L_2\cos\theta_3 + (\dot{\tilde{\theta}}_1 + \dot{\tilde{\theta}}_2 \\
&\quad + \dot{\tilde{\theta}}_3)L_3]EI\tilde{u}_0''' - EI\tilde{u}_0''(\dot{\tilde{\theta}}_1 + \dot{\tilde{\theta}}_2 + \dot{\tilde{\theta}}_3) \\
&= \dot{\tilde{\theta}}_1\{EI\tilde{u}_0'' - [L_1\cos(\theta_2+\theta_3) + L_2\cos\theta_3 + L_3]EI\tilde{u}_0'''\} \\
&\quad + \dot{\tilde{\theta}}_2\{EI\tilde{u}''_0 - (L_2\cos\theta_3 + L_3)EI\tilde{u}_0'''\} \\
&\quad + \dot{\tilde{\theta}}_3\{EI\tilde{u}_0'' - L_3EI\tilde{u}_0'''\}. \qquad \text{(B3)}
\end{aligned}
$$

Thus, we get

$$
\dot{E}_a = -k_d\dot{\tilde{\theta}}^{\mathrm{T}}\dot{\tilde{\theta}} \le 0. \qquad \text{(B4)}
$$

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