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ORIGINAL ARTICLE

New ranking method for normal intuitionistic sets under crisp, interval environments and its applications to multiple attribute decision making process

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Abstract
The aim of this paper is to present novel algorithms for solving the multiple attribute decision-making problems under the normal intuitionistic fuzzy set environment. Normal intuitionistic and interval-valued intuitionistic sets are the essential mechanisms for influencing the decision-making queries with anonymous and indeterminant data by engaging a degree of membership and non-membership of normal distribution data in quantitative terms. Holding these features in mind and united the idea of hesitation degree, this paper presents some improved score functions to rank the normal intuitionistic and interval-valued intuitionistic sets. The advantage of these proposed functions is to overwhelm the weakness of the existing functions and will aid to rank the given objects in a more consistent way. The numerous salient features of the proposed functions are studied. Later, we develop two new algorithms for interval-valued as well as crisp numbers based on the proposed functions to solve multiple attribute decision-making problems. The given approaches have been confirmed with numerical examples and the advantages, as well as comparative analysis, are furnished to shows its influence over existing approaches.

Keywords Multiple attribute decision-making · Interval-valued set · Normal distribution functions · Normal intuitionistic fuzzy set · Score function

Introduction

Multiple attribute decision making (MADM) belongs to the process of getting optimal alternatives in complicated situations via synthetically assessing the values of multiple criteria of all alternatives given by a group of domain experts [1,2]. In this process, there are two crucial tasks. The first one is to call the environment where the consequences of different criteria are measured adequately, while the second task is to aggregate the related information. However, in any case, because of the lack of learning and other factors, it is greatly troublesome- if not difficult express the data. Originally, the data which accesses the alternatives is ordinarily taken as a crisp number. However, in many cases, it is difficult for a person to opt for a suitable one due to the presence of several kinds of uncertainties in the data, which may occur due to a lack of knowledge or human error. To access it completely, a theory of fuzzy set (FS) [3] and its extension such as intuitionistic fuzzy set (IFS) [4], cubic intuitionistic fuzzy set [5], interval-valued IFS (IVIFS) [6], linguistic interval-valued IFS [1], are used by the researchers.

In all these existing theories, an object is assessed in terms of their membership grade ς and non-membership grade υ such that $\varsigma + \upsilon \leq 1$ for $\varsigma, \upsilon \in [0, 1]$. However, for IVIFSs, each object is followed by defining two grades of memberships named as membership, $[\underline{\varsigma}, \overline{\varsigma}]$ and non-membership, $[\underline{\upsilon}, \overline{\upsilon}]$ with the constraints that $\overline{\varsigma} + \overline{\upsilon} \leq 1$ for each number lying between [0, 1]. After their existence, many researchers have described the basic operational laws and aggregation operators to solve MADM problems. For instance, Xu and Yager [7] presented weighted averaging operators for the pairs of intuitionistic fuzzy numbers (IFNs). Garg [8] presented interactive aggregation operators for IFNs. In [9,10], authors have acted the averaging and geometric operators for IVIFSs. Ye [11] established the accuracy function to rank the different interval-valued IFNs (IVIFNs). Afterward, Nayagam et al. [12] overcome the weaknesses of Ye's method

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by establishing a new accuracy function for IVIFNs. Wang et al. [13] gave a score function for IVIFNs based on the prospect value function and employed them to solve the MADM problems. Garg [14] presented a generalized score function to overcome the deficiencies observed in the existing studies. Xing et al. [15] proposed a ranking approach for IFNs based on the Euclidean distance. Zhang et al. [16] presented a score function on IVIFS with preference parameters for different types of decision-makers. Wang and Chen [17] introduces a MADM method based on the linear programming methodology and the score, accuracy functions of IVIFNs. Zeng et al. [18] developed a MADM method and modified VIKOR ("VlseKriterijumska Optimizacija I Kompromisno Resenje") method based on the score function of IFNs. Nguyen [19] presented a generalized p-norm based score function for IVIFNs.

Traditionally, all the above current studies exhibited that the information is handled by the experts are either in the form of IFNs or IVIFNs. Although the aforementioned approaches to rank the given numbers hold great care by the researchers but still, some problems appear. Firstly, the approaches based on IFSs and IVIFSs are operating under the discrete set of the universe and hence do not tell the clear picture for the fuzzy concept such as "good", "bad" and so on. Secondly, whether all the membership degree values in the subintervals, $[\underline{\varsigma}, \overline{\varsigma}]$ for IVIFS and $[\varsigma, 1-\upsilon]$ for IFS, have the same probabilities? If not, then what kind of distribution will they be?. Furthermore, it is worth noting that in the real world, a lot of economic and social phenomenon conform to the normal distribution, such as "random measurement error", "the useful lifespan of productions", etc. Thus, there is a necessity to consider the behavior of these variables also. To address in a more detail manner, consider a case of IFS A which is characterized by a membership function $\varsigma(x_i)$ and non-membership $\upsilon(x_i)$ and hence the membership degree of $x_i \in A$ lying in the interval $[\varsigma(x_i), 1-\upsilon(x_i)]$. In general, the most influential value of membership value of x_i from $[\varsigma(x_i), 1-\upsilon(x_i)]$ which has the property of largest probability is derived as the median value given by $\varsigma = (\varsigma(x_i) + 1 - \upsilon(x_i))/2$. On the other hand, the probabilities of the other values are distributed according to this rule: "*With the distance away from ς becomes bigger, the probability becomes smaller.*" Therefore, the distribution of the probabilities to express the membership values of x_i on the subinterval $[\varsigma(x_i), 1-\upsilon(x_i)]$ follows a *normal distribution function*. Yang and Ko [20] defined the normal fuzzy number to express the uncertainties in the data while Li et al. [21] compares them with the triangular or trapezoidal fuzzy numbers. Prompted by this fuzzy number, Lv et al. [22] presented the concept of normal distribution fuzzy sets (NFSs) and a method to display the membership degrees in an IFSs by a series of normal distribution functions with parameters a (expected values) and σ (variance), and denoted as NFS(a, σ). The normal fuzzy numbers (NFNs) have broadly used and several advantages such as all the physical aspects and production activities are well represented by such numbers; the higher derivative of the normal membership function is continuous.

Based on the advantages of the normal fuzzy numbers, Wang and Li [23] extended the IFS to the normal IFS (NIFS) by considering the membership and non-membership degrees of the element as a normal fuzzy number, which have much more pragmatic sense than the other IFNs such as triangular, trapezoidal numbers, linguistic numbers and so on. For instance, concerning the stochastic happenings the life-length of the electric bulb, the average and the variance of a lifetime of production are 100 and 2 respectively, which are expressed as an NFN(100,2). However, due to the ambiguity present in the information, an expert gives a certain degree of membership as 80% and 10% as a negation degree to ensure the life-length of the electric bulb. Such information is drawn more closely with the normal intuitionistic fuzzy number (NIFN) as $((100, 2), 0.8, 0.1)$. Hence, NIFNs can show the stochastic phenomena better than NFNs as well as IFNs. After their existence, some researchers have continued their theories to solve the MADM problem by defining some different kinds of approaches. Wang and Li [23] defined the induced aggregation operators for NIFNs to solve the MADM problems. Wang and Li [24] presented the weighted aggregation operators for a different NIFNs. Wang et al. [25] developed the MADM approach for NIFNs by using the concept of score function based on the relative entropy. Wang et al. [26] proposed some generalized induced aggregation operators for NIFNs to solve the decision-making problems.

Later on, Liu and Teng [27] has extended the concept of NIFS to the normal IVIFS (NIVIFS) by considering the interval-valued degrees of the membership and non-membership functions which follows the normal fuzzy numbers. Since it is always better for an expert to design their information in terms of interval numbers rather than a crisp number. Hence, there is less loss of information in normal IVIFNs (NIVIFNs) than NIFNs. To aggregate the different pairs of the NIVIFNs, Liu and Teng [27] put forward the average and geometric aggregation operators. Further, they also defined some basic operational laws as well as some score and accuracy functions to rank the numbers. Hence, the study conducted under the normal fuzzy set environment is one of the most broadly practiced and authentic tasks to access the information. Although many other scholars have introduced the aggregation operators to aggregate the information, simultaneously an important phase during solving the MADM problems is to order the numbers. For it, some scholars have put forward the ranking methods by using score and accuracy functions, but it is observed that they have a vital shortcoming in some cases. In the existing score functions of NIFN and NIVIFN, it is observed that it requires only the pair of the membership and non-membership degrees but entirely ignores the hesitance degree into the analysis, which is illogical. Hence, under some cases, such existing score functions will inadequate to classify the numbers (demonstrated in Examples 1, 2). It is noted that in day-to-day situation, there is a need to consider simultaneously the membership and hesitancy degrees to define an efficient score function. For instance, in a NIFN $\mathcal{I} = ((a, \sigma), \varsigma_{\mathcal{I}}, \upsilon_{\mathcal{I}})$, when $\varsigma_{\mathcal{I}} > \upsilon_{\mathcal{I}}$ then it means a decision will more likely to support the statement, while when $\varsigma_{\mathcal{I}} < \upsilon_{\mathcal{I}}$ then an impact on the decision is more likely towards the oppose. Hence, there is a need to recognize the hesitation index during the analysis. So to address it completely, a more generalized score function for NIFS and NIVIFS has been proposed in this manuscript and subsequently MADM methods for solving the problems.

Thus, taking the advantages of the normal fuzzy set to express the information with wide confidence, the present work enhances the study on the normal intuitionistic and interval-valued intuitionistic set by defining more generalized score functions to rank their numbers. The following are the main objectives of this paper.

(1) to develop new score functions for NIFS and NIVIFS to rank them.
(2) to build two algorithms, based on score functions and aggregation operators, to interpret decision-making concerns.
(3) to demonstrate the approach with a numerical example to explore the study.

To complete the objective 1, we employ the normal fuzzy numbers to express the rating of the expert during their evaluation and hence present a more generalized score and accuracy functions. The proposed functions overcome the drawbacks of the exiting functions under certain cases. The salient features of the proposed functions are also studied. Objective 2 is accomplished by establishing the two MADM methods based on the proposed functions, in which preferences related to each alternative are expressed in terms of crisp and interval normal fuzzy numbers. Finally, the feasibility of the approaches has been demonstrated through numerical examples and validate it by comparing their results with the several existing approaches results for fulfilling the last objective.

The rest of the text is organized as follows. Section 2 presents basic concepts related to normal fuzzy set, NIFS and NIVIFS. In Sect. 3, we present improved score functions for normal intuitionistic sets under a crisp and interval environment. The effectiveness, as well as the properties of the proposed functions, are also described in this section. In Sect. 4, we offer two algorithms based on proposed functions to solve the MADM problems. The applicability of the presented algorithms is demonstrated in Sect. 5 and compare their results with several existing approaches. Finally, Sect. 6 concludes the paper.

Preliminaries

In this section, we define the basic features of the normal intuitionistic and interval-valued intuitionistic fuzzy sets over the universal set $\mathcal{X}$.

Normal intuitionistic fuzzy sets

Definition 1 [4] An intuitionistic fuzzy set "$\mathcal{I}$" in a set $\mathcal{X}$ is defined as :

$$\mathcal{I} = \{(x, \varsigma_{\mathcal{I}}(x), \upsilon_{\mathcal{I}}(x)) \mid x \in \mathcal{X}\} \tag{1}$$

where $\varsigma_{\mathcal{I}}, \upsilon_{\mathcal{I}} : \mathcal{X} \to [0, 1]$ such that $0 \leq \varsigma_{\mathcal{I}}(x) \leq 1$ and $0 \leq \upsilon_{\mathcal{I}}(x) \leq 1$ and $0 \leq \varsigma_{\mathcal{I}}(x) + \upsilon_{\mathcal{I}}(x) \leq 1$. The pair $\mathcal{I} = (\varsigma_{\mathcal{I}}, \upsilon_{\mathcal{I}})$ is called as an intuitionistic fuzzy number (IFN) [7].

Definition 2 [20] A normal distribution fuzzy set $\mathcal{I}$ is characterized by a set of normal distribution functions $\{\psi_i(z)\}$, for each $\psi_i(z)$, such that

$$a_i = E(\psi_i(z)), \qquad \sigma_i = Var(\psi_i(z)) \geq 1/\sqrt{2\pi} \tag{2}$$

Here a_i, σ_i represents the mean and variance of the normal distribution functions $\{\psi_i(z)\}$.

Remark 1 The null set ψ_N is defined as $\psi_N = \{(x_i, 0, 1/\sqrt{2\pi}) \mid x_i \in \mathcal{X}\}$, the universal set is defined as $\psi = \{(x_i, 1, 1/\sqrt{2\pi}) \mid x_i \in \mathcal{X}\}$.

Definition 3 [20] A normal distribution fuzzy set (NFS) $\mathcal{I}$ is express as

$$\mathcal{I} = \{(x_i, (a_i, \sigma_i), \varsigma_{\mathcal{I}}(x_i)) \mid x_i \in \mathcal{X}\} \tag{3}$$

where $\varsigma_{\mathcal{I}}$ represent the membership function presented as

$$\varsigma_{\mathcal{I}}(x) = e^{-\left(\frac{x-a}{\sigma}\right)^2} \quad ; \sigma > 0 \tag{4}$$

Definition 4 [23] A normal intuitionistic fuzzy set (NIFS) over $\mathcal{X}$ is defined as

$$\mathcal{I}_N = \{(x_i, (a_i, \sigma_i), \varsigma_{\mathcal{I}}(x_i), \upsilon_{\mathcal{I}}(x_i)) \mid x_i \in \mathcal{X}\}$$

where membership and non-membership functions are presented as

$$\varsigma_{\mathcal{I}}(x) = \varsigma_{\mathcal{I}} e^{-(\frac{x-a}{\sigma})^2}, \qquad x \in \mathcal{X} \tag{5}$$

and $$\upsilon_{\mathcal{I}}(x) = 1 - (1 - \upsilon_{\mathcal{I}})e^{-(\frac{x-a}{\sigma})^2}, \qquad x \in \mathcal{X} \tag{6}$$

such that $0 \leq \varsigma_{\mathcal{I}}, \upsilon_{\mathcal{I}} \leq 1$ and $0 \leq \varsigma_{\mathcal{I}} + \upsilon_{\mathcal{I}} \leq 1$ holds.

Remark 2 From the definition, we conclude

(1) When $\varsigma_{\mathcal{I}} = 1$ and $\upsilon_{\mathcal{I}} = 0$, then NIFS reduces to NFS.
(2) A pair $((a, \sigma), \varsigma, \upsilon)$ is called normal intuitionistic fuzzy number (NIFN).

Definition 5 [23] The complement of NIFS $\mathcal{I}_N$ is defined as

$$\mathcal{I}_N^c = \{(x_i, (a_i, \sigma_i), \upsilon_{\mathcal{I}}(x_i), \varsigma_{\mathcal{I}}(x_i)) \mid x_i \in \mathcal{X}\}$$

Definition 6 [23] For NIFN $\mathcal{I} = ((a, \sigma), \varsigma_{\mathcal{I}}, \upsilon_{\mathcal{I}})$, score functions are stated as

$$S_1(\mathcal{I}) = a\left(\varsigma_{\mathcal{I}} - \upsilon_{\mathcal{I}}\right), \qquad S_2(\mathcal{I}) = \sigma\left(\varsigma_{\mathcal{I}} - \upsilon_{\mathcal{I}}\right), \tag{7}$$

while accuracy functions are given as

$$H_1(\mathcal{I}) = a\left(\varsigma_{\mathcal{I}} + \upsilon_{\mathcal{I}}\right), \qquad H_2(\mathcal{I}) = \sigma\left(\varsigma_{\mathcal{I}} + \upsilon_{\mathcal{I}}\right) \tag{8}$$

Based on these functions, a comparison law between two different NIFNs are stated as follows:

Definition 7 [23] For two NIFNs $\mathcal{I}_1$ and $\mathcal{I}_2$, an order relation between them is stated as:

(1) If $S_1(\mathcal{I}_1) > S_1(\mathcal{I}_2)$, then $\mathcal{I}_1 > \mathcal{I}_2$.
(2) If $S_1(\mathcal{I}_1) = S_1(\mathcal{I}_2)$ and $H_1(\mathcal{I}_1) > H_1(\mathcal{I}_2)$, then $\mathcal{I}_1 > \mathcal{I}_2$.
(3) If $S_1(\mathcal{I}_1) = S_1(\mathcal{I}_2)$ and $H_1(\mathcal{I}_1) = H_1(\mathcal{I}_2)$, then
    (a) If $S_2(\mathcal{I}_1) < S_2(\mathcal{I}_2)$, then $\mathcal{I}_1 > \mathcal{I}_2$.
    (b) If $S_2(\mathcal{I}_1) = S_2(\mathcal{I}_2)$ and $H_2(\mathcal{I}_1) < H_2(\mathcal{I}_2)$, then $\mathcal{I}_1 > \mathcal{I}_2$.
    (c) If $S_2(\mathcal{I}_1) = S_2(\mathcal{I}_2)$ and $H_2(\mathcal{I}_1) = H_2(\mathcal{I}_2)$, then $\mathcal{I}_1 = \mathcal{I}_2$.

Definition 8 [26] For the collection of "n" NIFNs $\mathcal{I}_j = \left((a_j, \sigma_j), \varsigma_j, \upsilon_j\right)$, a normal intuitionistic fuzzy weighted geometric averaging (NIFWGA) operator is defined as

$$\begin{aligned}&\text{NIFWGA}(\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n)\\ &= \left(\left(\prod_{j=1}^{n} a_j^{\omega_j}, \prod_{j=1}^{n} a_j^{\omega_j}\sqrt{\sum_{j=1}^{n}\frac{\omega_j \sigma_j^2}{a_j^2}}\right), \prod_{j=1}^{n} \varsigma_j^{\omega_j},\right.\\ &\left. 1 - \prod_{j=1}^{n}\left(1-\upsilon_j\right)^{\omega_j}\right)\end{aligned} \tag{9}$$

where ω_j is the weight of $\mathcal{I}_j$ such that $\omega_j > 0$ and $\sum_{j=1}^{n} \omega_j = 1$

Normal interval-valued intuitionistic fuzzy sets

Definition 9 [6] An IVIFS $\mathcal{I}$ in $\mathcal{X}$ is described as

$$\mathcal{I} = \{\langle x, \varsigma_{\mathcal{I}}(x), \upsilon_{\mathcal{I}}(x)\rangle \mid x \in \mathcal{U}\}, \tag{10}$$

where $\varsigma_{\mathcal{I}}(x) = \left[\underline{\varsigma}_{\mathcal{I}}(x), \overline{\varsigma}_{\mathcal{I}}(x)\right]$ and $\upsilon_{\mathcal{I}}(x) = \left[\underline{\upsilon}_{\mathcal{I}}(x), \overline{\upsilon}_{\mathcal{I}}(x)\right]$ are the subsets of [0, 1], and represents the MDs ("membership degrees") and NMDs ("non-membership degrees") such that $0 \leq \overline{\varsigma}_{\mathcal{I}}(x) + \overline{\upsilon}_{\mathcal{I}}(x) \leq 1$ for all $x \in \mathcal{X}$. For accessibility, this pair is inscribed as $\mathcal{I} = ([\underline{\varsigma}_{\mathcal{I}}, \overline{\varsigma}_{\mathcal{I}}], [\underline{\upsilon}_{\mathcal{I}}, \overline{\upsilon}_{\mathcal{I}}])$ is called an IVIFN with the requirement that $[\underline{\varsigma}_{\mathcal{I}}, \overline{\varsigma}_{\mathcal{I}}], [\underline{\upsilon}_{\mathcal{I}}, \overline{\upsilon}_{\mathcal{I}}] \subseteq [0, 1]$ and $\overline{\varsigma}_{\mathcal{I}} + \overline{\upsilon}_{\mathcal{I}} \leq 1$.

Definition 10 [27] A normal interval-valued intuitionistic fuzzy set (NIVIFS) over $\mathcal{X}$ is defined as

$$\begin{aligned}\mathcal{I}_N = \Big\{ &\Big(x_i, (a_i, \sigma_i), \left[\underline{\varsigma}_{\mathcal{I}}(x), \overline{\varsigma}_{\mathcal{I}}(x)\right],\\ &\left[\underline{\upsilon}_{\mathcal{I}}(x),\ \overline{\upsilon}_{\mathcal{I}}(x)\right]\Big) \mid x_i \in \mathcal{X}\Big\}\end{aligned} \tag{11}$$

where a_i, σ_i represents the mean and variance of the normal distribution functions.

Remark 3 From the Definition 10, we find

(1) When $\underline{\varsigma}_{\mathcal{I}}(x) = \overline{\varsigma}_{\mathcal{I}}(x)$ and $\underline{\upsilon}_{\mathcal{I}}(x) = \overline{\upsilon}_{\mathcal{I}}(x)$ for all $x \in \mathcal{X}$ then NIVIFS reduces to NIFS.
(2) A pair $((a, \sigma), [\underline{\varsigma}_{\mathcal{I}}, \overline{\varsigma}_{\mathcal{I}}], [\underline{\upsilon}_{\mathcal{I}}, \overline{\upsilon}_{\mathcal{I}}])$ is called normal IVIFN (NIVIFN) with the requirement that $a_i, \sigma_i > 0$, $[\underline{\varsigma}_{\mathcal{I}}, \overline{\varsigma}_{\mathcal{I}}], [\underline{\upsilon}_{\mathcal{I}}, \overline{\upsilon}_{\mathcal{I}}] \subseteq [0, 1]$ and $\overline{\varsigma}_{\mathcal{I}} + \overline{\upsilon}_{\mathcal{I}} \leq 1$.

Definition 11 [27] For a NIVIFN $\mathcal{I} = \Big((a, \sigma), [\underline{\varsigma}_{\mathcal{I}}, \overline{\varsigma}_{\mathcal{I}}], [\underline{\upsilon}_{\mathcal{I}}, \overline{\upsilon}_{\mathcal{I}}]\Big)$, score functions are defined as

$$\begin{aligned}S_1'(\mathcal{I}) &= a\left(\frac{\underline{\varsigma}_{\mathcal{I}} + \overline{\varsigma}_{\mathcal{I}} - \underline{\upsilon}_{\mathcal{I}} - \overline{\upsilon}_{\mathcal{I}}}{2}\right),\\ S_2'(\mathcal{I}) &= \sigma\left(\frac{\underline{\varsigma}_{\mathcal{I}} + \overline{\varsigma}_{\mathcal{I}} - \underline{\upsilon}_{\mathcal{I}} - \overline{\upsilon}_{\mathcal{I}}}{2}\right)\end{aligned} \tag{12}$$

while the accuracy functions are given as

$$\begin{aligned}H_1'(\mathcal{I}) &= a\left(\frac{\underline{\varsigma}_{\mathcal{I}} + \overline{\varsigma}_{\mathcal{I}} + \underline{\upsilon}_{\mathcal{I}} + \overline{\upsilon}_{\mathcal{I}}}{2}\right),\\ H_2'(\mathcal{I}) &= \sigma\left(\frac{\underline{\varsigma}_{\mathcal{I}} + \overline{\varsigma}_{\mathcal{I}} + \underline{\upsilon}_{\mathcal{I}} + \overline{\upsilon}_{\mathcal{I}}}{2}\right)\end{aligned} \tag{13}$$

Definition 12 [27] An order relation between two NIVIFNs $\mathcal{I}_1$ and $\mathcal{I}_2$ is stated as:

(1) If either $S_1'(\mathcal{I}_1) > S_1'(\mathcal{I}_2)$, or $S_1'(\mathcal{I}_1) = S_1'(\mathcal{I}_2)$, $H_1'(\mathcal{I}_1) > H_1'(\mathcal{I}_2)$ holds, then $\mathcal{I}_1 > \mathcal{I}_2$.

(2) If $S_1'(\mathcal{I}_1) = S_1'(\mathcal{I}_2)$ and $H_1'(\mathcal{I}_1) = H_1'(\mathcal{I}_2)$, then
  (a) When $S_2'(\mathcal{I}_1) < S_2'(\mathcal{I}_2)$, implies $\mathcal{I}_1 > \mathcal{I}_2$.
  (b) When $S_2'(\mathcal{I}_1) = S_2'(\mathcal{I}_2)$ and $H_2'(\mathcal{I}_1) < H_2'(\mathcal{I}_2)$, implies that $\mathcal{I}_1 > \mathcal{I}_2$.

Definition 13 [27] For a collection of "n" NIVIFNs $\mathcal{I}_j = \left((a_j, \sigma_j), \left[\underline{\varsigma}_j, \overline{\varsigma}_j\right], \left[\underline{\upsilon}_j, \overline{\upsilon}_j\right]\right)$, a normal interval-valued intuitionistic fuzzy weighted geometric averaging (NIVIFWGA) operator is defined as

$$\begin{aligned}
&\text{NIVIFWGA}(\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n) \\
&= \left(\left(\prod_{j=1}^{n} a_j^{\omega_j}, \prod_{j=1}^{n} a_j^{\omega_j} \sqrt{\sum_{j=1}^{n} \frac{\omega_j \sigma_j^2}{a_j^2}}\right),\right. \\
&\quad \left[\prod_{j=1}^{n} \left(\underline{\varsigma}_j\right)^{\omega_j}, \prod_{j=1}^{n} \left(\overline{\varsigma}_j\right)^{\omega_j}\right], \\
&\quad \left.\left[1 - \prod_{j=1}^{n} \left(1 - \underline{\upsilon}_j\right)^{\omega_j}, 1 - \prod_{j=1}^{n} \left(1 - \overline{\upsilon}_j\right)^{\omega_j}\right]\right) \qquad (14)
\end{aligned}$$

where $\omega_j > 0$ is the weight of $\mathcal{I}_j$ such that $\sum_{j=1}^{n} \omega_j = 1$

Shortcoming of the existing score functions

In this section, we have presented some shortcomings of the existing score functions which are defined in Definition 6 and Definition 11 as below.

Example 1 Let $\mathcal{I}_1 = ((2, 0.2), 0.6, 0.3)$ and $\mathcal{I}_2 = ((3, 0.3), 0.4, 0.2)$ are two NIFNs. In order to find out the biggest NIFN among $\mathcal{I}_1$ and $\mathcal{I}_2$, we compute the score function values by using Eq. (7) and get $S_1(\mathcal{I}_1) = 0.6$, $S_1(\mathcal{I}_2) = 0.6$, $S_2(\mathcal{I}_1) = 0.06$, $S_2(\mathcal{I}_2) = 0.06$ and accuracy values by Eq. (8) are $H_1(\mathcal{I}_1) = 1.8$, $H_1(\mathcal{I}_2) = 1.8$, $H_2(\mathcal{I}_1) = 0.18$, $H_2(\mathcal{I}_2) = 0.18$. Therefore, based on comparison law defined in Definition 7, we conclude that $\mathcal{I}_1 = \mathcal{I}_2$. But it is clearly seen that $\mathcal{I}_1 \neq \mathcal{I}_2$ and hence these functions are unable to classify the given numbers.

Example 2 Let $\mathcal{I}_1 = ((6, 0.2), [0.15, 0.25], [0.05, 0.15])$ and $\mathcal{I}_2 = ((3, 0.1), [0.30, 0.50], [0.1, 0.3])$ be two NIVIFNs. To compute the biggest one among them, we calculate the score values of them by Eq. (12) and get $S_1'(\mathcal{I}_1) = 0.6$, $S_1'(\mathcal{I}_2) = 0.6$, $S_2'(\mathcal{I}_1) = 0.02$, $S_2'(\mathcal{I}_2) = 0.02$. Since $S_1'(\mathcal{I}_1) = S_1'(\mathcal{I}_2)$ and $S_2'(\mathcal{I}_1) = S_2'(\mathcal{I}_2)$, hence we calculate the accuracy values by Eq. (13) and get $H_1'(\mathcal{I}_1) = 1.8$, $H_1'(\mathcal{I}_2) = 1.8$, $H_2'(\mathcal{I}_1) = 0.06$, $H_2'(\mathcal{I}_2) = 0.06$. It is clearly seen from these values and by Definition 12 that $\mathcal{I}_1$ and $\mathcal{I}_2$ are equivalent. But this is not true. Therefore, the existing score functions are unable to distinguish between these two NIVIFNs.

Hence, from these two examples, we can see that the functions defined in Eqs. (7), (8) and in Eqs. (12), (13) fails to discriminate between the pairs of NIFNs and NIVIFNs in terms of ranking. Therefore, in order to handle it, an improved score function has been proposed in the next section by sufficiently considering the indeterminacy information between the pairs of the NIFNs and NIVIFNs.

Proposed improved score functions

In this section, we define a improved score function for the pairs of NIFNs as well as NIVIFNs to rank them. The salient features of the proposed measure are also investigated.

Proposed score functions for NIFNs

From Eq. (7), it has been observed that the existing score functions depend only on the degrees of the membership and non-membership of the NIFN. In day-to-day situation, there is a need to consider simultaneously the membership and hesitancy degrees to define an efficient score function. For a NIFN $\mathcal{I} = ((a, \sigma), \varsigma_{\mathcal{I}}, \upsilon_{\mathcal{I}})$, when $\varsigma_{\mathcal{I}} > \upsilon_{\mathcal{I}}$ then it means a decision will more likely to support the statement, while when $\varsigma_{\mathcal{I}} < \upsilon_{\mathcal{I}}$ then an impact on the decision is more likely towards the oppose. Hence, there is a need to add the hesitation index $\pi_{\mathcal{I}} = 1 - \varsigma_{\mathcal{I}} - \upsilon_{\mathcal{I}}$ relative to $\mathcal{I}$ such that $\varsigma_{\mathcal{I}} + \frac{\pi_{\mathcal{I}}}{2}$ and $\upsilon_{\mathcal{I}} + \frac{\pi_{\mathcal{I}}}{2}$ are called favorable and against degrees relative to $\mathcal{I}$ for $x \in \mathcal{X}$. This further means that when $\varsigma_{\mathcal{I}} > \upsilon_{\mathcal{I}}$ then $\pi_{\mathcal{I}}$ has positive impact on it, while when $\varsigma_{\mathcal{I}} < \upsilon_{\mathcal{I}}$ then $\pi_{\mathcal{I}}$ has a negative impact on it, which makes $S(\mathcal{I})$ increases and decreases respectively. Thus, by considering the importance of hesitation index, we define a new score function as follows:

Definition 14 Let $\mathcal{I} = ((a, \sigma), \varsigma_{\mathcal{I}}, \upsilon_{\mathcal{I}})$ be NIFN, then new improved score functions $\mathcal{M}_1(\mathcal{I})$ and $\mathcal{M}_2(\mathcal{I})$ based on the hesitation degree are defined as

$$\mathcal{M}_1(\mathcal{I}) = a\left(\frac{\varsigma_{\mathcal{I}} - 3\upsilon_{\mathcal{I}} + 1}{2}\right) \qquad (15)$$

$$\mathcal{M}_2(\mathcal{I}) = \sigma\left(\frac{\varsigma_{\mathcal{I}} - 3\upsilon_{\mathcal{I}} + 1}{2}\right) \qquad (16)$$

where $\mathcal{M}_1(\mathcal{I}) \in [-a, a]$ and $\mathcal{M}_2(\mathcal{I}) \in [-\sigma, \sigma]$.

Based on the proposed score functions, we give the following comparison law for any two NIFNs $\mathcal{I}_1$ and $\mathcal{I}_2$ as

Definition 15 Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be two NIFNs and $\mathcal{M}_1(\cdot)$, $\mathcal{M}_2(\cdot)$ be their score functions respectively. Then, a comparison rule $\mathcal{I}_1 > \mathcal{I}_2$ hold if either of the following condition met.

(1) $\mathcal{M}_1(\mathcal{I}_1) > \mathcal{M}_1(\mathcal{I}_2)$.
(2) $\mathcal{M}_1(\mathcal{I}_1) = \mathcal{M}_1(\mathcal{I}_2)$ and $\mathcal{M}_2(\mathcal{I}_1) < \mathcal{M}_2(\mathcal{I}_2)$.

Proposed score functions for NIVIFNs

From Definition 11, it is again concluded that the existing score functions depend only on the degrees of the membership. However, there is a need to consider the degree of hesitancy between them also during the analysis. For it, an interval hesitancy degree $[\underline{\pi}_{\mathcal{I}}, \overline{\pi}^U_{\mathcal{I}}] = [1-\overline{\varsigma}_{\mathcal{I}}-\overline{\upsilon}_{\mathcal{I}}, 1-\underline{\varsigma}_{\mathcal{I}}-\underline{\upsilon}_{\mathcal{I}}]$ relative to NIVIFN $\mathcal{I} = \left((a, \sigma), [\underline{\varsigma}_{\mathcal{I}}, \overline{\varsigma}_{\mathcal{I}}], [\underline{\upsilon}_{\mathcal{I}}, \overline{\upsilon}_{\mathcal{I}}]\right)$ is added into the analysis such that $\underline{\varsigma}_{\mathcal{I}}+\frac{\underline{\pi}_{\mathcal{I}}}{2}, \overline{\varsigma}_{\mathcal{I}}+\frac{\overline{\pi}_{\mathcal{I}}}{2}$ are called favorable degrees relative to $\mathcal{I}$, and $\underline{\upsilon}_{\mathcal{I}} + \frac{\underline{\pi}_{\mathcal{I}}}{2}, \overline{\upsilon}_{\mathcal{I}} + \frac{\overline{\pi}_{\mathcal{I}}}{2}$ are called against degrees relative to $\mathcal{I}$. Thus, due to considering the importance of membership degrees and the hesitancy index, we define an improved score function for NIVIFN as follows:

Definition 16 An improved score function $\mathcal{M}'_1(\mathcal{I})$ and $\mathcal{M}'_2(\mathcal{I})$ of an NIVIFN $\mathcal{I} = ((a, \sigma), [\underline{\varsigma}_{\mathcal{I}}, \overline{\varsigma}_{\mathcal{I}}], [\underline{\upsilon}_{\mathcal{I}}, \overline{\upsilon}_{\mathcal{I}}])$, based on the hesitation degrees, is defined as

$$\mathcal{M}'_1(\mathcal{I}) = a\left(\frac{\underline{\varsigma}_{\mathcal{I}} + \overline{\varsigma}_{\mathcal{I}} - 3(\underline{\upsilon}_{\mathcal{I}} + \overline{\upsilon}_{\mathcal{I}}) + 2}{4}\right) \tag{17}$$

$$\text{and } \mathcal{M}'_2(\mathcal{I}) = \sigma\left(\frac{\underline{\varsigma}_{\mathcal{I}} + \overline{\varsigma}_{\mathcal{I}} - 3(\underline{\upsilon}_{\mathcal{I}} + \overline{\upsilon}_{\mathcal{I}}) + 2}{4}\right) \tag{18}$$

Based on it, the following comparison law is defined as

Definition 17 Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be two NIVIFNs and $\mathcal{M}'_1(\cdot)$, $\mathcal{M}'_2(\cdot)$ be their score functions respectively, then

(i) If $\mathcal{M}'_1(\mathcal{I}_1) > \mathcal{M}'_1(\mathcal{I}_2)$, then $\mathcal{I}_1 > \mathcal{I}_2$.
(ii) If $\mathcal{M}'_1(\mathcal{I}_1) = \mathcal{M}'_1(\mathcal{I}_2)$ and $\mathcal{M}'_2(\mathcal{I}_1) < \mathcal{M}'_2(\mathcal{I}_2)$, then $\mathcal{I}_1 > \mathcal{I}_2$.

Remark 4 In particular, when $\underline{\varsigma}_{\mathcal{I}} = \overline{\varsigma}_{\mathcal{I}} = \varsigma_{\mathcal{I}}$ and $\underline{\upsilon}_{\mathcal{I}} = \overline{\upsilon}_{\mathcal{I}} = \upsilon_{\mathcal{I}}$, then $\mathcal{M}'_1(\cdot)$, $\mathcal{M}'_2(\cdot)$ reduces to $\mathcal{M}_1(\cdot)$ and $\mathcal{M}_2(\cdot)$ respectively.

Effectiveness of the proposed score functions

To illustrate the effectiveness of the proposed score functions for NIFNs and NIVIFNs, consider the Examples 1 and 2 again here.

Example 3 If we utilize the proposed score functions $\mathcal{M}_1$ and $\mathcal{M}_2$ on the consider data of the Example 1, then we get $\mathcal{M}_1(\mathcal{I}_1) = 0.7$, $\mathcal{M}_1(\mathcal{I}_2) = 1.2$. Since $\mathcal{M}_1(\mathcal{I}_1) \leq \mathcal{M}_2(\mathcal{I}_2)$ and thus according to the Definition 15, we get $\mathcal{I}_1 \prec \mathcal{I}_2$, i.e., the NIFN $\mathcal{I}_2$ is better than NIFN $\mathcal{I}_1$.

Example 4 Consider the data as given in Example 2 where we observed that the existing functions are unable to classify between the sets. In order to distinguish them, if we apply our proposed function $\mathcal{M}'_1$ to the data then we get $\mathcal{M}'_1(\mathcal{I}_1) = 2.7$, $\mathcal{M}'_1(\mathcal{I}_2) = 1.2$. Hence, by Definition 17, we conclude that $\mathcal{I}_1$ is the better than $\mathcal{I}_2$.

Properties of the proposed score functions

In this section, we studied some characteristics of the proposed score functions for NIFN and NIVIFN in detail.

Property 1 Let $\mathcal{I}_1 = ((a_1, \sigma_1), \varsigma_1, \upsilon_1)$ and $\mathcal{I}_2 = ((a_2, \sigma_2), \varsigma_2, \upsilon_2)$ be two comparable NIFNs such that $a_1 \leq a_2, \sigma_1 \geq \sigma_2$, $\varsigma_1 \leq \varsigma_2$ and $\upsilon_1 \geq \upsilon_2$, i.e., $\mathcal{I}_1 < \mathcal{I}_2$ then $\mathcal{M}_1(\mathcal{I}_1) \leq \mathcal{M}_1(\mathcal{I}_2)$ and $\mathcal{M}_2(\mathcal{I}_1) \geq \mathcal{M}_2(\mathcal{I}_2)$.

Proof Let $\mathcal{I}_1 = ((a_1, \sigma_1), \varsigma_1, \upsilon_1)$ and $\mathcal{I}_2 = ((a_2, \sigma_2), \varsigma_2, \upsilon_2)$ be two comparable NIFNs such that $a_1 \leq a_2$, $\sigma_1 \geq \sigma_2$, $\varsigma_1 \leq \varsigma_2$ and $\upsilon_1 \geq \upsilon_2$ which implies that $\mathcal{I}_1 < \mathcal{I}_2$. By the definition of the improved score functions $\mathcal{M}_1$, we have

$$\mathcal{M}_1(\mathcal{I}_1) = a_1\left(\frac{\varsigma_1 - 3\upsilon_1 + 1}{2}\right)$$

$$\text{and } \mathcal{M}_1(\mathcal{I}_2) = a_2\left(\frac{\varsigma_2 - 3\upsilon_2 + 1}{2}\right)$$

Since $\varsigma_1 \leq \varsigma_2$ and $\upsilon_1 \geq \upsilon_2$ which gives that $\varsigma_2 - 3\upsilon_2 + 1 \geq \varsigma_1 - 3\upsilon_1 + 1$ and hence $\mathcal{M}_1(\mathcal{I}_1) \leq \mathcal{M}_1(\mathcal{I}_2)$. Similarly, we can prove that $\mathcal{M}_2(\mathcal{I}_1) \geq \mathcal{M}_2(\mathcal{I}_2)$. □

This relation has been explained with a numerical example as follow.

Example 5 Let $\mathcal{I}_1 = ((8, 0.4), 0.8, 0.1)$ and $\mathcal{I}_2 = ((7, 0.6), 0.6, 0.3)$ be two comparable NIFNs. Clearly seen that $\mathcal{I}_1 > \mathcal{I}_2$. Now, by using the expression of the proposed score function $\mathcal{M}_1$, we get

$$\mathcal{M}_1(\mathcal{I}_1) = 8\left(\frac{0.8 - 3 \times 0.1 + 1}{2}\right) = 6 \text{ and}$$

$$\mathcal{M}_1(\mathcal{I}_2) = 7\left(\frac{0.6 - 3 \times 0.3}{2}\right) = 2.45$$

Hence $\mathcal{M}_1(\mathcal{I}_1) \geq \mathcal{M}_1(\mathcal{I}_2)$ holds.

Property 2 For NIFN $\mathcal{I} = ((a, \sigma), \varsigma_{\mathcal{I}}, \upsilon_{\mathcal{I}})$, the proposed improved score function $\mathcal{M}_1(\mathcal{I})$ and $\mathcal{M}_2(\mathcal{I})$, lies between $[-a, a]$ and $[-\sigma, \sigma]$ respectively.

Proof Since $\mathcal{I} = ((a, \sigma), \varsigma_{\mathcal{I}}, \upsilon_{\mathcal{I}})$ be NIFN which implies that $\varsigma_{\mathcal{I}}, \upsilon_{\mathcal{I}} \in [0, 1]$ and $\varsigma_{\mathcal{I}} + \upsilon_{\mathcal{I}} \leq 1$. Thus, $1 - \varsigma_{\mathcal{I}} - \upsilon_{\mathcal{I}} \geq 0$ and hence by Eq. (15),

$$\begin{aligned}\mathcal{M}_1(\mathcal{I}) &= a\left(\frac{\varsigma_{\mathcal{I}} - 3\upsilon_{\mathcal{I}} + 1}{2}\right)\\ &= a\left(\varsigma_{\mathcal{I}} - \upsilon_{\mathcal{I}} + \frac{1 - \varsigma_{\mathcal{I}} - \upsilon_{\mathcal{I}}}{2}\right)\\ &\geq a\left(\varsigma_{\mathcal{I}} - \upsilon_{\mathcal{I}}\right) \geq -a\end{aligned}$$

Also,

$$\mathcal{M}_1(\mathcal{I}) = a\left(\frac{\varsigma_{\mathcal{I}} - 3\upsilon_{\mathcal{I}} + 1}{2}\right) = a\left(\frac{\varsigma_{\mathcal{I}} + 1}{2} - \frac{3\upsilon_{\mathcal{I}}}{2}\right) \leq a\left(\frac{\varsigma_{\mathcal{I}} + 1}{2}\right) \leq a$$

Hence, $-a \leq \mathcal{M}_1(\mathcal{I}) \leq a$. Similarly, we can obtain $\mathcal{M}_2(\mathcal{I}) \in [-\sigma, \sigma]$ for NIFN $\mathcal{I} = ((a,\sigma), \varsigma_{\mathcal{I}}, \upsilon_{\mathcal{I}})$. □

Property 3 For a NIFN $\mathcal{I}$, the proposed score function $\mathcal{M}_1(\mathcal{I})$ and the existing score function $S_1(\mathcal{I})$ satisfy the inequality $\mathcal{M}_1(\mathcal{I}) \geq S_1(\mathcal{I})$.

Proof Consider $\mathcal{I} = ((a,\sigma), \varsigma_{\mathcal{I}}, \upsilon_{\mathcal{I}})$ be a NIFN such that $0 \leq \varsigma_{\mathcal{I}}, \upsilon_{\mathcal{I}} \leq 1$ and $\varsigma_{\mathcal{I}} + \upsilon_{\mathcal{I}} \leq 1$. Therefore, $1 - \varsigma_{\mathcal{I}} - \upsilon_{\mathcal{I}} \geq 0$. Hence, by expression of $\mathcal{M}_1(\mathcal{I})$ and $S_1(\mathcal{I})$, given in Eqs. (15) and (7) respectively, we have

$$\mathcal{M}_1(\mathcal{I}) - S_1(\mathcal{I}) = a\left(\frac{\varsigma_{\mathcal{I}} - 3\upsilon_{\mathcal{I}} + 1}{2}\right) - a\left(\varsigma_{\mathcal{I}} - \upsilon_{\mathcal{I}}\right) \geq a\left(\frac{1 - \varsigma_{\mathcal{I}} - \upsilon_{\mathcal{I}}}{2}\right) \geq 0$$

Hence, the result hold. □

Property 4 Let $\mathcal{I} = ((a,\sigma_{\mathcal{I}}), \varsigma_{\mathcal{I}}, \upsilon_{\mathcal{I}})$ be NIFN. Then $\mathcal{M}_1(\mathcal{I}) + \mathcal{M}_1(\mathcal{I}^c) \leq a$ and $\mathcal{M}_2(\mathcal{I}) + \mathcal{M}_2(\mathcal{I}^c) \leq \sigma$.

Proof Since $\mathcal{I}^c = ((a,\sigma_{\mathcal{I}}), \upsilon_{\mathcal{I}}, \varsigma_{\mathcal{I}})$ be the complement of the NIFN $\mathcal{I} = ((a,\sigma_{\mathcal{I}}), \varsigma_{\mathcal{I}}, \upsilon_{\mathcal{I}})$, then by the definition of the $\mathcal{M}_1(\mathcal{I})$, we have

$$\mathcal{M}_1(\mathcal{I}) + \mathcal{M}_1(\mathcal{I}^c) = a\left(\frac{\varsigma_{\mathcal{I}} - 3\upsilon_{\mathcal{I}} + 1}{2}\right) + a\left(\frac{\upsilon_{\mathcal{I}} - 3\varsigma_{\mathcal{I}} + 1}{2}\right) = a\left(\frac{2 - 2\varsigma_{\mathcal{I}} - 2\upsilon_{\mathcal{I}}}{2}\right) = a\left(1 - \varsigma_{\mathcal{I}} - \upsilon_{\mathcal{I}}\right) \leq a$$

Similarly, we get $\mathcal{M}_2(\mathcal{I}) + \mathcal{M}_2(\mathcal{I}^c) \leq \sigma$. □

Property 5 For a NIFN $\mathcal{I} = ((a,\sigma), \varsigma_{\mathcal{I}}, \upsilon_{\mathcal{I}})$, the proposed score function $\mathcal{M}_1(\mathcal{I})$ and existing function $S_1(\mathcal{I})$ satisfies the relation $\mathcal{M}_1(\mathcal{I}) = S_1(\mathcal{I}) + a\left(\frac{\pi_{\mathcal{I}}}{2}\right)$, where $\pi_{\mathcal{I}}$ is the hesitancy degree of $\mathcal{I}$.

Proof For NIFN $\mathcal{I} = ((a,\sigma), \varsigma_{\mathcal{I}}, \upsilon_{\mathcal{I}})$ and by Eqs. (7), (15), we have

$$\mathcal{M}_1(\mathcal{I}) - S_1(\mathcal{I}) = a\left(\frac{\varsigma_{\mathcal{I}} - 3\upsilon_{\mathcal{I}} + 1}{2}\right) - a\left(\varsigma_{\mathcal{I}} - \upsilon_{\mathcal{I}}\right) = a\left(\frac{\varsigma_{\mathcal{I}} - 3\upsilon_{\mathcal{I}} + 1 - 2\varsigma_{\mathcal{I}} + 2\upsilon_{\mathcal{I}}}{2}\right) = a\left(\frac{\pi_{\mathcal{I}}}{2}\right)$$

□

Property 6 (Zero Property) If NIFN $\mathcal{I} = ((a,\sigma), 0, 1)$ then $\mathcal{M}_1(\mathcal{I}) = -a$ and $\mathcal{M}_2(\mathcal{I}) = -\sigma$.

Proof For $\mathcal{I} = ((a,\sigma), 0, 1)$ and by Eqs. (15), (16), we get $\mathcal{M}_1(\mathcal{I}) = a\left(\frac{0-3+1}{2}\right) = -a$ and $\mathcal{M}_2(\mathcal{I}) = \sigma\left(\frac{0-3+1}{2}\right) = -\sigma$. □

Property 7 (One Property) If NIFN $\mathcal{I} = ((a,\sigma), 1, 0)$ then $\mathcal{M}_1(\mathcal{I}) = a$ and $\mathcal{M}_2(\mathcal{I}) = \sigma$.

Proof For NIFN $\mathcal{I} = ((a,\sigma), 1, 0)$, we get $\mathcal{M}_1(\mathcal{I}) = a\left(\frac{1-0+1}{2}\right) = a$ and $\mathcal{M}_2(\mathcal{I}) = \sigma\left(\frac{1-0+1}{2}\right) = \sigma$. □

Property 8 For a subset $\mathcal{I} = ((a,\sigma), \varsigma_{\mathcal{I}}, 1 - \varsigma_{\mathcal{I}})$ of NIFN, the proposed score functions reduce to $\mathcal{M}_1(\mathcal{I}) = a\left(2\varsigma_{\mathcal{I}} - 1\right)$ and $\mathcal{M}_2(\mathcal{I}) = \sigma\left(2\varsigma_{\mathcal{I}} - 1\right)$.

Proof As $\mathcal{I} = ((a,\sigma), \varsigma_{\mathcal{I}}, \upsilon_{\mathcal{I}})$ be NIFN then from Eq. (15), we have

$$\mathcal{M}_1(\mathcal{I}) = a\left(\frac{\varsigma_{\mathcal{I}} - 3(1 - \varsigma_{\mathcal{I}}) + 1}{2}\right) = a\left(\frac{4\varsigma_{\mathcal{I}} - 2}{2}\right) = a\left(2\varsigma_{\mathcal{I}} - 1\right)$$

Similarly, we have

$$\mathcal{M}_2(\mathcal{I}) = \sigma\left(\frac{\varsigma_{\mathcal{I}} - 3(1 - \varsigma_{\mathcal{I}}) + 1}{2}\right) = \sigma\left(\frac{4\varsigma_{\mathcal{I}} - 2}{2}\right) = \sigma\left(2\varsigma_{\mathcal{I}} - 1\right)$$

□

Remark 5 For an arbitrary NIFN $\mathcal{I} = ((a,\sigma), \varsigma_{\mathcal{I}}, \upsilon_{\mathcal{I}})$, we conclude the following observations:

(1) For NIFN, when $\varsigma_{\mathcal{I}} + \upsilon_{\mathcal{I}} = 1$, i.e., when there is no hesitancy between the pairs of membership degrees then the proposed function $\mathcal{M}_1(\mathcal{I})$ becomes the existing function $S_1(\mathcal{I})$ as defined by Wang and Li [23] in Eq. (7).
(2) For any two NIFNs $\mathcal{I}$ and $\mathcal{J}$ such that $\pi_{\mathcal{I}} = \pi_{\mathcal{J}}$ then $\mathcal{I} > \mathcal{J}$ if $\upsilon_{\mathcal{I}} < \upsilon_{\mathcal{J}}$ and $\varsigma_{\mathcal{I}} > \varsigma_{\mathcal{J}}$ holds.
(3) If $\upsilon_{\mathcal{I}}$ (or $\varsigma_{\mathcal{I}}$) be same for any two NIFNs then the one having the smaller (or the larger) $\pi_{\mathcal{I}}$ has higher priority.

On the other hand, the proposed score functions $\mathcal{M}'_1$ and $\mathcal{M}'_2$ also satisfy the Properties 1–8 for NIVIFN $\mathcal{I} = \left((a,\sigma), \left[\underline{\varsigma}_{\mathcal{I}}, \overline{\varsigma}_{\mathcal{I}}\right], \left[\underline{\upsilon}_{\mathcal{I}}, \overline{\theta}_{\mathcal{I}}\right]\right)$, which are stated (without proof) as below.

(P1) For any two comparable NIVIFNs $\mathcal{I}_1$ and $\mathcal{I}_2$, if $\mathcal{I}_1 < \mathcal{I}_2$ then $\mathcal{M}'_1(\mathcal{I}_1) < \mathcal{M}'_1(\mathcal{I}_2)$ and $\mathcal{M}'_2(\mathcal{I}_1) > \mathcal{M}'_2(\mathcal{I}_2)$.

(P2) For NIVIFN $\mathcal{I}$, the proposed improved score functions $\mathcal{M}_1'(\mathcal{I})$ lies between $[-a, a]$ and $\mathcal{M}_2'(\mathcal{I})$ lies between $[-\sigma, \sigma]$.
(P3) The functions $\mathcal{M}_1'$ and S_1' for a NIVIFN $\mathcal{I}$ satisfy the inequality $\mathcal{M}_1'(\mathcal{I}) \geq S_1'(\mathcal{I})$.
(P4) For NIVIFN $\mathcal{I}$, $\mathcal{M}_1'(\mathcal{I}) + \mathcal{M}_1'(\mathcal{I}^c) \leq a$ and $\mathcal{M}_2'(\mathcal{I}) + \mathcal{M}_2'(\mathcal{I}^c) \leq \sigma$.
(P5) The score functions $\mathcal{M}_1'(\mathcal{I})$ and $S_1'(\mathcal{I})$ for any NIVIFN $\mathcal{I}$ satisfy the relation, $\mathcal{M}_1'(\mathcal{I}) = S_1'(\mathcal{I}) + a\left(\frac{\pi_{\mathcal{I}}}{2}\right)$.
(P6) (Zero Property) If NIVIFN $\mathcal{I} = ((a, \sigma), [0, 0], [1, 1])$ then $\mathcal{M}_1'(\mathcal{I}) = -a$ and $\mathcal{M}_2'(\mathcal{I}) = -\sigma$.
(P7) (One Property) If NIVIFN $\mathcal{I} = ((a, \sigma), [1, 1], [0, 0])$ then $\mathcal{M}_1'(\mathcal{I}) = a$ and $\mathcal{M}_2'(\mathcal{I}) = \sigma$.
(P8) For a subset $\mathcal{I} = ((a, \sigma), [\varsigma_{\mathcal{I}}, \varsigma_{\mathcal{I}}], [1 - \varsigma_{\mathcal{I}}, 1 - \varsigma_{\mathcal{I}}])$ of NIVIFN, the proposed score functions becomes $\mathcal{M}_1'(\mathcal{I}) = a(2\varsigma_{\mathcal{I}} - 1)$ and $\mathcal{M}_2'(\mathcal{I}) = \sigma(2\varsigma_{\mathcal{I}} - 1)$.

Decision-making approach based on the proposed score functions

In this section, we present a decision-making algorithm for solving the MADM problems.

Consider a set of alternatives $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_m$ which are evaluated by an expert under the different attributes $\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n$ and gives their preferences either in terms of NIFNs or NIVIFNs. Let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ be the weight vector assigned to the given attributes $\mathfrak{B}_j$ such that $\omega_j > 0$ and $\sum_{j=1}^n \omega_j = 1$. Then, the method for determining the finest alternative(s) by employing the proposed functions are summarized in the following algorithms.

Algorithm 1: When ratings are given under the NIFS environment

When an expert evaluate the given alternatives $\mathcal{V}_i$ under different attributes $\mathfrak{B}_j$ and represent their values in terms of NIFNs $\widetilde{\beta}_{ij} = \left((\widetilde{a}_{ij}, \widetilde{\sigma}_{ij}), \widetilde{\varsigma}_{ij}, \widetilde{\upsilon}_{ij}\right)$ where $\widetilde{\varsigma}_{ij}$ and $\widetilde{\upsilon}_{ij}$, respectively, represent the degree that the alternative $\mathcal{V}_i$ satisfies and dissatisfaction the criteria $\mathfrak{B}_j$ such that $\widetilde{\varsigma}_{ij}, \widetilde{\upsilon}_{ij} \in [0, 1]$ and $\widetilde{\varsigma}_{ij} + \widetilde{\upsilon}_{ij} \leq 1$. The representation of the characteristic for alternatives $\mathcal{V}_i (i = 1, 2, \ldots, m)$ is given by

$$\mathcal{V}_i = \{(\mathfrak{B}_1, (\widetilde{a}_{i1}, \widetilde{\sigma}_{i1}), \widetilde{\varsigma}_{i1}, \widetilde{\upsilon}_{i1}), (\mathfrak{B}_2, (\widetilde{a}_{i2}, \widetilde{\sigma}_{i2}), \widetilde{\varsigma}_{i2}, \widetilde{\upsilon}_{i2}), \ldots, (\mathfrak{B}_n, (\widetilde{a}_{in}, \widetilde{\sigma}_{in}), \widetilde{\varsigma}_{in}, \widetilde{\upsilon}_{in})\} \tag{19}$$

Then, the following are the steps summarized based on the proposed score function to obtain the best alternative(s).

Step 1: Arrange the collective information of the expert in the decision matrix $\mathcal{M} = (\widetilde{\beta}_{ij})_{m\times n}$ as

$$\mathcal{M} = \begin{array}{c} \\ \mathcal{V}_1 \\ \mathcal{V}_2 \\ \vdots \\ \mathcal{V}_m \end{array} \begin{array}{c} \begin{array}{cccc} \mathfrak{B}_1 & \mathfrak{B}_2 & \ldots & \mathfrak{B}_n \end{array} \\ \begin{pmatrix} \widetilde{\beta}_{11} & \widetilde{\beta}_{12} & \ldots & \widetilde{\beta}_{1n} \\ \widetilde{\beta}_{21} & \widetilde{\beta}_{22} & \ldots & \widetilde{\beta}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{\beta}_{m1} & \widetilde{\beta}_{m2} & \ldots & \widetilde{\beta}_{mn} \end{pmatrix} \end{array}$$

Step 2: Normalize the decision matrix $\mathcal{M} = (\widetilde{\beta}_{ij})_{m\times n}$, if required, into the matrix $\mathcal{R} = (\odot_{ij})_{m\times n}$ where $\odot_{ij} = \left((a_{ij}, \sigma_{ij}), \varsigma_{ij}, \upsilon_{ij}\right)$ has been obtained as follows.

For benefit attribute:

$$a_{ij} = \frac{\widetilde{a}_{ij}}{\max_i (\widetilde{a}_{ij})}, \ \sigma_{ij} = \frac{\widetilde{\sigma}_{ij}}{\max_i (\widetilde{\sigma}_{ij})} \cdot \frac{\widetilde{\sigma}_{ij}}{\widetilde{a}_{ij}}, \quad \varsigma_{ij} = \widetilde{\varsigma}_{ij}, \ \upsilon_{ij} = \widetilde{\upsilon}_{ij} \tag{20}$$

For cost attribute:

$$a_{ij} = \frac{\min_i (\widetilde{a}_{ij})}{\widetilde{a}_{ij}}, \ \sigma_{ij} = \frac{\widetilde{\sigma}_{ij}}{\max_i (\widetilde{\sigma}_{ij})} \cdot \frac{\widetilde{\sigma}_{ij}}{\widetilde{a}_{ij}}, \quad \varsigma_{ij} = \widetilde{\upsilon}_{ij}, \ \upsilon_{ij} = \widetilde{\varsigma}_{ij} \tag{21}$$

Step 3: Aggregate the collection information of the alternative $\mathcal{V}_i$ with expert evaluation β_{ij}, $j = 1, 2, \ldots, n$ into β_i, $i = 1, 2, \ldots, m$ by using NIFWGA operator as defined in Eq. (22), as

$$\begin{aligned} &\text{NIFWGA}(\beta_{i1}, \beta_{i2}, \ldots, \beta_{in}) \\ &= \left(\left(\prod_{j=1}^n a_{ij}^{\omega_j}, \prod_{j=1}^n a_{ij}^{\omega_j} \sqrt{\sum_{j=1}^n \frac{\omega_j \sigma_{ij}^2}{a_{ij}^2}} \right), \prod_{j=1}^n (\varsigma_{ij})^{\omega_j}, 1 - \prod_{j=1}^n (1 - \upsilon_{ij})^{\omega_j} \right) \end{aligned} \tag{22}$$

Step 4: Compute the score value of the obtained aggregated number $\beta_i = ((a_i, \sigma_i), \varsigma_i, \upsilon_i)$, $i = 1, 2, \ldots, m$ as

$$\mathcal{M}_1(\beta_i) = a_i \left(\frac{\varsigma_i - 3\upsilon_i + 1}{2} \right) \tag{23}$$

If score values are equal for any two indices then compute the score function values for them by using Eq. (24).

$$\mathcal{M}_2(\beta_i) = \sigma_i \left(\frac{\varsigma_i - 3\upsilon_i + 1}{2} \right) \tag{24}$$

Step 5: Rank the given alternatives based on the descending values of score values of β_i and hence select the best one(s).

Algorithm 2: When ratings are given under the NIVIFS environment

When an expert evaluate the given alternatives $\mathcal{V}_i$ under different attributes $\mathfrak{B}_j$ and represent their values in terms of NIVIFNs $\widetilde{\gamma}_{ij} = \left((\widetilde{a}_{ij}, \widetilde{\sigma}_{ij}), \left[\underline{\widetilde{\varsigma}}_{ij}, \overline{\widetilde{\varsigma}}_{ij}\right], \left[\underline{\widetilde{\upsilon}}_{ij}, \overline{\widetilde{\upsilon}}_{ij}\right]\right)$ such that $\left[\underline{\widetilde{\varsigma}}_{ij}, \overline{\widetilde{\varsigma}}_{ij}\right], \left[\underline{\widetilde{\upsilon}}_{ij}, \overline{\widetilde{\upsilon}}_{ij}\right] \subseteq [0,1]$ and $\overline{\widetilde{\varsigma}}_{ij} + \overline{\widetilde{\upsilon}}_{ij} \leq 1$. Then, the following are the steps summarized to find the best alternative(s).

Step 1: Arrange the collective information of the expert in the decision matrix $\mathcal{M} = (\widetilde{\gamma}_{ij})_{m\times n}$ as

$$\mathcal{M} = \begin{array}{c} \\ \mathcal{V}_1 \\ \mathcal{V}_2 \\ \vdots \\ \mathcal{V}_m \end{array} \begin{array}{c} \begin{array}{cccc} \mathfrak{B}_1 & \mathfrak{B}_2 & \cdots & \mathfrak{B}_n \end{array} \\ \begin{pmatrix} \widetilde{\gamma}_{11} & \widetilde{\gamma}_{12} & \cdots & \widetilde{\gamma}_{1n} \\ \widetilde{\gamma}_{21} & \widetilde{\gamma}_{22} & \cdots & \widetilde{\gamma}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{\gamma}_{m1} & \widetilde{\gamma}_{m2} & \cdots & \widetilde{\gamma}_{mn} \end{pmatrix} \end{array}$$

Step 2: Normalize the matrix $\mathcal{M}$, if required, into the matrix $\mathcal{R} = (\gamma_{ij})_{m\times n}$ where $\gamma_{ij} = \left((a_{ij}, \sigma_{ij}), [\underline{\varsigma}_{ij}, \overline{\varsigma}_{ij}], [\underline{\upsilon}_{ij}, \overline{\upsilon}_{ij}]\right)$ has been obtained as
For benefit attribute:

$$a_{ij} = \frac{\widetilde{a}_{ij}}{\max\limits_i (\widetilde{a}_{ij})}, \quad \sigma_{ij} = \frac{\widetilde{\sigma}_{ij}}{\max\limits_i (\widetilde{\sigma}_{ij})} \cdot \frac{\widetilde{\sigma}_{ij}}{\widetilde{a}_{ij}},$$
$$\underline{\varsigma}_{ij} = \underline{\widetilde{\varsigma}}_{ij}, \ \overline{\varsigma}_{ij} = \overline{\widetilde{\varsigma}}_{ij}, \ \underline{\upsilon}_{ij} = \underline{\widetilde{\upsilon}}_{ij}, \ \overline{\upsilon}_{ij} = \overline{\widetilde{\upsilon}}_{ij} \quad (25)$$

For cost attribute:

$$a_{ij} = \frac{\min\limits_i (\widetilde{a}_{ij})}{\widetilde{a}_{ij}}, \quad \sigma_{ij} = \frac{\widetilde{\sigma}_{ij}}{\max\limits_i (\widetilde{\sigma}_{ij})} \cdot \frac{\widetilde{\sigma}_{ij}}{\widetilde{a}_{ij}},$$
$$\underline{\varsigma}_{ij} = \underline{\widetilde{\upsilon}}_{ij}, \ \overline{\varsigma}_{ij} = \overline{\widetilde{\upsilon}}_{ij}, \ \underline{\upsilon}_{ij} = \underline{\widetilde{\varsigma}}_{ij}, \ \overline{\upsilon}_{ij} = \overline{\widetilde{\varsigma}}_{ij} \quad (26)$$

Step 3: Aggregate the collection information of the alternative $\mathcal{V}_i$ with expert evaluation $\gamma_{ij}, j = 1, 2, \ldots, n$ into $\gamma_i, i = 1, 2, \ldots, m$ by using NIVIFWGA operator as defined in Eq. (27) as

$$\begin{aligned} &\text{NIVIFWGA}(\gamma_{i1}, \gamma_{i2}, \ldots, \gamma_{in}) \\ &= \left(\left(\prod_{j=1}^n a_{ij}^{\omega_j}, \prod_{j=1}^n a_{ij}^{\omega_j} \sqrt{\sum_{j=1}^n \frac{\omega_j \sigma_{ij}^2}{a_{ij}^2}} \right), \right. \\ &\left[\prod_{j=1}^n \left(\underline{\varsigma}_{ij}\right)^{\omega_j}, \prod_{j=1}^n (\overline{\varsigma}_{ij})^{\omega_j} \right], \\ &\left. \left[1 - \prod_{j=1}^n \left(1 - \underline{\upsilon}_{ij}\right)^{\omega_j}, 1 - \prod_{j=1}^n (1 - \overline{\upsilon}_{ij})^{\omega_j} \right] \right) \end{aligned} \quad (27)$$

Step 4: Compute the score value of the obtained aggregated number $\gamma_i = ((a_i, \sigma_i), [\underline{\varsigma}_i, \overline{\varsigma}_i], [\underline{\upsilon}_i, \overline{\upsilon}_i]), i = 1, 2, \ldots, m$ as

$$\mathcal{M}_1'(\gamma_i) = a_i \left(\frac{\underline{\varsigma}_i + \overline{\varsigma}_i - 3(\underline{\upsilon}_i + \overline{\upsilon}_i) + 2}{4} \right) \quad (28)$$

If score values are equal for any two indices then compute the score values for them by using Eq. (29).

$$\mathcal{M}_2'(\gamma_i) = \sigma_i \left(\frac{\underline{\varsigma}_i + \overline{\varsigma}_i - 3(\underline{\upsilon}_i + \overline{\upsilon}_i) + 2}{4} \right) \quad (29)$$

Step 5: Rank the given alternatives based on the descending values of score values of γ_i and hence select the best one(s) according to Definition 17.

Illustrate examples

To demonstrate the working of the above-defined algorithms, we illustrate them with case studies that can be read as follows.

When evaluations are taken in NIFNs

Example 6 Indian rural society is changing and transforming in many aspects, including jobs, business structures, transportation facilities, and communication systems. Therefore, people in rural areas migrate to big cities to find more opportunities. To stop this immigration, the Indian government wants to provide all facilities in the rural areas, for this good road connectivity of the rural areas with the cities must require. In that direction, the Government of India has started numerous projects to build or to repair the roads and hence issued the global tender to select the contractor for these projects in the newspaper and considered the four attributes required for its namely, tender price ($\mathfrak{B}_1$), completion time ($\mathfrak{B}_2$), contractor background ($\mathfrak{B}_3$), financial status ($\mathfrak{B}_4$)

with $\omega = (0.4, 0.2, 0.25, 0.15)^T$ as a weight vector. After screening, only four contractors (i.e., alternatives) namely, Jaihind Road Builders Pvt. Ltd. ($\mathcal{V}_1$), J.K. Construction ($\mathcal{V}_2$), Bull quick Infrastructure Pvt. Ltd. ($\mathcal{V}_3$), Relcon Infra Projects Ltd. ($\mathcal{V}_4$) remain present to bid these projects. The target of the problem is to find the best contractor for the required project. The evaluation of these strategies is in terms of normal fuzzy numbers taken by the evaluators/experts under the criteria defined above. To access the best alternatives among the given ones, we implemented the steps of the proposed Algorithm 1 here.

Step 1: Assume the given alternatives are evaluated by an expert and gives their rating in terms of NIFNs. Such values are represented in Table 1.

Step 2: The attributes $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are the cost types while others are the benefit types. Thus, by Eqs. (20) and (21), we obtain the normalized values of each NIFN and the results are represented in Table 2.

Step 3: By utilizing the NIFWGA operator as defined in Eq. (22), to aggregate the given information $\beta_{ij}, j = 1, 2, 3, 4$ of the Table 2, the collective values β_i's of each alternative $\mathcal{V}_i$ are obtained as

$$\beta_1 = ((0.9264, 0.0849), 0.3366, 0.5389),$$
$$\beta_2 = ((0.8001, 0.0527), 0.3495, 0.5496),$$
$$\beta_3 = ((0.8989, 0.0601), 0.4431, 0.5223),$$
$$\text{and } \beta_4 = ((0.7356, 0.0898), 0.3413, 0.5929).$$

Step 4: By Eq. (23), the score values of collective number obtained through NIFWGA operator are computed as $\mathcal{M}_1(\beta_1) = -0.1297$, $\mathcal{M}_1(\beta_2) = -0.1198$, $\mathcal{M}_1(\beta_3) = -0.0557$ and $\mathcal{M}_1(\beta_4) = -0.1608$.

Step 5: From the values of $\mathcal{M}_1$'s, we obtain that $\mathcal{M}_1(\beta_3) > \mathcal{M}_1(\beta_2) > \mathcal{M}_1(\beta_1) > \mathcal{M}_1(\beta_4)$ and hence by applying the Definition 15, the ranking order of the alternatives is taken as $\mathcal{V}_3 \succ \mathcal{V}_2 \succ \mathcal{V}_1 \succ \mathcal{V}_4$, where $\succ$ means "preferred to". From this ordering, we compute the $\mathcal{V}_3$ is the best alternative for the desired task.

The feasibility of the approach has been verified by carrying a comparative study with some existing approaches [24–26] under the NIFS environment and the outcomes are compiled in Table 3. From this investigation, it has been noted that the best alternative obtained by using [24,25] approaches is $\mathcal{V}_2$ rather than $\mathcal{V}_3$, which is due to the fact that their score function is unable to consider the degree of the hesitation during the analysis. Therefore, they are unable to completely justify the information during the decision-making process.

When evaluations are taken in NIVIFNs

In this section, we demonstrate the working of stated Algorithm 2 with a numerical example.

Example 7 Consider a well-recognized university that wants to appoint an outstanding professor in the research and development center. For it, they have issued a notice in a newspaper for the application regarding the desired post. Against the advertisement, the director of the Institute constitute a committee for short-listing the candidate based on the three major factors namely $\mathfrak{B}_1$, research publications, $\mathfrak{B}_2$: number of Ph.D. supervised and $\mathfrak{B}_3$: number of research project handled whose weight vector is $\omega = (0.35, 0.25, 0.40)^T$. Based on these preferences information, a committee has shortlisted the four applicants $\mathcal{V}_i, i = 1, 2, 3, 4$ and called them for the personal interview where they have evaluated them and rate their preferences in terms of NIVIFNs $\widetilde{\gamma}_{ij}$. Then to access the best alternatives among them, we implemented the steps of the proposed Algorithm 2 here.

Step 1: The rating values of each applicant is given by an expert and are summarized in Table 4.

Step 2: Since all the attributes are of benefit types, so we normalized the given information $\widetilde{\gamma}_{ij}$ into γ_{ij} by using Eq. (25). The obtained numbers are recorded in Table 5.

Step 3: Utilize NIVIFWGA operator defined in Eq. (27) to aggregate the rating of Table 5 and the collective values γ_i's of each alternative $\mathcal{V}_i$ are obtained as

$$\gamma_1 = ((0.7213, 0.1161), [0.2297, 0.4266], [0.3674, 0.4898]),$$
$$\gamma_2 = ((0.7298, 0.0300), [0.5102, 0.7384], [0.1614, 0.2616]),$$
$$\gamma_3 = ((0.8643, 0.0332), [0.3824, 0.5578], [0.2260, 0.3618]),$$
$$\text{and } \gamma_4 = ((0.6685, 0.0559), [0.4799, 0.5864], [0.1000, 0.2263]).$$

Step 4: By Eq. (28), we compute the score values of the numbers obtained by NIVIFWGA operator as $\mathcal{M}'_1(\mathcal{I}_1) = 0.0153$, $\mathcal{M}'_1(\mathcal{I}_2) = 0.3612$, $\mathcal{M}'_1(\mathcal{I}_3) = 0.2543$, $\mathcal{M}'_1(\mathcal{I}_4) = 0.3489$.

Step 5: Based on these score values, we obtain the ranking order of the given alternatives as $\mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_1$ and hence $\mathcal{V}_2$ is the best applicant for the post. Here $\succ$ refers "preferred to".

To examine the appearance of the stated Algorithm 2 with the existing approach [27], we execute the given information based on the various operators as suggested in Liu and Teng [27]. For instance, when we aggregate the given information corresponding to each alternative by using NIVIFWAA operator [27], then the aggregated values are obtained as

Table 1 Decision matrix in terms of NIFNs

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{B}_4$
$\mathcal{V}_1$	$((3.0, 0.4), 0.7, 0.2)$	$((7, 0.6), 0.6, 0.3)$	$((5, 0.4), 0.6, 0.2)$	$((7, 0.6), 0.6, 0.3)$
$\mathcal{V}_2$	$((4.0, 0.2), 0.6, 0.3)$	$((8, 0.4), 0.8, 0.1)$	$((6, 0.7), 0.8, 0.2)$	$((5, 0.3), 0.7, 0.3)$
$\mathcal{V}_3$	$((3.5, 0.3), 0.6, 0.4)$	$((6, 0.2), 0.6, 0.3)$	$((5.5, 0.6), 0.5, 0.5)$	$((6, 0.4), 0.8, 0.1)$
$\mathcal{V}_4$	$((5.0, 0.5), 0.8, 0.2)$	$((7, 0.5), 0.6, 0.2)$	$((4.5, 0.5), 0.8, 0.2)$	$((7, 0.2), 0.7, 0.1)$

Table 2 Normalized data

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{B}_4$
$\mathcal{V}_1$	$((1.0000, 0.1067), 0.2, 0.7)$	$((0.8571, 0.0857), 0.3, 0.6)$	$((0.8333, 0.0457), 0.6, 0.2)$	$((1.0000, 0.0857), 0.6, 0.3)$
$\mathcal{V}_2$	$((0.7500, 0.0200), 0.3, 0.6)$	$((0.7500, 0.0333), 0.1, 0.8)$	$((1.0000, 0.1167), 0.8, 0.2)$	$((0.7143, 0.0300), 0.7, 0.3)$
$\mathcal{V}_3$	$((0.8571, 0.0514), 0.4, 0.6)$	$((1.0000, 0.0111), 0.3, 0.6)$	$((0.9167, 0.0935), 0.5, 0.5)$	$((0.8571, 0.0444), 0.8, 0.1)$
$\mathcal{V}_4$	$((0.6000, 0.1000), 0.2, 0.8)$	$((0.8571, 0.0595), 0.2, 0.6)$	$((0.7500, 0.0794), 0.8, 0.2)$	$((1.0000, 0.0095), 0.7, 0.1)$

Table 3 Comparative study for the Example 6

	Overall score value of				Ranking order
	$\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$	
Wang and Li [24]	-0.1231	-0.0021	-0.0386	-0.0101	$\mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
Wang et al. [25]	-0.0964	0.0192	-0.0189	0.0122	$\mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
Wang et al. [26]	-0.1874	-0.1602	-0.0713	-0.1850	$\mathcal{V}_3 \succ \mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_1$

Table 4 Input data in the form of NIVIFNs

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$
$\mathcal{V}_1$	$((3, 0.4), [0.4, 0.5], [0.3, 0.4])$	$((5, 0.6), [0.4, 0.6], [0.2, 0.4])$	$((7, 0.6), [0.1, 0.3], [0.5, 0.6])$
$\mathcal{V}_2$	$((4, 0.2), [0.6, 0.7], [0.2, 0.3])$	$((6, 0.4), [0.6, 0.7], [0.2, 0.3])$	$((5, 0.3), [0.4, 0.8], [0.1, 0.2])$
$\mathcal{V}_3$	$((6, 0.3), [0.3, 0.6], [0.3, 0.4])$	$((5, 0.2), [0.5, 0.6], [0.3, 0.4])$	$((6, 0.4), [0.4, 0.5], [0.1, 0.3])$
$\mathcal{V}_4$	$((5, 0.5), [0.7, 0.8], [0.1, 0.2])$	$((7, 0.5), [0.6, 0.7], [0.1, 0.3])$	$((3, 0.2), [0.3, 0.4], [0.1, 0.2])$

Table 5 Normalized data in the form of NIVIFNs

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$
$\mathcal{V}_1$	$((0.5000, 0.1067), [0.4, 0.5], [0.3, 0.4])$	$((0.7143, 0.1200), [0.4, 0.6], [0.2, 0.4])$	$((1.0000, 0.0857), [0.1, 0.3], [0.5, 0.6])$
$\mathcal{V}_2$	$((0.6667, 0.0200), [0.6, 0.7], [0.2, 0.3])$	$((0.8571, 0.0444), [0.6, 0.7], [0.2, 0.3])$	$((0.7143, 0.0300), [0.4, 0.8], [0.1, 0.2])$
$\mathcal{V}_3$	$((1.0000, 0.0300), [0.3, 0.6], [0.3, 0.4])$	$((0.7143, 0.0133), [0.5, 0.6], [0.3, 0.4])$	$((0.8571, 0.0444), [0.4, 0.5], [0.1, 0.3])$
$\mathcal{V}_4$	$((0.8333, 0.1000), [0.7, 0.8], [0.1, 0.2])$	$((1.0000, 0.0595), [0.6, 0.7], [0.1, 0.3])$	$((0.4286, 0.0222), [0.3, 0.4], [0.1, 0.2])$

Table 6 Comparative analysis for the Example 7

Method	Score value of the alternatives				Ranking
	$\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$	order
NIVIFWAA [27]	-0.0187	0.3182	0.1777	0.3147	$\mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
NIVIFOWAA [27]	-0.0095	0.3148	0.1763	0.3147	$\mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
NIVIFWGA [27]	-0.0724	0.3012	0.1523	0.2474	$\mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_1$
NIVIFOWGA [27]	-0.0690	0.2986	0.1522	0.2474	$\mathcal{V}_2 \succ \mathcal{V}_4 \succ \mathcal{V}_3 \succ \mathcal{V}_1$

NIVIFWAA normal interval-valued intuitionistic fuzzy weighted arithmetic averaging, *NIVIFOWAA* normal interval-valued intuitionistic fuzzy ordered weighted arithmetic averaging, *NIVIFWGA* normal interval-valued intuitionistic fuzzy weighted geometric averaging, *NIVIFOWGA* normal interval-valued intuitionistic fuzzy ordered weighted geometric averaging

$\gamma_1 = ((0.7536, 0.1016), [0.2944, 0.4590], [0.3325, 0.4704])$, $\gamma_2 = ((0.7333, 0.0301), [0.5296, 0.7449], [0.1516, 0.2551])$, $\gamma_3 = ((0.8714, 0.0316), [0.3950, 0.5627], [0.1933, 0.3565])$ and $\gamma_4 = ((0.7131, 0.0588), [0.5476, 0.6565], [0.1000, 0.2213])$. By using the existing score function $\mathcal{S}_1'$, we get the score values of each alternative as $\mathcal{S}_1'(\gamma_1) = -0.0187$, $\mathcal{S}_1'(\gamma_2) = 0.3182$, $\mathcal{S}_1'(\gamma_3) = 0.1777$ and $\mathcal{S}_1'(\gamma_4) = 0.3147$. Similarly, we can perform the other operators to aggregate the numbers and the resultant values of the alternatives obtained through different operators and the final ranking order are listed in Table 6.

From the table, it can clearly be seen that the most optimal alternative is $\mathcal{V}_2$, and it coincides with the result of the proposed approach.

Conclusion

The key contribution of the work can be summarized below.

(1) The examined study employs the normal intuitionistic fuzzy numbers to express the vagueness in the data. The normal fuzzy numbers have widely used and several advantages such as all the natural phenomena and production activities are well expressed by such numbers; the higher derivative of the normal membership function is continuous. Thus, it will represent the data in a better manner than other fuzzy numbers.

(2) To rank the different NIFNs and/or NIVIFNs, we proposed some new improved score functions by adding the degree of the hesitancy into the given functions, to overcome the impediments of existing score and accuracy functions [23,27]. The various properties of the stated functions are discussed. On the other hand, the supremacies of the proposed functions to rank the given numbers over the existing functions are described through counter-intuitive cases (Examples 1, 2).

(3) Also, from the stated functions, it has been achieved that by setting a zero hesitancy degree corresponding to given numbers, the proposed functions defined in Eqs. (15) and (17) reduces to the existing functions as Eqs. (7) and (12) respectively, under the NIFS and NIVIFS environment.

(4) Two new algorithms based on the stated functions are defined to solve the MADM problems with NIFS and NIVIFS information. In these approaches, different values of the alternatives are aggregated by using weighted operators and then finally the obtained numbers are ranked by using the stated score functions.

(5) To demonstrate the appearance of the stated algorithms, a numerical example is given and compare their results with the existing studies [24–27]. It is concluded from this study that the proposed work gives more reasonable ways to handle the fuzzy information to solve the practical problems.

In the future, we shall lengthen the application of the proposed measures to the diverse fuzzy environment as well as different fields of application such as supply chain management, emerging decision problems, brain hemorrhage, risk evaluation, etc [28–31].

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