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ORIGINAL ARTICLE

Interval-valued Pythagorean fuzzy Einstein hybrid weighted averaging aggregation operator and their application to group decision making

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Abstract

The objective of the present work is divided into two folds. Firstly, interval-valued Pythagorean fuzzy Einstein hybrid weighted averaging aggregation operator has been introduced along with their several properties, namely idempotency, boundedness and monotonicity. Secondly, we apply the proposed operator to deal with multi-attribute group decision-making problem under Pythagorean fuzzy information. For this, we construct an algorithm for multi-attribute group decision making. At the last, we construct a numerical example for multi-attribute group decision making. The main advantage of using the proposed operator is that this operator provides more accurate and precise results is compared to the existing methods.

Keywords IVPFS · IVPFEHWA averaging operator · MAGDM problems

Introduction

Multi-criteria group decision making is one of the successful processes for finding the optimal alternative from all the feasible alternatives according to some criteria or attributes. Traditionally, it has been generally assumed that all the information that access the alternative in terms of criteria and their corresponding weights are expressed in the form of crisp numbers. But most of the decisions in the real-life situations are taken in the environment where the goals and constraints are generally imprecise or vague in nature. In order to handle the uncertainties and fuzziness intuitionistic fuzzy set [1] theory is one of the successful extensions of the fuzzy set theory [2], which is characterized by the degree of membership and degree of non-membership has been presented. After the successful and positive applications of intuitionistic fuzzy set, aggregation operators become more interesting topic for research. Thus, many scholars in [3–16] developed several aggregation operators for group decision making using intuitionistic fuzzy information.

However, there are many cases where the decision maker may provide the degree of membership and nonmembership of a particular attribute in such a way that their sum is greater than one. To solve these types of problems, Yager [17,18] introduced the concept of another set called Pythagorean fuzzy set. Pythagorean fuzzy set is more powerful tool to solve uncertain problems. Like intuitionistic fuzzy aggregation operators, Pythagorean fuzzy aggregation operators are also become an interesting and important area for research, after the advent of Pythagorean fuzzy set theory. Several researchers in [19–28] introduced many aggregation operators for decision using Pythagorean fuzzy information.

But, in some real decision-making problems, due to insufficiency in available information, it may be difficult for decision makers to exactly quantify their opinions with a crisp number, but they can be represented by an interval number within [0, 1]. Therefore, it is so important to present the idea of interval-valued Pythagorean fuzzy sets, which permit the membership degrees and non- membership degrees to a given set to have an interval value. Thus in [29] Peng and Yang introduced the concept of interval-valued Pythagorean fuzzy set. Rahman et al. [30–33] introduced many aggregation operators using interval-valued Pythagorean fuzzy

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numbers and applied them to multi-attribute group decision making.

Thus, keeping the advantages of these operators, in this paper, we introduce the notion of interval-valued Pythagorean fuzzy Einstein hybrid weighted averaging operator. Moreover, we introduce some of their basic properties such as idempotency, boundedness and monotonicity. This motivation comes from [32], in which the authors introduced the notion of IVPFEWA operator and IVPFEOWA operator and applied them to group decision making. But in this paper we introduce the notion of IVPFEHWA operator, which is the generalization of the above mention operators.

The remainder of this paper is structured as follows. In Sect. "Preliminaries", we give some basic definitions and results which will be used in our later sections. In Sect. "Interval-valued Pythagorean fuzzy Einstein hybrid weighted averaging aggregation operator", we introduce the notion of interval-valued Pythagorean fuzzy Einstein hybrid weighted averaging operator. In Sect. "An approach to multiple attribute group decision-making problems based on intervalvalued Pythagorean fuzzy information", we apply the proposed operator to multi-attribute group decision-making problem with Pythagorean fuzzy information. In Sect. "Illustrative example", we develop a numerical example. In Sect. "Conclusion", we have conclusion.

Preliminaries

Definition 1 [17,18] Let K be a fixed set, then a Pythagorean fuzzy set can be defined as:

$$P = \{\langle k, u_P(k), v_P(k)\rangle | k \in K\}, \tag{1}$$

where $u_P(k) : P \to [0, 1]$, $v_P(k) : K \to [0, 1]$ are called membership function and non-membership function, respectively, with condition $0 \leq (u_P(k))^2 + (v_P(k))^2 \leq 1$, for all $k \in K$.

Let

$$\pi_P(k) = \sqrt{1 - u_P^2(k) - v_P^2(k)}. \tag{2}$$

Then, it is called the Pythagorean fuzzy index of $k \in K$, with condition $0 \leq \pi_P(k) \leq 1$, for every $k \in K$.

Definition 2 [29] Let K be a fixed set, then an interval-valued Pythagorean fuzzy set can be defined as:

$$I = \{\langle k, u_I(k), v_I(k)\rangle | k \in K\}, \tag{3}$$

where

$$u_I(k) = [u_I^a(k), u_I^b(k)] \subset [0, 1], \tag{4}$$

and

$$v_I(k) = [v_I^a(k), v_I^b(k)] \subset [0, 1]. \tag{5}$$

Also

$$u_I^a(k) = \inf(u_I(k)), \tag{6}$$
$$u_I^b(k) = \sup(u_I(k)), \tag{7}$$
$$v_I^a(k) = \inf(v_I(k)), \tag{8}$$
$$v_I^b(k) = \sup(v_I(k)), \tag{9}$$

and

$$0 \leq \left(u_I^b(k)\right)^2 + \left(v_I^b(k)\right)^2 \leq 1. \tag{10}$$

If

$$\pi_I(k) = \left[\pi_I^a(k), \pi_I^b(k)\right], \text{ for all } k \in K. \tag{11}$$

Then, it is called the interval-valued Pythagorean fuzzy index of k to I, where

$$\pi_I^a(k) = \sqrt{1 - \left(u_I^b(k)\right)^2 - \left(v_I^b(k)\right)^2}, \tag{12}$$

and

$$\pi_I^b(k) = \sqrt{1 - \left(u_I^a(k)\right)^2 - \left(v_I^a(k)\right)^2}. \tag{13}$$

Definition 3 [29] Let $\lambda = ([u_\lambda, v_\lambda], [x_\lambda, y_\lambda])$ be an IVPFN, then the score function and accuracy function of λ can be defined as follows, respectively:

$$s(\lambda) = \frac{1}{2}\left[(u_\lambda)^2 + (v_\lambda)^2 - (x_\lambda)^2 - (y_\lambda)^2\right], \tag{14}$$

and

$$h(\lambda) = \frac{1}{2}\left[(u_\lambda)^2 + (v_\lambda)^2 + (x_\lambda)^2 + (y_\lambda)^2\right]. \tag{15}$$

If λ_1 and λ_2 are two IVPFNs, then

1. If $s(\lambda_1) \prec s(\lambda_2)$, then $\lambda_1 \prec \lambda_2$.
2. If $s(\lambda_1) = s(\lambda_2)$, then we have the following three conditions.
   1) If $h(\lambda_1) = h(\lambda_2)$, then $\lambda_1 = \lambda_2$.
   2) If $h(\lambda_1) \prec h(\lambda_2)$, then $\lambda_1 \prec \lambda_2$.
   3) If $h(\lambda_1) \succ h(\lambda_2)$, then $\lambda_1 \succ \lambda_2$.

Definition 4 [32] Let $\lambda = ([u, v], [x, y])$, $\lambda_1 = ([u_1, v_1], [x_1, y_1])$, $\lambda_2 = ([u_2, v_2], [x_2, y_2])$ are three IVPFNs, and $\delta \succ 0$, then some Einstein operations for $\lambda, \lambda_1, \lambda_2$ can be defined as follows:

1.

$$\lambda_1 \oplus_\varepsilon \lambda_2 = \left(\left[\frac{\sqrt{u_1^2 + u_2^2}}{\sqrt{1 + u_1^2 u_2^2}}, \frac{\sqrt{v_1^2 + v_2^2}}{\sqrt{1 + v_1^2 v_2^2}} \right], \left[\frac{x_1 x_2}{\sqrt{1 + \left(1 - x_1^2\right)\left(1 - x_2^2\right)}}, \frac{y_1 y_2}{\sqrt{1 + \left(1 - y_1^2\right)\left(1 - y_2^2\right)}} \right] \right)$$

2.

$$\lambda_1 \otimes_\varepsilon \lambda_2 = \left(\left[\frac{u_1 u_2}{\sqrt{1 + \left(1 - u_1^2\right)\left(1 - u_2^2\right)}}, \frac{v_1 v_2}{\sqrt{1 + \left(1 - v_1^2\right)\left(1 - v_2^2\right)}} \right], \left[\frac{\sqrt{x_1^2 + x_2^2}}{\sqrt{1 + x_1^2 x_2^2}}, \frac{\sqrt{y_1^2 + y_2^2}}{\sqrt{1 + y_1^2 y_2^2}} \right] \right)$$

3.

$$\delta\lambda = \left(\left[\frac{\sqrt{(1 + u^2)^\delta - (1 - u^2)^\delta}}{\sqrt{(1 + u^2)^\delta + (1 - u^2)^\delta}}, \frac{\sqrt{(1 + v^2)^\delta - (1 - v^2)^\delta}}{\sqrt{(1 + v^2)^\delta + (1 - v^2)^\delta}} \right], \left[\frac{\sqrt{2(x^2)^\delta}}{\sqrt{(2 - x^2)^\delta + (x^2)^\delta}}, \frac{\sqrt{2(y^2)^\delta}}{\sqrt{(2 - y^2)^\delta + (y^2)^\delta}} \right] \right)$$

4.

$$\lambda^\delta = \left(\left[\frac{\sqrt{2(u^2)^\delta}}{\sqrt{(2 - u^2)^\delta + (u^2)^\delta}}, \frac{\sqrt{2(v^2)^\delta}}{\sqrt{(2 - v^2)^\delta + (v^2)^\delta}} \right], \left[\frac{\sqrt{(1 + x^2)^\delta - (1 - x^2)^\delta}}{\sqrt{(1 + x^2)^\delta + (1 - x^2)^\delta}}, \frac{\sqrt{(1 + y^2)^\delta - (1 - y^2)^\delta}}{\sqrt{(1 + y^2)^\delta + (1 - y^2)^\delta}} \right] \right)$$

Definition 5 [32] Let $\lambda_j = ([u_j, v_j], [x_j, y_j])(j = 1, 2, 3, \ldots, n)$ be the collection of IVPFVs, then IVPFEWA operator can be defined as:

$$\text{IVPFEWA}_w(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n) = \left(\left[\frac{\sqrt{\prod_{j=1}^n \left(1+u_{\lambda_j}^2\right)^{w_j} - \prod_{j=1}^n \left(1-u_{\lambda_j}^2\right)^{w_j}}}{\sqrt{\prod_{j=1}^n \left(1+u_{\lambda_j}^2\right)^{w_j} + \prod_{j=1}^n \left(1-u_{\lambda_j}^2\right)^{w_j}}}, \frac{\sqrt{\prod_{j=1}^n \left(1+v_{\lambda_j}^2\right)^{w_j} - \prod_{j=1}^n \left(1-v_{\lambda_j}^2\right)^{w_j}}}{\sqrt{\prod_{j=1}^n \left(1+v_{\lambda_j}^2\right)^{w_j} + \prod_{j=1}^n \left(1-v_{\lambda_j}^2\right)^{w_j}}} \right], \left[\frac{\sqrt{2 \prod_{j=1}^n \left(x_{\lambda_j}^2\right)^{w_j}}}{\sqrt{\prod_{j=1}^n \left(2-x_{\lambda_j}^2\right)^{w_j} + \prod_{j=1}^n \left(x_{\lambda_j}^2\right)^{w_j}}}, \frac{\sqrt{2 \prod_{j=1}^n \left(y_{\lambda_j}^2\right)^{w_j}}}{\sqrt{\prod_{j=1}^n \left(2-y_{\lambda_j}^2\right)^{w_j} + \prod_{j=1}^n \left(y_{\lambda_j}^2\right)^{w_j}}} \right] \right), \tag{16}$$

where $w = (w_1, w_2, w_3, \ldots, w_n)^T$ is the weighted vector of $\lambda_j(j = 1, 2, 3, \ldots, n)$, such that $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$.

Definition 6 [32] Let $\lambda_j(j = 1, 2, 3, \ldots, n)$ be a collection of IVPFVs, then IVPFEOWA operator can be defined as:

$$\text{IVPFEOWA}_w(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n) = \left(\left[\frac{\sqrt{\prod_{j=1}^n \left(1+u_{\lambda_{\sigma(j)}}^2\right)^{w_j} - \prod_{j=1}^n \left(1-u_{\lambda_{\sigma(j)}}^2\right)^{w_j}}}{\sqrt{\prod_{j=1}^n \left(1+u_{\lambda_{\sigma(j)}}^2\right)^{w_j} + \prod_{j=1}^n \left(1-u_{\lambda_{\sigma(j)}}^2\right)^{w_j}}}, \frac{\sqrt{\prod_{j=1}^n \left(1+v_{\lambda_{\sigma(j)}}^2\right)^{w_j} - \prod_{j=1}^n \left(1-v_{\lambda_{\sigma(j)}}^2\right)^{w_j}}}{\sqrt{\prod_{j=1}^n \left(1+v_{\lambda_{\sigma(j)}}^2\right)^{w_j} + \prod_{j=1}^n \left(1-v_{\lambda_{\sigma(j)}}^2\right)^{w_j}}} \right], \left[\frac{\sqrt{2 \prod_{j=1}^n \left(x_{\lambda_{\sigma(j)}}^2\right)^{w_j}}}{\sqrt{\prod_{j=1}^n \left(2-x_{\lambda_{\sigma(j)}}^2\right)^{w_j} + \prod_{j=1}^n \left(x_{\lambda_{\sigma(j)}}^2\right)^{w_j}}}, \frac{\sqrt{2 \prod_{j=1}^n \left(y_{\lambda_{\sigma(j)}}^2\right)^{w_j}}}{\sqrt{\prod_{j=1}^n \left(2-y_{\lambda_{\sigma(j)}}^2\right)^{w_j} + \prod_{j=1}^n \left(y_{\lambda_{\sigma(j)}}^2\right)^{w_j}}} \right] \right), \tag{17}$$

where $(\sigma(1), \sigma(2), \sigma(3), \ldots, \sigma(n))$ is a permutation of $(1, 2, 3, \ldots, n)$ such that $\sigma(j) \leq \sigma(j-1)$ for all $j$and $w = (w_1, w_2, w_3, \ldots, w_n)^T$ is the weighted vector of $\lambda_{\sigma(j)}(j = 1, 2, 3, \ldots, n)$ such that $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$.

Interval-valued Pythagorean fuzzy Einstein hybrid weighted averaging aggregation operator

In this section, we introduce the notion of interval-valued Pythagorean fuzzy Einstein hybrid weighted averaging aggregation operator. We also discuss some desirable properties such as idempotency, boundedness and monotonicity.

Definition 7 An interval-valued Pythagorean fuzzy Einstein hybrid weighted averaging operator of dimension n is a mapping IVPFEHWA : $\Theta^n \rightarrow \Theta$,which has associated vector$w = (w_1, w_2, w_3, \ldots, w_n)^T$, such that $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$. Furthermore

$$\text{IVPFEHWA}_{\omega,w}(\lambda_1,\lambda_2,\lambda_3,...,\lambda_n) = \left(\begin{array}{l}\left[\sqrt{\frac{\prod_{j=1}^{n}\left(1+u^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}-\prod_{j=1}^{n}\left(1-u^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+u^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{n}\left(1-u^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}}}, \sqrt{\frac{\prod_{j=1}^{n}\left(1+v^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}-\prod_{j=1}^{n}\left(1-v^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+v^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{n}\left(1-v^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}}}\right],\\ \left[\frac{\sqrt{2\prod_{j=1}^{n}\left(x^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{n}\left(2-x^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{n}\left(x^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}}}, \frac{\sqrt{2\prod_{j=1}^{n}\left(y^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{n}\left(2-y^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{n}\left(y^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}}}\right]\end{array}\right), \quad (18)$$

where $\dot\lambda_{\sigma(j)}$ is the jth largest of the weighted interval-valued Pythagorean fuzzy values, $\dot\lambda_{\sigma(j)}(\dot\lambda_{\sigma(j)} = n\omega_j\lambda_j)$. $\omega = (\omega_1, \omega_2, \omega_3, ..., \omega_n)^T$ is the weighted vector of $\lambda_j (j = 1, 2, 3, ..., n)$ such that $\omega_j \in [0, 1]$, $\sum_{j=1}^{n}\omega_j = 1$, and n is the balancing coefficient, which plays a role of balance. If the vector $w = (w_1, w_2, w_3, ..., w_n)^T$ approaches to $\left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)^T$, then the vector $(n\omega_1\lambda_1, n\omega_2\lambda_2, ..., n\omega_n\lambda_n)^T$ approaches to $(\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n)^T$.

Theorem 1 *Let $\lambda, \lambda_1, \lambda_2$ be the three interval-valued Pythagorean fuzzy numbers and $\delta, \delta_1, \delta_2 \succ 0$, then the following conditions always hold:*

1. $\lambda_1 \oplus_\varepsilon \lambda_2 = \lambda_2 \oplus_\varepsilon \lambda_1$,
2. $\lambda_1 \otimes_\varepsilon \lambda_2 = \lambda_2 \otimes_\varepsilon \lambda_1$,
3. $\delta(\lambda_1 \oplus_\varepsilon \lambda_2) = \delta\lambda_1 \oplus_\varepsilon \delta\lambda_2$,
4. $(\lambda_1 \otimes_\varepsilon \lambda_2)^\delta = (\lambda_1)^\delta \otimes_\varepsilon (\lambda_2)^\delta$,
5. $\delta_1(\lambda) \oplus_\varepsilon \delta_2(\lambda) = (\delta_1 \oplus_\varepsilon \delta_2)\lambda$,
6. $(\lambda)^{\delta_1} \otimes_\varepsilon (\lambda)^{\delta_2} = \lambda^{(\delta_1 \otimes_\varepsilon \delta_2)}$.

Proof The proof is trivial, so it is omitted here.

Theorem 2 *Let $\lambda_j = ([u_j, v_j], [x_j, y_j])(j = 1, 2, 3, ..., n)$ be a collection of IVPFVs, then their aggregated value using the IVPFEHWA operator is also an IVPFV, and*

$$\mathit{IVPFEHWA}_{\omega,w}(\lambda_1,\lambda_2,\lambda_3,...,\lambda_n) = \left(\begin{array}{l}\left[\sqrt{\frac{\prod_{j=1}^{n}\left(1+u^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}-\prod_{j=1}^{n}\left(1-u^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+u^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{n}\left(1-u^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}}}, \sqrt{\frac{\prod_{j=1}^{n}\left(1+v^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}-\prod_{j=1}^{n}\left(1-v^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+v^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{n}\left(1-v^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}}}\right],\\ \left[\frac{\sqrt{2\prod_{j=1}^{n}\left(x^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{n}\left(2-x^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{n}\left(x^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}}}, \frac{\sqrt{2\prod_{j=1}^{n}\left(y^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{n}\left(2-y^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{n}\left(y^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}}}\right]\end{array}\right), \quad (19)$$

where $\dot\lambda_{\sigma(j)}$ is the jth largest of the weighted interval-valued Pythagorean fuzzy values, $\dot\lambda_{\sigma(j)}(\dot\lambda_{\sigma(j)} = n\omega_j\lambda_j)$, $w = (w_1, w_2, w_2, \ldots, w_n)^T$ is the weighted vector of IVPFEHWA, such that $w_j \in [0, 1]$, $\sum_{j=1}^{n} w_j = 1$. $\omega = (\omega_1, \omega_2, \omega_2, \ldots, \omega_n)^T$ is the weighted vector of $\lambda_j (j = 1, 2, 3, \ldots, n)$ such that $\omega_j \in [0, 1]$, $\sum_{j=1}^{n}\omega_j = 1$, and n is the balancing coefficient, which plays a role of balance. If the vector $w = (w_1, w_2, w_2, \ldots, w_n)^T$ approaches $\left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$, then the vector $(nw\omega\lambda_1, n\omega_2\lambda_2, \ldots, n\omega_n\lambda_n)^T$ approaches$(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n)^T$.

Proof We can prove this theorem by mathematical induction on n.

For $n = 2$

$$w_1\dot\lambda_1 = \left(\left[\sqrt{\frac{\left(1+u^2_{\dot\lambda_1}\right)^{w_1}-\left(1-u^2_{\dot\lambda_1}\right)^{w_1}}{\left(1+u^2_{\dot\lambda_1}\right)^{w_1}+\left(1-u^2_{\dot\lambda_1}\right)^{w_1}}}, \sqrt{\frac{\left(1+v^2_{\dot\lambda_1}\right)^{w_1}-\left(1-v^2_{\dot\lambda_1}\right)^{w_1}}{\left(1+v^2_{\dot\lambda_1}\right)^{w_1}+\left(1-v^2_{\dot\lambda_1}\right)^{w_1}}}\right], \left[\frac{\sqrt{2\left(x^2_{\dot\lambda_1}\right)^{w_1}}}{\sqrt{\left(2-x^2_{\dot\lambda_1}\right)^{w_1}+\left(x^2_{\dot\lambda_1}\right)^{w_1}}}, \frac{\sqrt{2\left(y^2_{\dot\lambda_1}\right)^{w_1}}}{\sqrt{\left(2-y^2_{\dot\lambda_1}\right)^{w_1}+\left(y^2_{\dot\lambda_1}\right)^{w_1}}}\right]\right)$$

and

$$w_2\dot\lambda_2 = \left(\left[\sqrt{\frac{\left(1+u^2_{\dot\lambda_2}\right)^{w_2}-\left(1-u^2_{\dot\lambda_2}\right)^{w_2}}{\left(1+u^2_{\dot\lambda_2}\right)^{w_2}+\left(1-u^2_{\dot\lambda_2}\right)^{w_2}}}, \sqrt{\frac{\left(1+v^2_{\dot\lambda_2}\right)^{w_2}-\left(1-v^2_{\dot\lambda_2}\right)^{w_2}}{\left(1+v^2_{\dot\lambda_2}\right)^{w_2}+\left(1-v^2_{\dot\lambda_2}\right)^{w_2}}}\right], \left[\frac{\sqrt{2\left(x^2_{\dot\lambda_2}\right)^{w_2}}}{\sqrt{\left(2-x^2_{\dot\lambda_2}\right)^{w_2}+\left(x^2_{\dot\lambda_2}\right)^{w_2}}}, \frac{\sqrt{2\left(y^2_{\dot\lambda_2}\right)^{w_2}}}{\sqrt{\left(2-y^2_{\dot\lambda_2}\right)^{w_2}+\left(y^2_{\dot\lambda_2}\right)^{w_2}}}\right]\right)$$

Then

$$\mathrm{IVPFEHWA}_{\omega,w}(\lambda_1,\lambda_2)$$
$$=\left(\left[\frac{\sqrt{\prod_{j=1}^{2}\left(1+u^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}-\prod_{j=1}^{2}\left(1-u^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{2}\left(1+u^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{2}\left(1-u^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}},\frac{\sqrt{\prod_{j=1}^{2}\left(1+v^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}-\prod_{j=1}^{2}\left(1-v^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{2}\left(1+v^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{2}\left(1-v^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}}\right],\left[\frac{\sqrt{2\prod_{j=1}^{2}\left(x^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{2}\left(2-x^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{2}\left(x^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}},\frac{\sqrt{2\prod_{j=1}^{2}\left(y^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{2}\left(2-y^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{2}\left(y^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}}\right]\right).$$

Thus, the result is true for $n=2$, now we assume that Eq. (19) holds for $n=k$. Thus

$$\mathrm{IVPFEHWA}_{\omega,w}(\lambda_1,\lambda_2,\lambda_3,...,\lambda_k)$$
$$=\left(\left[\frac{\sqrt{\prod_{j=1}^{k}\left(1+u^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}-\prod_{j=1}^{k}\left(1-u^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{k}\left(1+u^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{k}\left(1-u^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}},\frac{\sqrt{\prod_{j=1}^{k}\left(1+v^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}-\prod_{j=1}^{k}\left(1-v^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{k}\left(1+v^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{k}\left(1-v^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}}\right],\left[\frac{\sqrt{2\prod_{j=1}^{k}\left(x^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{k}\left(2-x^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{k}\left(x^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}},\frac{\sqrt{2\prod_{j=1}^{k}\left(y^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{k}\left(2-y^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{k}\left(y^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}}\right]\right).$$

If Eq. (19) holds for $n=k$, then we show that Eq. (19) holds for $n=k+1$. Thus

$$\mathrm{IVPFEHWA}_{\omega,w}(\lambda_1,\lambda_2,\lambda_3,...,\lambda_{k+1})$$
$$=\left(\left[\frac{\sqrt{\prod_{j=1}^{k}\left(1+u^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}-\prod_{j=1}^{k}\left(1-u^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{k}\left(1+u^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{n}\left(1-u^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}},\frac{\sqrt{\prod_{j=1}^{k}\left(1+v^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}-\prod_{j=1}^{k}\left(1-v^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{k}\left(1+v^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{k}\left(1-v^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}}\right],\left[\frac{\sqrt{2\prod_{j=1}^{k}\left(x^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{k}\left(2-x^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{k}\left(x^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}},\frac{\sqrt{2\prod_{j=1}^{k}\left(y^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{k}\left(2-y^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{k}\left(y^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}}\right]\right)$$
$$\oplus_{\varepsilon}\left(\left[\frac{\sqrt{\left(1+u^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}}-\left(1-u^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}}}}{\sqrt{\left(1+u^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}}+\left(1-u^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}}}},\frac{\sqrt{\left(1+v^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}}-\left(1-v^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}}}}{\sqrt{\left(1+v^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}}+\left(1-v^2_{\dot{\lambda}_{k+1}}\right)^{wk+1}}}\right],\left[\frac{\sqrt{2\left(x^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}}}}{\sqrt{\left(2-x^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}}+\left(x^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}}}},\frac{\sqrt{2\left(y^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}}}}{\sqrt{\left(2-y^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}}+\left(y^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}}}}\right]\right). \tag{20}$$

Let

$$t_1=\sqrt{\prod_{j=1}^{k}\left(1+u^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}-\prod_{j=1}^{k}\left(1-u^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}$$

$$t_2=\sqrt{\prod_{j=1}^{k}\left(1+u^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{n}\left(1-u^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}$$

$$p_1=\sqrt{\prod_{j=1}^{k}\left(1+v^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}-\prod_{j=1}^{k}\left(1-v^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}$$

$$p_2=\sqrt{\prod_{j=1}^{k}\left(1+v^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{k}\left(1-v^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}$$

$$m_1=\sqrt{\left(1+u^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}}-\left(1-u^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}}}$$

$$m_2=\sqrt{\left(1+u^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}}+\left(1-u^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}}}$$

$$a_1=\sqrt{\left(1+v^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}}-\left(1-v^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}}}$$

$$a_2=\sqrt{\left(1+v^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}}+\left(1-v^2_{\dot{\lambda}_{k+1}}\right)^{wk+1}}$$

$$r_2=\sqrt{\prod_{j=1}^{k}\left(2-x^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{k}\left(x^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}$$

$$r_1=\sqrt{2\prod_{j=1}^{k}\left(x^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}},\ s_1=\sqrt{2\prod_{j=1}^{k}\left(y^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}$$

$$s_2=\sqrt{\prod_{j=1}^{k}\left(2-y^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{k}\left(y^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}$$

$$b_2=\sqrt{\left(2-x^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}}+\left(x^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}}}$$

$$b_1=\sqrt{2\left(x^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}}},\ c_1=\sqrt{2\left(y^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}}}$$

$$c_2=\sqrt{\left(2-y^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}}+\left(y^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}}}$$

Now putting these values in Eq. (20), we have

$$\mathrm{IVPFEHWA}_{\omega,w}(\lambda_1,\lambda_2,\lambda_3,...,\lambda_{k+1})$$
$$=\left(\left[\frac{t_1}{t_2},\frac{p_1}{p_2}\right],\left[\frac{r_1}{r_2},\frac{s_1}{s_2}\right]\right)\oplus_{\varepsilon}\left(\left[\frac{m_1}{m_2},\frac{a_1}{a_2}\right],\left[\frac{b_1}{b_2},\frac{c_1}{c_2}\right]\right)$$
$$=\left(\left[\frac{\sqrt{\left(\frac{t_1}{t_2}\right)^2+\left(\frac{m_1}{m_2}\right)^2}}{\sqrt{1+\left(\frac{t_1}{t_2}\right)^2\left(\frac{m_1}{m_2}\right)^2}},\frac{\sqrt{\left(\frac{p_1}{p_2}\right)^2+\left(\frac{a_1}{a_2}\right)^2}}{\sqrt{1+\left(\frac{p_1}{p_2}\right)^2\left(\frac{a_1}{a_2}\right)^2}}\right],\left[\frac{\left(\frac{r_1}{r_2}\right)\left(\frac{b_1}{b_2}\right)}{\sqrt{1+\left(1-\left(\frac{r_1}{r_2}\right)^2\right)\left(1-\left(\frac{b_1}{b_2}\right)^2\right)}},\frac{\left(\frac{s_1}{s_2}\right)\left(\frac{c_1}{c_2}\right)}{\sqrt{1+\left(1-\left(\frac{s_1}{s_2}\right)\right)^2\left(1-\left(\frac{c_1}{c_2}\right)\right)^2}}\right]\right)$$
$$=\left(\left[\frac{\sqrt{(t_1m_2)^2+(t_2m_1)^2}}{\sqrt{(t_2m_2)^2+(t_1m_1)^2}},\frac{\sqrt{(p_1a_2)^2+(a_1p_2)^2}}{\sqrt{(p_2a_2)^2+(p_1a_1)^2}}\right],\right.$$

$$\left[\frac{r_1 b_1}{\sqrt{2r_2^2 b_2^2 + r_1^2 b_1^2 - r_2^2 b_1^2 - r_1^2 b_2^2}}, \frac{s_1 c_1}{\sqrt{2s_2^2 c_2^2 + s_1^2 c_1^2 - s_2^2 c_1^2 - s_1^2 c_2^2}}\right]\Bigg). \tag{21}$$

Again putting the values of $(t_1 m_2)^2 + (t_2 m_1)^2$, $(t_2 m_2)^2 + (t_1 m_1)^2$, $(p_1 a_2)^2 + (a_1 p_2)^2$, $(p_2 a_2)^2 + (p_1 a_1)^2$, $r_1 b_1, 2r_2^2 b_2^2 + r_1^2 b_1^2 - r_2^2 b_1^2 - r_1^2 b_2^2$, $s_1 c_1$, $2s_2^2 c_2^2 + s_1^2 c_1^2 - s_2^2 c_1^2 - s_1^2 c_2^2$, in Eq. (21), then

$$\mathrm{IVPFEHWA}_{\omega,w}(\lambda_1, \lambda_2, \lambda_3, ..., \lambda_{k+1})$$

$$= \left(\begin{array}{c} \left[\dfrac{\sqrt{\prod_{j=1}^{k+1}\left(1+u^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j} - \prod_{j=1}^{k+1}\left(1-u^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{k+1}\left(1+u^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j} + \prod_{j=1}^{k+1}\left(1-u^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}}, \dfrac{\sqrt{\prod_{j=1}^{k+1}\left(1+v^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j} - \prod_{j=1}^{k+1}\left(1-v^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{k+1}\left(1+v^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j} + \prod_{j=1}^{k+1}\left(1-v^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}} \right], \\ \left[\dfrac{\sqrt{2\prod_{j=1}^{k+1}\left(x^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{k+1}\left(2-x^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j} + \prod_{j=1}^{k+1}\left(x^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}}, \dfrac{\sqrt{2\prod_{j=1}^{k+1}\left(y^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{k+1}\left(2-y^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j} + \prod_{j=1}^{k+1}\left(y^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}} \right] \end{array} \right).$$

Hence, Eq. (19) holds for $n = k + 1$. Thus, Eq. (19) holds for all n.

Remark 1 In the following, let us look $\delta\lambda$ and λ^δ some special cases of δ and λ.

1. If $\lambda = ([u, v], [x, y]) = ([1, 1], [0, 0])$ i. e,. $u = v = 1$ and $u = v = 1$, then

$$\lambda^\delta = \left(\left[\frac{\sqrt{2\left(u^2\right)^\delta}}{\sqrt{\left(2-u^2\right)^\delta + \left(u^2\right)^\delta}}, \frac{\sqrt{2\left(v^2\right)^\delta}}{\sqrt{\left(2-v^2\right)^\delta + \left(v^2\right)^\delta}}\right], \left[\frac{\sqrt{\left(1+x^2\right)^\delta - \left(1-x^2\right)^\delta}}{\sqrt{\left(1+x^2\right)^\delta + \left(1-x^2\right)^\delta}}, \frac{\sqrt{\left(1+y^2\right)^\delta - \left(1-y^2\right)^\delta}}{\sqrt{\left(1+y^2\right)^\delta + \left(1-y^2\right)^\delta}}\right]\right)$$
$$= \left(\left[\frac{\sqrt{2(1)^\delta}}{\sqrt{(2-1)^\delta + (1)^\delta}}, \frac{\sqrt{2(1)^\delta}}{\sqrt{(2-1)^\delta + (1)^\delta}}\right], \left[\frac{\sqrt{(1+0)^\delta - (1-0)^\delta}}{\sqrt{(1+0)^\delta + (1-0)^\delta}}, \frac{\sqrt{(1+0)^\delta - (1-0)^\delta}}{\sqrt{(1+0)^\delta + (1-0)^\delta}}\right]\right)$$
$$= ([1, 1], [0, 0]).$$

Thus $\lambda^\delta = ([1, 1], [0, 0])$ and $\delta\lambda = ([0, 0], [1, 1])$.

2. If $\lambda = ([u, v], [x, y]) = ([0, 0], [1, 1])$ i. e,. $u = v = 0$ and $x = y = 1$, then

$$\lambda^\delta = \left(\left[\frac{\sqrt{2\left(u^2\right)^\delta}}{\sqrt{\left(2-u^2\right)^\delta + \left(u^2\right)^\delta}}, \frac{\sqrt{2\left(v^2\right)^\delta}}{\sqrt{\left(2-v^2\right)^\delta + \left(v^2\right)^\delta}}\right], \left[\frac{\sqrt{\left(1+x^2\right)^\delta - \left(1-x^2\right)^\delta}}{\sqrt{\left(1+x^2\right)^\delta + \left(1-x^2\right)^\delta}}, \frac{\sqrt{\left(1+y^2\right)^\delta - \left(1-y^2\right)^\delta}}{\sqrt{\left(1+y^2\right)^\delta + \left(1-y^2\right)^\delta}}\right]\right)$$
$$= \left(\left[\frac{\sqrt{2(0)^\delta}}{\sqrt{(2-0)^\delta + (0)^\delta}}, \frac{\sqrt{2(0)^\delta}}{\sqrt{(2-0)^\delta + (0)^\delta}}\right], \left[\frac{\sqrt{(1+1)^\delta - (1-1)^\delta}}{\sqrt{(1+1)^\delta + (1-1)^\delta}}, \frac{\sqrt{(1+1)^\delta - (1-1)^\delta}}{\sqrt{(1+1)^\delta + (1-1)^\delta}}\right]\right)$$
$$= ([0, 0], [1, 1]).$$

Thus $\lambda^\delta = ([0, 0], [1, 1])$ and $\delta\lambda = ([1, 1], [0, 0])$.

3. If $\lambda = ([u, v], [x, y]) = ([0, 0], [0, 0])$ i. e,. $u = v = 0$ and $x = y = 0$, then

$$\lambda^\delta = \left(\left[\frac{\sqrt{2\left(u^2\right)^\delta}}{\sqrt{\left(2-u^2\right)^\delta + \left(u^2\right)^\delta}}, \frac{\sqrt{2\left(v^2\right)^\delta}}{\sqrt{\left(2-v^2\right)^\delta + \left(v^2\right)^\delta}}\right], \left[\frac{\sqrt{\left(1+x^2\right)^\delta - \left(1-x^2\right)^\delta}}{\sqrt{\left(1+x^2\right)^\delta + \left(1-x^2\right)^\delta}}, \frac{\sqrt{\left(1+y^2\right)^\delta - \left(1-y^2\right)^\delta}}{\sqrt{\left(1+y^2\right)^\delta + \left(1-y^2\right)^\delta}}\right]\right)$$
$$= \left(\left[\frac{\sqrt{2(0)^\delta}}{\sqrt{(2-0)^\delta + (0)^\delta}}, \frac{\sqrt{2(0)^\delta}}{\sqrt{(2-0)^\delta + (0)^\delta}}\right], \left[\frac{\sqrt{(1+0)^\delta - (1-0)^\delta}}{\sqrt{(1+0)^\delta + (1-0)^\delta}}, \frac{\sqrt{(1+0)^\delta - (1-0)^\delta}}{\sqrt{(1+0)^\delta + (1-0)^\delta}}\right]\right)$$
$$= ([0, 0], [0, 0]).$$

Thus $\lambda^\delta = ([0, 0], [0, 0])$ and $\delta\lambda = ([0, 0], [0, 0])$.

4. If $\delta \to 0$ and $0 \leq u, v, x, y \leq 1$, then

$$\lambda^\delta = \left(\left[\frac{\sqrt{2\left(u^2\right)^\delta}}{\sqrt{\left(2-u^2\right)^\delta + \left(u^2\right)^\delta}}, \frac{\sqrt{2\left(v^2\right)^\delta}}{\sqrt{\left(2-v^2\right)^\delta + \left(v^2\right)^\delta}}\right], \left[\frac{\sqrt{\left(1+x^2\right)^\delta - \left(1-x^2\right)^\delta}}{\sqrt{\left(1+x^2\right)^\delta + \left(1-x^2\right)^\delta}}, \frac{\sqrt{\left(1+y^2\right)^\delta - \left(1-y^2\right)^\delta}}{\sqrt{\left(1+y^2\right)^\delta + \left(1-y^2\right)^\delta}}\right]\right)$$
$$= \left(\left[\frac{\sqrt{2(u^2)^0}}{\sqrt{(2-u^2)^0 + (u^2)^0}}, \frac{\sqrt{2(v^2)^0}}{\sqrt{(2-v^2)^0 + (v^2)^0}}\right],\right.$$

$$\left[\frac{\sqrt{(1+x^2)^0-(1-x_1^2)^0}}{\sqrt{(1+x^2)^0+(1-x^2)^0}},\ \frac{\sqrt{(1+y^2)^0-(1-y^2)^0}}{\sqrt{(1+y^2)^0+(1-y^2)^0}}\right]\Bigg) = ([1,1],[0,0]).$$

Thus $\lambda^\delta = ([1,1],[0,0])$ and $\delta\lambda = ([0,0],[1,1])$.

5. If $\delta \to +\infty$ and $0 \le u, v, x, y \le 1$, then

$$\lambda^\delta = \left(\left[\frac{\sqrt{2\left(u^2\right)^\delta}}{\sqrt{\left(2-u^2\right)^\delta+\left(u^2\right)^\delta}}, \frac{\sqrt{2\left(v^2\right)^\delta}}{\sqrt{\left(2-v^2\right)^\delta+\left(v^2\right)^\delta}}\right], \left[\frac{\sqrt{\left(1+x^2\right)^\delta-\left(1-x^2\right)^\delta}}{\sqrt{\left(1+x^2\right)^\delta+\left(1-x^2\right)^\delta}}, \frac{\sqrt{\left(1+y^2\right)^\delta-\left(1-y^2\right)^\delta}}{\sqrt{\left(1+y^2\right)^\delta+\left(1-y^2\right)^\delta}}\right]\right)$$

$$= \left(\left[\frac{\sqrt{2(u^2)^\infty}}{\sqrt{(2-u^2)^\infty+(u^2)^\infty}}, \frac{\sqrt{2(v^2)^\infty}}{\sqrt{(2-v^2)^\infty+(u^2)^\infty}}\right], \left[\frac{\sqrt{(1+x^2)^\infty-(1-x^2)^\infty}}{\sqrt{(1+x^2)^\infty+(1-x^2)^\infty}}, \frac{\sqrt{(1+y^2)^\infty-(1-y^2)^\infty}}{\sqrt{(1+y^2)^\infty+(1-y^2)^\infty}}\right]\right) = ([[0,0],1,1]).$$

Thus, $\lambda^\delta = ([0,0],[1,1])$ and $\delta\lambda = ([1,1],[0,0])$.

6. If $\delta = 1$ and $0 \le u, v, x, y \le 1$, then

$$\lambda^\delta = \left(\left[\frac{\sqrt{2\left(u^2\right)^\delta}}{\sqrt{\left(2-u^2\right)^\delta+\left(u^2\right)^\delta}}, \frac{\sqrt{2\left(v^2\right)^\delta}}{\sqrt{\left(2-v^2\right)^\delta+\left(v^2\right)^\delta}}\right], \left[\frac{\sqrt{\left(1+x^2\right)^\delta-\left(1-x^2\right)^\delta}}{\sqrt{\left(1+x^2\right)^\delta+\left(1-x^2\right)^\delta}}, \frac{\sqrt{\left(1+y^2\right)^\delta-\left(1-y^2\right)^\delta}}{\sqrt{\left(1+y^2\right)^\delta+\left(1-y^2\right)^\delta}}\right]\right)$$

$$= \left(\left[\frac{\sqrt{2\left(u^2\right)^1}}{\sqrt{\left(2-u^2\right)^1+\left(u^2\right)^1}}, \frac{\sqrt{2\left(v^2\right)^1}}{\sqrt{\left(2-v^2\right)^1+\left(u^2\right)^1}}\right], \left[\frac{\sqrt{\left(1+x^2\right)^1-\left(1-x^2\right)^1}}{\sqrt{\left(1+x^2\right)^1+\left(1-x^2\right)^1}}, \frac{\sqrt{\left(1+y^2\right)^1-\left(1-y^2\right)^1}}{\sqrt{\left(1+y^2\right)^1+\left(1-y^2\right)^1}}\right]\right) = \lambda.$$

Thus, $\lambda^\delta = \lambda$ and $\delta\lambda = \lambda$.

Lemma 1 [6] *Let $\lambda_j \succ 0$, $w_j \succ 0 (j = 1, 2, 3, ..., n)$ and $\sum_{j=1}^n w_j = 1$, then*

$$\prod_{j=1}^n (\lambda_j)^{w_j} \le \sum_{j=1}^n w_j \lambda_j, \tag{22}$$

where the equality holds if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_n$.

Theorem 3 *Let $\lambda_j = ([u_j, v_j], [x_j, y_j]) (j = 1, 2, 3, ..., n)$ be a collection of IVPFVs, where the $w = (w_1, w_2, w_3, ..., w_n)^T$ is the weighted vector of IVPFEHWA and IVPFHWA, such that $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$. $\omega = (\omega_1, \omega_2, \omega_2, ..., \omega_n)^T$ is the weighted vector of $\lambda_j (j = 1, 2, 3, ..., n)$ such that $\omega_j \in [0, 1]$, $\sum_{j=1}^n \omega_j = 1$, then*

$$IVPFEHWA_{\omega,w}(\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n) \le IVPFHWA_{\omega,w}(\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n). \tag{23}$$

Proof Straight forward.

Theorem 4 *Idempotency: If $\dot{\lambda}_{\sigma(j)} = \dot{\lambda}$ for all $j (j = 1, 2, 3, ..., n)$, where $\lambda = ([u, v], [x, y])$, then*

$$IVPFEHWA_{\omega,w}(\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n) = \dot{\lambda}. \tag{24}$$

Proof Since $\dot{\lambda}_{\sigma(j)} = \dot{\lambda}$ for all j, then we have

$$\mathrm{IVPFEHWA}_{\omega,w}(\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n)$$

$$= \left(\left[\frac{\sqrt{\prod_{j=1}^n\left(1+u^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}-\prod_{j=1}^n\left(1-u^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^n\left(1+u^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^n\left(1-u^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}}, \frac{\sqrt{\prod_{j=1}^n\left(1+v^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}-\prod_{j=1}^n\left(1-v^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^n\left(1+v^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^n\left(1-v^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}}\right], \left[\frac{\sqrt{2\prod_{j=1}^n\left(x^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^n\left(2-x^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^n\left(x^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}}, \frac{\sqrt{2\prod_{j=1}^n\left(y^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^n\left(2-y^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^n\left(y^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}}\right]\right)$$

$$= \left(\left[\frac{\sqrt{\left(1+u^2_{\dot{\lambda}}\right)^{\sum_{j=1}^n w_j}-\left(1-u^2_{\dot{\lambda}}\right)^{\sum_{j=1}^n w_j}}}{\sqrt{\left(1+u^2_{\dot{\lambda}}\right)^{\sum_{j=1}^n w_j}+\left(1-u^2_{\dot{\lambda}}\right)^{\sum_{j=1}^n w_j}}}, \frac{\sqrt{\left(1+v^2_{\dot{\lambda}}\right)^{\sum_{j=1}^n w_j}-\left(1-v^2_{\dot{\lambda}}\right)^{\sum_{j=1}^n w_j}}}{\sqrt{\left(1+v^2_{\dot{\lambda}}\right)^{\sum_{j=1}^n w_j}+\left(1-v^2_{\dot{\lambda}}\right)^{\sum_{j=1}^n w_j}}}\right], \left[\frac{\sqrt{2\left(x^2_{\dot{\lambda}}\right)^{\sum_{j=1}^n w_j}}}{\sqrt{\left(2-x^2_{\dot{\lambda}}\right)^{\sum_{j=1}^n w_j}+\left(x^2_{\dot{\lambda}}\right)^{\sum_{j=1}^n w_j}}}, \frac{\sqrt{2\left(y^2_{\dot{\lambda}}\right)^{\sum_{j=1}^n w_j}}}{\sqrt{\left(2-y^2_{\dot{\lambda}}\right)^{\sum_{j=1}^n w_j}+\left(y^2_{\dot{\lambda}}\right)^{\sum_{j=1}^n w_j}}}\right]\right)$$

$$= \left(\left[\frac{\sqrt{\left(1+u^2_{\dot{\lambda}}\right)-\left(1-u^2_{\dot{\lambda}}\right)}}{\sqrt{\left(1+u^2_{\dot{\lambda}}\right)+\left(1-u^2_{\dot{\lambda}}\right)}}, \frac{\sqrt{\left(1+v^2_{\dot{\lambda}}\right)-\left(1-v^2_{\dot{\lambda}}\right)}}{\sqrt{\left(1+v^2_{\dot{\lambda}}\right)+\left(1-v^2_{\dot{\lambda}}\right)}}\right], \left[\frac{\sqrt{2\left(x^2_{\dot{\lambda}}\right)}}{\sqrt{\left(2-x^2_{\dot{\lambda}}\right)+\left(x^2_{\dot{\lambda}}\right)}}, \frac{\sqrt{2\left(y^2_{\dot{\lambda}}\right)}}{\sqrt{\left(2-y^2_{\dot{\lambda}}\right)+\left(y^2_{\dot{\lambda}}\right)}}\right]\right) = \dot{\lambda}.$$

The proof is completed.

Theorem 5 *Boundedness: Let* $\lambda_j = ([u_{\lambda_j}, v_{\lambda_j}], [x_{\lambda_j}, y_{\lambda_j}])$ $(j = 1, 2, 3, ..., n)$ *be a collection of IVPFNs, then*

$$\dot{\lambda}_{\min} \leq IVPFEHWA_{\omega,w}(\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n) \leq \dot{\lambda}_{\max}, \quad (25)$$

$$\dot{\lambda}_{\max} = \max_j(\dot{\lambda}_{\sigma(j)}), \quad (26)$$

$$\dot{\lambda}_{\min} = \min_j(\dot{\lambda}_{\sigma(j)}). \quad (27)$$

Proof Proof is easy so it is omitted here.

Theorem 6 *Monotonicity: If* $\lambda_j \leq \lambda_j^*$ *for all* $j(j = 1, 2, 3, ..., n)$*, then*

$$IVPFEHWA_{\omega,w}(\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n) \leq IVPFEHWA_{\omega,w}(\lambda_1^*, \lambda_2^*, \lambda_3^*, ..., \lambda_n^*). \quad (28)$$

Proof As we know that.

$$\text{IVPFEHWA}_{\omega,w}(\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n) = w_1\dot{\lambda}_{\sigma(1)} \oplus_\varepsilon w_2\dot{\lambda}_{\sigma(2)} \oplus_\varepsilon w_3\dot{\lambda}_{\sigma(3)} \oplus_\varepsilon \cdots \oplus_\varepsilon w_n\dot{\lambda}_{\sigma(n)}, \quad (29)$$

and

$$\text{IVPFEHWA}_{\omega,w}(\lambda_1^*, \lambda_2^*, \lambda_3^*, ..., \lambda_n^*) = w_1\dot{\lambda}^*_{\sigma(1)} \oplus_\varepsilon w_2\dot{\lambda}^*_{\sigma(2)} \oplus_\varepsilon w_3\dot{\lambda}^*_{\sigma(3)} \oplus_\varepsilon \cdots \oplus_\varepsilon w_n\dot{\lambda}^*_{\sigma(n)}. \quad (30)$$

Since $\lambda_j \leq \lambda_j^*$ for all j, thus Eq. (28) always holds.

Theorem 7 *Interval-valued Pythagorean fuzzy Einstein weighted averaging operator is a special case of the interval-valued Pythagorean fuzzy Einstein hybrid weighted averaging operator.*

Proof Let $\omega = \left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n},\right)^T$, then we have

$$\begin{aligned}
&\text{IVPFEHWA}_{\omega,w}(\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n)\\
&= w_1\dot{\lambda}_{\sigma(1)} \oplus_\varepsilon w_2\dot{\lambda}_{\sigma(2)} \oplus_\varepsilon \cdots \oplus_\varepsilon w_n\dot{\lambda}_{\sigma(n)}\\
&= \frac{1}{n}(\dot{\lambda}_{\sigma(1)} \oplus_\varepsilon \dot{\lambda}_{\sigma(2)} \oplus_\varepsilon \cdots \oplus_\varepsilon \dot{\lambda}_{\sigma(n)})\\
&= \frac{1}{n}(n\omega_1\lambda_1 \oplus_\varepsilon n\omega_2\lambda_2 \oplus_\varepsilon \cdots \oplus_\varepsilon n\omega_n\lambda_n)\\
&= \omega_1\lambda_1 \oplus_\varepsilon \omega_2\lambda_2 \oplus_\varepsilon \cdots \oplus_\varepsilon \omega_n\lambda_n\\
&= \text{IVPFEWA}_w(\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n).
\end{aligned}$$

The proof is completed.

Theorem 8 *Interval-valued Pythagorean fuzzy Einstein ordered weighted averaging operator is a special case of the interval-valued Pythagorean fuzzy Einstein hybrid weighted averaging operator.*

Proof Let $w = \left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n},\right)^T$, and $\dot{\lambda}_{\sigma(j)} = \lambda_{\sigma(j)}$, then we have

$$\begin{aligned}
&\text{IVPFEHWA}_{\omega,w}(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n)\\
&= w_1\dot{\lambda}_{\sigma(1)} \oplus_\varepsilon w_2\dot{\lambda}_{\sigma(2)} \oplus_\varepsilon \cdots \oplus_\varepsilon w_n\dot{\lambda}_{\sigma(n)}\\
&= w_1\lambda_{\sigma(1)} \oplus_\varepsilon w_2\lambda_{\sigma(2)} \oplus_\varepsilon \cdots \oplus_\varepsilon w_n\lambda_{\sigma(n)}\\
&= \text{IVPFEOWA}_w(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n).
\end{aligned}$$

The proof completed.

An approach to multiple attribute group decision-making problems based on interval-valued Pythagorean fuzzy information

Algorithm Let $X = \{X_1, X_2, X_3, ..., X_m\}$ be a finite set of m alternatives and $C = \{C_1, C_2, C_3, ..., C_n\}$ be a finite set of n attributes. Suppose the grade of the alternatives$X_i(i = 1, 2, 3, ..., m)$on attribute$C_j(j = 1, 2, 3, ..., n)$ given by decision makers is interval-valued Pythagorean fuzzy numbers. Let $D = \{D_1, D_2, D_3, ..., D_k\}$ be the set of k decision makers, and let $w = (w_1, w_2, w_3, ..., w_n)^T$ be the weighted vector of the attributes $C_j(j = 1, 2, 3, ..., n)$, such that $w_j \in [0, 1],\sum_{j=1}^n w_j = 1$, and let $\omega = (\omega_1, \omega_2, \omega_3, ..., \omega_k)^T$ be the weighted vector of the decision makers $D^s(s = 1, 2, 3, ..., k)$, such that $\omega_s \in [0, 1]$ and $\sum_{s=1}^k \omega_s = 1$. Let $D = (a_{ji}) = \langle[u_{ji}, v_{ji}], [x_{ji}, y_{ji}]\rangle(i = 1, 2, 3, ..., m, j = 1, 2, 3, ..., n)$ where $[u_{ji}, v_{ji}]$ indicates the interval degree that the alternative $X_i(i = 1, 2, 3, ..., m)$ satisfies the attribute $C_j(j = 1, 2, 3, ..., n)$ and $[x_{ji}, y_{ji}]$ indicates the interval degree that the alternative $X_i(i = 1, 2, 3, ..., m)$ does not satisfy the attribute $C_j(j = 1, 2, 3, ..., n)$, And also $[u_{ji}, v_{ji}] \in [0, 1], [x_{ji}, y_{ji}] \in [0, 1]$ with condition $0 \leq (v_{ji})^2 + (y_{ji})^2 \leq 1, (i = 1, 2, 3, ..., m, j = 1, 2, 3, ..., n)$. This method has the following steps.

Step 1 Utilize the given information in the form of matrices, $D^s = \left[a_{ji}^{(s)}\right]_{n\times m} (s = 1, 2, 3, ..., k)$.

Step 2 If the criteria have two types, such as benefit criteria and cost criteria, then the interval-valued Pythagorean fuzzy decision matrices, $D^s = \left[a_{ji}^{(s)}\right]_{n\times m}$ $(s = 1, 2, 3, ..., k)$ can be converted into the normalized interval-valued Pythagorean fuzzy decision matrices, $R^s = \left[r_{ji}^{(s)}\right]_{n\times m}$ $(s = 1, 2, 3, ..., n)$, where

$$r_{ji}^{(s)} = \begin{cases} a_{ji}^{(s)}, & \text{for benefit criteria } C_j \\ \bar{a}_{ji}^{(s)}, & \text{for cost criteria } C_j, \end{cases} \begin{pmatrix} j = 1, 2, 3, ..., n \\ i = 1, 2, 3, ..., m \end{pmatrix},$$

and $\bar{a}_{ji}^{(s)}$ is the complement of α_{ji}^{s}. If all the criteria have the same type, then there is no need of normalization.

Step 3 Utilize the IVPFEWA operator to aggregate all the individual normalized interval-valued Pythagorean fuzzy decision matrices, $R^s = \left[r_{ji}^{(s)}\right]_{n\times m}$ $(s = 1, 2, 3, ..., k)$ into a single interval-valued Pythagorean fuzzy decision-matrix, $R = [r_{ji}]_{n\times m}$, where $r_{ji} = \langle [u_{ji}, v_{ji}], [x_{ji}, y_{ji}] \rangle$.

Step 4 In this step, we calculate $\dot{r}_{ji} = nw_j r_{ji}$.

Step 5 Calculate the scores function of $\dot{r}_{ji}(i = 1, 2, 3, ..., m, j = 1, 2, 3, ..., n)$. If there is no difference between two or more than two scores, then we must find out the accuracy degrees of the collective overall preference values.

Step 6 Utilize the IVPFEHWA operator to aggregate all preference values.

Step 7 Arrange the scores of the all alternatives in the form of descending order and select that alternative which has the highest score function.

Illustrative example

Suppose in Hazara University, the IT department wants to select a new information system for the purpose of the best productivity. After the first selection, there are only three $X_i(i = 1, 2, 3)$ alternatives have been short listed. There are three experts $D^s(s = 1, 2, 3)$ from a group to act as decision makers, whose weight vector is $\omega = (0.2, 0.3, 0.5)^T$. There are many factors that must be considered while selecting the most suitable system, but here, we have consider only the following four criteria, whose weighted vector is $w = (0.1, 0.2, 0.3, 0.4)^T$

1. C_1 : Costs of hardware.
2. C_2 : Support of the organization.
3. C_3 : Effort to transform from current systems.
4. C_4 : Outsourcing software developer reliability,

where C_1, C_3, are cost type criteria and C_2, C_4 are benefit type criteria, i.e., the attributes have two types of criteria; thus, we must change the cost type criteria into benefit type criteria.

Step 1 Construct the decision-making matrices (Tables 1, 2 and 3).

Step 2 Construct the normalized decision making matrices (Tables 4, 5 and 6).

Step 3 Utilize the IVPFEWA operator to aggregate all the individual normalized interval-valued Pythagorean fuzzy decision matrices, $R^s = \left[r_{ji}^{(s)}\right]_{n\times m}$ into a single interval-valued Pythagorean fuzzy decision matrix, $R = [r_{ji}]_{n\times m}$ (Table 7).

Table 1 Interval-valued Pythagorean fuzzy decision matrix of D[1]

	X_1	X_2	X_3
C_1	([0.5, 0.8], [0.3, 0.4])	([0.6, 0.7], [0.3, 0.6])	([0.3, 0.7], [0.3, 0.5])
C_2	([0.3, 0.5], [0.6, 0.7])	([0.3, 0.7], [0.2, 0.6])	([0.3, 0.6], [0.4, 0.7])
C_3	([0.5, 0.7], [0.3, 0.7])	([0.5, 0.6], [0.3, 0.7])	([0.2, 0.6], [0.3, 0.7])
C_4	([0.3, 0.6], [0.6, 0.7])	([0.6, 0.5], [0.2, 0.7])	([0.3, 0.4], [0.5, 0.6])

Table 2 Interval-valued Pythagorean fuzzy decision matrix of D[2]

	X_1	X_2	X_3
C_1	([0.5, 0.6], [0.3, 0.5])	([0.5, 0.7], [0.3, 0.6])	([0.2, 0.8], [0.3, 0.4])
C_2	([0.3, 0.4], [0.6, 0.8])	([0.3, 0.8], [0.2, 0.6])	([0.3, 0.6], [0.3, 0.7])
C_3	([0.4, 0.5], [0.3, 0.8])	([0.5, 0.7], [0.3, 0.6])	([0.2, 0.6], [0.3, 0.8])
C_4	([0.3, 0.6], [0.5, 0.7])	([0.3, 0.4], [0.2, 0.8])	([0.3, 0.5], [0.5, 0.7])

Table 3 Interval-valued Pythagorean fuzzy decision matrix of D[3]

	X_1	X_2	X_3
C_1	([0.3, 0.8], [0.5, 0.6])	([0.3, 0.5], [0.5, 0.7])	([0.2, 0.4], [0.5, 0.7])
C_2	([0.5, 0.7], [0.3, 0.4])	([0.4, 0.6], [0.5, 0.8])	([0.5, 0.7], [0.2, 0.5])
C_3	([0.3, 0.6], [0.4, 0.6])	([0.3, 0.5], [0.5, 0.6])	([0.2, 0.8], [0.4, 0.6])
C_4	([0.5, 0.7], [0.3, 0.4])	([0.5, 0.7], [0.2, 0.4])	([0.5, 0.6], [0.3, 0.5])

Table 4 Normalized Pythagorean fuzzy decision matrix R^1

	X_1	X_2	X_3
C_1	([0.3, 0.4], [0.5, 0.8])	([0.3, 0.6], [0.6, 0.7])	([0.3, 0.5], [0.3, 0.7])
C_2	([0.3, 0.5], [0.6, 0.7])	([0.3, 0.7], [0.2, 0.6])	([0.3, 0.6], [0.4, 0.7])
C_3	([0.3, 0.7], [0.5, 0.7])	([0.3, 0.7], [0.5, 0.6])	([0.3, 0.7], [0.2, 0.6])
C_4	([0.3, 0.6], [0.6, 0.7])	([0.6, 0.5], [0.2, 0.7])	([0.3, 0.4], [0.5, 0.6])

Table 5 Normalized Pythagorean fuzzy decision matrix R^2

	X_1	X_2	X_3
C_1	([0.3, 0.5], [0.5, 0.6])	([0.3, 0.6], [0.5, 0.7])	([0.3, 0.4], [0.2, 0.8])
C_2	([0.3, 0.4], [0.6, 0.8])	([0.3, 0.8], [0.2, 0.6])	([0.3, 0.6], [0.3, 0.7])
C_3	([0.3, 0.8], [0.4, 0.5])	([0.3, 0.6], [0.5, 0.7])	([0.3, 0.8], [0.2, 0.6])
C_4	([0.3, 0.6], [0.5, 0.7])	([0.3, 0.4], [0.2, 0.8])	([0.3, 0.5], [0.5, 0.7])

Table 6 Normalized Pythagorean fuzzy decision matrix R^3

	X_1	X_2	X_3
C_1	([0.5, 0.6], [0.3, 0.8])	([0.5, 0.7], [0.3, 0.5])	([0.5, 0.7], [0.2, 0.4])
C_2	([0.5, 0.7], [0.3, 0.4])	([0.4, 0.6], [0.5, 0.8])	([0.5, 0.7], [0.2, 0.5])
C_3	([0.4, 0.6], [0.3, 0.6])	([0.5, 0.6], [0.3, 0.5])	([0.4, 0.6], [0.2, 0.8])
C_4	([0.5, 0.7], [0.3, 0.4])	([0.5, 0.7], [0.2, 0.4])	([0.5, 0.6], [0.3, 0.5])

Table 7 Collective interval-valued Pythagorean fuzzy decision matrix R

	X_1	X_2	X_3
C_1	([0.413, 0.537], [0.389, 0.738])	([0.413, 0.653], [0.405, 0.595])	([0.413, 0.593], [0.216, 0.562])
C_2	([0.413, 0.593], [0.429, 0.563])	([0.352, 0.692], [0.320, 0.697])	([0.413, 0.653], [0.260, 0.595])
C_3	([0.352, 0.692], [0.363, 0.587])	([0.413, 0.622], [0.389, 0.576])	([0.352, 0.692], [0.200, 0.697])
C_4	([0.413, 0.653], [0.405, 0.536])	([0.475, 0.593], [0.200, 0.563])	([0.413, 0.537], [0.389, 0.576])

Step 4 Calculate $\dot{\lambda}_{ji} = nw\lambda_{ji}$.

$$\dot{\lambda}_{11} = ([0.262, 0.343], [0.733, 0.897]),$$
$$\dot{\lambda}_{21} = ([0.370, 0.534], [0.523, 0.645])$$
$$\dot{\lambda}_{31} = ([0.385, 0.745], [0.281, 0.513]),$$
$$\dot{\lambda}_{41} = ([0.518, 0.788], [0.201, 0.329])$$
$$\dot{\lambda}_{12} = ([0.262, 0.424], [0.742, 0.837]),$$
$$\dot{\lambda}_{22} = ([0.315, 0.628], [0.420, 0.757])$$
$$\dot{\lambda}_{32} = ([0.452, 0.665], [0.307, 0.501]),$$
$$\dot{\lambda}_{42} = ([0.593, 0.726], [0.061, 0.359])$$
$$\dot{\lambda}_{13} = ([0.262, 0.382], [0.605, 0.823]),$$
$$\dot{\lambda}_{23} = ([0.370, 0.590], [0.357, 0.672])$$
$$\dot{\lambda}_{33} = ([0.385, 0.745], [0.136, 0.638]),$$
$$\dot{\lambda}_{43} = ([0.518, 0.664], [0.188, 0.374]).$$

Step 5 Calculate the score functions (Table 8).

$$s(\dot{\lambda}_{11}) = -0.57, s(\dot{\lambda}_{21}) = -0.13, s(\dot{\lambda}_{31}) = 0.18, s(\dot{\lambda}_{41}) = 0.37$$
$$s(\dot{\lambda}_{12}) = -0.50, s(\dot{\lambda}_{22}) = -0.12, s(\dot{\lambda}_{32}) = 0.15, s(\dot{\lambda}_{42}) = 0.37$$
$$s(\dot{\lambda}_{13}) = -0.41, s(\dot{\lambda}_{23}) = -0.04, s(\dot{\lambda}_{33}) = 0.13, s(\dot{\lambda}_{43}) = 0.26.$$

Step 6 Utilize the IVPFEHWA aggregation operator to aggregate all preference values.

$$r_1 = ([0.354, 0.567], [0.550, 0.674])$$

Table 8 Pythagorean fuzzy hybrid decision matrix R

	X_1	X_2	X_3
C_1	([0.518, 0.788], [0.201, 0.329])	([0.593, 0.726], [0.061, 0.359])	([0.518, 0.664], [0.188, 0.374]),
C_2	([0.385, 0.745], [0.281, 0.513])	([0.525, 0.665], [0.307, 0.501])	([0.585, 0.745], [0.136, 0.638]),
C_3	([0.370, 0.534], [0.523, 0.645])	([0.315, 0.628], [0.420, 0.757])	([0.370, 0.590], [0.357, 0.672]),
C_4	([0.262, 0.343], [0.733, 0.897])	([0.262, 0.424], [0.742, 0.837])	([0.262, 0.382], [0.605, 0.823]).

$$r_2 = ([0.367, 0.581], [0.422, 0.686])$$
$$r_3 = ([0.354, 0.571], [0.347, 0.695]).$$

Step 7 Calculate the score functions.

$$s(r_1) = -0.154, s(r_1) = -0.088, s(r_1) = -0.076.$$

Step 8 Arrange the scores of the all alternatives in the form of descending order and select that alternative which has the highest score function.

$$s(r_3) \succ s(r_2) \succ s(r_1)$$

Thus, the best alternative is X_3.

Conclusion

In this paper, we have developed the notion of interval-valued Pythagorean fuzzy Einstein hybrid weighted averaging aggregation operator along with their some desirable properties such as idempotency, boundedness, and monotonicity. Actually interval-valued Pythagorean fuzzy Einstein weighted averaging aggregation operator weights only the Pythagorean fuzzy arguments and interval-valued Pythagorean fuzzy Einstein ordered weighted averaging aggregation operator weights only the ordered positions of the Pythagorean fuzzy arguments instead of weighting the Pythagorean fuzzy arguments themselves. To overcome these limitations, we have introduced an interval-valued Pythagorean fuzzy Einstein hybrid weighted averaging aggregation operator, which weights both the given Pythagorean fuzzy value and its ordered position. Finally, the proposed operator has been applied to decision-making problems to show the validity, practicality and effectiveness of the new approach.

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