Complex & Intelligent Systems (2019) 5:65–78
https://doi.org/10.1007/s40747-018-0074-z

ORIGINAL ARTICLE

Intuitionistic trapezoidal fuzzy multi-numbers and its application to multi-criteria decision-making problems

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Received: 7 August 2017 / Accepted: 23 May 2018 / Published online: 13 June 2018

Abstract
Intuitionistic trapezoidal fuzzy multi-numbers (ITFM-numbers) are a special intuitionistic fuzzy multiset on a real number set, which are very useful for decision makers to depict their intuitionistic fuzzy multi-preference information. In the ITFM-numbers, the occurrences are more than one with the possibility of the same or the different membership and non-membership functions. In this paper, we define ITFM-numbers based on multiple criteria decision-making problems in which the ratings of alternatives are expressed with ITFM-numbers. Firstly, some operational laws using t-norm and t-conorm are proposed. Then, some aggregation operators on ITFM-numbers are developed. Also, the ranking order of alternative is given according to the similarity of the alternative with respect to the positive ideal solution. Finally, a numerical example is given to verify the developed approach and to demonstrate its practicality and effectiveness.

Keywords Intuitionistic fuzzy set · Intuitionistic fuzzy number · Intuitionistic fuzzy multiset · Intuitionistic fuzzy multi-numbers · Multi-criteria decision making

Introduction

In 1986, the theory of intuitionistic fuzzy set was first presented by Atanassov [1] to deal with uncertainty of imperfect information. Since the intuitionistic fuzzy set theory was proposed by Atanassov [1], many researches treating imprecision and uncertainty have been developed and studied: for example, trapezoidal fuzzy multi-number [2], intuitionistic fuzzy sets [3–11], methodology for ranking triangular intuitionistic fuzzy numbers [12–23], intuitionistic trapezoidal fuzzy aggregation operator [21,24–29], interval-valued trapezoidal fuzzy numbers aggregation operator [30–33], interval-valued generalized intuitionistic fuzzy numbers [26,34], entropy and similarity measure of intuitionistic fuzzy sets [35,36] and so on.

"Many fields of modern mathematics have been emerged by violating a basic principle of a given theory only because useful structures could be defined this way. Set is a well-defined collection of distinct objects, that is, the elements of a set are pair wise different. If we relax this restriction and allow repeated occurrences of any element, then we can get a mathematical structure that is known as Multisets or Bags. For example, the prime factorization of an integer $n > 0$ is a Multiset whose elements are primes. The number 120 has the prime factorization $120 = 2^3 3^1 5^1$ which gives the Multiset {2, 2, 2, 3, 5}" [37]. As a generalization of multiset, Yager [38] proposed fuzzy multiset which can occur more than once with possibly the same or different membership values. Then, Shinoj and John [37,39,40] proposed intuitionistic fuzzy multiset as a new research area. Many researchers studied intuitionistic fuzzy multisets. Ibrahim and Ejegwa [41] and Ejegwa [42] extended the idea of modal operators to intuitionistic fuzzy multisets. Rajarajeswari and Uma [43] developed normalized geometric and normalized hamming distance measures on intuitionistic fuzzy multisets. Ejegwa [44] gave a method to convert intuitionistic fuzzy multisets to fuzzy sets. Ejegwa and Awolola [45] proposed a application of intuitionistic fuzzy multisets in binomial dis-

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tributions. Deepa [46] examined some implication results and Das et al. [47] proposed a group decision-making method. Rajarajeswari and Uma [48–52] introduced some measure for intuitionistic fuzzy multisets. Also, Shinoj and Sunil [53] and Ejegwa and Awolola [54] gave algebraic structures of intuitionistic fuzzy multisets, called intuitionistic fuzzy multigroups, and its various properties were examined. Also, the same authors proposed the topological structures of the sets in [55].

From the existing research results, we cannot see any study on intuitionistic trapezoidal fuzzy multi-numbers (ITFM-numbers). The ITFM-numbers are a generalization of trapezoidal fuzzy numbers and intuitionistic trapezoidal fuzzy numbers which are commonly used in real decision problems, because the lack of information or imprecision of the available information in real situations is more serious. So the research of ranking ITFM-numbers is very necessary and the ranking problem is more difficult than ranking ITFM-numbers due to additional multi-membership functions and multi-non-membership functions. Therefore, the remainder of this article is organized as follows. In "Preliminary", some preliminary background on intuitionistic fuzzy multiset and intuitionistic fuzzy numbers is given. In "Intuitionistic trapezoidal fuzzy multi-number", ITFM-numbers and operations are proposed. In "Some aggregation operators on ITFM-numbers", some aggregation operators on ITFM-numbers by using algebraic sum and algebraic product is given in Definition 2.3. In "An approach to MADM problems with ITFM-numbers", we introduce a multi-criteria making method, called ITFM-numbers multi-criteria decision-making method, by using the aggregation operator. In "Application", case studies are proposed to verify the developed approach and to demonstrate its practicality and effectiveness. In "Comparison analysis and discussion", some conclusions and directions for future work are initiated.

Preliminary

Let us start with some basic concepts related to fuzzy set, multi-fuzzy set, intuitionistic fuzzy set, intuitionistic fuzzy multiset and intuitionistic fuzzy numbers.

Definition 2.1 [56] Let E be a universe. Then a fuzzy set X over E is defined by

$$X = \{(\mu_X(x)/x) : x \in E\},$$

where μ_X is called membership function of X and defined by $\mu_X : E \to [0.1]$. For each $x \in E$, the value $\mu_X(x)$ represents the degree of x belonging to the fuzzy set X.

Definition 2.2 [57] t-norms are associative, monotonic and commutative two-valued functions t that map from $[0, 1] \times [0, 1]$ into $[0, 1]$. These properties are formulated with the following conditions:

1. $t(0, 0) = 0$ and $t(\mu_{X_1}(x), 1) = t(1, \mu_{X_1}(x)) = \mu_{X_1}(x)$, $x \in E$,
2. If $\mu_{X_1}(x) \le \mu_{X_3}(x)$ and $\mu_{X_2}(x) \le \mu_{X_4}(x)$, then

   $t(\mu_{X_1}(x), \mu_{X_2}(x)) \le t(\mu_{X_3}x), \mu_{X_4}(x))$,
3. $t(\mu_{X_1}(x), \mu_{X_2}(x)) = t(\mu_{X_2}(x), \mu_{X_1}(x))$,
4. $t(\mu_{X_1}(x), t(\mu_{X_2}(x), \mu_{X_3}(x))) = t(t(\mu_{X_1}(x), \mu_{X_2})(x), \mu_{X_3}(x))$.

Definition 2.3 [57] s-norm are associative, monotonic and commutative two-placed functions s which map from $[0, 1] \times [0, 1]$ into $[0, 1]$. These properties are formulated with the following conditions:

1. $s(1, 1) = 1$ and $s(\mu_{X_1}(x), 0) = s(0, \mu_{X_1}(x)) = \mu_{X_1}(x), x \in E$,
2. if $\mu_{X_1}(x) \le \mu_{X_3}(x)$ and $\mu_{X_2}(x) \le \mu_{X_4}(x)$, then $s(\mu_{X_1}(x), \mu_{X_2}(x)) \le s(\mu_{X_3}(x), \mu_{X_4}(x))$,
3. $s(\mu_{X_1}(x), \mu_{X_2}(x)) = s(\mu_{X_2}(x), \mu_{X_1}(x))$,
4. $s(\mu_{X_1}(x), s(\mu_{X_2}(x), \mu_{X_3}(x))) = s(s(\mu_{X_1}(x), \mu_{X_2})(x), \mu_{X_3}(x))$.

t-norm and t-conorm are related in a sense of logical duality. Typical dual pairs of non-parameterized t-norm and t-conorm are compiled below [57]:

1. Drastic product:

$$t_w(\mu_{X_1}(x), \mu_{X_2}(x)) = \begin{cases} \min\{\mu_{X_1}(x), \mu_{X_2}(x)\}, & \max\{\mu_{X_1}(x)\mu_{X_2}(x)\} = 1 \\ 0, & \text{otherwise} \end{cases}.$$

2. Drastic sum:

$$s_w(\mu_{X_1}(x), \mu_{X_2}(x)) = \begin{cases} \max\{\mu_{X_1}(x), \mu_{X_2}(x)\}, & \min\{\mu_{X_1}(x)\mu_{X_2}(x)\} = 0 \\ 1, & \text{otherwise} \end{cases}.$$

3. Bounded product:

$$t_1(\mu_{X_1}(x), \mu_{X_2}(x)) = \max\{0, \mu_{X_1}(x) + \mu_{X_2}(x) - 1\}.$$

4. Bounded sum:

$$s_1(\mu_{X_1}(x), \mu_{X_2}(x)) = \min\{1, \mu_{X_1}(x) + \mu_{X_2}(x)\}.$$

5. Einstein product:

$$t_{1.5}(\mu_{X_1}(x), \mu_{X_2}(x)) = \frac{\mu_{X_1}(x) \cdot \mu_{X_2}(x)}{2 - [\mu_{X_1}(x) + \mu_{X_2}(x) - \mu_{X_1}(x) \cdot \mu_{X_2}(x)]}.$$

6. Einstein sum:

$$s_{1.5}(\mu_{X_1}(x), \mu_{X_2}(x)) = \frac{\mu_{X_1}(x) + \mu_{X_2}(x)}{1 + \mu_{X_1}(x) \cdot \mu_{X_2}(x)}.$$

7. Algebraic product:

$$t_2(\mu_{X_1}(x), \mu_{X_2}(x)) = \mu_{X_1}(x) \cdot \mu_{X_2}(x).$$

8. Algebraic sum:

$$s_2(\mu_{X_1}(x), \mu_{X_2}(x)) = \mu_{X_1}(x) + \mu_{X_2}(x) - \mu_{X_1}(x) \cdot \mu_{X_2}(x).$$

9. Hamacher product:

$$t_{2.5}(\mu_{X_1}(x), \mu_{X_2}(x)) = \frac{\mu_{X_1}(x) \cdot \mu_{X_2}(x)}{\mu_{X_1}(x) + \mu_{X_2}(x) - \mu_{X_1}(x) \cdot \mu_{X_2}(x)}.$$

10. Hamacher sum:

$$s_{2.5}(\mu_{X_1}(x), \mu_{X_2}(x)) = \frac{\mu_{X_1}(x) + \mu_{X_2}(x) - 2.\mu_{X_1}(x) \cdot \mu_{X_2}(x)}{1 - \mu_{X_1}(x) \cdot \mu_{X_2}(x)}.$$

11. Minumum:

$$t_3(\mu_{X_1}(x), \mu_{X_2}(x)) = \min\{\mu_{X_1}(x), \mu_{X_2}(x)\}.$$

12. Maximum:

$$s_3(\mu_{X_1}(x), \mu_{X_2}(x)) = \max\{\mu_{X_1}(x), \mu_{X_2}(x)\}.$$

Definition 2.4 [58] Let X be a non-empty set. A multi-fuzzy set A on X is defined as:

$$A = \{\langle x, \mu_A^1(x), \mu_A^2(x), \ldots, \mu_A^P(x)\rangle : x \in E\}.$$

where $\mu_i : X \to [0, 1]$ for all $i \in \{1, 2, \ldots, p\}$ such that $\mu_A^1(x) \geq \mu_A^2(x) \geq \cdots \geq \mu_A^P(x)$for $x \in E$.

Definition 2.5 [1] Let X be a nonempty set. An intuitionistic fuzzy set (IFS) A is an object having the form $A = \{\langle x; \mu_A(x), \nu_A(x)\rangle : x \in X\}$, where the function $\mu_A : X \to [0.1]$,$\nu_A : X \to [0.1]$ defines, respectively, the degree of membership and the degree of non-membership of the element $x \in X$ to the set A with $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$.

Definition 2.6 [39,40] Let X be a non-empty set. A intuitionistic fuzzy multiset IFM on X is defined as:

$$\text{IFM} = \{\langle x : (\mu_A^1(x), \mu_A^2(x), \ldots, \mu_A^P(x)), (\nu_A^1(x), \nu_A^2(x), \ldots, \nu_A^P(x))\rangle : x \in X\},$$

where $\mu_i : X \to [0, 1]$ and $\nu_i : X \to [0, 1]$ such that $0 \leq \mu_A^i(x) + \nu_A^i(x) \leq 1$ for all $i \in \{1, 2, \ldots, p\}$ and $x \in X$.

Also, the membership sequence is defined as a decreasingly ordered sequence of elements, that is, $(\mu_A^1(x), \mu_A^2(x), \ldots, \mu_A^P(x))$, where $\mu_A^1(x) \geq \mu_A^2(x) \geq \cdots \geq \mu_A^P(x)$ and the corresponding non-membership sequence will be denoted by $(\nu_A^1(x), \nu_A^2(x), \ldots, \nu_A^P(x))$ such that neither decreasing nor increasing function $x \in X$ and $i = (1, 2, \ldots, p)$

Definition 2.7 [7] Let $\tilde{\alpha}$ be an intuitionistic trapezoidal fuzzy number; its membership function and non-membership function are given, respectively, as

$$\mu_{\tilde{\alpha}}(x) = \begin{cases} \frac{(x-a)}{(b-a)}\eta_{\tilde{\alpha}}, & a \leq x < b \\ \eta_{\tilde{\alpha}}, & b \leq x \leq c \\ \frac{(d-x)}{(d-c)}\eta_{\tilde{\alpha}}, & c < x \leq d \\ 0, & \text{otherwise,} \end{cases} \quad \text{and}$$

$$\nu_{\tilde{\alpha}}(x) = \begin{cases} \frac{(b-x)+\nu_{\tilde{\alpha}}(x-a_1)}{(b-a_1)}, & a_1 \leq x < b \\ \nu_{\tilde{\alpha}}, & b \leq x \leq c \\ \frac{(x-c)+\nu_{\tilde{\alpha}}(d_1-x)}{(d_1-c)}, & c < x \leq d_1 \\ 1, & \text{otherwise,} \end{cases}$$

where $0 \leq \mu_{\tilde{a}} \leq 1$; $0 \leq \nu_{\tilde{a}} \leq 1$; $\mu_{\tilde{a}} + \nu_{\tilde{a}} \leq 1$; $a, b, c, d \in R$. Then, $\tilde{\alpha} = \langle([a, b, c, d]; \mu_{\tilde{\alpha}}), [a_1, b, c, d_1]; \nu_{\tilde{\alpha}})\rangle$ is called an intuitionistic trapezoidal fuzzy number. For convenience, let $\tilde{\alpha} = \langle[a, b, c, d]; \mu_{\tilde{\alpha}}, \nu_{\tilde{\alpha}}\rangle$.

Definition 2.8 [7] Let $\tilde{\alpha} = \langle[a, b, c, d]; \mu_{\tilde{\alpha_1}}, \nu_{\tilde{\alpha_1}}\rangle$ and $\tilde{\alpha} = \langle[a, b, c, d]; \mu_{\tilde{\alpha_2}}, \nu_{\tilde{\alpha_2}}\rangle$ be two intuitionistic trapezoidal fuzzy numbers, and $\lambda \geq 0$, then

1. $\alpha_1 \oplus \alpha_2 = \langle[a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2]; \mu_{\tilde{\alpha_1}} + \mu_{\tilde{\alpha_2}} - \mu_{\tilde{\alpha_1}}\mu_{\tilde{\alpha_2}}, \nu_{\tilde{\alpha_1}}\nu_{\tilde{\alpha_2}}\rangle$;
2. $\alpha_1 \otimes \alpha_2 = \langle[a_1a_2, b_1b_2, c_1c_2, d_1d_2]; \mu_{\tilde{\alpha_1}}\mu_{\tilde{\alpha_2}}, \nu_{\tilde{\alpha_1}} + \nu_{\tilde{\alpha_2}} - \nu_{\tilde{\alpha_1}}\nu_{\tilde{\alpha_2}}\rangle$;
3. $\lambda\tilde{\alpha} = \langle[\lambda a, \lambda b, \lambda c, \lambda d]; 1 - (1 - \mu_{\tilde{\alpha}})^{\lambda}, (\nu_{\tilde{\alpha}})^{\lambda}\rangle$;
4. $\tilde{\alpha}^{\lambda} = \langle[a^{\lambda}, b^{\lambda}, c^{\lambda}, d^{\lambda}]; (\mu_{\tilde{\alpha}})^{\lambda}, 1 - (1 - \nu_{\tilde{\alpha}})^{\lambda}\rangle$.

Definition 2.9 [59] Let $A = \langle(a_1, b_1, c_1, d_1); \eta_A\rangle$, $B = \langle(a_2, b_2, c_2, d_2); \eta_B\rangle$, $0 \leq a_1 \leq b_1 \leq c_1 \leq d_1 \leq 1$, $0 \leq a_2 \leq b_2 \leq c_2 \leq d_2 \leq 1$ Then, the degree of similarity $S(A, B)$ between the generalized trapezoidal fuzzy numbers $P(A)$ and $P(B)$ is calculated as follows:

$$S(A, B) = \left(1 - \frac{\sum_{i=1}^{4} |a_i - b_i|}{4}\right) \times \frac{\min\{P(A), P(B)\} + \min\{\eta_A, \eta_B\}}{\max\{P(A), P(B)\} + \max\{\eta_A, \eta_B\}},$$

where $S(A, B) \in [0, 1]$; $P(A)$ and $P(B)$ are defined as follows:

$$P(A) = \sqrt{(a_1 - a_2)^2 + (\eta_A)^2} + \sqrt{(a_3 - a_4)^2) + (\eta_A)^2} + (a_3 - a_2) + (a_4 - a_1),$$

$$P(B) = \sqrt{(b_1 - b_2)^2 + (\eta_B)^2} + \sqrt{(b_3 - b_4)^2 + (\eta_B)^2} + (b_3 - b_2) + (b_4 - b_1).$$

$P(A)$ and $P(B)$ denote the perimeters of the generalized trapezoidal fuzzy numbers A and B , respectively.

Intuitionistic trapezoidal fuzzy multi-number

Definition 3.1 Let $\eta_A^i, \vartheta_A^i \in [0,1]$ $(i \in \{1,2,\ldots,p\})$and $a,b,c,d \in \mathbb{R}$ such that $a \le b \le c \le d$. Then, an intuitionistic trapezoidal fuzzy multi-number (ITFM-numbers) $\tilde{a} = \langle [a,b,c,d]; (\eta_A^1, \eta_A^2, \ldots, \eta_A^P), (\vartheta_A^1, \vartheta_A^2, \ldots, \vartheta_A^P)\rangle$ is a special intuitionistic fuzzy multiset on the real number set $\mathbb{R}$, whose membership functions and non-membership functions are defined as follows, respectively:

$$\mu_{\tilde{\alpha}}^i(x) = \begin{cases} \frac{(x-a)}{(b-a)}\eta_{\tilde{\alpha}}^i, & a \le x < b \\ \eta_{\tilde{\alpha}}^i, & b \le x \le c \\ \frac{(d-x)}{(d-c)}\eta_{\tilde{\alpha}}^i, & c < x \le d \\ 0, & \text{otherwise,} \end{cases} \quad \text{and}$$

$$\nu_{\tilde{\alpha}}^i(x) = \begin{cases} \frac{(b-x)+\vartheta_{\tilde{\alpha}}^i(x-a_1)}{(b-a_1)}, & a_1 \le x < b \\ \vartheta_{\tilde{\alpha}}^i, & b \le x \le c \\ \frac{(x-c)+\vartheta_{\tilde{\alpha}}^i(d_1-x)}{(d_1-c)}, & c < x \le d_1 \\ 1, & \text{otherwise.} \end{cases}$$

Note that the set of all ITFM-numbers on $\mathbb{R}$ will be denoted by Γ.

Example 3.2 The ITFM-numbers function

$$\eta_A^1(x) = \begin{cases} \frac{(x-1)}{2}(0.3) & 1 \le x < 3 \\ 0.3 & 3 \le x \le 6 \\ \frac{(8-x)}{2}(0.3) & 6 < x \le 8 \\ 0 & \text{otherwise,} \end{cases}$$

$$\vartheta_A^1(x) = \begin{cases} \frac{(3-x)+0.5(x-2)}{1} & 2 \le x < 3 \\ 0.5 & 3 \le x \le 6 \\ \frac{(x-5)+0.5(7-x)}{1} & 6 < x \le 7 \\ 1 & \text{otherwise,} \end{cases}$$

$$\eta_A^2(x) = \begin{cases} \frac{(x-1)}{2}(0.6) & 1 \le x < 3 \\ 0.6 & 3 \le x \le 6 \\ \frac{(8-x)}{2}(0.6) & 6 < x \le 8 \\ 0 & \text{otherwise,} \end{cases}$$

$$\vartheta_A^2(x) = \begin{cases} \frac{(3-x)+0.2(x-2)}{1} & 2 \le x < 3 \\ 0.2 & 3 \le x \le 6 \\ \frac{(x-5)+0.2(7-x)}{1} & 6 < x \le 7 \\ 1 & \text{otherwise,} \end{cases}$$

$$\vdots$$

$$\eta_A^p(x) = \begin{cases} \frac{(x-1)}{2}(0.2) & 1 \le x < 3 \\ 0.2 & 3 \le x \le 6 \\ \frac{(8-x)}{2}(0.2) & 6 < x \le 8 \\ 0 & \text{otherwise,} \end{cases}$$

$$\vartheta_A^p(x) = \begin{cases} \frac{(3-x)+0.7(x-2)}{1} & 2 \le x < 3 \\ 0.7 & 3 \le x \le 6 \\ \frac{(x-5)+0.7(7-x)}{1} & 6 < x \le 7 \\ 1 & \text{otherwise} \end{cases}$$

is the ITFM-numbers with $\langle [2,3,6,8]; (0.3, 0.6, \ldots, 0.2), (0.5, 0.2, \ldots, 0.7)\rangle$

Definition 3.3 Let $A = \langle [a_1,b_1,c_1,d_1]; (\eta_A^1, \eta_A^2, \ldots, \eta_A^P), (\vartheta_A^1, \vartheta_A^2, \ldots, \vartheta_A^P)\rangle, B = \langle [a_2,b_2,c_2,d_2]; (\eta_A^1, \eta_A^2, \ldots, \eta_A^P), (\vartheta_A^1, \vartheta_A^2, \ldots, \vartheta_A^P)\rangle \in \Lambda$ and $\gamma \ne 0$ be any real number. Then,

1. $A + B = \langle [a_1+a_2, b_1+b_2, c_1+c_2, d_1+d_2]; (s(\eta_A^1, \eta_B^1), s(\eta_A^2, \eta_B^2), \ldots, s(\eta_A^p, \eta_B^p)), (t(\vartheta_A^1, \vartheta_B^1), t(\vartheta_A^2, \vartheta_B^2), \ldots, t(\vartheta_A^p, \vartheta_B^p))\rangle$.
2. $A - B = \langle [a_1-d_2, b_1-c_2, c_1-b_2, d_1-a_2]; (s(\eta_A^1, \eta_B^1), s(\eta_A^2, \eta_B^2), \ldots, s(\eta_A^p, \eta_B^p)), (t(\vartheta_A^1, \vartheta_B^1), t(\vartheta_A^2, \vartheta_B^2), \ldots, t(\vartheta_A^p, \vartheta_B^p))\rangle$.
3. $A \cdot B = \begin{cases} \langle [a_1a_2, b_1b_2, c_1c_2, d_1d_2]; (t(\eta_A^1, \eta_B^1), \ldots, t(\eta_A^p, \eta_B^p)), (s(\vartheta_A^1, \vartheta_B^1), \ldots, s(\vartheta_A^p, \vartheta_B^p))\rangle & (d_1 > 0, d_2 > 0) \\ \langle [a_1d_2, b_1c_2, c_1b_2, d_1a_2]; (t(\eta_A^1, \eta_B^1), \ldots, t(\eta_A^p, \eta_B^p)), (s(\vartheta_A^1, \vartheta_B^1), \ldots, s(\vartheta_A^p, \vartheta_B^p))\rangle & (d_1 < 0, d_2 > 0) \\ \langle [d_1d_2, c_1c_2, b_1b_2, a_1a_2]; (t(\eta_A^1, \eta_B^1), \ldots, t(\eta_A^p, \eta_B^p)), (s(\vartheta_A^1, \vartheta_B^1), \ldots, s(\vartheta_A^p, \vartheta_B^p))\rangle & (d_1 < 0, d_2 < 0) \end{cases}$.
4. $A/B = \begin{cases} \langle [a_1/d_2, b_1/c_2, c_1/b_2, d_1/a_2]; (t(\eta_A^1, \eta_B^1), \ldots, t(\eta_A^p, \eta_B^p)), (s(\vartheta_A^1, \vartheta_B^1), \ldots, s(\vartheta_A^p, \vartheta_B^p))\rangle & (d_1 > 0, d_2 > 0) \\ \langle [(d_1/d_2, c_1/c_2, b_1/b_2, a_1/a_2]; (t(\eta_A^1, \eta_B^1), \ldots, t(\eta_A^p, \eta_B^p)), (s(\vartheta_A^1, \vartheta_B^1), \ldots, s(\vartheta_A^p, \vartheta_B^p))\rangle & (d_1 < 0, d_2 > 0) \\ \langle [d_1/a_2, c_1/b_2, b_1/c_2, a_1/d_2]; (t(\eta_A^1, \eta_B^1), \ldots, t(\eta_A^p, \eta_B^p)), (s(\vartheta_A^1, \vartheta_B^1), \ldots, s(\vartheta_A^p, \vartheta_B^p))\rangle & (d_1 < 0, d_2 < 0) \end{cases}$.
5. $\gamma A = \langle [\gamma a_1, \gamma b_1, \gamma c_1, \gamma d_1]; (1-(1-\eta_A^1)^\gamma, 1-(1-\eta_A^2)^\gamma), 1-(1-\eta_A^p)^\gamma), ((\vartheta_A^1)^\gamma, (\vartheta_A^2)^\gamma, \ldots, (\vartheta_A^p)^\gamma)\rangle (\gamma \ge 0)$.
6. $A^\gamma = \langle [a_1^\gamma, b_1^\gamma, c_1^\gamma, d_1^\gamma]; ((\eta_A^1)^\gamma, (\eta_A^2)^\gamma, \ldots, (\eta_A^p)^\gamma), (1-(1-\vartheta_A^1)^\gamma, 1-(1-\vartheta_A^2)^\gamma, \ldots, 1-(1-\vartheta_A^p)^\gamma)\rangle (\gamma \ge 0)$.

In the following example, we use the Einstein sum and Einstein product is given in Definition 2.3.

Example 3.4 Let $A = \langle[2, 4, 7, 9]; (0.2, 0.5, \ldots, 0.7), (0.6, 0.3, \ldots, 0.1)\rangle$, $B = \langle[1, 2, 3, 6]; (0.6, 0.1, \ldots, 0.9), (0.3, 0.8, \ldots, 0.01)\rangle \in \Gamma$.

1. $A + B = \langle[3, 6, 10, 15]; (0.71428, 0.57142, \ldots, 0.98159), (0.14062, 0.21052, \ldots, 0.00052)\rangle$.
2. $A - B = \langle[1, 2, 4, 3]; (0.71428, 0.57142, \ldots, 0.98159), (0.14062, 0.21052, \ldots, 0.00052))\rangle$.
3. $A \cdot B = \langle[2, 8, 21, 54]; (0.0909, 0.0344, \ldots, 0.61165), (0.76271, 0.88709, \ldots, 0.10989)\rangle$.
4. $A/B = \langle[2/6, 4/3, 7/2, 9]; (0.0909, 0.03448, \ldots, 0.61165), (0.76271, 0.88709, \ldots, 0.10989)\rangle$.
5. $4 \cdot A = \langle[8, 16, 28, 36]; (0.5904, 0.9375, \ldots, 0.9919), (0.1296, 0.0081, \ldots, 0.0001)\rangle$. $3 \cdot B = \langle[3, 6, 9, 18]; (0.936, 0.271, \ldots, 0.999), (0.027, 0.512, \ldots, 0.000001)\rangle$.
6. $A^2 = \langle[4, 16, 49, 81]; (0.04, 0.25, \ldots, 0.49), (0.84, 0.51, \ldots, 0.19)\rangle$.

Definition 3.5 Let $A = \tilde{a} = \langle[a, b, c, d]; (\eta_A^1, \eta_A^2, \ldots, \eta_A^P), (\vartheta_A^1, \vartheta_A^2, \ldots, \vartheta_A^P)\rangle \in \Gamma$. Then,

1. A is called positive ITFM-numbers if $a > 0$,
2. A is called negative ITFM-numbers if $d < 0$,
3. A is called neither positive nor negative ITFM-numbers if $a > 0$ and $d < 0$.

Note 3.6 *A negative ITFM-number can be written as the negative multiplication of a positive ITFM-number.*

Example 3.7 $A = \langle(-7, -4, -3, -1); (0.03, 0.45, \ldots, 0.59), (0.64, 0.81, \ldots, 0.39)\rangle$ is a negative ITFM-numbers this can be written as $A = -\langle(1, 3, 4, 7); (0.03, 0.45, \ldots, 0.59), (0.64, 0.81, \ldots, 0.39)\rangle$.

Theorem 3.8 *Let* $A = \langle[a_1, b_1, c_1, d_1]; (\eta_A^1, \eta_A^2, \ldots, \eta_A^P), (\vartheta_A^1, \vartheta_A^2, \ldots, \vartheta_A^P)\rangle$, $B = \langle[a_2, b_2, c_2, d_2]; (\eta_B^1, \eta_B^2, \ldots, \eta_B^P), (\vartheta_B^1, \vartheta_B^2, \ldots, \vartheta_B^P)\rangle$ *and* $C = \langle[a_3, b_3, c_3, d_3]; (\eta_C^1, \eta_C^2, \ldots, \eta_C^P), (\vartheta_C^1, \vartheta_C^2, \ldots, \vartheta_C^P)\rangle \in \Gamma$ *Then, we have*

1. $A + B = B + A$,
2. $(A + B) + C = A + (B + C)$,
3. $A \cdot B = B \cdot A$,
4. $(A \cdot B) \cdot C = A \cdot (B \cdot C)$,
5. $\lambda_1 \cdot A + \lambda_2 \cdot A = (\lambda_1 + \lambda_2) \cdot A, \lambda_1 + \lambda_2) \geq 0$,
6. $\lambda \cdot (A + B) = \lambda \cdot A + \lambda \cdot B, \lambda \geq 0$.

Proof In the following proof, we use the Einstein sum and Einstein product is given in Definition 2.3.

1. Based on Definition 3.3, it can be seen that

$$\begin{aligned} A + B &= \Bigg\langle (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2); \\ &\quad \left(\frac{\eta_A^1 + \eta_B^1}{1 + (\eta_A^1 \cdot \eta_B^1)}, \ldots, \frac{\eta_A^p + \eta_B^p}{1 + (\eta_A^p \cdot \eta_B^p)} \right), \\ &\quad \left(\frac{\vartheta_A^1 + \vartheta_B^1}{2 - [\vartheta_A^1 + \vartheta_B^1 - \vartheta_A^1 \cdot \vartheta_B^1]}, \ldots, \right. \\ &\quad \left. \frac{\vartheta_A^p + \vartheta_B^p}{2 - [\vartheta_A^p + \vartheta_B^p - \vartheta_A^p \cdot \vartheta_B^p]} \right) \Bigg\rangle \\ &= \langle (a_2 + a_1, b_2 + b_1, c_2 + c_1, d_2 + d_1); \\ &\quad \left(\frac{\eta_B^1 + \eta_A^1}{1 + (\eta_B^1 \cdot \eta_A^1)}, \ldots, \frac{\eta_B^p + \eta_A^p}{1 + (\eta_B^p \cdot \eta_A^p)} \right), \\ &\quad \left(\frac{\vartheta_B^1 + \vartheta_A^1}{2 - [\vartheta_B^1 + \vartheta_A^1 - \vartheta_B^1 \cdot \vartheta_A^1]}, \ldots, \right. \\ &\quad \left. \frac{\vartheta_B^p + \vartheta_A^p}{2 - [\vartheta_B^p + \vartheta_A^p - \vartheta_B^p \cdot \vartheta_A^p]} \right) \Bigg\rangle \\ &= B + A. \end{aligned}$$

2. Based on Definition 3.3, it can be seen that

$$\begin{aligned} A \cdot B &= \Bigg\langle (a_1 a_2, b_1 b_2, c_1 c_2, d_1 d_2); \\ &\quad \left(\frac{\eta_A^1 + \eta_B^1}{2 - [\eta_A^1 + \eta_B^1 - \eta_A^1 \cdot \eta_B^1]}, \ldots, \right. \\ &\quad \left. \frac{\eta_A^p + \eta_B^p}{2 - [\eta_A^p + \eta_B^p - \eta_A^p \cdot \eta_B^p]} \right), \\ &\quad \left(\frac{\vartheta_A^1 + \vartheta_B^1}{1 + (\vartheta_A^1 \cdot \vartheta_B^1)}, \ldots, \frac{\vartheta_A^p + \vartheta_B^p}{1 + (\vartheta_A^p \cdot \vartheta_B^p)} \right) \Bigg\rangle \\ &= \langle (a_2 a_1, b_2 b_1, c_2 c_1, d_2 d_1); \\ &\quad \left(\frac{\eta_B^1 + \eta_A^1}{2 - [\eta_B^1 + \eta_A^1 - \eta_B^1 \cdot \eta_A^1]}, \ldots, \right. \\ &\quad \left. \frac{\eta_B^p + \eta_A^p}{2 - [\eta_B^p + \eta_A^p - \eta_B^p \cdot \eta_A^p]} \right), \\ &\quad \left(\frac{\vartheta_B^1 + \vartheta_A^1}{1 + (\vartheta_B^1 \cdot \vartheta_A^1)}, \ldots, \frac{\vartheta_B^p + \vartheta_A^p}{1 + (\vartheta_B^p \cdot \vartheta_A^p)} \right) \Bigg\rangle \\ &= B \cdot A. \end{aligned}$$

The proofs of (2), (4), (5) and (6) can be obtained similarly. □

Definition 3.9 Let $A = \langle(a_1, b_1, c_1, d_1); (\eta_A^1, \eta_A^2, \ldots, \eta_A^P), (\vartheta_A^1, \vartheta_A^2, \ldots, \vartheta_A^P)\rangle \in \Gamma$. Then, the normalized ITFM-numbers of A is given by:

$$\overline{A} = \left\langle \left[\frac{a_1}{a_1+b_1+c_1+d_1}, \frac{b_1}{a_1+b_1+c_1+d_1}, \frac{c_1}{a_1+b_1+c_1+d_1}, \frac{d_1}{a_1+b_1+c_1+d_1} \right]; (\eta_A^1, \eta_A^2, \ldots, \eta_A^P), (\vartheta_A^1, \vartheta_A^2, \ldots, \vartheta_A^P) \right\rangle.$$

Example 3.10 Assume that $A = \langle (2, 5, 6, 8); (0.01, 0.35, \ldots, 0.79), (0.14, 0.19, \ldots, 0.43) \rangle \in \Gamma$. Then, normalized ITFM-numbers of A can be written as:

$$\overline{A} = \left\langle \left(\frac{2}{21}, \frac{5}{21}, \frac{6}{21}, \frac{8}{21} \right); (0.01, 0.35, \ldots, 0.79), (0.14, 0.19, \ldots, 0.43) \right\rangle.$$

Definition 3.11 Let $\overline{A} = \langle (a_1, b_1, c_1, d_1); (\eta_{\overline{A}}^1, \eta_{\overline{A}}^2, \ldots, \eta_{\overline{A}}^P), (\vartheta_{\overline{A}}^1, \vartheta_{\overline{A}}^2, \ldots, \vartheta_{\overline{A}}^P) \rangle$, $\overline{B} = \langle (a_2, b_2, c_2, d_2); (\eta_{\overline{B}}^1, \eta_{\overline{B}}^2 \ldots, \eta_{\overline{B}}^P), (\vartheta_{\overline{B}}^1, \vartheta_{\overline{B}}^2 \ldots, \vartheta_{\overline{B}}^P) \rangle \in \Gamma$. Then, the normalized similarity measure between $\overline{A}$ and $\overline{B}$ is defined as:

$$S(\overline{A}, \overline{B}) = \frac{1}{p} \cdot \left[\left(1 - \frac{|a_2 - a_1| + |b_2 - b_1| + |c_2 - c_1| + |d_2 - d_1|}{4} \right) \times \frac{(\min\{P(A)^1, P(A)^2, P(A)^3, P(A)^4, P(B)^1, P(B)^2, P(B)^3, P(B)^4\}) + \min\{(\eta_{\overline{A}}^1, \ldots, \eta_{\overline{A}}^P), (\eta_{\overline{B}}^1, \eta_{\overline{B}}^2, \ldots, \eta_{\overline{B}}^P)\} + \max\{(\vartheta_{\overline{A}}^1, \ldots, \vartheta_{\overline{A}}^P), (\vartheta_{\overline{B}}^1, \ldots, \vartheta_{\overline{B}}^P)\}}{(\max\{P(A)^1, P(A)^2, P(A)^3, P(A)^4, P(B)^1, P(B)^2, P(B)^3, P(B)^4\}) + \max\{(\eta_{\overline{A}}^1, \ldots, \eta_{\overline{A}}^P), (\eta_{\overline{B}}^1, \eta_{\overline{B}}^2, \ldots, \eta_{\overline{B}}^P)\} + \min\{(\vartheta_{\overline{A}}^1, \ldots, \vartheta_{\overline{A}}^P), (\vartheta_{\overline{B}}^1, \ldots, \vartheta_{\overline{B}}^P)\}} \right],$$

where $S(\overline{A}, \overline{B}) \in [0, 1]$; $P(A)$ and $P(B)$ are defined as follows:

$$P(A) = \sqrt{(a_1 - a_2)^2 + (\eta_{\overline{A}}^i - \vartheta_{\overline{A}}^i)^2} + \sqrt{(a_3 - a_4)^2) + (\eta_{\overline{A}}^i - \vartheta_{\overline{A}}^i)^2} + (a_3 - a_2) + (a_4 - a_1),$$

$$P(B) = \sqrt{(b_1 - b_2)^2 + (\eta_{\overline{B}}^i - \vartheta_{\overline{B}}^i)^2} + \sqrt{(b_3 - b_4)^2) + (\eta_{\overline{B}}^i - \vartheta_{\overline{B}}^i)^2} + (b_3 - b_2) + (b_4 - b_1).$$

Let $\overline{A} = \langle (a_1, b_1, c_1, d_1); (\eta_{\overline{A}}^1, \ldots, \eta_{\overline{A}}^P), (\vartheta_{\overline{A}}^1, \ldots, \vartheta_{\overline{A}}^P) \rangle$ *and* $\overline{B} = \langle (a_2, b_2, c_2, d_2); (\eta_{\overline{B}}^1, \ldots, \eta_{\overline{B}}^P), (\vartheta_{\overline{A}}^1, \ldots, \vartheta_{\overline{A}}^P) \rangle$ be two normalized ITFM-numbers.

Now, we give a theorem for ITFM-numbers inspired by [59].

Theorem 3.12 *Let* $A = \langle [a_1, b_1, c_1, d_1]; (\eta_A^1, \eta_A^2, \ldots, \eta_A^P), (\vartheta_A^1, \vartheta_A^2, \ldots, \vartheta_A^P) \rangle$, $B = \langle [a_2, b_2, c_2, d_2]; (\eta_B^1, \eta_B^2, \ldots, \eta_B^P), (\vartheta_B^1, \vartheta_B^2, \ldots, \vartheta_B^P) \rangle$ *and* $C = \langle [a_3, b_3, c_3, d_3]; (\eta_C^1, \eta_C^2, \ldots, \eta_C^P), (\vartheta_C^1, \vartheta_C^2, \ldots, \vartheta_C^P) \rangle \in \Gamma$. *Then,* $S(\overline{A}, \overline{B})$ *satisfies the following properties:*

i. *Two normalized ITFM-numbers* $\overline{A}$ *and* $\overline{B}$ *are identical if and only if* $S(\overline{A}, \overline{B}) = 1$.
ii. $S(\overline{A}, \overline{B}) = S(\overline{B}, \overline{A})$.
iii. *Let* $\overline{A}$ *and* $\overline{B}$ *be two normalized ITFM-numbers with the same shape, the same scale (i.e.,*$\eta_{\overline{A}}^1 = \eta_{\overline{B}}^1, \eta_{\overline{A}}^2 = \eta_{\overline{B}}^2, \ldots \eta_{\overline{A}}^p = \eta_{\overline{B}}^p, \vartheta_{\overline{A}}^1 = \vartheta_{\overline{B}}^1, \vartheta_{\overline{A}}^2 = \vartheta_{\overline{B}}^2, \ldots \vartheta_{\overline{A}}^p = \vartheta_{\overline{B}}^p$*) and the same set d, where* $d = a_2 - a_1 = b_2 - b_1 = c_2 - c_1 = d_2 - d_1$, *then* $S(\overline{A}, \overline{B}) = 1 - |d|$.
iv. *If* $\overline{A} \subseteq \overline{B} \subseteq \overline{C}$, *then* $S(\overline{A}, \overline{C}) \leq S(\overline{A}, \overline{B})$ *and* $S(\overline{A}, \overline{C}) \leq S(\overline{B}, \overline{C})$.

Proof i.⇒ If $\overline{A}$ and $\overline{B}$ are identical, then $a_1 = a_2, b_1 = b_2, c_1 = c_2, d_1 = d_2$ and $\eta_{\overline{A}}^1 = \eta_{\overline{B}}^1, \eta_{\overline{A}}^2 = \eta_{\overline{B}}^2, \ldots, \eta_{\overline{A}}^p = \eta_{\overline{B}}^p, \vartheta_{\overline{A}}^1 = \vartheta_{\overline{B}}^1, \vartheta_{\overline{A}}^2 = \vartheta_{\overline{B}}^2, \ldots, \vartheta_{\overline{A}}^p = \vartheta_{\overline{B}}^p$. Thus, $(\min\{P(A)^1, P(A)^2, P(A)^3, P(A)^4, P(B)^1, P(B)^2, P(B)^3, P(B)^4\}) = (\max\{P(A)^1, P(A)^2, P(A)^3, P(A)^4, P(B)^1, P(B)^2, P(B)^3, P(B)^4\})$ and $\min\{(\eta_{\overline{A}}^1, \ldots, \eta_{\overline{A}}^P), (\eta_{\overline{B}}^1, \eta_{\overline{B}}^2, \ldots, \eta_{\overline{B}}^P)\} = \max\{(\eta_{\overline{A}}^1, \ldots, \eta_{\overline{A}}^P), (\eta_{\overline{B}}^1, \eta_{\overline{B}}^2, \ldots, \eta_{\overline{B}}^P)\}$ and $\max\{(\vartheta_{\overline{A}}^1, \ldots, \vartheta_{\overline{A}}^P), (\vartheta_{\overline{B}}^1, \ldots, \vartheta_{\overline{B}}^P)\} = \min\{(\vartheta_{\overline{A}}^1, \ldots, \vartheta_{\overline{A}}^P), (\vartheta_{\overline{B}}^1, \ldots, \vartheta_{\overline{B}}^P)\}$. The degree of similarity between $\overline{A}$ and $\overline{B}$ is calculated as follows:

$$\begin{aligned} S(\overline{A}, \overline{B}) &= \frac{1}{p} \cdot \left[\left(1 - \frac{|a_2 - a_1| + |b_2 - b_1| + |c_2 - c_1| + |d_2 - d_1|}{4} \right) \times \frac{(\min\{P(A)^1, P(A)^2, P(A)^3, P(A)^4, P(B)^1, P(B)^2, P(B)^3, P(B)^4\}) + \min\{(\eta_{\overline{A}}^1, \ldots, \eta_{\overline{A}}^P), (\eta_{\overline{B}}^1, \eta_{\overline{B}}^2, \ldots, \eta_{\overline{B}}^P)\} + \max\{(\vartheta_{\overline{A}}^1, \ldots, \vartheta_{\overline{A}}^P), (\vartheta_{\overline{B}}^1, \ldots, \vartheta_{\overline{B}}^P)\}}{(\max\{P(A)^1, P(A)^2, P(A)^3, P(A)^4, P(B)^1, P(B)^2, P(B)^3, P(B)^4\}) + \max\{(\eta_{\overline{A}}^1, \ldots, \eta_{\overline{A}}^P), (\eta_{\overline{B}}^1, \eta_{\overline{B}}^2, \ldots, \eta_{\overline{B}}^P)\} + \min\{(\vartheta_{\overline{A}}^1, \ldots, \vartheta_{\overline{A}}^P), (\vartheta_{\overline{B}}^1, \ldots, \vartheta_{\overline{B}}^P)\}} \right] \\ &= (1 - 0) \times 1 \\ &= 1 \end{aligned}$$

$\Leftarrow S(\overline{A}, \overline{B}) = 1$, then

$$S(\overline{A}, \overline{B}) = \frac{1}{p} \cdot \left[\left(1 - \frac{|a_2 - a_1| + |b_2 - b_1| + |c_2 - c_1| + |d_2 - d_1|}{4} \right) \times \frac{(\min\{P(A)^1, P(A)^2, P(A)^3, P(A)^4, P(B)^1, P(B)^2, P(B)^3, P(B)^4\}) + \min\{(\eta_{\overline{A}}^1, \ldots, \eta_{\overline{A}}^P), (\eta_{\overline{B}}^1, \eta_{\overline{B}}^2, \ldots, \eta_{\overline{B}}^P)\} + \max\{(\vartheta_{\overline{A}}^1, \ldots, \vartheta_{\overline{A}}^P), (\vartheta_{\overline{B}}^1, \ldots, \vartheta_{\overline{B}}^P)\}}{(\max\{P(A)^1, P(A)^2, P(A)^3, P(A)^4, P(B)^1, P(B)^2, P(B)^3, P(B)^4\}) + \max\{(\eta_{\overline{A}}^1, \ldots, \eta_{\overline{A}}^P), (\eta_{\overline{B}}^1, \eta_{\overline{B}}^2, \ldots, \eta_{\overline{B}}^P)\} + \min\{(\vartheta_{\overline{A}}^1, \ldots, \vartheta_{\overline{A}}^P), (\vartheta_{\overline{B}}^1, \ldots, \vartheta_{\overline{B}}^P)\}} \right] = 1.$$

In a similar way, it is easy to prove $S(\overline{A}, \overline{C}) \leq S(\overline{B}, \overline{C})$. □

It implies that $a_1 = a_2, b_1 = b_2, c_1 = c_2, d_1 = d_2$ and $\eta_{\overline{A}}^1 = \eta_{\overline{B}}^1, \eta_{\overline{A}}^2 = \eta_{\overline{B}}^2, \ldots, \eta_{\overline{A}}^p = \eta_{\overline{B}}^p, \vartheta_{\overline{A}}^1 = \vartheta_{\overline{B}}^1, \vartheta_{\overline{A}}^2 = \vartheta_{\overline{B}}^2, \ldots, \vartheta_{\overline{A}}^p = \vartheta_{\overline{B}}^p$. Thus, $(\min\{P(A)^1, P(A)^2, P(A)^3, P(A)^4, P(B)^1, P(B)^2, P(B)^3, P(B)^4\}) = (\max\{P(A)^1, P(A)^2, P(A)^3, P(A)^4, P(B)^1, P(B)^2, P(B)^3, P(B)^4\})$ and $\min\{(\eta_{\overline{A}}^1, \ldots, \eta_{\overline{A}}^P), (\eta_{\overline{B}}^1, \eta_{\overline{B}}^2, \ldots, \eta_{\overline{B}}^P)\} = \max\{(\eta_{\overline{A}}^1, \ldots, \eta_{\overline{A}}^P), (\eta_{\overline{B}}^1, \eta_{\overline{B}}^2, \ldots, \eta_{\overline{B}}^P)\}$ and $\max\{(\vartheta_{\overline{A}}^1, \ldots, \vartheta_{\overline{A}}^P), (\vartheta_{\overline{B}}^1, \ldots, \vartheta_{\overline{B}}^P)\} = \min\{(\vartheta_{\overline{A}}^1, \ldots, \vartheta_{\overline{A}}^P), (\vartheta_{\overline{B}}^1, \ldots, \vartheta_{\overline{B}}^P)\}$. Therefore, normalized ITFM-numbers $\overline{A}$ and $\overline{B}$ are identical.

Example 3.13 Suppose that $\overline{A} = \langle [0.2, 0.3, 0.4, 0.5]; (0.15, 0.32, 0.36, 0.43, 0.59), (0.44, 0.37, 0.42, 0.53, 0.23) \rangle$, $\overline{B} = \langle [0.1, 0.4, 0.5, 0.6]; (0.2, 0.23, 0.34, 0.41, 0.63), (0.04, 0.17, 0.27, 0.29, 0.38) \rangle \in \Gamma$. Then,

$$\begin{aligned} P(\overline{A})^1 &= \sqrt{(0.2 - 0.3)^2 + (0.15 - 0.44)^2} \\ &\quad + \sqrt{(0.4 - 0.5)^2 + (0.15 - 0.44)^2} \\ &\quad + (0.4 - 0.3) + (0.5 - 0.2) \\ &= 1.01351, \end{aligned}$$

ii.

$$\begin{aligned} S(\overline{A}, \overline{B}) &= \frac{1}{p} \cdot \left[\left(1 - \frac{|a_2 - a_1| + |b_2 - b_1| + |c_2 - c_1| + |d_2 - d_1|}{4} \right) \times \frac{(\min\{P(A)^1, P(A)^2, P(A)^3, P(A)^4, P(B)^1, P(B)^2, P(B)^3, P(B)^4\}) + \min\{(\eta_{\overline{A}}^1, \ldots, \eta_{\overline{A}}^P), (\eta_{\overline{B}}^1, \eta_{\overline{B}}^2, \ldots, \eta_{\overline{B}}^P)\} + \max\{(\vartheta_{\overline{A}}^1, \ldots, \vartheta_{\overline{A}}^P), (\vartheta_{\overline{B}}^1, \ldots, \vartheta_{\overline{B}}^P)\}}{(\max\{P(A)^1, P(A)^2, P(A)^3, P(A)^4, P(B)^1, P(B)^2, P(B)^3, P(B)^4\}) + \max\{(\eta_{\overline{A}}^1, \ldots, \eta_{\overline{A}}^P), (\eta_{\overline{B}}^1, \eta_{\overline{B}}^2, \ldots, \eta_{\overline{B}}^P)\} + \min\{(\vartheta_{\overline{A}}^1, \ldots, \vartheta_{\overline{A}}^P), (\vartheta_{\overline{B}}^1, \ldots, \vartheta_{\overline{B}}^P)\}} \right] \\ &= \frac{1}{p} \cdot \left[\left(1 - \frac{|a_1 - a_2| + |b_1 - b_2| + |c_1 - c_2| + |d_1 - d_2|}{4} \right) \times \frac{(\min\{P(A)^1, P(A)^2, P(A)^3, P(A)^4, P(B)^1, P(B)^2, P(B)^3, P(B)^4\}) + \min\{(\eta_{\overline{A}}^1, \ldots, \eta_{\overline{A}}^P), (\eta_{\overline{B}}^1, \eta_{\overline{B}}^2, \ldots, \eta_{\overline{B}}^P)\} + \max\{(\vartheta_{\overline{A}}^1, \ldots, \vartheta_{\overline{A}}^P), (\vartheta_{\overline{B}}^1, \ldots, \vartheta_{\overline{B}}^P)\}}{(\max\{P(A)^1, P(A)^2, P(A)^3, P(A)^4, P(B)^1, P(B)^2, P(B)^3, P(B)^4\}) + \max\{(\eta_{\overline{A}}^1, \ldots, \eta_{\overline{A}}^P), (\eta_{\overline{B}}^1, \eta_{\overline{B}}^2, \ldots, \eta_{\overline{B}}^P)\} + \min\{(\vartheta_{\overline{A}}^1, \ldots, \vartheta_{\overline{A}}^P), (\vartheta_{\overline{B}}^1, \ldots, \vartheta_{\overline{B}}^P)\}} \right] \\ &= S(\overline{B}, \overline{A}). \end{aligned}$$

iii.

$$\begin{aligned} S(\overline{A}, \overline{B}) &= \frac{1}{p} \cdot \left[\left(1 - \frac{|a_2 - a_1| + |b_2 - b_1| + |c_2 - c_1| + |d_2 - d_1|}{4} \right) \times \frac{(\min\{P(A)^1, P(A)^2, P(A)^3, P(A)^4, P(B)^1, P(B)^2, P(B)^3, P(B)^4\}) + \min\{(\eta_{\overline{A}}^1, \ldots, \eta_{\overline{A}}^P), (\eta_{\overline{B}}^1, \eta_{\overline{B}}^2, \ldots, \eta_{\overline{B}}^P)\} + \max\{(\vartheta_{\overline{A}}^1, \ldots, \vartheta_{\overline{A}}^P), (\vartheta_{\overline{B}}^1, \ldots, \vartheta_{\overline{B}}^P)\}}{(\max\{P(A)^1, P(A)^2, P(A)^3, P(A)^4, P(B)^1, P(B)^2, P(B)^3, P(B)^4\}) + \max\{(\eta_{\overline{A}}^1, \ldots, \eta_{\overline{A}}^P), (\eta_{\overline{B}}^1, \eta_{\overline{B}}^2, \ldots, \eta_{\overline{B}}^P)\} + \min\{(\vartheta_{\overline{A}}^1, \ldots, \vartheta_{\overline{A}}^P), (\vartheta_{\overline{B}}^1, \ldots, \vartheta_{\overline{B}}^P)\}} \right] \\ &= \frac{1}{p} \cdot \left[\left(1 - \frac{|a_2 - a_1| + |b_2 - b_1| + |c_2 - c_1| + |d_2 - d_1|}{4} \right) \times 1 \right] \\ &= 1 - |d|. \end{aligned}$$

iv. If $\overline{A}, \overline{B}, \overline{C} \in \Gamma$, then $\overline{A} \subseteq \overline{B} \subseteq \overline{C} \Leftrightarrow \eta_{\overline{A}}^i \leq \eta_{\overline{B}}^i \leq \eta_{\overline{C}}^i$ and $\vartheta_{\overline{A}}^i \geq \vartheta_{\overline{B}}^i \geq \vartheta_{\overline{C}}^i$. Therefore,

$$\begin{aligned} S(\overline{A}, \overline{C}) &= \frac{1}{p} \cdot \left[\left(1 - \frac{|a_3 - a_1| + |b_3 - b_1| + |c_3 - c_1| + |d_3 - d_1|}{4} \right) \times \frac{(\min\{P(A)^1, P(A)^2, P(A)^3, P(A)^4, P(C)^1, P(C)^2, P(C)^3, P(C)^4\}) + \min\{(\eta_{\overline{A}}^1, \ldots, \eta_{\overline{A}}^P), (\eta_{\overline{C}}^1, \eta_{\overline{C}}^2, \ldots, \eta_{\overline{C}}^P)\} + \max\{(\vartheta_{\overline{A}}^1, \ldots, \vartheta_{\overline{A}}^P), (\vartheta_{\overline{C}}^1, \ldots, \vartheta_{\overline{C}}^P)\}}{(\max\{P(A)^1, P(A)^2, P(A)^3, P(A)^4, P(C)^1, P(C)^2, P(C)^3, P(C)^4\}) + \max\{(\eta_{\overline{A}}^1, \ldots, \eta_{\overline{A}}^P), (\eta_{\overline{C}}^1, \eta_{\overline{C}}^2, \ldots, \eta_{\overline{C}}^P)\} + \min\{(\vartheta_{\overline{A}}^1, \ldots, \vartheta_{\overline{A}}^P), (\vartheta_{\overline{C}}^1, \ldots, \vartheta_{\overline{C}}^P)\}} \right] \\ &\leq \frac{1}{p} \cdot \left[\left(1 - \frac{|a_2 - a_1| + |b_2 - b_1| + |c_2 - c_1| + |d_2 - d_1|}{4} \right) \times \frac{(\min\{P(A)^1, P(A)^2, P(A)^3, P(A)^4, P(B)^1, P(B)^2, P(B)^3, P(B)^4\}) + \min\{(\eta_{\overline{A}}^1, \ldots, \eta_{\overline{A}}^P), (\eta_{\overline{B}}^1, \eta_{\overline{B}}^2, \ldots, \eta_{\overline{B}}^P)\} + \max\{(\vartheta_{\overline{A}}^1, \ldots, \vartheta_{\overline{A}}^P), (\vartheta_{\overline{B}}^1, \ldots, \vartheta_{\overline{B}}^P)\}}{(\max\{P(A)^1, P(A)^2, P(A)^3, P(A)^4, P(B)^1, P(B)^2, P(B)^3, P(B)^4\}) + \max\{(\eta_{\overline{A}}^1, \ldots, \eta_{\overline{A}}^P), (\eta_{\overline{B}}^1, \eta_{\overline{B}}^2, \ldots, \eta_{\overline{B}}^P)\} + \min\{(\vartheta_{\overline{A}}^1, \ldots, \vartheta_{\overline{A}}^P), (\vartheta_{\overline{B}}^1, \ldots, \vartheta_{\overline{B}}^P)\}} \right] \\ &= S(\overline{A}, \overline{B}). \end{aligned}$$

$$
\begin{aligned}
P(\overline{A})^2 &= \sqrt{(0.2-0.3)^2+(0.32-0.37)^2} \\
&\quad +\sqrt{(0.4-0.5)^2+(0.32-0.37)^2} \\
&\quad +(0.4-0.3)+(0.5-0.2) \\
&= 0.62360, \\
P(\overline{A})^3 &= \sqrt{(0.2-0.3)^2+(0.36-0.42)^2} \\
&\quad +\sqrt{(0.4-0.5)^2+(0.36-0.42)^2} \\
&\quad +(0.4-0.3)+(0.5-0.2) \\
&= 0.63323, \\
P(\overline{A})^4 &= \sqrt{(0.2-0.3)^2+(0.43-0.53)^2} \\
&\quad +\sqrt{(0.4-0.5)^2+(0.43-0.53)^2} \\
&\quad +(0.4-0.3)+(0.5-0.2) \\
&= 0.68284, \\
P(\overline{A})^5 &= \sqrt{(0.2-0.3)^2+(0.59-0.23)^2} \\
&\quad +\sqrt{(0.4-0.5)^2+(0.59-0.23)^2} \\
&\quad +(0.4-0.3)+(0.5-0.2) \\
&= 1.14726, \\
P(\overline{B})^1 &= \sqrt{(0.2-0.3)^2+(0.59-0.23)^2} \\
&\quad +\sqrt{(0.4-0.5)^2+(0.59-0.23)^2} \\
&\quad +(0.4-0.3)+(0.5-0.2) \\
&= 0.77735, \\
P(\overline{B})^2 &= \sqrt{(0.1-0.4)^2+(0.23-0.17)^2} \\
&\quad +\sqrt{(0.5-0.6)^2+(0.23-0.17)^2} \\
&\quad +(0.5-0.4)+(0.6-0.1) \\
&= 0.63323, \\
P(\overline{B})^3 &= \sqrt{(0.1-0.4)^2+(0.34-0.27)^2} \\
&\quad +\sqrt{(0.5-0.6)^2+(0.34-0.27)^2} \\
&\quad +(0.5-0.4)+(0.6-0.1) \\
&= 0.64413, \\
P(\overline{B})^4 &= \sqrt{(0.1-0.4)^2+(0.41-0.29)^2} \\
&\quad +\sqrt{(0.5-0.6)^2+(0.41-0.29)^2} \\
&\quad +(0.4-0.4)+(0.6-0.1) \\
&= 0.71240, \\
P(\overline{B})^5 &= \sqrt{(0.1-0.4)^2+(0.63-0.38)^2} \\
&\quad +\sqrt{(0.5-0.6)^2+(0.63-0.38)^2} \\
&\quad +(0.5-0.4)+(0.6-0.1) \\
&= 0.93851,
\end{aligned}
$$

$$
\begin{aligned}
S(\overline{A},\overline{B}) &= \frac{1}{5}\cdot\left(1-\frac{|0.2-0.1|+|0.3-0.4|+|0.4-0.5|+|0.5-0.6|}{4}\right) \\
&\times\left(\begin{array}{l}\min\{1.01351, 0.62360, 0.63323, 0.68284, 1.14726, 0.77735,\\ 0.63323, 0.64413, 0.71240, 0.93851\})+\\ \min\{(0.15, 0.32, 0.36, 0.43, 0.59), (0.2, 0.23, 0.34, 0.41, 0.63)\}+\\ \max\{(0.44, 0.37, 0.42, 0.53, 0.23), (0.04, 0.17, 0.27, 0.29, 0.38)\}\end{array}\right) \\
&\quad\frac{\max\{1.01351, 0.62360, 0.63323, 0.68284, 1.14726, 0.77735,}{0.63323, 0.64413, 0.71240, 0.93851\})+} \\
&\quad\frac{\max\{(0.15, 0.32, 0.36, 0.43, 0.59), (0.2, 0.23, 0.34, 0.41, 0.63)\}+}{\min\{(0.44, 0.37, 0.42, 0.53, 0.23), (0.04, 0.17, 0.27, 0.29, 0.38)\}} \\
&= 0.38375.
\end{aligned}
$$

Definition 3.14 Let $\overline{A} = \langle (a_1, b_1, c_1, d_1); (\eta_A^1, \eta_A^2, \ldots, \eta_A^P), (\vartheta_A^1, \vartheta_A^2, \ldots, \vartheta_A^P)\rangle$, $\overline{B} = \langle (a_2, b_2, c_2, d_2); (\eta_B^1, \eta_B^2 \ldots, \eta_B^P), (\vartheta_B^1, \vartheta_B^2 \ldots, \vartheta_B^P)\rangle \in \Gamma$. Then, to compare $\overline{A}$ and $\overline{B}$, the ITFM-numbers positive ideal solution and ITFM-numbers negative solution are defined as:

$$
\begin{aligned}
r_A^+ &= \langle [a_1^+, b_1^+, c_1^+, d_1^+]; ((\eta_A^1)^+, (\eta_A^2)^+, (\eta_A^3)^+, \ldots, (\eta_A^P)^+), \\
&\quad ((\vartheta_A^1)^+, (\vartheta_A^2)^+, (\vartheta_A^3)^+, \ldots, (\vartheta_A^P)^+)\rangle \\
&= \langle [1,1,1,1]; (1,1,\ldots,1), (0,0,\ldots,0)\rangle, \\
r_A^- &= \langle [a_1^-, b_1^-, c_1^-, d_1^-]; ((\eta_A^1)^-, (\eta_A^2)^-, (\eta_A^3)^-, \ldots, \\
&\quad (\eta_A^P)^-), ((\vartheta_A^1)^-, (\vartheta_A^2)^-, (\vartheta_A^3)^-, \ldots, (\vartheta_A^P)^-)\rangle \\
&= \langle [0,0,0,0]; (0,0,\ldots,0), (1,1,\ldots,1)\rangle,
\end{aligned}
$$

respectively.

Definition 3.15 Let $A = \langle (a_1, b_1, c_1, d_1); (\eta_A^1, \eta_A^2, \ldots, \eta_A^P), (\vartheta_A^1, \vartheta_A^2, \ldots, \vartheta_A^P)\rangle \in \Gamma$ and r^+ and r^- be an ITFM-numbers positive ideal solution and ITFM-numbers negative ideal solution, respectively. Then,

1. If $S(A, r^+) > S(B, r^+)$, then B is smaller than A, denoted by $A \succ B$,
2. If $S(A, r^+) = S(B, r^+) \wedge S(A, r^-) < S(B, r^-)$, then A is smaller than B, denoted by $A \prec B$,
3. If $S(A, r^+) = S(B, r^+) \wedge S(A, r^-) = S(B, r^-)$, then A is similar to B, denoted by $A \simeq B$.

Example 3.16 Suppose that

$$
\begin{aligned}
\overline{A} &= \langle [0.1, 0.4, 0.6, 0.7]; (0.2, 0.3, 0.5, 0.7), \\
&\quad (0.7, 0.5, 0.4, 0.2)\rangle, \\
\overline{B} &= \langle [0.3, 0.5, 0.7, 0.9]; (0.4, 0.2, 0.1, 0.6), \\
&\quad (0.5, 0.4, 0.7, 0.3)\rangle \in \Gamma.
\end{aligned}
$$

Then, $S(\overline{A}, r^+) = 0,30412$ and $S(\overline{B}, r^+) = 0,39050 \Rightarrow S(\overline{A}, r^+) < S(\overline{B}, r^+) \Rightarrow \overline{A} \prec \overline{B}$.

Some aggregation operators on ITFM-numbers

In the section, we use the algebraic sum and algebraic product is given in Definition 2.3.

From now on, we use $I_n = \{1, 2, \ldots, n\}$ and $I_m = \{1, 2, \ldots, m\}$ as an index set for $n \in \mathbb{N}$ and $m \in \mathbb{N}$, respectively.

Definition 4.1 Let $A_j \in \Gamma$, $j \in I_n$ be a collection of ITFM-number. For ITFMWG : $\varphi^n \to \varphi$, if

$$\begin{aligned}&\text{ITFMWG}_w(A_1, A_2, A_3, \ldots, A_n)\\&= (A_1^{w_1} \times A_2^{w_2} \times A_3^{w_3} \times \ldots \times A_n^{w_n}),\end{aligned}$$

then ITFMWG is called ITFM-numbers weighted geometric operator of dimension n, where $w = (w_1, w_2, w_3, \ldots, w_n)^T$ is the weight vector of A_j, $j \in I_n$, with $w_1 \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$. Especially, if $w = (1/n, 1/n, 1/n, \ldots, 1/n)^T$, then the TFMWG operator is reduced to an intuitionistic trapezoidal fuzzy multiset geometric averaging (ITFMWG) operator of dimension n, which is defined follows:

$$\begin{aligned}&\text{ITFMWG}_w(A_1, A_2, A_3, \ldots, A_n)\\&= (A_1 \times A_2 \times A_3 \times \ldots \times A_n)^{1/n}.\end{aligned}$$

Theorem 4.2 *Let $A_j, j \in I_n$ be a collection of ITFM-numbers, then their aggregated value by using the ITFMWG operator is also an ITFM-number and*

$$\begin{aligned}\text{ITFMWG} &= \prod_{j=1}^n A_j^{w_j}\\&= \left\langle \left[\prod_{j=1}^n a_j^{w_j}, \prod_{j=1}^n b_j^{w_j}, \prod_{j=1}^n c_j^{w_j}, \prod_{j=1}^n d_j^{w_j}\right]; \left(\left(\prod_{j=1}^n (\eta_{A_j}^1)^{w_j}, \prod_{j=1}^n (\eta_{A_j}^2)^{w_j}, \ldots, \prod_{j=1}^n (\eta_{A_j}^p)^{w_j}\right)\right), \left(\left(\sum_{j=1}^n (\vartheta_{A_j}^1)^{w_j} - \prod_{j=1}^n (\vartheta_{A_j}^1)^{w_j}, \ldots, \left(\sum_{j=1}^n (\vartheta_{A_j}^p)^{w_j} - \prod_{j=1}^n (\vartheta_{A_j}^p)^{w_j}\right)\right)\right\rangle. \quad (1)\end{aligned}$$

Proof The first result follows quickly from Definition 3.3 and Theorem 3.8. In the following, we prove the second result by using mathematical induction on n. We first prove that Eq. (1) holds for $n = 2$. Since

$$\begin{aligned}(A_1)^{w_1} &= \langle (a_1^{w_1}, b_1^{w_1}, c_1^{w_1}, d_1^{w_1}); ((\eta_{A_1}^1)^{w_1}, (\eta_{A_1}^2)^{w_1}, \ldots, (\eta_{A_1}^p)^{w_1}), ((\vartheta_{A_1}^1)^{w_1}, (\vartheta_{A_1}^2)^{w_1}, \ldots, (\vartheta_{A_1}^p)^{w_1})\rangle\\(A_2)^{w_2} &= \langle (a_2^{w_2}, b_2^{w_2}, c_2^{w_2}, d_2^{w_2}); ((\eta_{A_2}^1)^{w_2}, (\eta_{A_2}^2)^{w_2}, \ldots, (\eta_{A_2}^p)^{w_2}), ((\vartheta_{A_2}^1)^{w_2}, (\vartheta_{A_2}^2)^{w_2}, \ldots, (\vartheta_{A_2}^p)^{w_2})\rangle,\end{aligned}$$

we have

$$\begin{aligned}&(\text{TFMWG})(A_1, A_2) = A_1 \times A_2\\&= \langle [a_1^{w_1} a_2^{w_2}, b_1^{w_1} b_2^{w_2}, c_1^{w_1} c_2^{w_2}, d_1^{w_1} d_2^{w_2}];\\&\quad (\eta_{A_1}^1)^{w_1} \cdot (\eta_{A_2}^1)^{w_2}, (\eta_{A_2}^1)^{w_2} \cdot (\eta_{A_2}^2)^{w_2}, \ldots,\\&\quad (\eta_{A_1}^p)^{w_1} \cdot (\eta_{A_2}^p)^{w_2}, (\vartheta_{A_1}^1)^{w_1} + (\vartheta_{A_2}^1)^{w_2}\\&\quad -(\vartheta_{A_1}^1)^{w_1} \cdot (\vartheta_{A_2}^1)^{w_2}, (\vartheta_{A_2}^1)^{w_2}\\&\quad +(\vartheta_{A_2}^2)^{w_2} - (\vartheta_{A_2}^1)^{w_2} \cdot (\vartheta_{A_2}^2)^{w_2}, \ldots,\\&\quad (\vartheta_{A_1}^p)^{w_1} + (\vartheta_{A_2}^p)^{w_2} - (\vartheta_{A_1}^p)^{w_1} \cdot (\vartheta_{A_2}^p)^{w_2}\rangle\end{aligned}$$

if Eq. (1) holds for $n = k$, that is,

$$\begin{aligned}\text{ITFMWG} &= \prod_{j=1}^k A_j^{w_j}\\&= \left\langle \left[\prod_{j=1}^k a_j^{w_j}, \prod_{j=1}^k b_j^{w_j}, \prod_{j=1}^k c_j^{w_j}, \prod_{j=1}^k d_j^{w_j}\right]; \left(\left(\prod_{j=1}^k (\eta_{A_j}^1)^{w_j}, \prod_{j=1}^k (\eta_{A_j}^2)^{w_j}, \ldots, \prod_{j=1}^k (\eta_{A_j}^p)^{w_j}\right)\right), \left(\left(\sum_{j=1}^k (\vartheta_{A_j}^1)^{w_j} - \prod_{j=1}^k (\vartheta_{A_j}^1)^{w_j}, \ldots, \left(\sum_{j=1}^k (\vartheta_{A_j}^p)^{w_j} - \prod_{j=1}^k (\vartheta_{A_j}^p)^{w_j}\right)\right)\right\rangle,\end{aligned}$$

then both sides of the equation are multiplied by A_{k+1} and by the operational laws in Definition 3 we have

$$\begin{aligned}\text{ITFMWG} &= \prod_{j=1}^{k+1} A_j^{w_j}\\&= \left\langle \left[\prod_{j=1}^{k+1} a_j^{w_j}, \prod_{j=1}^{k+1} b_j^{w_j}, \prod_{j=1}^{k+1} c_j^{w_j}, \prod_{j=1}^{k+1} d_j^{w_j}\right]; \left(\left(\prod_{j=1}^{k+1} (\eta_{A_j}^1)^{w_j}, \prod_{j=1}^{k+1} (\eta_{A_j}^2)^{w_j}, \ldots, \prod_{j=1}^{k+1} (\eta_{A_j}^p)^{w_j}\right)\right), \left(\left(\sum_{j=1}^{k+1} (\vartheta_{A_j}^1)^{w_j} - \prod_{j=1}^{k+1} (\vartheta_{A_j}^1)^{w_j}, \ldots, \left(\sum_{j=1}^{k+1} (\vartheta_{A_j}^p)^{w_j} - \prod_{j=1}^{k+1} (\vartheta_{A_j}^p)^{w_j}\right)\right)\right\rangle,\end{aligned}$$

i.e., that Eq. (1) holds for $n = k+1$. Therefore, Eq. (1) holds for all n, which completes the proof of Theorem 4.2 □

Definition 4.3 Let $A_j, j \in I_n$ be a collection of ITFM-numbers and let ITFMWA : $\varphi^n \to \varphi$. If

$$\begin{aligned}&\text{ITFMWA}_w(A_1, A_2, A_3, \ldots, A_n)\\&= (w_1 A_1 + w_2 A_2 + w_3 A_3 + \cdots + w_n A_n),\end{aligned}$$

then ITFMWA is called intuitionistic trapezoidal fuzzy multiset weighted arithmetic operator of dimension n, where

$w = (w_1, w_2, w_3, \ldots, w_n)^T$ is the weight vector of A_j, $j \in I_n$, with $w_1 \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. Especially, if $w = (1/n, 1/n, 1/n, \ldots, 1/n)^T$, then the ITFMWA operator is reduced to an intuitionistic trapezoidal fuzzy multiset arithmetic averaging (ITFMWA) operator of dimension n, which is defined as follows:

$$\begin{aligned}&\mathrm{ITFMWG}_w(A_1, A_2, A_3, \ldots, A_n)\\ &= \frac{1}{n}(A_1 + A_2 + A_3 + \cdots + A_n).\end{aligned}$$

An approach to MADM problems with ITFM-numbers

In this section, we define a multi-criteria making method, called ITFM-numbers multi-criteria decision-making method, by using the ITFMWG and (ITFMWA) operators.

Definition 5.1 Let $X = (x_1, x_2, \ldots, x_m)$ be a set of alternatives, $U = (u_1, u_2, \ldots, u_n)$ be the set of attributes and $[A_{ij}] = \langle [a_{ij}, b_{ij}, c_{ij}, d_{ij}]; (\eta_{ij}^1, \eta_{ij}^2, \eta_{ij}^3, \ldots, \eta_{ij}^p), (\vartheta_{ij}^1, \vartheta_{ij}^2, \vartheta_{ij}^3, \ldots, \vartheta_{ij}^p)\rangle$ be an ITFM-number for all $i \in I_m$ and $j \in I_n$. For a normalized ITFM-numbers decision-making matrix $R = (r_{ij})_{m\times n} = \langle [a_{ij}, b_{ij}, c_{ij}, d_{ij}]; (\eta_{ij}^1, \eta_{ij}^2, \eta_{ij}^3, \ldots, \eta_{ij}^P), (\vartheta_{ij}^1, \vartheta_{ij}^2, \vartheta_{ij}^3, \ldots, \vartheta_{ij}^P)\rangle_{m\times n}$ where $0 \le a_{ij} \le b_{ij} \le c_{ij} \le d_{ij} \le 1$, $0 \le \eta_{ij}^1, \eta_{ij}^2, \eta_{ij}^3, \ldots, \eta_{ij}^P, \vartheta_{ij}^1, \vartheta_{ij}^2, \vartheta_{ij}^3, \ldots, \vartheta_{ij}^P \le 1$. Then,

$$[A_{ij}]_{m\times n} = \begin{array}{c} \\ u_1 \\ u_2 \\ \vdots \\ u_m \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & \cdots & x_n \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array}$$

is called an ITFM-number multi-criteria decision matrix of the decision maker.

Now, we can give algorithm of the ITFM-numbers multi-criteria decision-making method as follows:

5.2 Algorithm:

Step 1 Construct the ITFM-numbers multi-criteria decision matrix $A = (a_{ij})_{m\times n}$, for decision;

Step 2 Compute overall values

$$r_i = \mathrm{ITFMWG}_w(a_{i1}, a_{i2}, a_{i3}, a_{i4});\ (i = 1, 2, 3, 4, 5).$$

Note that if r_i for all $i \in I_m$ is not normalized ITFM-numbers, then we compute the normalized ITFM-numbers according to Definition 3.9.

Step 3 Calculate the distances between collective overall values $r_i = \langle [a_i, b_i, c_i, d_i]; (\eta_i^1, \eta_i^2, \eta_i^3, \ldots, \eta_i^p), (\vartheta_i^1, \vartheta_i^2, \vartheta_i^3, \ldots, \vartheta_i^p)\rangle$ and positive ideal solution r_i^+ (or negative ideal solution r_i^-)

$$S(\overline{A}, \overline{B}) = \frac{1}{p} \cdot \left[\left(1 - \frac{|a_2-a_1|+|b_2-b_1|+|c_2-c_1|+|d_2-d_1|}{4}\right) \times \frac{(\min\{P(A)^1, P(A)^2, P(A)^3, P(A)^4, P(B)^1, P(B)^2, P(B)^3, P(B)^4\}) + \min\{(\eta_A^1, \ldots, \eta_A^P), (\eta_B^1, \ldots, \eta_B^P)\} + \max\{(\vartheta_A^1, \ldots, \vartheta_A^P), (\vartheta_B^1, \ldots, \vartheta_B^P)\}}{(\max\{P(A)^1, P(A)^2, P(A)^3, P(A)^4, P(B)^1, P(B)^2, P(B)^3, P(B)^4\}) + \max\{(\eta_A^1, \ldots, \eta_A^P), (\eta_B^1, \ldots, \eta_B^P)\} + \min\{(\vartheta_A^1, \ldots, \vartheta_A^P), (\vartheta_B^1, \ldots, \vartheta_B^P)\}}\right] \cdot$$

Step 4 Rank all the alternatives $A_i (i = 1, 2, 3, \ldots, m)$ and select the best one(s) in accordance with $S(r_i, r^+)$. The bigger the distance $S(r_i, r_i^+)$, the better are the alternatives A_i, $i \in I_m$.

Step 5 End.

5.3 Algorithm:

Step 1 Construct the ITFM-numbers multi-criteria decision matrix $A = (a_{ij})_{m\times n}$; for decision;

Step 2 Compute overall values

$$r_i = \mathrm{ITFMWA}_w(a_{i1}, a_{i2}, a_{i3}, a_{i4});\ (i = 1, 2, 3, 4, 5).$$

Note that if r_i for all $i \in I_m$ is not normalized ITFM-numbers, then we compute the normalized ITFM-numbers according to Definition 3.9.

Step 3 Calculate the distances between collective overall values $r_i = \langle [a_i, b_i, c_i, d_i]; (\eta_i^1, \eta_i^2, \eta_i^3, \ldots, \eta_i^p), (\vartheta_i^1, \vartheta_i^2, \vartheta_i^3, \ldots, \vartheta_i^p)\rangle$ and positive ideal solution r_i^+ (or negative ideal solution r_i^-)

$$S(\overline{A}, \overline{B}) = \frac{1}{p} \cdot \left[\left(1 - \frac{|a_2-a_1|+|b_2-b_1|+|c_2-c_1|+|d_2-d_1|}{4}\right) \times \frac{(\min\{P(A)^1, P(A)^2, P(A)^3, P(A)^4, P(B)^1, P(B)^2, P(B)^3, P(B)^4\}) + \min\{(\eta_A^1, \ldots, \eta_A^P), (\eta_B^1, \ldots, \eta_B^P)\} + \max\{(\vartheta_A^1, \ldots, \vartheta_A^P), (\vartheta_B^1, \ldots, \vartheta_B^P)\}}{(\max\{P(A)^1, P(A)^2, P(A)^3, P(A)^4, P(B)^1, P(B)^2, P(B)^3, P(B)^4\}) + \max\{(\eta_A^1, \ldots, \eta_A^P), (\eta_B^1, \ldots, \eta_B^P)\} + \min\{(\vartheta_A^1, \ldots, \vartheta_A^P), (\vartheta_B^1, \ldots, \vartheta_B^P)\}}\right] \cdot$$

Step 4 Rank all the alternatives $A_i (i = 1, 2, 3, \ldots, m)$ and select the best one(s) in accordance with $S(r_i, r^+)$. The bigger the distance $S(r_i, r_i^+)$, the better is the alternatives $A_i, i \in I_m$.

Step 5 End.

Application

The anonymous review of the doctoral dissertation in Turkey universities.

In many Turkey universities, doctoral dissertation will be reviewed by three experts anonymously and they have same importance in this review process. They will review dissertation according to five criteria, including topic selection and literature review, innovation, theory basis and special knowledge, capacity of scientific research and theses writing. Different weights are given to different criteria and the standards for those principles are as follows. After thorough investigation, four universities (alternatives) are taken into consideration, i.e., $\{x_1, x_2, x_3, x_4\}$. There are many factors that affect the review process and five factors are considered based on the experience of the department personnel, including u_1; topic selection and literature review (such as belonging to the leading edge of the subject or the hot research point has important theoretic significance and applied value; familiar with the research status and process for subject.), u_2; innovation (such as have theoretical breakthrough; have positive influence and impact on the development of social economy and culture; creativity points, u_3; theory basis and special knowledge (such as solid and broad theoretical foundation, also have specialized knowledge for the subject and related area) and u_4; capacity of scientific research (such as independently scientific research ability; informative citing information; subject to be explored in depth) u_5; theses writing(such as clear concept and logistics, smooth sentences, format specification, good school ethos) $\cdot$ (whose weighted vector $\omega = (0.1, 0.3, 0.2, 0.3, 0.1)$) Our solution is to examine the university at different time intervals (four times a year: autumn, spring, winter, summer), which in turn gives rise to different membership functions for each university.

Step 1 Construct the decision-making matrix $A = (a_{ij})_{m \times n}$ for decision as:

$$
\begin{array}{c}
\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{array}
\begin{pmatrix}
x_1 & x_2 \\
\langle [0.2, 0.3, 0.5, 0.7]; (0.6, 0.3, 0.5, 0.7), (0.1, 0.5, 0.4, 0.1) \rangle & \langle [0.3, 0.5, 0.7, 0.8]; (0.4, 0.3, 0.6, 0.2), (0.01, 0.6, 0.3, 0.7) \rangle \\
\langle [0.1, 0.4, 0.6, 0.7]; (0.2, 0.5, 0.1, 0.8), (0.7, 0.3, 0.8, 0.1) \rangle & \langle [0.1, 0.4, 0.6, 0.9]; (0.1, 0.4, 0.3, 0.6), (0.1, 0.3, 0.5, 0.2) \rangle \\
\langle [0.2, 0.4, 0.5, 0.6]; (0.1, 0.3, 0.5, 0.2), (0.2, 0.6, 0.1, 0.6) \rangle & \langle [0.2, 0.3, 0.6, 0.7]; (0.3, 0.2, 0.5, 0.4), (0.5, 0.6, 0.3, 0.5) \rangle \\
\langle [0.1, 0.3, 0.4, 0.6]; (0.3, 0.2, 0.4, 0.6), (0.6, 0.3, 0.5, 0.2) \rangle & \langle [0.3, 0.4, 0.6, 0.8]; (0.2, 0.1, 0.3, 0.6), (0.6, 0.5, 0.4, 0.3) \rangle \\
\langle [0.2, 0.3, 0.5, 0.8]; (0.4, 0.3, 0.2, 0.5), (0.5, 0.6, 0.7, 0.3) \rangle & \langle [0.2, 0.5, 0.7, 0.9]; (0.3, 0.2, 0.4, 0.5), (0.5, 0.7, 0.6, 0.4) \rangle
\end{pmatrix}
\end{array}
$$

$$
\begin{pmatrix}
x_3 & x_4 \\
\langle [0.2, 0.3, 0.5, 0.7]; (0.7, 0.5, 0.2, 0.6), (0.02, 0.3, 0.5, 0.2) \rangle & \langle [0.1, 0.2, 0.4, 0.5]; (0.2, 0.3, 0.5, 0.4), (0.03, 0.1, 0.2, 0.3) \rangle \\
\langle [0.4, 0.5, 0.7, 0.8]; (0.5, 0.6, 0.2, 0.3), (0.2, 0.3, 0.1, 0.6) \rangle & \langle [0.3, 0.4, 0.5, 0.6]; (0.6, 0.8, 0.4, 0.5), (0.1, 0.3, 0.2, 0.4) \rangle \\
\langle [0.3, 0.6, 0.8, 0.9]; (0.6, 0.5, 0.1, 0.5), (0.3, 0.4, 0.5, 0.1) \rangle & \langle [0.1, 0.3, 0.4, 0.5]; (0.7, 0.4, 0.6, 0.5), (0.2, 0.5, 0.3, 0.4) \rangle \\
\langle [0.1, 0.2, 0.3, 0.4]; (0.4, 0.3, 0.7, 0.5), (0.5, 0.6, 0.1, 0.4) \rangle & \langle [0.2, 0.4, 0.5, 0.7]; (0.6, 0.5, 0.4, 0.8), (0.2, 0.3, 0.4, 0.1) \rangle \\
\langle [0.2, 0.4, 0.6, 0.8]; (0.8, 0.6, 0.1, 0.4), (0.1, 0.3, 0.5, 0.3) \rangle & \langle [0.3, 0.5, 0.6, 0.8]; (0.8, 0.7, 0.6, 0.5), (0.1, 0.2, 0.3, 0.4) \rangle
\end{pmatrix}
$$

Step 2 Applying the ITFMWG operator to derive the collective overall preference intuitionistic trapezoidal fuzzy multiset r_i :

$$
\begin{aligned}
r_1 = & \langle [0.16817, 0.33178, 0.48255, 0.65678]; \\
& (0.27735, 0.28378, 0.40257, 0.45370), \\
& (0.96232, 0.92592, 0.97938, 0.69384) \rangle, \\
r_2 = & \langle [0.24145, 0.40568, 0.64309, 0.78228]; \\
& (0.28958, 0.20773, 0.45870, 0.35515), \\
& (0.93228, 0.94370, 0.92592, 0.94945) \rangle, \\
r_3 = & \langle [0.20356, 0.35958, 0.53834, 0.68854]; \\
& (0.59552, 0.46395, 0.19472, 0.50804), \\
& (0.95285, 0.88175, 0.88080, 0.92949) \rangle, \\
r_4 = & \langle [0.13544, 0.29438, 0.44045, 0.56567]; \\
& (0.44028, 0.41406, 0.50864, 0.50801), \\
& (0.96397, 0.80973, 0.81451, 0.90565) \rangle.
\end{aligned}
$$

Step 3 Calculate the distances between collective overall values r_i and intuitionistic trapezoidal fuzzy positive ideal solution r^+.

$S(r_1, r^+) = 0.29410,$
$S(r_2, r^+) = 0.41407,$
$S(r_3, r^+) = 0.34198,$
$S(r_4, r^+) = 0.27944.$

Step 4 Rank all the alternatives $A_i (i = 1, 2, 3, 4)$ in accordance with the ascending order of $S(r_i, r^+)$: $A_4 < A_1 < A_3 < A_2$, thus the most desirable alternative is A_2.

Step 5 End

Step 1 Construct the decision-making matrix $A = (a_{ij})_{m\times n}$, for decision as:

$$\begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{array} \begin{array}{c} x_1 \\ \left(\begin{array}{l} \langle[0.2, 0.3, 0.5, 0.7]; (0.6, 0.3, 0.5, 0.7), (0.1, 0.5, 0.4, 0.1)\rangle \\ \langle[0.1, 0.4, 0.6, 0.7]; (0.2, 0.5, 0.1, 0.8), (0.7, 0.3, 0.8, 0.1)\rangle \\ \langle[0.2, 0.4, 0.5, 0.6]; (0.1, 0.3, 0.5, 0.2), (0.2, 0.6, 0.1, 0.6)\rangle \\ \langle[0.1, 0.3, 0.4, 0.6]; (0.3, 0.2, 0.4, 0.6), (0.6, 0.3, 0.5, 0.2)\rangle \\ \langle[0.2, 0.3, 0.5, 0.8]; (0.4, 0.3, 0.2, 0.5), (0.5, 0.6, 0.7, 0.3)\rangle \end{array}\right. \end{array} \begin{array}{c} x_2 \\ \left.\begin{array}{l} \langle[0.3, 0.5, 0.7, 0.8]; (0.4, 0.3, 0.6, 0.2), (0.01, 0.6, 0.3, 0.7)\rangle \\ \langle[0.1, 0.4, 0.6, 0.9]; (0.1, 0.4, 0.3, 0.6), (0.1, 0.3, 0.5, 0.2)\rangle \\ \langle[0.2, 0.3, 0.6, 0.7]; (0.3, 0.2, 0.5, 0.4), (0.5, 0.6, 0.3, 0.5)\rangle \\ \langle[0.3, 0.4, 0.6, 0.8]; (0.2, 0.1, 0.3, 0.6), (0.6, 0.5, 0.4, 0.3)\rangle \\ \langle[0.2, 0.5, 0.7, 0.9]; (0.3, 0.2, 0.4, 0.5), (0.5, 0.7, 0.6, 0.4)\rangle \end{array}\right) \end{array}$$

$$\begin{array}{c} x_3 \\ \left(\begin{array}{l} \langle[0.2, 0.3, 0.5, 0.7]; (0.7, 0.5, 0.2, 0.6), (0.02, 0.3, 0.5, 0.2)\rangle \\ \langle[0.4, 0.5, 0.7, 0.8]; (0.5, 0.6, 0.2, 0.3), (0.2, 0.3, 0.1, 0.6)\rangle \\ \langle[0.3, 0.6, 0.8, 0.9]; (0.6, 0.5, 0.1, 0.5), (0.3, 0.4, 0.5, 0.1)\rangle \\ \langle[0.1, 0.2, 0.3, 0.4]; (0.4, 0.3, 0.7, 0.5), (0.5, 0.6, 0.1, 0.4)\rangle \\ \langle[0.2, 0.4, 0.6, 0.8]; (0.8, 0.6, 0.1, 0.4), (0.1, 0.3, 0.5, 0.3)\rangle \end{array}\right. \end{array} \begin{array}{c} x_4 \\ \left.\begin{array}{l} \langle[0.1, 0.2, 0.4, 0.5]; (0.2, 0.3, 0.5, 0.4), (0.03, 0.1, 0.2, 0.3)\rangle \\ \langle[0.3, 0.4, 0.5, 0.6]; (0.6, 0.8, 0.4, 0.5), (0.1, 0.3, 0.2, 0.4)\rangle \\ \langle[0.1, 0.3, 0.4, 0.5]; (0.7, 0.4, 0.6, 0.5), (0.2, 0.5, 0.3, 0.4)\rangle \\ \langle[0.2, 0.4, 0.5, 0.7]; (0.6, 0.5, 0.4, 0.8), (0.2, 0.3, 0.4, 0.1)\rangle \\ \langle[0.3, 0.5, 0.6, 0.8]; (0.8, 0.7, 0.6, 0.5), (0.1, 0.2, 0.3, 0.4)\rangle \end{array}\right) \end{array}$$

Step 2 Applying the ITFMWA operator to derive the collective overall preference intuitionistic trapezoidal fuzzy multiset r_i :

$r_1 = \langle[0.71198, 0.80871, 0.86744, 0.92073];$
$(0.79171, 0.78605, 0.83439, 0.86489),$
$(0.77639, 0.86405, 0.81024, 0.76144)\rangle,$
$r_2 = \langle[0.76035, 0.84044, 0.91663, 0.95280];$
$(0.78501, 0.74213, 0.85771, 0.82793),$
$(0.74778, 0.89425, 0.82033, 0.86320)\rangle,$
$r_3 = \langle[0.74037, 0.82344, 0.88767, 0.93018];$
$(0.90339, 0.86156, 0.74373, 0.87462),$
$(0.70774, 0.82936, 0.81041, 0.75288)\rangle,$
$r_4 = \langle[0.70015, 0.79739, 0.85439, 0.89614];$
$(0.086468, 0.84804, 0.87609, 0.87864),$
$(0.66409, 0.76757, 0.78159, 0.78288)\rangle.$

Step 3 Calculate the distances between collective overall values r_i and intuitionistic trapezoidal fuzzy positive ideal solution r^+.

$S(r_1, r^+) = 0.535,$
$S(r_2, r^+) = 0.55736,$
$S(r_3, r^+) = 0.53694,$
$S(r_4, r^+) = 0.54450.$

Step 4 Rank all the alternatives $A_i (i = 1, 2, 3, 4)$ in accordance with the ascending order of $S(r_i, r^+)$: $A_1 < A_3 < A_4 < A_2$, thus the most desirable alternative is A_2.

Step 5 End

Comparison analysis and discussion

To verify the feasibility and effectiveness of the proposed decision-making approach, a comparison analysis with TFM-numbers multi-criteria decision-making method, used by Ulucay et al. [49], is given, based on the same illustrative example. Clearly, the ranking order results are consistent with the result obtained in [49] (Table 1).

Conclusion

In this study, we have defined ITFM-numbers and operational laws, which are mainly based on t norm and t conorm. The ITFM-numbers are a generalization of trapezoidal fuzzy numbers, and intuitionistic trapezoidal fuzzy numbers which are commonly used in real decision problems with the lack of information or imprecision of the available information in real situations is more serious. So the research of ranking ITFM-numbers is very necessary and the ranking problem is more difficult than ranking ITFM-numbers due

Table 1 The ranking results of different methods

Methods	The final ranking	The best alternative(s)	The worst alternative(s)
Method 1	$A_4 < A_1 < A_3 < A_2$	A_2	A_4
Method 2	$A_1 < A_3 < A_4 < A_2$	A_2	A_1
Ulucay et al. [49]	$A_4 < A_3 < A_1 < A_2$	A_2	A_4
The proposed method	$A_4 < A_1 < A_3 < A_2$	A_2	A_4

to additional multi-membership functions and multi-non-membership functions. So, some aggregation operators on ITFM-numbers by using algebraic sum and algebraic product is given in Definition 2.3. Based on the aggregation operators, we developed a multi-criteria making method, called ITFM-numbers multi-criteria decision-making method, by using the ITFMWG operator. Finally, we have proposed a practical example to discuss the applicability of ITFM-numbers multi-criteria decision-making method. In future work, we shall develop some new method and apply our theory to other fields, such as medical diagnosis, game theory, investment decision making, military system efficiency evaluation, and so on.

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