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Comparison Between Dependent and Independent Ranked Set Sampling Designs for Parametric Estimation with Applications

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Abstract
This paper is concerned with the estimation problem using maximum likelihood method of estimation for the unknown parameters of exponetiated gumbel distribution based on neoteric ranked set sampling (NRSS) as a new modification of the usual ranked set sampling (RSS) technique. Numerical study is conducted to compare NRSS as a dependent ranked set sampling technique, with RSS, and median ranked set sampling as independent sampling techniques, and then the performance of RSS and its modifications will be compared with simple random sampling based on their mean square errors and efficiencies.

Keywords Neoteric ranked set sampling · Median ranked set sampling · Ranked set sampling · Maximum likelihood estimation

1 Introduction

In 1952, McIntyre introduced a new sampling technique which he called RSS. Takahasi and Wakimoto [18] derived a very essential statistical basis for the theory of RSS introduced by McIntyre [9]. Dell and Clutter [4] pointed out the role of ranking errors in RSS which cause loss in efficiency for estimating the population mean. Stokes and Sager [17] used RSS to estimate distribution functions, they showed that the empirical distribution function of a RSS is an unbiased estimator of the distribution function and has a smaller variance than that from a simple random sampling (SRS). For some real applications of RSS, See Patil [12], Yu and Lam [20], Al-Saleh and Al-Shrafat [3], Al-Saleh and Al-Hadrami [1], Al-Saleh and Al-Omari [2], Husby et al. [6], Wang et al. [19], Samawi [15] and references therein.

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Muttlak [10] proposed median ranked set sampling (MRSS) as a modification for the RSS technique, he demonstrated that MRSS has the potential to reduce errors in ranking and gives efficient estimate than RSS estimate in case of symmetric distribution. Sinha and Purkayastha [16] used the median ranked set sampling to modify the RSS estimators of the population mean if the underling distribution known to be normal or exponential.

RSS method as suggested by McIntyre [9] may be modified to come up with new sampling methods that can be made more efficient than the usual RSS method. A recently developed extension of RSS, Zamanzade and Al-Omari [21] was proposed a new neoteric ranked set sampling (NRSS).They proved that the NRSS estimators perform much better than their counterparts using RSS and SRS techniques whatever the ranking perfect or imperfect. Under this setup, Sabry and Shaaban [13] derived the likelihood function for NRSS and double neoteric ranked set sampling (DNRSS), they compared maximum likelihood estimators (MLEs) based on RSS, NRSS, and DNRSS schemes with the MLEs based on the SRS technique for inverse Weibull distribution.

Exponetiated gumbel (EG) distribution was introduced by Nadarajah [11] in the same way Gupta et al. [5] generalized the standard exponential distribution. Applications of gumbel distribution in various areas including accelerated life testing, earthquakes, floods, horse racing, rainfall, sea currents, wind speeds and track race records can be seen in Kotz and Nadarajah [8]. The probability density function (PDF) and, the cumulative distribution function (CDF) of the EG distribution are, respectively, given by

$$F(x;\lambda,\alpha) = [G(x,\lambda)]^{\alpha} = \left[\exp\left(-\exp\left(-\lambda x\right)\right)\right]^{\alpha}, \tag{1}$$

and

$$f(x;\lambda,\alpha) = \lambda\alpha e^{-\lambda x}\left[exp(-exp(-\lambda x))\right]^{\alpha}, \tag{2}$$

where $\alpha > 0, -\infty < x < \infty, \lambda > 0$, α and λ are the shape and scale parameters, respectively.

In this paper, an attempt has been made to compare the performance of NRSS, MRSS, and RSS schemes using maximum likelihood (ML) method of estimation with the usual SRS technique for EG distribution parameters. The remaining part of this paper is organized as follows: Sect. 2 introduced some sampling techniques. In Sect. 3, MLHs are derived for the shape and scale parameters of EG distribution using different sampling schemes. Section 4 is devoted to extensive numerical study to compare the performance of NRSS, MRSS with unknown estimators based on RSS and SRS techniques. Conclusions are derived in Sect. 5.

2 Some Ranked Set Sampling Techniques

In this section, various sampling procedures for selection of units in the sample will be considered; brief descriptions of RSS, MRSS and NRSS schemes will be introduced.

2.1 Ranked Set Sampling

In (1952) McIntyre introduced RSS technique as a useful procedure, when quantification of all sampling units is costly but a small set of units can be easily ranked, according to the characteristics under investigation, without actual quantification. This ordering criterion may be based, for example, on values of a concomitant variable or personal judgment. Several studies have proved the higher efficiency of RSS, relative to SRS, for the estimation of a large number of population parameters. The RSS scheme can be described as follows:

Step 1 Randomly select m^2 sample units from the population.
Step 2 Allocate the m^2 selected units as randomly as possible into m sets, each of size m.
Step 3 Choose a sample for actual quantification by including the smallest ranked unit in the first set, the second smallest ranked unit in the second set, the process is continues in this way until the largest ranked unit is selected from the last set.
Step 4 Repeat steps 1 through 4 for r cycles to obtain a sample of size mr (Fig. 1).

Let $\{X_{(ii)s},\ i = 1, 2, \dots, m;\ j = 1, 2, \dots, r\}$ be a ranked set sample where m is the set size and r is the number of cycles. Then the probability density function (PDF) of $X_{(ii)j}$ is given by

$$f_i\left(x_{(ii)j}\right) = C_1\left[F\left(x_{(ii)j}\right)\right]^{i-1}\left[1 - F\left(x_{(ii)j}\right)\right]^{m-i} f\left(x_{(ii)j}\right), \quad -\infty < x_{(ii)j} < \infty, \tag{3}$$

where $C_1 = \frac{m!}{(i-1)!(m-1)!}$ using Eq. (3) the likelihood function corresponding to RSS scheme is given by:

$$L_{RSS}(\theta|x) = \prod_{j=1}^{r}\prod_{i=1}^{m} C_1 f\left(x_{(ii)j};\theta\right)\left[F\left(x_{(ii)j};\theta\right)\right]^{i-1}\left[1 - F\left(x_{(ii)j};\theta\right)\right]^{m-i} \tag{4}$$

2.2 Median Ranked Set Sampling

Muttlak [10] proposed median ranked set sampling (MRSS) as a modification of the RSS to reduce loss of efficiency in RSS due to errors in ranking and an

cycle 1			cycle 2			cycle 3		
$\boxed{X_{(11)1}}$	$X_{(12)1}$	$X_{(13)1}$	$\boxed{X_{(11)2}}$	$X_{(12)2}$	$X_{(13)2}$	$\boxed{X_{(11)3}}$	$X_{(12)3}$	$X_{(13)3}$
$X_{(21)1}$	$\boxed{X_{(22)1}}$	$X_{(23)1}$	$X_{(21)2}$	$\boxed{X_{(22)2}}$	$X_{(23)2}$	$X_{(21)3}$	$\boxed{X_{(22)3}}$	$X_{(23)3}$
$X_{(31)1}$	$X_{(32)1}$	$\boxed{X_{(33)1}}$	$X_{(31)2}$	$X_{(32)2}$	$\boxed{X_{(33)2}}$	$X_{(31)3}$	$X_{(32)3}$	$\boxed{X_{(33)3}}$

Fig. 1 RSS design [14]

improvement upon the efficiency of the estimator of the population mean. MRSS procedure can be summarized as follows:

Step 1 Select m^2 random samples of size m units from the target population.
Step 2 Rank the units within each sample with respect to a variable of interest.
Step 3 If the sample size m is odd, from each sample select for measurement the $\left(\frac{m+1}{2}\right)$th smallest ranked unit, i.e., the median of the sample (see Fig. 2).

From Fig. 2, let the measured MRSS units in case of odd set size is $\left\{x_{(1\,g)j}, x_{(2\,g)j}, \ldots, x_{(g\,g)j}, \ldots, x_{(m\,g)j}\right\}$ where $g = (m+1)/2$. The PDF of (g)th order statistics can be obtained as follows:

$$f_g\left(x_{(i\,g)j}\right) = \frac{m!}{\left[(g-1)!\right]^2}\left[F\left(x_{(i\,g)j}\right)\right]^{g-1}\left[1 - F\left(x_{(i\,g)j}\right)\right]^{g-1} f\left(x_{(i\,g)j}\right), \quad \infty < x_{(i\,g)j} < \infty, \tag{5}$$

Then, using Eq. (5) the likelihood function corresponding to MRSS scheme for odd set sizes and with r cycles is given as follows:

$$L_{MRSS}(\theta|x) = \prod_{j=1}^{r}\prod_{i=1}^{m}\frac{m!}{[(g-1)!]^2} f\left(x_{(ig)j};\theta\right)\left[F\left(x_{(ig)j};\theta\right)\right]^{g-1}\left[1 - F\left(x_{(ig)j};\theta\right)\right]^{g-1} \tag{6}$$

Step 4 If the sample size m is even, select for the measurement from the first $\frac{m}{2}$ samples the $\left(\frac{m}{2}\right)$th smallest ranked unit and from the second $\frac{m}{2}$ samples the $\left(\frac{m}{2}+1\right)$th smallest ranked unit (see Fig. 3).

Step 5 The cycle can be repeated r times if needed to get a sample of size $n = mr$ units from MRSS data.

$$\begin{bmatrix} x_{(1\,1)j} & \cdots & x_{(1\,g)j} & \cdots & x_{(1\,m)j} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{(g\,1)j} & \cdots & x_{(gg)j} & \cdots & x_{(g\,m)j} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{(m\,1)j} & \cdots & x_{(m\,g)j} & \cdots & x_{(m\,m)j} \end{bmatrix} \xrightarrow{for\ m\ odd\ and\ r\ times} \begin{bmatrix} x_{(1\,1)j} & \cdots & \boxed{x_{(1\,g)j}} & \cdots & x_{(1\,m)j} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{(g\,1)j} & \cdots & \boxed{x_{(gg)j}} & \cdots & x_{(g\,m)j} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{(m\,1)j} & \cdots & \boxed{x_{(m\,g)j}} & \cdots & x_{(m\,m)j} \end{bmatrix}$$

Fig. 2 MRSS design in case of odd sample size [14]

From Fig. 3, let the measured MRSS units in case of even set size will be as follows: $x_{(1\,u)j}, x_{(2\,u)j}, \ldots, x_{(u\,u)j}, x_{(u+1\ u)j}, \ldots, x_{(n\,u)j}$, then the PDFs of (u)th and $(u+1)$th order statistics when m. is even are given as follows:

$$f_u\left(x_{(i\,u)j}\right) = C_2\left[F\left(x_{(i\,u)j}\right)\right]^{u-1}\left[1 - F\left(x_{(i\,u)j}\right)\right]^{u} f\left(x_{(i\,u)j}\right), \quad -\infty < x_{(i\,u)j} < \infty, \tag{7}$$

and

$$f_{u+1}\left(x_{(i\,u+1)j}\right) = C_2\left[F\left(x_{(i\,u+1)j}\right)\right]^{u}\left[1 - F\left(x_{(i\,u+1)j}\right)\right]^{u-1} f\left(x_{(i\,u+1)j}\right), \quad -\infty < x_{(i\,u+1)j} < \infty, \tag{8}$$

where $u = m/2$., and $C_2 = \frac{m!}{(u-1)!(u)!}$, using Eqs. (7) and (8), the likelihood function corresponding to MRSS scheme for even set sizes and with r. cycles given as follows:

$$L_{MRSS}(\theta|x) = \left[\prod_{j=1}^{r}\prod_{i=1}^{u} C_2 f\left(x_{(iu)j};\boldsymbol{\theta}\right)\left[F\left(x_{(iu)j};\boldsymbol{\theta}\right)\right]^{u-1}\left[1 - F\left(x_{(iu)j};\boldsymbol{\theta}\right)\right]^{u}\right] \cdot \left[\prod_{j=1}^{r}\prod_{i=u+1}^{n} C_2 f\left(x_{(iu+1)j};\boldsymbol{\theta}\right)\left[F\left(x_{(iu+1)j};\boldsymbol{\theta}\right)\right]^{u} \cdot \left[1 - F\left(x_{(iu+1)j};\boldsymbol{\theta}\right)\right]^{u-1}\right], \tag{9}$$

2.3 Neoteric Ranked Set Sampling

Zamanzade and Al-Omari [21] have defined a NRSS. NRSS technique differs from the original RSS scheme by the composition of a single set of m^2 units, instead of m sets of size m. this strategy has been shown to be effective, producing more efficient estimators for the population mean and variance.

$$\begin{bmatrix} x_{(1\,1)j} & \cdots & x_{(1\ \frac{m}{2})j} & x_{(1\ \frac{m}{2}+1)j} & \cdots & x_{(1m\,)j} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{(\frac{m}{2}\ 1)j} & \cdots & x_{(\frac{m}{2}\ \frac{m}{2})j} & x_{(\frac{m}{2}\ \frac{m}{2}+1)j} & \cdots & x_{(\frac{m}{2}\ m)j} \\ x_{(\frac{m}{2}+1\ \ 1)j} & \cdots & x_{(\frac{m}{2}+1\ \frac{m}{2})j} & x_{(\frac{m}{2}+1\ \ \frac{m}{2}+1)j} & \cdots & x_{(\frac{m}{2}+1\ \ m)j} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{(m\ 1)j} & \cdots & x_{(m\ \frac{m}{2})j} & x_{(m\ \frac{m}{2}+1)j} & \cdots & x_{(\,mm)j} \end{bmatrix}$$

$$\xrightarrow{for\ m\ even\ and\ r\ times} \begin{bmatrix} x_{(1\,1)j} & \cdots & \boxed{x_{(1\,\frac{m}{2})j}} & x_{(1\ \ \frac{m}{2}+1)j} & \cdots & x_{(1\,m)j} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{(\frac{m}{2}\ 1)j} & \cdots & \boxed{x_{(\frac{m}{2}\ \frac{m}{2})j}} & x_{(\frac{m}{2}\ \ \frac{m}{2}+1)j} & \cdots & x_{(\frac{m}{2}\ m)j} \\ x_{(\frac{m}{2}+1\ \ 1)j} & \cdots & x_{(\frac{m}{2}+1\ \frac{m}{2})j} & \boxed{x_{(\frac{m}{2}+1\ \frac{m}{2}+1)j}} & \cdots & x_{(\frac{m}{2}+1\ \ m)j} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{(m1)j} & \cdots & x_{(m\ \frac{m}{2})j} & \boxed{x_{(m\ \ \frac{m}{2}+1)j}} & \cdots & x_{(mm)j} \end{bmatrix}$$

Fig. 3 MRSS design in case of even sample size [14]

The following steps describe the NRSS sampling design:

Step 1 Select a simple random sample of size m^2 units from the target finite population.
Step 2 Ranked the m^2 selected units in an increasing magnitude based on a visual inspection or any other cost free method with respect to a variable of interest.
Step 3 If m is an odd, then select the $\left[g + (i-1)m\right]$th ranked unit. If m is an even, then select the $[u + (i-1)m]$th ranked unit, if i is an even and $[(u+1) + (i-1)m]$th if i is an odd where $\left(u = \frac{m}{2}, g = \frac{m+1}{2},\ and\, i = 1, 2, ..., m\right)$.
Step 4 Repeat steps 1 through 3 r cycles if needed to obtain a NRSS osize $n = rm$

The NRSS scheme can be described as follows (Fig. 4):

Using NRSS method, we have to choose the units with the rank 2, 5, 8 for actual quantification, then the measured NRSS units are $\left\{\boxed{X_{(2)1}}, \boxed{X_{(5)1}}, \boxed{X_{(8)1}}\right\}$ for one cycle.

Let $\{X_{(i)j}, i = 1, 2, \ldots, m; j = 1, 2, \ldots, r\}$ be a neoteric ranked set sample where m is the set size and r is the number of cycles. Then the likelihood function corresponding to NRSS scheme that proposed by Sabry and Shaaban [13], is given by

$$\begin{aligned} L\left(\boldsymbol{\theta}|x_{k(i)j}\right) = {} & \frac{w!}{\prod_{i=1}^{m+1}(k(i) - k(i-1) - 1)!}\prod_{i=1}^{m} f\left(x_{(k(i))j};\boldsymbol{\theta}\right) \\ & \times \prod_{i=1}^{m+1}\left[F\left(x_{(k(i))j};\boldsymbol{\theta}\right) - F\left(x_{(k(i-1))j};\boldsymbol{\theta}\right)\right]^{k(i)-k(i-1)-1} \end{aligned} \tag{10}$$

where m is the set size, r is the number of cycles, $w = m^2!$, and $k(i)$ is chosen as

$$k(i) = \begin{cases} g + (i-1)m, & m\text{odd} \\ u + (i-1)m, & m\,even, i\,even \\ (u+1) + (i-1)m, & m\,even, i\,odd \end{cases}$$

where $k(0) = 0, k(m+1) = w + 1$ and $x_{(k(0))} = -\infty, x_{(k(m+1))} = \infty$.

3 Estimation of the Exponentiated Gumbel Distribution Pameters

In this Section MLEs for the unknown parameters of EG distribution based on SRS and RSS will be reviewed, moreover we will derive MLHs for EG distribution based on MRSS and NRSS.

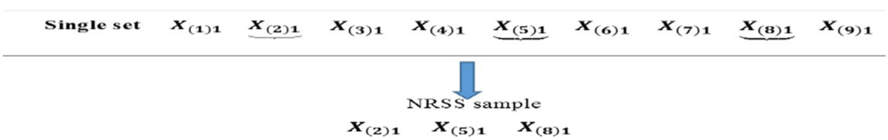


Fig. 4 NRSS design in case of odd sample size [13]

3.1 Estimation Based on SRS

Jabbari and Ravandeh [7] introduced MLHs for EG distribution parameters based on SRS. In this Subsection, MLHs based on SRS will be reviewed. Let $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_{\boldsymbol{n}}$ be a random sample of size $\boldsymbol{n}$ from $\boldsymbol{EG}(\alpha, \lambda)$, then the likelihood function can be written as follows

$$L_{SRS}(\lambda, \alpha|x) = \lambda^n \alpha^n e^{-\lambda \sum_{i=1}^{n} x_i} \prod_{i=1}^{n} \left[exp\left(-exp\left(-\lambda x_i\right)\right)\right]^{\alpha}, \tag{11}$$

The first derivatives of the log-likelihood function denoted by l_{SRS} with respect to λ and α respectively are as follows

$$\frac{\partial l_{SRS}}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i + \alpha \sum_{i=1}^{n} x_i exp\left(-\lambda x_i\right) = 0. \tag{12}$$

and

$$\frac{\partial l_{SRS}}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} exp\left(-\lambda x_i\right) = 0, \tag{13}$$

From Eq. (13), Jabbari and Ravandeh [7] showed that MLE of α say $\hat{\alpha}(\lambda)$ can be obtained as follows

$$\hat{\alpha}(\lambda) = \left(\frac{\sum_{i=1}^{n} exp\left(-\lambda x_i\right)}{n}\right)^{-1}, \tag{14}$$

by substituting $\hat{\alpha}(\lambda)$ in Eq. (11), the profile log-likelihood can be obtained of λ as follows

$$L(\hat{\alpha}(\lambda), \lambda) = c - nln\left(\sum_{i=1}^{n} e^{-\lambda x_i}\right) + nln\lambda - \lambda \sum_{i=1}^{n} x_i = 0. \tag{15}$$

Therefore the MLEs of λ, say $\hat{\lambda}_{MLE}$, can be obtained by maximizing Eq. (15) with respect to λ. It can be shown that the maximum likelihood of Eq. (15) can be obtained as a fixed point solution

$$v(\lambda) = \lambda = \left(\frac{\sum_{i=1}^{n} x_i(1 - n\lambda)}{n}\right)^{-1}.$$

3.2 Estimation Based on RSS

In this subsection, MLHs of EG distribution obtained by Jabbari and Ravandeh [7] will be considered, Suppose $\left\{X_{(11)j}, X_{(22)j,\dots,X_{(mm)j}}; j = 1, 2, \dots, r\right\}$. denotes the ranked set sample of size $n = mr$ from EG (α, λ), where m is the set size and r is the number of cycles. By substituting Eqs. (1) and (2) into Eq. (4), then the likelihood function based on RSS data is given by:

$$L_{RSS}(\lambda,\alpha|x) \propto \prod_{j=1}^{r}\prod_{i=1}^{m}\left(\lambda\alpha e^{-\lambda x_{(ii)j}}\right)\left(exp\left(-exp\left(-\lambda x_{(ii)j}\right)\right)\right)^{\alpha i} \times \left(1-\left[exp\left(-exp\left(-\lambda x_{(ii)j}\right)\right)\right]^{\alpha}\right)^{m-i}$$

The log likelihood function denoted by l_{RSS} can be derived directly as follows

$$l_{RSS} \propto n\log\lambda + nlog\alpha - \lambda\sum_{j=1}^{r}\sum_{i=1}^{m}x_{(ii)j} - \sum_{j=1}^{r}\sum_{i=1}^{m}(\alpha i)exp\left(-\lambda x_{(ii)j}\right) + \sum_{j=1}^{r}\sum_{i=1}^{m}(m-i)log\left(1-\left[exp\left(-exp\left(-\lambda x_{(ii)j}\right)\right)\right]^{\alpha}\right)$$

and the first derivatives of l_{RSS} with respect to λ *and* α respectively are given by

$$\frac{\partial l_{RSS}}{\partial\lambda} = \frac{n}{\lambda} - \sum_{j=1}^{r}\sum_{i=1}^{m}x_{(ii)j} + \sum_{j=1}^{r}\sum_{i=1}^{m}(\alpha i)x_{(ii)j}\exp\left(-\lambda x_{(ii)j}\right) - \alpha\sum_{j=1}^{r}\sum_{i=1}^{m}(m-i)x_{(ii)j}\frac{\left[exp\left(-exp\left(-\lambda x_{(ii)j}\right)\right)\right]^{\alpha}exp\left(-\lambda x_{(ii)j}\right)}{1-\left[exp\left(-exp\left(-\lambda x_{(ii)j}\right)\right)\right]^{\alpha}}, \tag{16}$$

and

$$\frac{\partial l_{RSS}}{\partial\alpha} = \frac{n}{\alpha} - \sum_{j=1}^{r}\sum_{i=1}^{m}(i)\exp\left(-\lambda x_{(ii)j}\right) + \sum_{j=1}^{r}\sum_{i=1}^{m}(m-i)\frac{\left[exp\left(-exp\left(-\lambda x_{(ii)j}\right)\right)\right]^{\alpha}exp\left(-\lambda x_{(ii)j}\right)}{1-\left[exp\left(-exp\left(-\lambda x_{(ii)j}\right)\right)\right]^{\alpha}}, \tag{17}$$

It is clear, it is not easy to obtain a closed form of the non linear Eqs. (16) and (17), so an iterative technique can be used to obtain MLEs of λ and α.

3.3 Estimation Based on MRSS

In the following subsection MLEs based on MRSS for unknown parameters of EG distribution will be obtained. Let $\{X_{(1g)j}, X_{(2g)j,\ldots,X_{(mg)j}};\ j=1,2,\ldots,r, g=\frac{m+1}{2}\}$ is a MRSS in case of odd set size from EG (α,λ) with sample size $n=mr$, where m is the set size, r is the number of cycles. By substituting Eqs. (1) and (2) into Eq. (6), then the likelihood function of the MRSS in case of odd set size is given by:

$$L_{OMRSS}(\lambda,\alpha|x)=\prod_{j=1}^{r}\prod_{i=1}^{m}\frac{m!}{\left[(g-1)!\right]^2}\cdot\left(\lambda\alpha e^{-\lambda x_{(ig)j}}\right)\left(exp(-exp\left(-\lambda x_{(ig)j}\right)\right)^{\alpha g}$$
$$\times\left(1-\left[exp\left(-exp\left(-\lambda x_{(ig)j}\right)\right)\right]^{\alpha}\right)^{g-1},$$

The log likelihood function denoted by l_{OMRSS} is given as:

$$l_{OMRSS}\propto n\log\lambda+n\log\alpha-\sum_{j=1}^{r}\sum_{i=1}^{m}\lambda x_{(ig)j}-(\alpha g)\sum_{j=1}^{r}\sum_{i=1}^{m}\exp\left(-\lambda x_{(ig)j}\right)$$
$$+(g-1)\sum_{j=1}^{r}\sum_{i=1}^{m}1-\left[\exp\left(-\exp\left(-\lambda x_{(ig)j}\right)\right)\right]^{\alpha}.$$

The first derivatives of l_{MRSS} with respect to λ and α respectively, are given by

$$\begin{aligned}\frac{\partial l_{OMRSS}}{\partial\lambda}&=\frac{n}{\lambda}-\sum_{j=1}^{r}\sum_{i=1}^{m}x_{(ig)j}+(\alpha g)\sum_{j=1}^{r}\sum_{i=1}^{m}x_{(ig)j}\exp\left(-\lambda x_{(ig)j}\right)\\&\quad+(g-1)\sum_{j=1}^{r}\sum_{i=1}^{m}x_{(ig)j}\frac{\left[\exp\left(-\exp\left(-\lambda x_{(ig)j}\right)\right)\right]^{\alpha}\exp\left(-\lambda x_{(ig)j}\right)}{1-\left[\exp\left(-\exp\left(-\lambda x_{(ig)j}\right)\right)\right]^{\alpha}},\end{aligned}\tag{18}$$

and

$$\begin{aligned}\frac{\partial l_{OMRSS}}{\partial\alpha}&=\frac{n}{\alpha}-\sum_{j=1}^{r}\sum_{i=1}^{m}(g)\exp\left(-\lambda x_{(ig)j}\right)\\&\quad+\sum_{j=1}^{r}\sum_{i=1}^{m}(g-1)\frac{\left[\exp\left(-\exp\left(-\lambda x_{(ig)j}\right)\right)\right]^{\alpha}\exp\left(-\lambda x_{(ig)j}\right)}{1-\left[\exp\left(-\exp\left(-\lambda x_{(ig)j}\right)\right)\right]^{\alpha}},\end{aligned}\tag{19}$$

MLEs of λ and α in case of odd set size cannot be obtained in a closed form, so an iterative technique will be used to solve Eqs. (18) and (19) numerically.

To obtain MLEs in case of even set size based on MRSS scheme, maximum likelihood function can be obtained by substituting Eqs. (1) and (2) into Eq. (9), as follows:

$$L_{EMRSS}(\lambda,\alpha|x)\propto\left[\prod_{j=1}^{r}\prod_{i=1}^{u}\lambda\alpha e^{-\lambda x_{(iu)j}}\left[exp\left(-exp\left(-\lambda x_{(iu)j}\right)\right)\right]^{\alpha}\left[\left[exp\left(-exp\left(-\lambda x_{(iu)j}\right)\right)\right]^{\alpha}\right]^{u-1}\cdot\right.$$
$$\left.\left[1-\left[exp\left(-exp\left(-\lambda x_{(iu)j}\right)\right)\right]^{\alpha}\right]^{u}\right]$$
$$\times\left[\prod_{j=1}^{r}\prod_{i=u+1}^{m}\lambda\alpha e^{-\lambda x_{(iu+1)j}}\left[exp\left(-exp\left(-\lambda x_{(iu+1)j}\right)\right)\right]^{\alpha}\right.$$
$$\left.\left[\left[exp\left(-exp\left(-\lambda x_{(iu+1)j}\right)\right)\right]^{\alpha}\right]^{u}\times\left[1-\left[exp\left(-exp\left(-\lambda x_{(iu+1)j}\right)\right)\right]^{\alpha}\right]^{u-1}\right]$$

$$L_{EMRSS}(\lambda,\alpha|x) \propto \left[\prod_{j=1}^{r}\prod_{i=1}^{u}\lambda\alpha e^{-\lambda x_{(iu)j}}\left[\left[exp\left(-exp\left(-\lambda x_{(iu)j}\right)\right)\right]^{\alpha}\right]^{u}\cdot\left[1-\left[exp\left(-exp\left(-\lambda x_{(iu)j}\right)\right)\right]^{\alpha}\right]^{u}\right] \times\left[\prod_{j=1}^{r}\prod_{i=u+1}^{m}\lambda\alpha e^{-\lambda x_{(iu+1)j}}\left[\left[exp\left(-exp\left(-\lambda x_{(iu+1)j}\right)\right)\right]^{\alpha}\right]^{u+1}\times\left[1-\left[exp\left(-exp\left(-\lambda x_{(iu+1)j}\right)\right)\right]^{\alpha}\right]^{u-1}\right]$$

The log likelihood function denoted by l_{EMRSS} is given as:

$$l_{EMRSS} \propto rm\log\lambda + rm\log\alpha - \sum_{j=1}^{r}\sum_{i=1}^{u}\lambda x_{(iu)j} - (\alpha u)\sum_{j=1}^{r}\sum_{i=1}^{u}\exp\left(-\lambda x_{(iu)j}\right) + (u)\sum_{j=1}^{r}\sum_{i=1}^{u}\log\left(1-\left[\exp\left(-\exp\left(-\lambda x_{(iu)j}\right)\right)\right]^{\alpha}\right) + \sum_{j=1}^{r}\sum_{i=u+1}^{m}\lambda x_{(iu+1)j} - (\alpha(u+1)) \times\sum_{j=1}^{r}\sum_{i=u+1}^{m}\exp\left(-\lambda x_{(iu+1)j}\right) + (u-1)\sum_{j=1}^{r}\sum_{i=u+1}^{m}\log\left(1-\left[\exp\left(-\exp\left(-\lambda x_{(iu+1)j}\right)\right)\right]^{\alpha}\right)$$

The first derivatives of l_{EMRSS} with respect to λ and α respectively, are given by

$$\frac{\partial l_{EMRSS}}{\partial\lambda} = \frac{n}{\lambda} - \sum_{j=1}^{r}\sum_{i=1}^{u}x_{(iu)j} + (\alpha u)\sum_{j=1}^{r}\sum_{i=1}^{u}x_{(iu)j}\exp\left(-\lambda x_{(iu)j}\right) + (u)\sum_{j=1}^{r}\sum_{i=1}^{u}\propto x_{(iu)j}\frac{\left[\exp\left(-\exp\left(-\lambda x_{(iu)j}\right)\right)\right]^{\alpha}\exp\left(-\lambda x_{(iu)j}\right)}{1-\left[\exp\left(-\exp\left(-\lambda x_{(iu)j}\right)\right)\right]^{\alpha}} + \sum_{j=1}^{r}\sum_{i=u+1}^{m}x_{(iu+1)j} + (\alpha(u+1))\sum_{j=1}^{r}\sum_{i=u+1}^{m}x_{(iu+1)j}\exp\left(-\lambda x_{(iu+1)j}\right) + (u-1)\sum_{j=1}^{r}\sum_{i=u+1}^{m}\alpha x_{(iu+1)j}\frac{\left[\exp\left(-\exp\left(-\lambda x_{(iu+1)j}\right)\right)\right]^{\alpha}\exp\left(-\lambda x_{(iu+1)j}\right)}{1-\left[\exp\left(-\exp\left(-\lambda x_{(iu+1)j}\right)\right)\right]^{\alpha}}, \tag{20}$$

and

$$\frac{\partial l_{EMRSS}}{\partial\alpha} = \frac{n}{\alpha} - (u)\sum_{j=1}^{r}\sum_{i=1}^{u}\exp\left(-\lambda x_{(iu)j}\right) + (u)\sum_{j=1}^{r}\sum_{i=1}^{u}\frac{\left[exp\left(-exp\left(-\lambda x_{(iu)j}\right)\right)\right]^{\alpha}exp\left(-\lambda x_{(iu)j}\right)}{1-\left[exp\left(-exp\left(-\lambda x_{(iu)j}\right)\right)\right]^{\alpha}} - (u+1)\sum_{j=1}^{r}\sum_{i=u+1}^{m}\exp\left(-\lambda x_{(iu+1)j}\right) + (u-1)\sum_{j=1}^{r}\sum_{i=u+1}^{m}\frac{\left[\exp\left(-\exp\left(-\lambda x_{(iu+1)j}\right)\right)\right]^{\alpha}\exp\left(-\lambda x_{(iu+1)j}\right)}{1-\left[\exp\left(-\exp\left(-\lambda x_{(iu+1)j}\right)\right)\right]^{\alpha}}. \tag{21}$$

MLEs of λ and α in case of even set size cannot be obtained in a closed form, so an iterative technique will be used to solve Eqs. (20) and (21) numerically.

3.4 Estimation Based on NRSS

In this subsection, we will derive MLEs for EG (λ, α) based on NRSS technique by substituting Eqs. (1) and (2) in Eq. (10). Let $\left\{X_{(i)j}, i = 1, 2, \dots, m; j = 1, 2, \dots, r\right\}$ be a neoteric ranked set sample where m is the set size and r is the number of cycles, then the likelihood function corresponding to NRSS scheme is given by

$$\begin{aligned} L_{NRSS}(\lambda, \alpha|x) &= \prod_{j=1}^{r}\Bigg(h\prod_{i=1}^{m}\left(\lambda\alpha e^{-\lambda x_{(k(i))j}}\left[exp\left(-exp\left(-\lambda x_{(k(i))j}\right)\right)\right]^{\alpha}\right) \\ &\times\prod_{i=1}^{m+1}\left[\left[exp\left(-exp\left(-\lambda x_{(k(i))j}\right)\right)\right]^{\alpha}-\left[exp\left(-exp\left(-\lambda x_{(k(i-1))j}\right)\right)\right]^{\alpha}\right]^{k(i)-k(i-1)-1}\Bigg) \end{aligned}$$

where $h = \frac{w!}{\prod_{i=1}^{m+1}(k(i)-k(i-1)-1)!}, w = m^2$.

The associated log-likelihood function denoted by l_{NRSS} is as follows

$$\begin{aligned} l_{NRSS} &= r\log h + mr\log\lambda + mrlog\alpha - \lambda\sum_{j=1}^{r}\sum_{i=1}^{m}x_{(k(i))j} - \alpha\sum_{j=1}^{r}\sum_{i=1}^{m}exp\left(-\lambda x_{(k(i))j}\right) \\ &+\sum_{j=1}^{r}\sum_{i=1}^{m+1}(k(i)-k(i-1)-1)log\left(\left[exp\left(-exp\left(-\lambda x_{(k(i))j}\right)\right)\right]^{\alpha}-\left[exp\left(-exp\left(-\lambda x_{(k(i-1))j}\right)\right)\right]^{\alpha}\right) \end{aligned}$$

and the first derivatives of the l_{NRSS} are given by

$$\begin{aligned} \frac{\partial l_{NRSS}}{\partial\lambda} &= \frac{mr}{\lambda} - \sum_{j=1}^{r}\sum_{i=1}^{m}x_{(k(i))j} + -\alpha\sum_{j=1}^{r}\sum_{i=1}^{m}x_{(k(i))j}exp\left(-\lambda x_{(k(i))j}\right) + \sum_{j=1}^{r}\sum_{i=1}^{m+1}(k(i)-k(i-1)-1) \\ &\times\left(\frac{\alpha x_{(k(i))j}\left(exp\left(-\lambda x_{(k(i))j}\right)\right)\left[\exp\left(-\exp\left(-\lambda x_{(k(i))j}\right)\right)\right]^{\alpha} - \alpha x_{(k(i-1))j}\left(exp\left(-\lambda x_{(k(i-1))j}\right)\right)\left[\exp\left(-\exp\left(-\lambda x_{(k(i-1))j}\right)\right)\right]^{\alpha}}{\left[exp\left(-exp\left(-\lambda x_{(k(i))j}\right)\right)\right]^{\alpha}-\left[exp\left(-exp\left(-\lambda x_{(k(i-1))j}\right)\right)\right]^{\alpha}}\right) \end{aligned} \tag{22}$$

$$\begin{aligned} \frac{\partial l_{NRSS}}{\partial\alpha} &= \frac{mr}{\alpha} - \sum_{j=1}^{r}\sum_{i=1}^{m}\exp\left(-\lambda x_{(k(i))j}\right) + \lambda\sum_{j=1}^{r}\sum_{i=1}^{m}\left(x_{(k(i))j}\right)^{-\beta}\log x_{(k(i))j} \\ &+\sum_{j=1}^{r}\sum_{i=1}^{m+1}(k(i)-k(i-1)-1) \\ &\times\left(\frac{-\left(exp\left(-\lambda x_{(k(i))j}\right)\right)\left[\exp\left(-\exp\left(-\lambda x_{(k(i))j}\right)\right)\right]^{\alpha} + \left(exp\left(-\lambda x_{(k(i-1))j}\right)\right)\left[\exp\left(-\exp\left(-\lambda x_{(k(i-1))j}\right)\right)\right]^{\alpha}}{\left[exp\left(-exp\left(-\lambda x_{(k(i))j}\right)\right)\right]^{\alpha}-\left[exp\left(-exp\left(-\lambda x_{(k(i-1))j}\right)\right)\right]^{\alpha}}\right) \end{aligned} \tag{23}$$

MLHs of EG distribution parameters based on NRSS can be obtained by solving Eqs. (22) and (23) using iterative technique.

4 Simulation Study

In this section, a simulation study is conducted to compare the maximum likelihood estimators of the shape and scale parameters of EG distribution based on different sampling schemes. The simulation is applied for 10,000 replications and different sample sizes, $m = \{3, 5, 9\}$. The simulation is made for different parameters values $\mathrm{EG}(\lambda, \alpha) = \{\mathrm{EG}(0.25, 0.5), \mathrm{EG}(0.5, 1.5), \mathrm{EG}(2, 1)\}$. Comparison between the proposed estimators for $\lambda and \alpha$ using SRS, RSS, MRSS, and NRSS are carried out using mean square errors (MSEs) and efficiencies criteria. The efficiency between all estimators with respect to the MLE based on SRS are calculated. The efficiency of the estimator is defined as

$$eff\left(\hat{\theta}_1, \hat{\theta}_2\right) = \frac{MSE\left(\hat{\theta}_1\right)}{MSE\left(\hat{\theta}_2\right)},$$

if $eff\left(\hat{\theta}_1, \hat{\theta}_2\right) > 1$, then $\hat{\theta}_2$ is better than $\hat{\theta}_1$.

The results of Biases and MSEs for the different estimators are listed in Tables 1 and 2, and the results of the efficiencies are reported in Table 3, Figs. 5, 6, 7, 8 and 9 are represented to clarify the simulation results. The following conclusions can be observed From Tables 1 and 2:

1. In almost all cases, the biases are small.
2. In all cases, MSEs of the estimators for (λ, α) based on SRS data are greater than MSEs of the estimators based on RSS, MRSS, and NRSS data (see Fig. 5).
3. In almost all cases, MSEs of all estimators based on SRS, RSS, MRSS, and NRSS decrease as the set sizes increase (see Fig. 6).
4. In almost all cases, MSEs of all estimators based on SRS, RSS, MRSS, and NRSS increase as the value of λ increases.
5. In almost all cases, MSEs of all estimators based on SRS, RSS, MRSS, and NRSS decrease as the value of α increases.
6. MSEs of the estimators for (λ) based on NRSS have the smallest MSEs in all cases comparing with the other estimators and MSEs of the estimators for (α) based on NRSS have the smallest MSEs in all cases comparing with the other estimators except in the case of EG (2, 1) when {m = 3, 5}, (see Fig. 7).
7. In almost all cases, MSEs of the estimators for $(\lambda and \propto)$ based on MRSS are smaller than MSEs of the estimators based on RSS (see Fig. 7).

From Table 3, it can be observed that:

8. In almost all cases, efficiencies of all estimators based on RSS, MRSS, and NRSS increase as the set size increase (see Fig. 8).
9. Efficiencies of the estimators for λ based on NRSS have the largest efficiencies in all cases, except when $\lambda = 2 and m = 3$. (see Fig. 8).

Table 1 Biases of the estimators for (λ,α) based on SRS, RSS, MRSS, and NRSS

m	SRS		RSS		MRSS		NRSS	
	λ	α	λ	α	λ	α	λ	α
EG (0.25, 0.5)								
3	−0.329	−0.961	−0.134	−1.419	−0.316	−0.137	−0.022	−0.200
5	−0.108	−0.171	−0.255	−1.051	−0.106	−0.532	0.031	0.178
9	−0.044	−0.128	−0.009	−1.020	−0.041	−0.209	0.056	0.346
EG (0.5, 1.5)								
3	−0.642	−0.167	−0.036	0.251	−0.603	−21.63	−0.282	0.214
5	−0.215	0.701	0.004	0.066	−0.206	0.727	−0.153	0.134
9	−0.088	0.787	0.143	0.014	−0.094	0.824	−0.091	0.041
EG (2, 1)								
3	−0.603	−0.471	−0.744	0.520	−0.441	−256	−0.966	0.557
5	−0.863	−0.523	−0.349	0.724	−0.827	−0.197	−0.472	0.773
9	−0.353	−0.13	−0.952	0.716	−0.359	−0.075	−0.222	0.886

Table 2 MSEs of the estimators for (λ,α) based on SRS, RSS, MRSS, and NRSS

m	SRS		RSS		MRSS		NRSS	
	λ	α	λ	α	λ	α	λ	α
EG (0.25, 0.5)								
3	0.1525	1.8878	0.0387	0.9013	0.1285	0.8007	0.0242	0.2999
5	0.1072	1.1552	0.0276	0.8263	0.0343	0.4194	0.0139	0.1033
9	0.0737	0.8367	0.0017	0.5880	0.0135	0.2686	0.0081	0.0512
EG (0.5, 1.5)								
3	0.6109	1.2929	0.1301	0.5655	0.1328	0.5769	0.0560	0.1683
5	0.4286	0.9026	0.0102	0.4137	0.0809	0.2574	0.0582	0.0924
9	0.2935	0.7068	0.0949	0.3190	0.0522	0.1713	0.0145	0.0283
EG (2, 1)								
3	0.9774	0.9120	0.5539	0.2710	0.1658	0.3048	0.1935	0.4340
5	0.6862	0.8058	0.1219	0.1524	0.1020	0.2014	0.0493	0.1662
9	0.4708	0.7385	0.09077	0.0913	0.0662	0.1334	0.0259	0.0798

10. Efficiencies of the estimators for $\propto$ based on NRSS have the largest efficiencies in all cases except in the case of EG (2,1) when $\{m = 3, 5\}$ (see Fig. 9).
11. In almost all cases, efficiencies of the estimators for (λand $\propto$) on MRSS are greater than the efficiencies based on RSS.

Table 3 Efficiencies of the estimators for (λ,α) based on RSS, MRSS, and NRSS

m	RSS		MRSS		NRSS	
	λ	α	λ	α	λ	α
EG (0.25,0.5)						
3	3.935	2.094	1.186	2.357	6.303	6.293
5	3.877	1.398	3.124	2.754	7.669	11.174
9	4.317	1.422	5.457	3.114	9.049	16.341
EG (0.5,1.5)						
3	4.695	2.285	4.598	2.240	10.892	7.680
5	4.193	2.181	5.295	3.506	7.356	9.765
9	3.090	2.215	5.620	4.126	20.165	24.908
EG (2,1)						
3	1.764	3.365	5.895	2.991	5.050	2.101
5	5.625	5.286	6.725	3.999	13.916	4.847
9	5.186	8.080	7.112	5.533	18.169	9.249

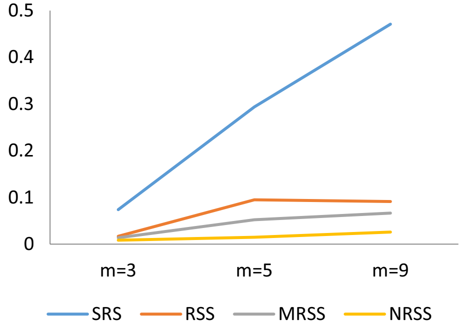


Fig. 5 MSEs of the estimators for (λ) at $m = 9$

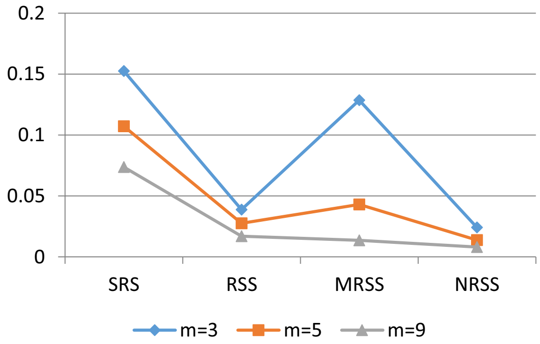


Fig. 6 MSEs of the estimators based on SRS, RSS, MRSS, and NRSS at $(\lambda = 0.25)$

Fig. 7 MSEs of the estimators based on SRS, RSS, MRSS, and NRSS at $\alpha = 1.5$

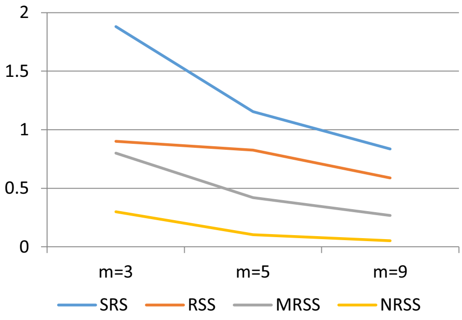


Fig. 8 Efficiencies of the estimators based on RSS, MRSS, and NRSS at $\lambda = 2$.

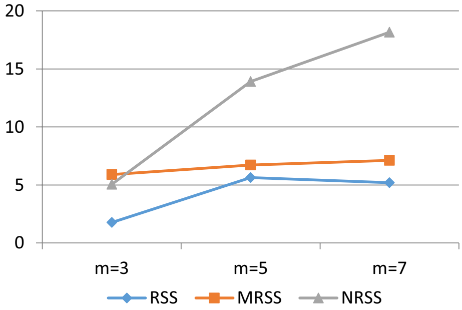
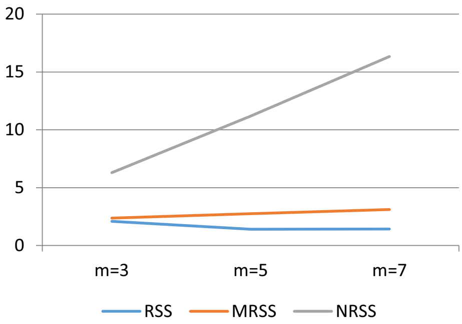


Fig. 9 Efficiencies of the estimators based on RSS, MRSS, and NRSS at $\alpha = 0.5$



5 Conclusions

On the basis of numerical results, it can be concluded that, MSEs based SRS data has the largest MSEs comparing to RSS and its modifications schemes. It can be noted that, in almost all cases, MSEs decrease as the set sizes increase and the efficiencies increase as the set sizes increase. This study revealed that MRSS is better than RSS. Also, NRSS technique has the superior over the rest of other sampling schemes. In almost all cases, NRSS has the smallest MSEs and largest efficiencies. Generally the estimators based NRSS, MRSS, and RSS based on RSS techniques are more efficient than the estimators based on SRS technique.

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