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Dependent Ranked Set Sampling Designs for Parametric Estimation with Applications

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Abstract
In this paper, we derive the likelihood function of the neoteric ranked set sampling (NRSS) as dependent in sampling method and double neoteric ranked set sampling (DNRSS) designs as combine between independent sampling method in the first stage and dependent sampling method in the second stage and they compared for the estimation of the parameters of the inverse Weibull (IW) distribution. An intensive simulation has been made to compare the one and the two stages designs. The results showed that likelihood estimation based on ranked set sampling (RSS) as independent sampling method, NRSS and DNRSS designs provide more efficient estimators than the usual simple random sampling design. Moreover, the DNRSS is slightly more efficient than the NRSS and RSS designs in the case of estimating the IW distribution parameters.

Keywords Simple random sampling · Ranked set sampling · Neoteric ranked set sampling · Double neoteric ranked set sampling · Maximum likelihood estimation

1 Introduction

McIntyre [1] first introduced the ranked set sampling (RSS) in the estimation of the mean of pasture yields as a method of improving precision of estimates by a method related to two-phase sampling. He proposed a method of sampling to estimate mean pasture yields with greater efficiency than simple random sampling (SRS). RSS is a useful alternative to SRS when measurements for the variable of interest are expensive or difficult to obtain, the method is shown to be at least as efficient as SRS with the same number of quantification. The RSS has wide applications in many scientific problems, especially in environmental and ecological studies where the main focus

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is on economical and efficient sampling strategies. A recently developed extension of RSS, Zamanzade and Al-Omari [2] was proposed a new neoteric ranked set sampling (NRSS). NRSS differs from the original RSS scheme by the composition of a single set of m^2 units, instead of m sets of size m, this strategy has been shown to be effective, producing more efficient estimators for the population mean and variance.

The inverse Weibull (IW) distribution can be readily applied to modeling processes in reliability, ecology, medicine, branching processes and biological studies. The properties and applications of IW distribution in several areas can be seen in the literature Keller et al. [3], Calabria and Pulcini [4], Johnson et al. [5].

A random variable X has an IW distribution if the probability density function (PDF) is given by

$$f(x;\lambda,\beta) = \lambda\beta x^{-(\beta+1)}e^{-\lambda x^{-\beta}}, \tag{1}$$

If $\beta = 1$, the IW pdf becomes inverse exponential pdf, and when $\beta = 2$; the IW PDF is referred to as the inverse Raleigh pdf. The IW cumulative distribution function (CDF) is given by

$$F(x;\lambda,\beta) = e^{-\lambda x^{-\beta}}, \tag{2}$$

where $x > 0$, $\lambda > 0$, $\beta > 0$ and $0 < u < 1$. β and λ are the shape and scale parameters, respectively.

2 Some Ranked Set Sampling

In this section, we give brief descriptions of RSS, NRSS, and double neoteric ranked set sampling (DNRSS) schemes.

Some notation frequently used in this section and in the rest of the paper are given as follows.

2.1 Ranked Set Sampling

RSS is an alternative design of SRS, The RSS design has some advantages according SRS. For example, by this design, efficient estimates can be obtained using less data in the sample.

This ordering criterion may be based, for example, on values of a concomitant variable or personal judgment. Several studies have proved the higher efficiency of RSS, relative to SRS, for the estimation of a large number of population parameters.

The RSS scheme can be described as follows:

Step 1 Randomly select m^2 sample units from the population.
Step 2 Allocate the m^2 selected units as randomly as possible into m sets, each of size m.

Step 3 Choose a sample for actual quantification by including the smallest ranked unit in the first set, the second smallest ranked unit in the second set, the process is continues in this way until the largest ranked unit is selected from the last set.
Step 4 Repeat steps 1 through 3 for r cycles to obtain a sample of size mr (Fig. 1).

Let $\{X_{(i)j}, i = 1, 2, \dots, m; j = 1, 2, \dots, r\}$ be a RSS where m is the set size, r is the number of cycle, Then the CDF and the PDF of $X_{(ii)j}$ is given by

$$F_m\left(x_{(i)j}; \theta\right) = \sum_{t=i}^{m} \binom{m}{t} \left[F\left(x_{(i)j}; \theta\right)\right]^t \times \left[1 - F\left(x_{(i)j}; \theta\right)\right]^{m-t},$$

and

$$f_m\left(x_{(i)j}; \theta\right) = C_i f\left(x_{(i)j}; \theta\right) \left[F\left(x_{(i)j}; \theta\right)\right]^{i-1} \cdot \left[1 - F\left(x_{(ii)j}; \theta\right)\right]^{m-i}.$$

The Likelihood function corresponding to RSS scheme is as follows:

$$L(x;\theta) = \prod_{j=1}^{r} \prod_{i=1}^{m} C_i f\left(x_{(i)j};\theta\right) \left[F\left(x_{(i)j};\theta\right)\right]^{i-1} \left[1 - F\left(x_{(i)j};\theta\right)\right]^{m-i} \tag{3}$$

where $C_i = \frac{m!}{(i-1)!(m-i)!}$.

2.2 Noetric Ranked Set Sampling

Zamanzade and Al-Omari [2] have defined a new NRSS. A recently developed extension of RSS. NRSS differs from the original RSS scheme by the composition of a single set of m^2 units, instead of m sets of size m. this strategy has been shown to be effective, producing more efficient estimators for the population mean and variance.

In this section, a steps for applying a NRSS scheme will be showed.

The following steps describe the NRSS sampling design:

Step 1 Select a simple random sample of size m^2 units from the target finite population.
Step 2 Ranked the m^2 selected units in an increasing magnitude based on a visual inspection or any other cost free method with respect to a variable of interest.

cycle 1			cycle 2			cycle 3		
$\boxed{X_{(11)1}}$	$X_{(12)1}$	$X_{(13)1}$	$\boxed{X_{(11)2}}$	$X_{(12)2}$	$X_{(13)2}$	$\boxed{X_{(11)3}}$	$X_{(12)3}$	$X_{(13)3}$
$X_{(21)1}$	$\boxed{X_{(22)1}}$	$X_{(23)1}$	$X_{(21)2}$	$\boxed{X_{(22)2}}$	$X_{(23)2}$	$X_{(21)3}$	$\boxed{X_{(22)3}}$	$X_{(23)3}$
$X_{(31)k}$	$X_{(32)k}$	$\boxed{X_{(33)1}}$	$X_{(31)2}$	$X_{(32)2}$	$\boxed{X_{(33)2}}$	$X_{(31)3}$	$X_{(32)3}$	$\boxed{X_{(33)3}}$

Fig. 1 RSS design [6]

Step 3 If m is an odd, then select the $\left[\frac{m+1}{2}+(i-1)m\right]$ th ranked unit for $(i=1,2,\ldots,m)$.

If m is an even, then select the $[l+(i-1)m]$th ranked unit, where $\left[l=m/2\right]$ if i is an even and $\left[l=\frac{m+2}{2}\right]$ if i is an odd for $(i=1,2,\ldots,m)$.

Step 4 Repeat steps 1 through 3 r cycles if needed to obtain a NRSS of size $n=rm$.

The NRSS scheme can be described as follows: (Fig. 2)

Where $m=3$ and $r=1$.

Using NRSS method, we have to choose the units with the rank 2, 5, 8 for actual quantification, then the measured NRSS units are $\left\{\boxed{X_{(2)1}}, \boxed{X_{(5)1}}, \boxed{X_{(8)1}}\right\}$ for one cycle.

2.3 Double Neoteric Ranked Set Sampling

DNRSS is defined by Taconeli and Cabral [7] which defined to be a two-stage design in which the first stage is defined by as RSS scheme, while the NRSS procedure should be applied in the second stage. To draw a DRSS sample of size n, the following steps must be implemented:

Step 1 Identify m^3 elements from the target population and divide them, randomly, into m blocks with m sets of size m.

Step 2 Apply the RSS method to each block to obtain m RSS samples of size n.

Step 3 Employ the NRSS procedure to the m^2 elements selected in step 2 to obtain a DNRSS sample of size m. Only these sample units must be measured for the variable of interest.

Step 4 Steps 1–3 can be repeated r times to draw a sample of size mr.

In Fig. 3, we show how to select a sample of size $m=3$ and $r=1$, then we have to select $m^3=27$ units as.

3 Maximum Likelihood Function Based on NRSS and DNRSS

In this section, Maximum likelihood function based on NRSS and DNRSS will be derived.

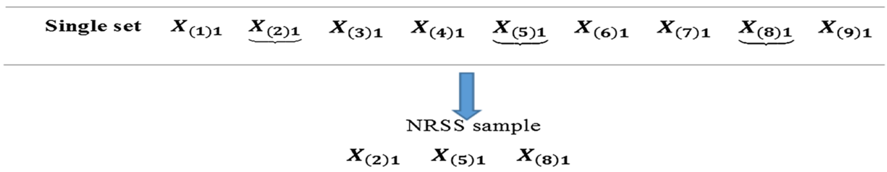


Fig. 2 NRSS design in case of odd sample size and one cycle [6]

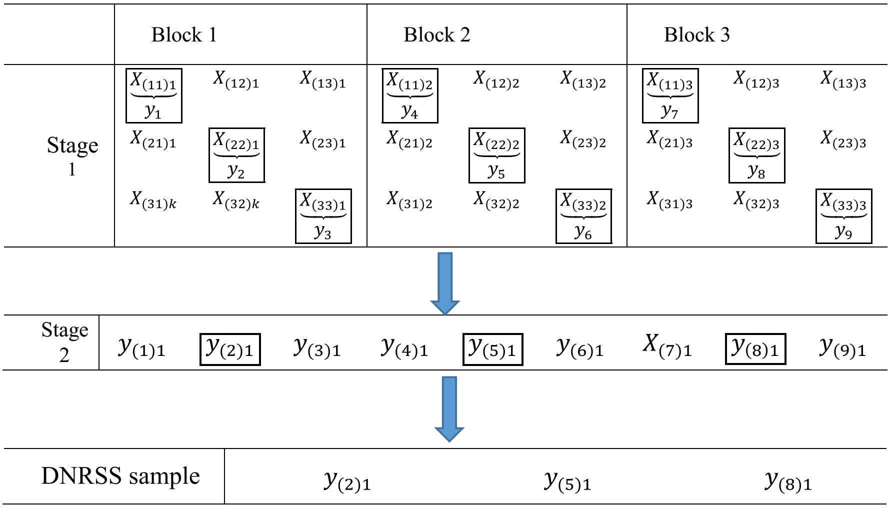


Fig. 3 DNRSS design in case of odd sample size and one cycle

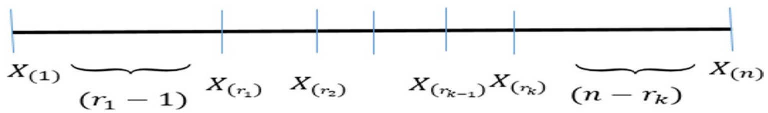
3.1 Maximum Likelihood Function Based on NRSS

In this section, we will define the likelihood function for NRSS scheme using the order statistics theory through Proposition 1 and depend on Lemma 1.

Lemma 1 *Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a continuous population and $x_{r_1} < X_{r_1:n} \le x_{r_1} + \delta x_{r_1}, x_{r_2} < X_{r_2:n} \le x_{r_2} + \delta x_{r_2}, \ldots, x_{r_k} < X_{r_k:n} \le x_{r_k} + \delta x_k$ denote the corresponding order statistics, Then the joint probability density function (PDF) of $X_{r_i:n}$ is given by*

$$f_{r_i:n}\left(x_{r_1}, x_{r_2}, \ldots, x_{r_k}\right) = \frac{n!}{\prod_{i=1}^{n+1}\left(r_i - r_{i-1} - 1\right)!} \prod_{i=1}^{n} f\left(x_{r_i}\right) \times \prod_{i=1}^{n+1}\left[F\left(x_{r_i}\right) - F\left(x_{r_{i-1}}\right)\right]^{r_i - r_{i-1} - 1}$$

where $r_0 = 0, r_{k-1} = n + 1$ and $i = 1, 2, \ldots, k$.



To proof the joint probability density function (PDF) of $X_{r_i:n}$ by the multinomial method [8]*, we could derive the joint PDF of $X_{r_i:n}$ for $(1 \le r_1 < r_2 < \cdots < r_k \le n)$ as*

$$
\begin{aligned}
&f_{r_i:n}(x_{r_1}, x_{r_2}, \dots, x_{r_k})\delta x_{r_1}\delta x_{r_2} \dots \delta x_{r_k} \\
&\quad \approx p\big(x_{r_1} < X_{r_1:n} \le x_{r_1} + \delta x_{r_1}; x_{r_2} < X_{r_2:n} \le x_{r_2} + \delta x_{r_2}; \dots ; x_{r_k} \\
&\quad < X_{r_k:n} \le x_{r_k} + \delta x_{r_k}\big)
\end{aligned}
$$

$$
\begin{aligned}
\approx p\big(&(r_1 - 1) \text{ of the } X_{r_i:n} \\
&\le x_{r_1}; \textit{one } X_{r_i:n} \textit{ in } (x_{r_1}, x_{r_1} + \delta x_{r_1}); (r_2 - r_1 - 1) \textit{ of the } X_{r_i:n} \textit{ in}(x_{r_1} \\
&\quad + \delta x_{r_1}, x_{r_2}); \textit{ one } X_{r_i:n} \textit{ in } (x_{r_2}, x_{r_2} + \delta x_{r_2}); \dots ; (r_k - r_{k-1} \\
&\quad - 1) \textit{ of the } X_{r_i:n} \textit{ in}\big(x_{r_{k-1}} + \delta x_{r_{k-1}}, x_{r_k}\big); \textit{one } X_{r_i:n} \textit{ in}(x_{r_k}, x_{r_k} \\
&\quad + \delta x_{r_k}); (n - r_k) \textit{ of the } X_{r_i:n} > x_{r_k} + \delta x_{r_k}\big)
\end{aligned}
$$

$$
\begin{aligned}
&f_{r_i:n}\big(x_{r_1}, x_{r_2}, \dots, x_{r_k}\big)\delta x_{r_1}\delta x_{r_2} \dots \delta x_{r_k} \\
&\quad \approx \frac{n!}{\big(r_1 - 1\big)!\big(r_2 - r_1 - 1\big)! \dots \big(n - r_k\big)!}\big[F\big(x_{r_1}\big)\big]^{r_1 - 1} f\big(x_{r_1}\big)\delta x_{r_1} \\
&\qquad \times \big[F\big(x_{r_2}\big) - F\big(x_{r_1} + \delta x_{r_1}\big)\big]^{r_2 - r_1 -} f\big(x_2\big)\delta x_{r_2} \dots \\
&\qquad \times \big[F\big(x_{r_k}\big) - F\big(x_{r_{k-1}} + \delta x_{r_{k-1}}\big)\big]^{r_k - r_{k-1} - 1} f\big(x_{r_k}\big)\delta x_{r_k} \\
&\qquad \times \big[1 - F\big(x_{r_k} + \delta x_{r_k}\big)\big]^{n - r_k}
\end{aligned}
$$

$$
\begin{aligned}
&f_{r_i:n}\big(x_{r_1}, x_{r_2}, \dots, x_{r_k}\big)\delta x_{r_1}\delta x_{r_2} \dots \delta x_{r_k} \\
&\quad \approx \frac{n!}{\big(r_1 - 1\big)!\big(r_2 - r_1 - 1\big)! \dots \big(n - r_k\big)!}\big[F\big(x_{r_1}\big)\big]^{r_1 - 1} \\
&\qquad \times \big[F\big(x_{r_2}\big) - F\big(x_{r_1}\big)\big]^{r_2 - r_1 - 1} \dots \big[F\big(x_{r_k}\big) - F\big(x_{r_{k-1}}\big)\big]^{r_k - r_{k-1} - 1} \\
&\qquad \times \big[1 - F\big(x_{r_k}\big)\big]^{n - r_k} f\big(x_{r_1}\big) f\big(x_2\big) \dots f\big(x_{r_k}\big)
\end{aligned}
$$

$$
\begin{aligned}
&f_{r_i:n}\big(x_{r_1}, x_{r_2}, \dots, x_{r_k}\big)\delta x_{r_1}\delta x_{r_2} \dots \delta x_{r_k} \\
&\quad = \frac{n!}{\prod_{i=1}^{n+1}\big(r_i - r_{i-1} - 1\big)!}\prod_{i=1}^{n} f\big(x_{r_i}\big) \\
&\qquad \times \prod_{i=1}^{n+1}\big[F\big(x_{r_i}\big) - F\big(x_{r_{i-1}}\big)\big]^{r_i - r_{i-1} - 1}
\end{aligned}
$$

Proposition 1 *Depending on the previous lemma, to find the maximum likelihood for NRSS. Let* $\{X_{(k(i))j}, i = 1, 2, \dots, m; j = 1, 2, \dots, r\}$ *and* $w = m^2$ *be a NRSS where* m^2 *is the set size,* r *is the number of cycles and* $k(i)$ *is chosen as*

$$k(i) = \begin{cases} \frac{m+1}{2} + (i-1)m, & m \, \text{odd} \\ \frac{m}{2} + (i-1)m, & m \, even, i \, even \\ \frac{m+2}{2} + (i-1)m, & m \, even, i \, odd \end{cases}$$

Then, the joint probability density function (PDF) of $X_{(k(i))j}$ *is given by*

$$\begin{aligned} L\left(x_{k(i)j};\boldsymbol{\theta}\right) = & \frac{w!}{\prod_{i=1}^{m+1}(k(i)-k(i-1)-1)!}\prod_{i=1}^{m} f\left(x_{(k(i))j};\boldsymbol{\theta}\right) \\ & \cdot \prod_{i=1}^{m+1}\left[F\left(x_{(k(i))j};\boldsymbol{\theta}\right) - F\left(x_{(k(i-1))j};\boldsymbol{\theta}\right)\right]^{k(i)-k(i-1)-1} \end{aligned} \tag{4}$$

where $k(0) = 0, k(m+1) = w+1$ *and* $x_{(k(0))} = -\infty, x_{(k(i+1))} = \infty$.

3.2 Maximum Likelihood Function Based on DNRSS

The likelihood function corresponding to DNRSS scheme will be derived in the same DRSS scheme based on the joint of order statistics but two stage.

Proposition 2 *Let* $\left\{Y_{(i)j}, i = 1,2,..,m^2; j = 1,2,..,r\right\}$ *and* $w = m^2$ *be a neoteric ranked set sample where* m^2 *is the set size and r is the number of cycles in the second stage, where in the first stage select* m^3 *elements from the target population and divide these elements randomly into m sets (of size* m^2*). Then the PDF of* $X_{(k(i))j}$ *is given by:*

$$\begin{aligned} L\left(x_{k(i)j};\theta\right) = & \frac{w!}{\prod_{i=1}^{m+1}(k(i)-k(i-1)-1)!}\prod_{i=1}^{m} f_m\left(y_{(k(i))j};\theta\right) \\ & \times \prod_{i=1}^{m+1}\left[F_m\left(y_{(k(i))j};\theta\right) - F_m\left(y_{(k(i-1))j};\theta\right)\right]^{k(i)-k(i-1)-1} \end{aligned}. \tag{5}$$

4 Estimation of the Inverse Weibull Distribution Parameters

This section is devoted to the MLE for the unknown parameters of IW distribution based on RSS, NRSS and DNRSS designs.

4.1 Estimation Based on SRS

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables from IW distribution with pdf given in Eq. (1). The likelihood function of λ and β is given by

$$L(x;\lambda,\beta)=\prod_{i=1}^{n}\lambda\beta x_i^{-(\beta+1)}e^{-\lambda x_i^{-\beta}},$$

and the log likelihood function is then derived as

$$\ell(\lambda,\beta)=n\log\lambda+n\log\beta-(\beta-1)\sum_{i=1}^{n}\log x_i-\sum_{i=1}^{n}\lambda x_i^{-\beta},$$

Let

$$\frac{\partial\ell(\lambda,\beta)}{\partial\lambda}=\frac{n}{\lambda}-\sum_{i=1}^{n}x_i^{-\beta}=0,$$

and

$$\frac{\partial\ell(\lambda,\beta)}{\partial\beta}=\frac{n}{\beta}-\sum_{i=1}^{n}\log x_i-\sum_{i=1}^{n}\lambda x_i^{-\beta}\log x_i=0.$$

4.2 Estimation Based on RSS

According to the Eq. (3) the Likelihood function for set sizes m and with r cycles based on RSS is given by

$$\begin{aligned}L(x;\lambda,\beta)=&\prod_{j=1}^{r}\prod_{i=1}^{m}C_i\left(\lambda\beta\left(x_{(i)j}\right)^{-(\beta+1)}e^{-\lambda\left(x_{(i)j}\right)^{-\beta}}\right)\left(e^{-\lambda\left(x_{(i)j}\right)^{-\beta}}\right)^{i-1}\\&\cdot\left(1-e^{-\lambda\left(x_{(i)j}\right)^{-\beta}}\right)^{m-i}\end{aligned}$$

The log likelihood function can be derived directly as follows

$$\begin{aligned}\ell(\lambda,\beta)=&rm\,logc+mr\log\lambda+mr\ \log\beta-(\beta+1)\sum_{j=1}^{r}\sum_{i=1}^{m}\log x_{(i)j}\\&-\lambda\sum_{j=1}^{r}\sum_{i=1}^{m}(i)\left(x_{(i)j}\right)^{-\beta}+\sum_{j=1}^{r}\sum_{i=1}^{m}(m-i)log\left(1-e^{-\lambda\left(x_{(i)j}\right)^{-\beta}}\right)\end{aligned}$$

and the first derivatives of the $\ell(\lambda,\beta)$ are given by

$$\frac{\partial\ell(\lambda,\beta)}{\partial\lambda}=\frac{mr}{\lambda}-\sum_{j=1}^{r}\sum_{i=1}^{m}(i)\left(x_{(i)j}\right)^{-\beta}+\sum_{j=1}^{r}\sum_{i=1}^{m}(m-i)\frac{e^{-\lambda\left(x_{(i)j}\right)^{-\beta}}\left(x_{(i)j}\right)^{-\beta}}{1-e^{-\lambda\left(x_{(i)j}\right)^{-\beta}}}$$

and

$$\frac{\partial \ell(\lambda,\beta)}{\partial \beta} = \frac{mr}{\beta} - \sum_{j=1}^{r}\sum_{i=1}^{m} \log x_{(i)j} + \lambda \sum_{j=1}^{r}\sum_{i=1}^{m} (i)\left(x_{(i)j}\right)^{-\beta} \log x_{(i)j} - \sum_{j=1}^{r}\sum_{i=1}^{m} (m-i)\frac{e^{-\lambda\left(x_{(i)j}\right)^{-\beta}} \lambda\left(x_{(i)j}\right)^{-\beta} \log x_{(i)j}}{1-e^{-\lambda\left(x_{(i)j}\right)^{-\beta}}}$$

These two nonlinear equations can't be solved analytically and will be solved numerically.

4.3 Estimation Based on NRSS

By substitution in Eq. (4) based on IW distribution the Likelihood function for set sizes m and with r cycles based on NRSS is given by

$$L(x;\lambda,\beta) = \prod_{j=1}^{r}\left(h \prod_{i=1}^{m}\left(\lambda\beta\left(x_{(k(i))j}\right)^{-(\beta+1)} e^{-\lambda\left(x_{(k(i))j}\right)^{-\beta}}\right) \times \prod_{i=1}^{m+1}\left[e^{-\lambda\left(x_{(k(i))j}\right)^{-\beta}} - e^{-\lambda\left(x_{(k(i-1))j}\right)^{-\beta}}\right]^{k(i)-k(i-1)-1}\right)$$

where $h = \frac{w!}{\prod_{i=1}^{m+1}(k(i)-k(i-1)-1)!}$.

The associated log-likelihood function is as follows

$$\ell(\lambda,\beta) = r\log h + mr\log\lambda + mr\ \log\beta - (\beta+1)\sum_{j=1}^{r}\sum_{i=1}^{m}\log x_{(k(i))j} - \lambda\sum_{j=1}^{r}\sum_{i=1}^{m}\left(x_{(k(i))j}\right)^{-\beta} + \sum_{j=1}^{r}\sum_{i=1}^{m+1}(k(i)-k(i-1)-1)log\left(e^{-\lambda\left(x_{(k(i))j}\right)^{-\beta}} - e^{-\lambda\left(x_{(k(i-1))j}\right)^{-\beta}}\right)$$

and the first derivatives of the $\ell\ell(\lambda,\beta)$ are given by

$$\frac{\partial \ell(\lambda,\beta)}{\partial \lambda} = \frac{mr}{\lambda} - \sum_{j=1}^{r}\sum_{i=1}^{m}\left(x_{(k(i))j}\right)^{-\beta} + \sum_{j=1}^{r}\sum_{i=1}^{m+1}(k(i)-k(i-1)-1) \times \left(\frac{-e^{-\lambda\left(x_{(k(i))j}\right)^{-\beta}}\left(x_{(k(i))j}\right)^{-\beta} + e^{-\lambda\left(x_{(k(i-1))j}\right)^{-\beta}}\left(x_{(k(i-1))j}\right)^{-\beta}}{e^{-\lambda\left(x_{(k(i))j}\right)^{-\beta}} - e^{-\lambda\left(x_{(k(i-1))j}\right)^{-\beta}}}\right)$$

and

$$\frac{\partial \ell(\lambda,\beta)}{\partial \beta} = \frac{mr}{\beta} - \sum_{j=1}^{r}\sum_{i=1}^{m} \log x_{(k(i))j} + \lambda \sum_{j=1}^{r}\sum_{i=1}^{m} \left(x_{(k(i))j}\right)^{-\beta} \log x_{(k(i))j} + \sum_{j=1}^{r}\sum_{i=1}^{m+1} (k(i) - k(i-1) - 1) \times \left(\frac{e^{-\lambda\left(x_{(k(i))j}\right)^{-\beta}} \lambda \left(x_{(k(i))j}\right)^{-\beta} \log x_{(k(i))j} + e^{-\lambda\left(x_{(k(i-1))j}\right)^{-\beta}} \lambda \left(x_{(k(i-1))j}\right)^{-\beta} \log x_{(k(i-1))j}}{e^{-\lambda\left(x_{(k(i))j}\right)^{-\beta}} - e^{-\lambda\left(x_{(k(i-1))j}\right)^{-\beta}}} \right)$$

4.4 Estimation Based on DNRSS

By substitution in Eq. (5) based on IW distribution the Likelihood function for set sizes m and with r cycles based on DNRSS is given by

$$L(\lambda,\beta;x) = \prod_{j=1}^{r} \left(h \prod_{i=1}^{m} \left(cf\left(x_{(ii)j};\boldsymbol{\theta}\right) \left[F\left(x_{(ii)j};\boldsymbol{\theta}\right)\right]^{i-1} \left[1 - F\left(x_{(ii)j};\boldsymbol{\theta}\right)\right]^{m-i} \right) \times \prod_{i=1}^{m+1} \left[\sum_{t=i}^{m} \binom{m}{t} \left[F\left(x_{(ii)j};\boldsymbol{\theta}\right)\right]^{t} \left[1 - F\left(x_{(ii)j};\boldsymbol{\theta}\right)\right]^{m-t} - \sum_{t=i-1}^{m} \binom{m}{t} \left[F\left(x_{(ii)j};\boldsymbol{\theta}\right)\right]^{t} \left[1 - F\left(x_{(ii)j};\boldsymbol{\theta}\right)\right]^{m-t} \right]^{k(i)-k(i-1)-1} \right)$$

$$L(\lambda,\beta;x) = \prod_{j=1}^{r} \left(h \prod_{i=1}^{m} \left(c\lambda\beta x_{(i)j}^{-(\beta+1)} e^{-\lambda x_{(ii)j}^{-\beta}} \left[e^{-\lambda x_{(i)j}^{-\beta}}\right]^{i-1} \left[1 - e^{-\lambda x_{(i)j}^{-\beta}}\right]^{m-i} \right) \times \prod_{i=1}^{m+1} \left[\sum_{t=i}^{m} \binom{m}{t} \left[e^{-\lambda x_{(i)j}^{-\beta}}\right]^{t} \left[1 - e^{-\lambda x_{(i)j}^{-\beta}}\right]^{m-t} - \sum_{t=i-1}^{m} \binom{m}{t} \left[e^{-\lambda x_{(i)j}^{-\beta}}\right]^{t} \left[1 - e^{-\lambda x_{(i)j}^{-\beta}}\right]^{m-t} \right]^{k(i)-k(i-1)-1} \right)$$

The associated log-likelihood function is as follows

$$
\begin{aligned}
\ell(\lambda,\beta) =& r\, logh + rm\, logc + mr\log\lambda + mr\,\log\beta - (\beta+1)\sum_{j=1}^{r}\sum_{i=1}^{m}\log x_{(i)j} \\
& - \lambda\sum_{j=1}^{r}\sum_{i=1}^{m}(i-1)\left(x_{(i)j}\right)^{-\beta} + \sum_{j=1}^{r}\sum_{i=1}^{m}(m-i)log\left(1-e^{-\lambda\left(x_{(i)j}\right)^{-\beta}}\right) \\
& + \sum_{j=1}^{r}\sum_{i=1}^{m}(k(i)-k(i-1)-1) \\
& \cdot \log\left[\sum_{t=i}^{m}\binom{m}{t}\left[e^{-\lambda x_{(i)j}^{-\beta}}\right]^{t}\left[1-e^{-\lambda x_{(i)j}^{-\beta}}\right]^{m-t} - \sum_{t=i-1}^{m}\binom{m}{t}\left[e^{-\lambda x_{(i)j}^{-\beta}}\right]^{t}\left[1-e^{-\lambda x_{(i)j}^{-\beta}}\right]^{m-t}\right]
\end{aligned}
$$

and the first derivatives of the $\ell(\theta)$ are given by

$$
\begin{aligned}
\frac{\partial\ell(\lambda,\beta)}{\partial\lambda} =& \frac{mr}{\lambda} - \sum_{j=1}^{r}\sum_{i=1}^{m}(i-1)\left(x_{(i)j}\right)^{-\beta} + \sum_{j=1}^{r}\sum_{i=1}^{m}(m-i)\frac{e^{-\lambda\left(x_{(i)j}\right)^{-\beta}}\left(x_{(i)j}\right)^{-\beta}}{1-e^{-\lambda\left(x_{(i)j}\right)^{-\beta}}} \\
& + \sum_{j=1}^{r}\sum_{i=1}^{m}(k(i)-k(i-1)-1)\times\frac{\partial Q}{\partial\lambda}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial\ell(\lambda,\beta)}{\partial\beta} =& \frac{mr}{\beta} - \sum_{j=1}^{r}\sum_{i=1}^{m}\log x_{(i)j} + \lambda\sum_{j=1}^{r}\sum_{i=1}^{m}(i)\left(x_{(i)j}\right)^{-\beta}\log x_{(i)j} \\
& - \sum_{j=1}^{r}\sum_{i=1}^{m}(m-i)\frac{e^{-\lambda\left(x_{(i)j}\right)^{-\beta}}\lambda\left(x_{(i)j}\right)^{-\beta}\log x_{(i)j}}{1-e^{-\lambda\left(x_{(i)j}\right)^{-\beta}}} \\
& + \sum_{j=1}^{r}\sum_{i=1}^{m}(k(i)-k(i-1)-1)\frac{\partial Q}{\partial\beta}
\end{aligned}
$$

where $Q = \log\left[\sum_{t=i}^{m}\binom{m}{t}\left[e^{-\lambda x_{(i)j}^{-\beta}}\right]^{t}\left[1-e^{-\lambda x_{(i)j}^{-\beta}}\right]^{m-t} - \sum_{t=i-1}^{m}\binom{m}{t}\left[e^{-\lambda x_{(i)j}^{-\beta}}\right]^{t}\left[1-e^{-\lambda x_{(i)j}^{-\beta}}\right]^{m-t}\right]$

5 Simulation Study

Sample units generated by the proposed sampling designs only become order statistics when the ranking process is done without any error (perfect ranking). Because of this, the RSS-based designs will produce sample units that are neither independent nor identically distributed which makes it difficult to analytically derive some of the properties of their respective estimators (see [7]). Therefore, an extensive simulation study was conducted to evaluate the derived MLEs performance and compare their performance with other RSS-based designs estimators' performance. The Monte Carlo simulation is made for the IW distribution with different parameter values to ensure a wide range of shapes of the IW distribution, namely IW(0.5,0.5), IW(0.5, 1.5), IW(1.5, 1.5) and IW(1, 4). Figure 4 shows the density function for the IW distribution for the initial parameter values used in the simulation. The simulation is made for samples of sizes

3, 4, 5 and 6 and 10,000 replications. Let $\hat{\theta}_k$·be the kth sample estimator generated by a particular RSS based sampling design $k = 1, 2, ..., 10,000$. The comparison were made using two criteria's, the relative bias (RB) and mean square errors (MSE), which are calculated as follows:

$$RB = \sum_{i=1}^{10,000} \frac{\hat{\theta} - \theta}{\theta}; \; MSE\left(\hat{\theta}\right) = \frac{1}{10,000} \sum_{k=1}^{10,000} \left(\hat{\theta} - \theta\right).$$

The relative efficiency (RE) to SRS estimators was calculated for each RSS-based design, by

$$RE\left(\hat{\theta}\right) = \frac{MSE_{RSS}\left(\hat{\theta}\right)}{MSE_{SRS}\left(\hat{\theta}\right)}.$$

All simulations were performed using routines developed by the authors in the R environment for statistical computing. Simulation results are shown in Tables 1 and 2. Also Fig. 5 shows the performance of the different RSS designs for different parameters.

From figures and tables it can be noticed that:

1. DNRSS presents a slightly better performance than NRSS and RSS.
2. As the sample size increases the relative bias decreases for β for all scheme.
3. As λ and α decreases and the sample sizes increase, the performance of the estimators of λ, and β for different designs become higher.
4. DNRSS design provide more efficient estimator than NRSS and RSS estimator for all the distribution parameters.

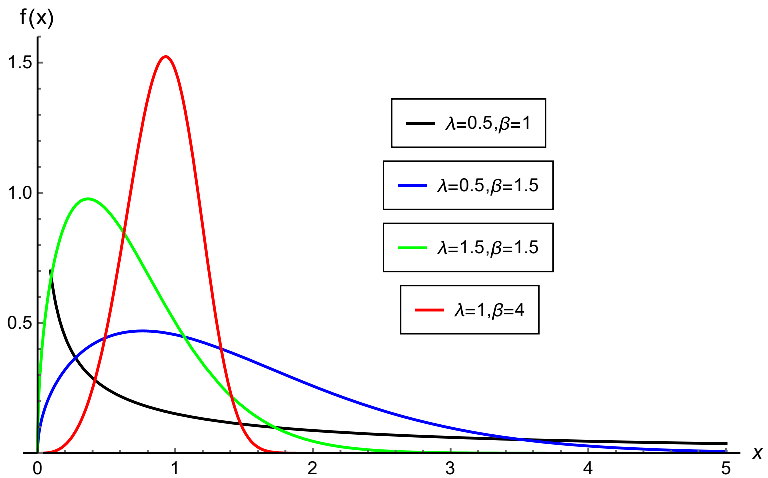


Fig. 4 The density function of the IW distribution for different parameter values

Table 1 Relative efficiency for RSS-based estimators compared to SRS based estimators under perfect ranking

IW(λ, β)	m	RSS		NRSS		DNRSS	
		λ	β	λ	β	λ	β
IW(0.5,0.5)	3	1.46630	1.21132	2.81259	3.78537	3.88623	5.82571
	4	2.14430	2.03357	3.98699	4.15773	5.96501	7.48505
	5	2.94103	2.49280	5.092747	5.22924	9.68038	11.11363
	6	2.72526	3.11749	5.73776	6.63662	11.63555	17.72417
IW(0.5,1.5)	3	1.27266	1.21184	3.85065	2.45862	3.38222	5.53623
	4	2.78971	1.35217	4.10789	3.18146	6.48985	8.39856
	5	2.85852	1.53746	4.95179	4.53329	7.47875	13.04239
	6	3.66195	2.05133	5.37870	5.39773	9.48357	18.85748
IW(1.5,1.5)	3	1.54470	1.50546	2.44404	2.62619	5.44733	2.65695
	4	2.26704	2.13102	2.74003	3.32870	8.58636	5.38930
	5	3.2982	3.03945	3.07941	4.32797	9.69485	8.09901
	6	4.28763	5.64471	4.63618	6.60479	14.14814	10.43904
IW(1, 4)	3	1.61446	1.63172	2.81621	3.83168	4.89325	5.50302
	4	1.75145	2.51913	3.17727	4.27652	6.18853	9.75309
	5	2.76019	2.92192	4.21255	4.42390	11.05959	12.82770
	6	3.51309	4.05778	6.20407	5.83183	16.73780	23.11162

Table 2 Relative bias for RSS-based estimators under perfect ranking

IW(λ, β)	m	RSS		NRSS		DNRSS	
		λ	β	λ	β	λ	β
IW(0.5,0.5)	3	0.27122	− 0.22457	0.38039	0.06428	0.03968	− 0.06339
	4	0.09601	− 0.19283	0.43373	0.03593	0.05381	− 0.05225
	5	− 0.30402	− 0.16946	0.43982	− 0.02678	0.05446	− 0.05267
	6	0.26262	− 0.16223	0.44115	− 0.02196	0.04867	− 0.04424
IW(0.5,1.5)	3	0.40368	− 0.98366	0.42522	− 0.66853	0.03109	− 0.16927
	4	0.01718	0.88135	0.43788	− 0.53550	0.04939	− 0.16079
	5	0.12139	− 0.65108	0.43722	− 0.41244	0.05332	− 0.15272
	6	0.07200	− 0.61769	0.47304	− 0.30432	0.05385	− 0.14612
IW(1.5,1.5)	3	− 0.14385	− 0.63255	0.65418	− 0.85589	− 0.04055	− 0.18596
	4	− 0.14363	− 0.54678	0.66840	− 0.71787	0.06603	− 0.17516
	5	− 0.24544	− 0.41382	0.70064	− 0.39076	0.06118	− 0.16921
	6	− 0.16207	0.26941	0.71791	− 0.39026	0.04126	− 0.14706
IW(1, 4)	3	− 0.29394	− 0.83746	0.93404	− 1.67601	0.01389	− 0.41074
	4	0.70578	− 0.77391	0.86085	− 0.99475	0.04934	− 0.39255
	5	− 0.18131	− 0.61650	0.88302	− 0.73519	0.058804	− 0.35885
	6	0.17585	− 0.31411	0.91377	− 0.62382	0.05521	− 0.35025

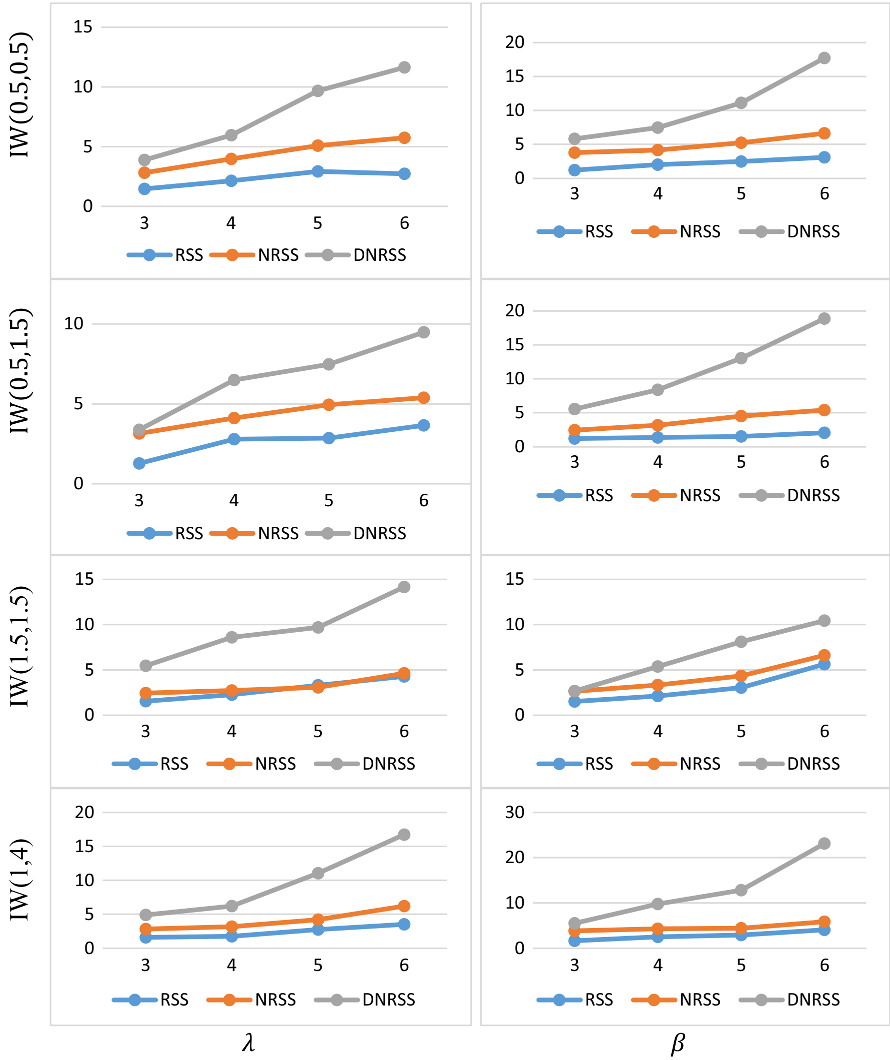


Fig. 5 Shows the RE of the different RSS designs for the different parameters

5. The relative efficiency from NRSS to the design with best performance more than RSS.
6. The efficiency of both RSS and NRSS for some sample sizes are nearly close but the overall performance of DNRSS is higher than the NRSS design.
7. Regarding the distribution shape, as the distribution becomes almost symmetric the RE is always higher than the RE for the other shapes of the distribution.

6 Conclusions

In this paper, we have derived the likelihood function for the DNRSS design and compare it with the RSS and DNRSS designs. Moreover, the MLE for IW distribution based on SRS, RSS, NRSS and DNRSS has been done. An intensive numerical comparison between the SRS and different RSS deigns is done and showed that the DNRSS is more efficient for all values for the scale parameter and the two shape parameters of the IW distribution. we found that the maximum likelihood estimation based on DNRSS proposed by Taconeli and Cabral [7] provides slightly more efficient estimators than the likelihood estimation based on the NRSS designs proposed by Zamanzade and Al-Omari [2] in case of inverted Weibull distribution.

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