Annals of Data Science (2020) 7(1):163–193
https://doi.org/10.1007/s40745-019-00197-5

Bivariate Weibull Distribution: Properties and Different Methods of Estimation

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Received: 4 October 2018 / Revised: 20 March 2019 / Accepted: 21 March 2019 /
Published online: 28 March 2019

Abstract
The bivariate Weibull distribution is an important lifetime distribution in survival analysis. In this paper, Farlie–Gumbel–Morgenstern (FGM) copula and Weibull marginal distribution are used for creating bivariate distribution which is called FGM bivariate Weibull (FGMBW) distribution. FGMBW distribution is used for describing bivariate data that have weak correlation between variables in lifetime data. It is a good alternative to bivariate several lifetime distributions for modeling real-valued data in application. Some properties of the FGMBW distribution are obtained such as product moment, skewness, kurtosis, moment generation function, reliability function and hazard function. Three different estimation methods for parameters estimation are discussed for FGMBW distribution namely; maximum likelihood estimation, inference function for margins method and semi-parametric method. To evaluate the performance of the estimators, a Monte Carlo simulations study is conducted to compare the preferences between estimation methods. Also, a real data set is introduced, analyzed to investigate the model and useful results are obtained for illustrative purposes.

Keywords Weibull distribution · FGM copula · Maximum likelihood estimation · Inference function for margins and semi-parametric

1 Introduction

The Weibull distribution has been attained more attention in the literature and has inherent flexibility. The univariate Weibull distribution has the following cdf and pdf respectively

$$F(y;\alpha,\beta) = 1 - e^{-\left(\frac{y}{\beta}\right)^{\alpha}}; \quad y > 0, \quad \alpha, \beta > 0, \tag{1.1}$$

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and

$$f(y;\alpha,\beta) = \frac{\alpha}{\beta}\left(\frac{y}{\beta}\right)^{\alpha-1} e^{-\left(\frac{y}{\beta}\right)^{\alpha}}; \quad y > 0, \quad \alpha,\beta > 0, \tag{1.2}$$

where β, α are the scale and shape parameters respectively.

In bivariate Weibull, recent researches have been made for the bivariate Weibull distribution. Galiani [6] concluded that bivariate Weibull are specifically oriented towards applications in economics, finance and risk management. Flores [4] used Weibull marginal to construct bivariate Weibull distributions. Kundu and Gupta [13] introduced the Marshall–Olkin bivariate Weibull distribution.

A copula is a convenient approach for description of a multivariate distribution. Nelsen [16] introduced Copulas as following; copula is function that join multivariate distribution functions with uniform [0, 1] margins. A copula is a convenient approach to describe a multivariate distribution with dependence structure. The n-dimensional copula (C) exists for all $y_1, \ldots, y_n$, $F(y_1, \ldots, y_n) = C(F_1(y_1), \ldots, F_n(y_n))$, if F is continuous, then C is uniquely defined.

Sklar [18] states that, considered the two random variables Y_1 and Y_2, with distribution functions $F_1(y_1)$ and $F_2(y_2)$ the following cdf and pdf for copula are respectively

$$F(y_1, y_2) = C(F_1(y_1), F_2(y_2)), \tag{1.3}$$

and

$$f(y_1, y_2) = f_1(y_1) f_2(y_2) c(F_1(y_1), F_2(y_2)). \tag{1.4}$$

Farlie–Gumbel–Morgenstern (FGM) is one of the most popular parametric families of copulas, the family was discussed by Gumbel [8] The joint cdf and joint pdf for FGM copula as following respectively

$$C(y_1, y_2) = F_1(y_1)F_2(y_2)\left(1 + \theta(1 - F_1(y_1))(1 - F_2(y_2))\right); \quad -1 < \theta < 1 \tag{1.5}$$

and

$$c(y_1, y_2) = \left(1 + \theta(1 - 2F_1(y_1))(1 - 2F_2(y_2))\right). \tag{1.6}$$

Figure 1 3-dimension for the pdf and cdf of FGM copula with different value of parameter of copula θ.

Fredricks and Nelsen [5] drives the formula for Spearman's and Kendall's correlation coefficient as follows

$$\rho_{sperman} = \left(12 \int\int uv(1 + \theta(1-u)(1-v))dudv\right) - 3 = \frac{\theta}{3}, \tag{1.7}$$

$$\rho_{\text{Kendall}} = 1 - 4 \int\int \frac{\partial C}{\partial u} C(u,v) \frac{\partial C}{\partial v} C(u,v) dudv = \frac{2}{9}\theta, \tag{1.8}$$

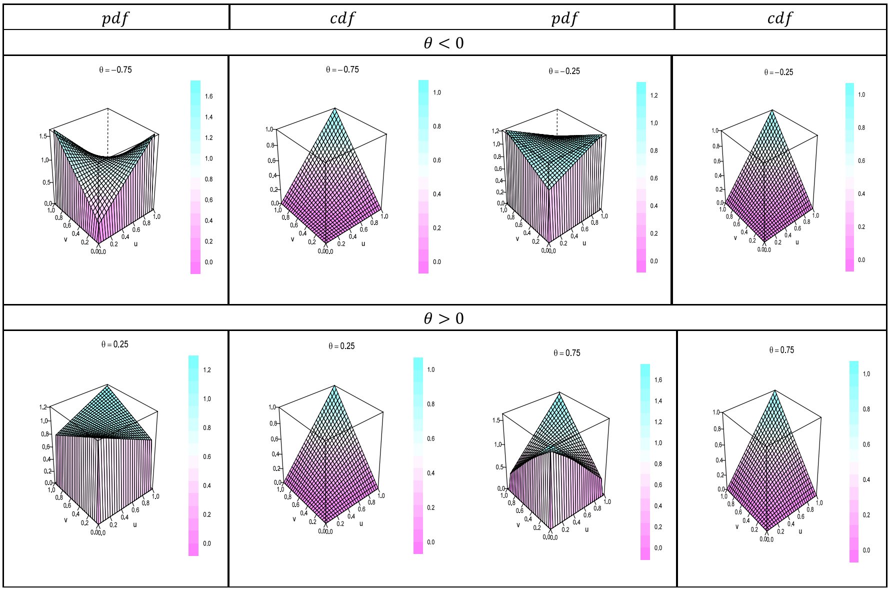

Fig. 1 FGM copula with various value of θ

Such that $\frac{-1}{3} \leq \rho_{sperman} \leq \frac{1}{3}, \frac{-2}{9} \leq \rho_{\text{Kendall}} \leq \frac{2}{9}$.

In this article, we study the bivariate extension of the Weibull distribution based on FGM copula function (FGMBW) and discuss its statistical properties. FGMBW distribution is used for describing bivariate data that have weak correlation between variables in lifetime data. It is a good alternative to bivariate several lifetime distributions for modeling non-negative real-valued data in application.

The objective of this article is twofold: to study the properties of the FGMBW distribution, and to estimate the parameters of the model by different estimation methods. The attractive feature of the marginal function of FGMBW distribution is the same as the basic distribution (Weibull). Other features of the FGMBW distribution: it contains closed forms for its cdf, product moment, moment generation function, and hazard rate function. The final motivation of the article is to develop a guideline for introducing the best estimation method for the FGMBW distribution, which we think would be of deep interest to statisticians. A simulation study is conducted to compare the preferences between estimation methods. Also, a real data set is introduced and analyzed to investigate the model. The uniqueness of this study comes from the fact that we introduce a comprehensive description of mathematical and statistical properties of FGMBW distribution with the hope that they will attract wider applications in medicine, economics, life testing and other areas of research.

The rest of this paper is organized as follows: FGM bivariate Weibull distribution is obtain in Sect. 2. Some statistical properties of FGMBW distribution in Sect. 3. Parameter estimation methods for the FGMBW distribution based in copula in Sect. 4. In Sect. 5, asymptotic confidence intervals are discussed. In Sect. 6, the potentiality of the new model is illustrated by simulation study. In Sect. 7, Application of real data are discussed. Finally, Conclusion of some remarks for FGMBW model are addressed in Sect. 8.

2 FGM Bivariate Weibull Distribution

According to Sklar theorem the joint pdf of bivariate Weibull distribution for any copula is as follows

$$f(y_1,y_2)=\frac{\alpha_1}{\beta_1}\left(\frac{y_1}{\beta_1}\right)^{\alpha_1-1}e^{-\left(\frac{y_1}{\beta_1}\right)^{\alpha_1}}\frac{\alpha_2}{\beta_2}\left(\frac{y_2}{\beta_2}\right)^{\alpha_2-1} e^{-\left(\frac{y_2}{\beta_2}\right)^{\alpha_2}}c\left(\left(1-e^{-\left(\frac{y_1}{\beta_1}\right)^{\alpha_1}}\right),\quad\left(1-e^{-\left(\frac{y_2}{\beta_2}\right)^{\alpha_2}}\right)\right) \tag{2.1}$$

The cdf of a FGMBW distribution can be expressed as

$$F(y_1,y_2)=\left(1-e^{-\left(\frac{y_1}{\beta_1}\right)^{\alpha_1}}\right)\left(1-e^{-\left(\frac{y_2}{\beta_2}\right)^{\alpha_2}}\right)\times\left[1+\theta\left(1-\left(1-e^{-\left(\frac{y_1}{\beta_1}\right)^{\alpha_1}}\right)\right)\left(1-\left(1-e^{-\left(\frac{y_2}{\beta_2}\right)^{\alpha_2}}\right)\right)\right] \tag{2.2}$$

The pdf of a FGMBW distribution is defined as

$$f(y_1,y_2)=\frac{\alpha_1}{\beta_1}\left(\frac{y_1}{\beta_1}\right)^{\alpha_1-1}e^{-\left(\frac{y_1}{\beta_1}\right)^{\alpha_1}}\frac{\alpha_2}{\beta_2}\left(\frac{y_2}{\beta_2}\right)^{\alpha_2-1}e^{-\left(\frac{y_2}{\beta_2}\right)^{\alpha_2}}\times\left[1+\theta\left(1-2\left(1-e^{-\left(\frac{y_1}{\beta_1}\right)^{\alpha_1}}\right)\right)\left(1-2\left(1-e^{-\left(\frac{y_2}{\beta_2}\right)^{\alpha_2}}\right)\right)\right] \tag{2.3}$$

Figure 2 show the plot 3-dimension for the pdf and cdf of FGMBW distribution with different value of $\alpha_1,\beta_1,\alpha_2,\beta_2$ and θ (Fig. 3).

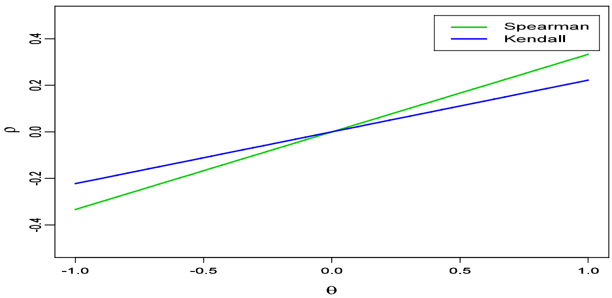

Fig. 2 Correlation of FGM copula with various value of copula parameter

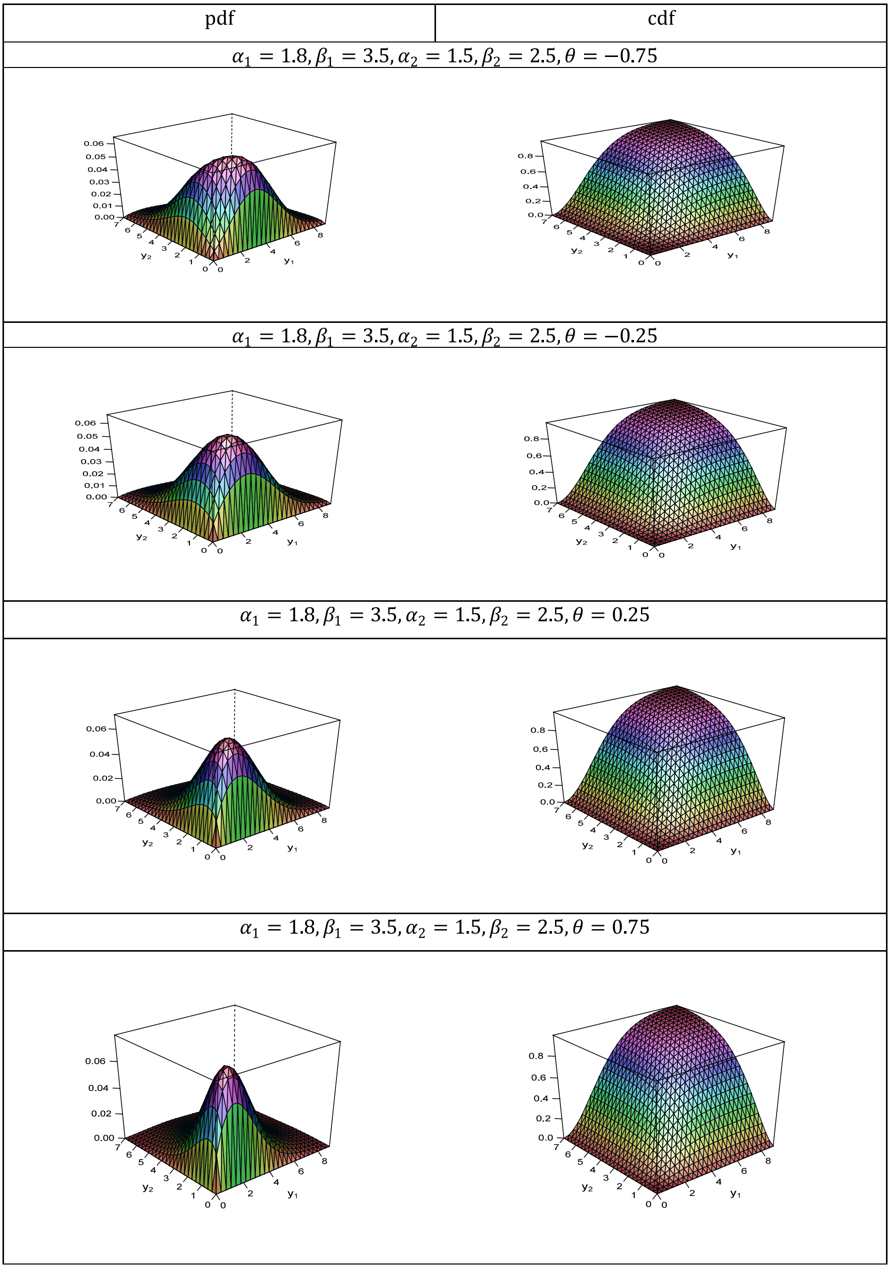

Fig. 3 The pdf and cdf of FGMBW distribution with various value of the parameters

3 Properties of FGMBW Distribution

In this section, we give some important statistical properties of the FGMBW distribution such as Marginal Distributions, product moments, moment generating function, conditional distribution, generating random variables, reliability function. Establishing algebraic expressions to determine some statistical properties of the FGMBW distribution can be more efficient than computing them directly by numerical simulation.

3.1 The Marginal Distributions

The marginal density functions for Y_1 and Y_2 respectively,

$$f(y_1;\alpha_1,\beta_1) = \frac{\alpha_1}{\beta_1}\left(\frac{y_1}{\beta_1}\right)^{\alpha_1-1} e^{-\left(\frac{y_1}{\beta_1}\right)^{\alpha_1}}; \quad y_1 > 0, \quad \alpha_1, \quad \beta_1 > 0, \tag{3.1}$$

$$f(y_2;\alpha_2,\beta_2) = \frac{\alpha_2}{\beta_2}\left(\frac{y_2}{\beta_2}\right)^{\alpha_2-1} e^{-\left(\frac{y_2}{\beta_2}\right)^{\alpha_2}}; \quad y_2 > 0, \quad \alpha_2, \quad \beta_2 > 0, \tag{3.2}$$

which are Weibull distributed, where the marginal distribution of Y_1 and Y_2 can be calculated directly by

$$f(y_i) = \int_{ally_j} f(y_1,y_2)dy_j; \quad i,j = 1,2, \quad i \neq j.$$

3.2 Conditional Distribution

The conditional probability distribution of Y_2 given Y_1 is given as follows

$$f(y_2|y_1) = \frac{\alpha_2}{\beta_2}\left(\frac{y_2}{\beta_2}\right)^{\alpha_2-1} e^{-\left(\frac{y_2}{\beta_2}\right)^{\alpha_2}} \left[1+\theta\left(1-2u(y_1)\right)\left(1-2u(y_2)\right)\right], \tag{3.3}$$

and the conditional cdf is

$$F(y_2|y_1) = u(y_2)\left[1-\theta+2\theta u(y_1)\right] + \theta v(y_2) - 2\theta u(y_1)v(y_2), \tag{3.4}$$

where $u(y_i) = 1 - e^{-\left(\frac{y_i}{\beta_i}\right)^{\alpha_i}}$ $i = 1,2$ and $v(y_i) = 1 - e^{-2\left(\frac{y_i}{\beta_i}\right)^{\alpha_i}}$ $i = 1,2$

3.3 Generating Random Variables

Nelsen [16] discussed generating a sample from a specified joint distribution. By conditional distribution method, the joint distribution function is as follows

$$f(y_1,y_2) = f(y_1)f(y_2|y_1)$$

By using the following steps, we can generate a bivariate sample by using the conditional approach:

1. Generate U and V independently from a uniform(0, 1) distribution.
2. Set $Y_1 = \beta_1[-\ln(1-U)]^{\alpha_1}$.
3. Set $F\left(y_2 \middle| y_1\right) = V$ to find Y_2 by numerical simulation.
4. Repeat Steps 1–3 (n) times to obtain $\left(y_{1i}, y_{2i}\right), i = 1, 2, \dots, n$.

3.4 Moment Generating Function

Let $\left(Y_1, Y_2\right)$ denote a random variable with the probability density function (2.3). Then, the moment generating function of $\left(Y_1, Y_2\right)$ is given by,

$$M_{\left(y_1, y_2\right)}\left(t_1 t_2\right) = \sum_{n=0}^{\infty}\left(\frac{t_1^n \beta_1^n}{n!}\Gamma\left(1+\frac{n}{\alpha_1}\right)\right)\sum_{m=0}^{\infty}\left(\frac{t_2^m \beta_2^m}{m!}\Gamma\left(1+\frac{m}{\alpha_2}\right)\right) \times\left[1+\theta-2\theta\frac{1}{2^{\left(1+\frac{m}{\alpha_2}\right)}}-2\theta\frac{1}{2^{\left(1+\frac{n}{\alpha_1}\right)}}+4\theta\frac{1}{2^{\left(1+\frac{n}{\alpha_1}\right)}}\frac{1}{2^{\left(1+\frac{m}{\alpha_2}\right)}}\right], \tag{3.5}$$

To prove the moment generating function start with

$$\mathrm{M}_{\left(y_1, y_2\right)}\left(t_1 t_2\right) = E\left(e^{t_1 y_1} e^{t_2 y_2}\right) = \int_0^{\infty}\int_0^{\infty} e^{t_1 y_1} e^{t_2 y_2} f\left(y_1, y_2\right) dy_1 dy_2$$

3.5 Product Moments

If the random variable $\left(Y_1, Y_2\right)$ is distributed as FGMBW, then its rth and sth moments around zero can be expressed as follows

$$\mu'_{rs} = \left(\beta_1^r \Gamma\left(\frac{r}{\alpha_1}+1\right)\beta_2^s \Gamma\left(\frac{s}{\alpha_2}+1\right)\right)\left[1+\theta-\frac{\theta}{2^{\left(s/\alpha_2\right)}}-\frac{\theta}{2^{\left(r/\alpha_1\right)}}+\frac{\theta}{2^{\left(s/\alpha_2\right)}2^{\left(r/\alpha_1\right)}}\right] \tag{3.6}$$

To prove that start with

$$\mu'_{rs} = E\left(Y_1^r Y_2^s\right) = \int_0^{\infty}\int_0^{\infty} y_1^r y_2^s f\left(y_1, y_2\right) dy_1 dy_2$$

We use Mardia's [14] measures of multivariate and bivariate skewness (SK) and kurtosis (KU) (Table 1). Mardia defined bivariate SK and KU, respectively, as

$$\mathrm{SK} = \left(1-\rho^2\right)^{-3}\left[\gamma_{30}^2+\gamma_{03}^2+3\left(1+2\rho^2\right)\left(\gamma_{12}^2+\gamma_{21}^2\right)-2\rho^3\gamma_{30}\gamma_{03}+6\rho\left\{\gamma_{30}\left(\rho\gamma_{12}-\gamma_{21}\right)+\gamma_{03}\left(\rho\gamma_{21}-\gamma_{12}\right)-\left(2+\rho^2\right)\gamma_{21}\gamma_{12}\right\}\right] \tag{3.7}$$

$$\mathrm{KU} = \frac{\gamma_{40}+\gamma_{04}+2\gamma_{22}+4\rho\left(\rho\gamma_{22}-\gamma_{13}-\gamma_{31}\right)}{\left(1-\rho^2\right)^2} \tag{3.8}$$

Table 1 Covariance, skewness, and kurtosis of FGMBW distribution

θ	cov	ρ	SK	KU
$\alpha_1 = 1.8,$	$\beta_1 = 3.5,$	$\alpha_2 = 1.5,\quad \beta_2 = 2.5$		
−1	−0.8308	−0.3030	1.8771	−39.8120
−0.8	−0.6646	−0.2424	1.8317	−27.0918
−0.6	−0.4985	−0.1818	1.7982	−16.3708
−0.4	−0.3323	−0.1212	1.7747	−6.9536
−0.2	−0.1662	−0.0606	1.7605	1.6790
0	0.0000	0.0000	1.7556	9.9473
0.2	0.1662	0.0606	1.7608	18.2257
0.4	0.3323	0.1212	1.7777	26.8865
0.6	0.4985	0.1818	1.8086	36.3420
0.8	0.6646	0.2424	1.8572	47.0963
1	0.8308	0.3030	1.9292	59.8180
$\alpha_1 = 1.5,$	$\beta_1 = 1.2,$	$\alpha_2 = 2.1,\quad \beta_2 = 2.8$		
−1	−0.2795	−0.3063	1.4634	−20.5682
−0.8	−0.2236	−0.2450	1.4755	−12.8286
−0.6	−0.1677	−0.1838	1.4774	−6.3369
−0.4	−0.1118	−0.1225	1.4751	−0.6539
−0.2	−0.0559	−0.0613	1.4722	4.5456
0	0.0000	0.0000	1.4709	9.5227
0.2	0.0559	0.0613	1.4725	14.5085
0.4	0.1118	0.1225	1.4773	19.7317
0.6	0.1677	0.1838	1.4852	25.4453
0.8	0.2236	0.2450	1.4948	31.9583
1	0.2795	0.3063	1.5027	39.6819

where $\gamma_{rs} = \frac{\mu_{rs}}{\sigma_1^r \sigma_2^s}$, $\rho = corr\left(Y_1, Y_2\right) = \frac{E((Y_1 - E(Y_1))((Y_2 - E(Y_2))))}{\sigma_1 \sigma_2}$ where μ_{rs}, is the central moment of order (r, s) of $\left(Y_1, Y_2\right)$, $\sigma_1 = \sqrt{E\left(Y_1 - E\left(Y_1\right)\right)^2}$ and $\sigma_2 = \sqrt{E\left(Y_2 - E\left(Y_2\right)\right)^2}$

$$\mu_{rs} = E\left(\left(Y_1 - E\left(Y_1\right)\right)^r \left(\left(Y_2 - E\left(Y_2\right)\right)\right)^s\right)$$
$$= \int_0^\infty \int_0^\infty \left(y_1 - E\left(y_1\right)\right)^r \left(\left(y_2 - E\left(y_2\right)\right)\right)^s f\left(y_1, y_2\right) dy_1 dy_2$$

3.6 Reliability Function

Osmetti and Chiodini [17] discussed that the reliability function is more convenient to express a joint survival function as a copula of its marginal survival functions, where Y_1 and Y_2 be random variable with survival functions $\bar{F}\left(y_1\right)$ and $\bar{F}\left(y_2\right)$ as following.

The reliability function of the marginal distributions is defined as

$$R(y_j;\alpha_j,\beta_j)=1-F(y_j;\alpha_j,\beta_j)=e^{-\left(\frac{y_j}{\beta_j}\right)^{\alpha_j}};\quad y>0,\quad \alpha,\beta>0,\quad j=1,2$$

The expression of the joint survival function for copula is as following

$$R(y_1,y_2)=C(R(y_1),R(y_2))$$

Then the reliability function of FGMBW distribution is

$$R(y_1,y_2)=e^{-\left(\frac{y_1}{\beta_1}\right)^{\alpha_1}}e^{-\left(\frac{y_2}{\beta_2}\right)^{\alpha_2}}\left[1+\theta\left(1-e^{-\left(\frac{y_1}{\beta_1}\right)^{\alpha_1}}\right)\left(1-e^{-\left(\frac{y_2}{\beta_2}\right)^{\alpha_2}}\right)\right] \tag{3.9}$$

Basu [1] defined the bivariate failure rate function for the first time as

$$h(y_1,y_2)=\frac{f(y_1,y_2)}{R(y_1,y_2)} \quad (.)$$

Then the hazard rate function of FGMBW distribution is

$$h(y_1,y_2)=\frac{\frac{\alpha_1}{\beta_1}\left(\frac{y_1}{\beta_1}\right)^{\alpha_1-1}\frac{\alpha_2}{\beta_2}\left(\frac{y_2}{\beta_2}\right)^{\alpha_2-1}\left[1+\theta\left(-1+2e^{-\left(\frac{y_1}{\beta_1}\right)^{\alpha_1}}\right)\left(-1+2e^{-\left(\frac{y_2}{\beta_2}\right)^{\alpha_2}}\right)\right]}{\left[1+\theta\left(1-e^{-\left(\frac{y_1}{\beta_1}\right)^{\alpha_1}}\right)\left(1-e^{-\left(\frac{y_2}{\beta_2}\right)^{\alpha_2}}\right)\right]} \tag{3.10}$$

4 Estimation Based on Copulas

In the section, we introduce different estimation methods that used to estimate the parameters of FGMBW distribution, such as: maximum likelihood estimation (MLE), inference functions for margins (IFM) and semi-parametric method (SP). To more information about these methods see Chen [2], Tsukahara [19] and Weiß [20].

4.1 Maximum Likelihood Estimation (MLE)

Elaal and Jarwan [3], discussed the maximum likelihood estimator to estimate all model parameters jointly, it is a one-step parametric method. Therefore, the log-likelihood is given as

$$\ln L=\sum_{j=1}^{n}\left[\ln\left(f_1(y_{1j})f_2(y_{2j})c(F_1(y_{1j},\delta_1),F_2(y_{2j},\delta_2);\theta)\right)\right]$$

The parameter estimates are obtained by maximizing the log-likelihood function with expect to each parameter separately. Considering Eqs. (2.3) and (1.6), let

$$a(y_j;\alpha_j,\beta_j)=1-2\left(1-e^{-\left(\frac{y_j}{\beta_j}\right)^{\alpha_j}}\right),\quad j=1,2.$$

The likelihood function of a FGMBW distribution is defined as

$$L=\left(\frac{\alpha_2\alpha_1}{\beta_2\beta_1}\right)^n\prod_{i=1}^{n}\left(\left(\frac{y_{1i}}{\beta_1}\right)^{\alpha_1-1}\left(\frac{y_{2i}}{\beta_2}\right)^{\alpha_2-1}\right)e^{-\sum_{i=1}^{n}\left(\frac{y_{1i}}{\beta_1}\right)^{\alpha_1}-\sum_{i=1}^{n}\left(\frac{y_{2i}}{\beta_2}\right)^{\alpha_2}}$$
$$\times\prod_{i=1}^{n}\left(1+\theta\left(a\left(y_{1i},\alpha_1,\beta_1\right)\right)\left(a\left(y_{2i},\alpha_2,\beta_2\right)\right)\right)$$

and the log-likelihood function can be written as

$$\begin{aligned}\ln L=&\,n(\ln\alpha_1-\ln\beta_1)+n(\ln\alpha_2-\ln\beta_2)+\left(\alpha_1-1\right)\sum_{i=1}^{n}\ln\left(\frac{y_{1i}}{\beta_1}\right)\\&-\sum_{i=1}^{n}\left(\frac{y_{1i}}{\beta_1}\right)^{\alpha_1}+\left(\alpha_2-1\right)\sum_{i=1}^{n}\ln\left(\frac{y_{2i}}{\beta_2}\right)\\&-\sum_{i=1}^{n}\left(\frac{y_{2i}}{\beta_2}\right)^{\alpha_2}+\sum_{i=1}^{n}\ln\left(1+\theta\left(a\left(y_{1i},\alpha_1,\beta_1\right)\right)\left(a\left(y_{2i},\alpha_2,\beta_2\right)\right)\right)\end{aligned}\tag{4.1}$$

The estimates of all parameters are obtained by differentiating the log-likelihood function in (4.1) with respect to each parameter separately, as following

$$\begin{aligned}\frac{\partial L}{\partial\alpha_1}=&\,\frac{n}{\alpha_1}+\sum_{i=1}^{n}\ln\left(\frac{y_{1i}}{\beta_1}\right)-\sum_{i=1}^{n}\left(\frac{y_{1i}}{\beta_1}\right)^{\alpha_1}\ln\frac{y_{1i}}{\beta_1}\\&+\sum_{i=1}^{n}\frac{-2\theta\left(a\left(y_{2i},\alpha_2,\beta_2\right)\right)e^{-\left(\frac{y_{1i}}{\beta_1}\right)^{\alpha_1}}\left(\frac{y_{1i}}{\beta_1}\right)^{\alpha_1}\ln\frac{y_{1i}}{\beta_1}}{\left(1+\theta\left(a\left(y_{1i},\alpha_1,\beta_1\right)\right)\left(a\left(y_{2i},\alpha_2,\beta_2\right)\right)\right)}\\\frac{\partial L}{\partial\alpha_2}=&\,\frac{n}{\alpha_2}+\sum_{i=1}^{n}\ln\left(\frac{y_{2i}}{\beta_2}\right)-\sum_{i=1}^{n}\left(\frac{y_{2i}}{\beta_2}\right)^{\alpha_2}\ln\frac{y_{2i}}{\beta_2}\\&+\sum_{i=1}^{n}\frac{-2\theta\left(a\left(y_{1i},\alpha_1,\beta_1\right)\right)e^{-\left(\frac{y_{2i}}{\alpha_2}\right)^{\beta_2}}\left(\frac{y_{2i}}{\beta_2}\right)^{\alpha_2}\ln\frac{y_{2i}}{\beta_2}}{\left(1+\theta\left(a\left(y_{1i},\alpha_1,\beta_1\right)\right)\left(a\left(y_{2i},\alpha_2,\beta_2\right)\right)\right)}\\\frac{\partial L}{\partial\beta_1}=&\,\frac{-n}{\beta_1}-\frac{\left(\alpha_1-1\right)}{\beta_1}-\frac{\alpha_1}{\beta_1}\sum_{i=1}^{n}\left(\frac{y_{1i}}{\beta_1}\right)^{\alpha_1}\\&+\sum_{i=1}^{n}\frac{2\alpha_1\theta y_{1i}\left(\frac{y_{1i}}{\beta_1}\right)^{\alpha_1-1}e^{-\left(\frac{y_{1i}}{\beta_1}\right)^{\alpha_1}}\left(a\left(y_{2i},\alpha_2,\beta_2\right)\right)}{\beta_1^2\left(1+\theta\left(a\left(y_{1i},\alpha_1,\beta_1\right)\right)\left(a\left(y_{2i},\alpha_2,\beta_2\right)\right)\right)}\\\frac{\partial L}{\partial\beta_2}=&\,\frac{-n}{\beta_2}-\frac{n\left(\alpha_2-1\right)}{\beta_2}-\frac{\alpha_2}{\beta_2}\sum_{i=1}^{n}\left(\frac{y_{2i}}{\beta_2}\right)^{\alpha_2}\\&+\sum_{i=1}^{n}\frac{2\alpha_2\theta y_{2i}\left(\frac{y_{2i}}{\beta_2}\right)^{\alpha_2-1}e^{-\left(\frac{y_{2i}}{\beta_2}\right)^{\alpha_2}}\left(a\left(y_{1i},\alpha_1,\beta_1\right)\right)}{\beta_2^2\left(1+\theta\left(a\left(y_{1i},\alpha_1,\beta_1\right)\right)\left(a\left(y_{2i},\alpha_2,\beta_2\right)\right)\right)}\end{aligned}$$

and

$$\frac{\partial L}{\partial \theta}=\sum_{i=1}^{n}\frac{\left(a\left(y_{1i},\alpha_1,\beta_1\right)\right)\left(a\left(y_{2i},\alpha_2,\beta_2\right)\right)}{\left(1+\theta\left(a\left(y_{1i},\alpha_1,\beta_1\right)\right)\left(a\left(y_{2i},\alpha_2,\beta_2\right)\right)\right)}.$$

The MLE $\hat{\delta}=\left(\hat{\alpha}_1,\hat{\beta}_1,\hat{\alpha}_2,\hat{\beta}_2,\hat{\theta}\right)$ can be obtained by solving simultaneously the likelihood equations

$$\left.\frac{\partial L}{\partial \theta}\right|_{\theta=\hat{\theta}}=0,\quad \left.\frac{\partial L}{\partial \beta_j}\right|_{\beta_j=\hat{\beta}_j}=0,\quad \left.\frac{\partial L}{\partial \alpha_j}\right|_{\alpha_j=\hat{\alpha}_j}=0,\quad j=1,2$$

But the equations has to be performed numerically using a nonlinear optimization algorithm.

4.2 Estimation by Inference Functions for Margins (IFM)

Joe [9] introduced this parametric method with two-step of estimation. In the first step, each marginal distribution is estimated separately.

$$\ln L_1=\sum_{j=1}^{n}\ln f_1\left(y_{1j},\delta_1\right);\quad \ln L_2=\sum_{i=1}^{n}\ln f_2\left(y_{2j},\delta_2\right) \tag{4.2}$$

Then, in the second step the copula parameter is estimated by maximizing the log-likelihood function of the copula density using the ML estimates of the marginal $\hat{F}_1\left(y_{1j},\delta\right)$ and $\hat{F}_2\left(y_{2j},\delta\right)$. Considering the Eq. (2.3), the log likelihood function of a Weibull distribution is defined as

$$\ln L_j=n(\ln\alpha_j-\ln\beta_j)+\left(\alpha_j-1\right)\sum_{i=1}^{n}\ln\left(\frac{y_{ji}}{\beta_j}\right)-\sum_{i=1}^{n}\left(\frac{y_{ji}}{\beta_j}\right)^{\alpha_j};j=1,2 \tag{4.3}$$

The MLEs $\left(\hat{\alpha}_1,\hat{\beta}_1,\hat{\alpha}_2,\hat{\beta}_2\right)$ can be obtained by solving simultaneously the likelihood equations

$$\left.\frac{\partial \ln L}{\partial \beta_j}\right|_{\beta_j=\hat{\beta}_j}=0,\quad \left.\frac{\partial \ln L}{\partial \alpha_j}\right|_{\alpha_j=\hat{\alpha}_j}=0,\quad j=1,2$$

then

$$\hat{F}_j\left(y_j\right)=1-e^{-\left(\frac{y_j}{\hat{\beta}_j}\right)^{\hat{\alpha}_j}};\quad j=1,2$$

and considering the previous step, the IFM estimate of a FGMBW distribution is defined as

$$\ln L_{IFM} = \sum_{i=1}^{n} \ln \left(1 + \theta\left(1 - 2\hat{F}_1\left(y_{1\mathrm{i}}\right)\left(1 - 2\hat{F}_2\left(y_{2i}\right)\right)\right)\right) \tag{4.4}$$

The estimates of all parameters are obtained by differentiating the log-likelihood function in (4.4) with respect to each parameter separately. Basing on this, differentiating the log-likelihood function with respect to θ is given as

$$\frac{\partial \ln L_{IFM}}{\partial \theta} = \sum_{i=1}^{n} \frac{\left(a\left(y_{1i}, \hat{\alpha}_1, \hat{\beta}_1\right)\right)\left(a\left(y_{2i}, \hat{\alpha}_2, \hat{\beta}_2\right)\right)}{\left(1 + \theta\left(a\left(y_{1i}, \hat{\alpha}_1, \hat{\beta}_1\right)\right)\left(a\left(y_{2i}, \hat{\alpha}_2, \hat{\beta}_2\right)\right)\right)}$$

The estimates of parameters are handled numerically simultaneously the likelihood equations

$$\left.\frac{\partial \ln L_{IFM}}{\partial \theta}\right|_{\theta=\hat{\theta}} = 0$$

There is no closed-form expression for the MLE $\hat{\theta}$ and its computation has to be performed numerically using a nonlinear optimization algorithm.

4.3 Estimation by Semi-Parametric Method [SP]

Kim et al. [10] introduced Estimation that carried out in two stages as in IFM, but the difference is that the marginal distributions are estimated non-parametrically by their sample empirical distributions. In this method, the observations are transformed into pseudo-observations using the empirical distribution function of each marginal distribution. The empirical distribution function is defined as

$$\tilde{F}_i\left(y_i\right) = \frac{\sum_{j=1}^{n} I\left(Y_{i,j} \leq y_i\right)}{n+1}; \quad i = 1, 2 \tag{4.5}$$

Then, θ is estimated by the maximizer of the pseudo loglikelihood,

$$\sum_{i=1}^{n} \ln c\left(\tilde{F}_1\left(Y_{1\mathrm{i}}\right), \tilde{F}_2\left(Y_{2\mathrm{i}}\right); \theta\right) \tag{4.6}$$

Considering the Eq. (4.6), the log likelihood function of a FGMBW distribution is defined as

$$\ln L_{SP} = \sum_{i=1}^{n} \ln \left(1 + \theta\left(1 - 2\tilde{F}_1\left(y_{1\mathrm{i}}\right)\left(1 - 2\tilde{F}_2\left(y_{2\mathrm{i}}\right)\right)\right)\right) \tag{4.7}$$

There is no closed-form expression for the MLE $\hat{\theta}$ by using of Eq. (4.7) and it computation has to be performed numerically using a statistical software.

5 Asymptotic Confidence Intervals

In this section, we propose the asymptotic confidence intervals using methods of estimations. Keeping this in mind, we may propose the asymptotic confidence intervals using ML, IFM, and SP methods can be used to construct the confidence intervals for the parameters. We first obtain $I\left(\hat{\alpha}_1, \hat{\beta}_1, \hat{\alpha}_2, \hat{\beta}_2, \hat{\theta}\right)$ which is the observed inverse Fishers information matrix and it is defined as:

$$
\begin{aligned}
I\left(\hat{\alpha}_1, \hat{\beta}_1, \hat{\alpha}_2, \hat{\beta}_2, \hat{\theta}\right) &= \begin{bmatrix} -L''_{\alpha_1\alpha_1} & -L''_{\alpha_1\beta_1} & -L''_{\alpha_1\alpha_2} & -L''_{\alpha_1\beta_2} & -L''_{\alpha_1\theta} \\ -L''_{\beta_1\alpha_1} & -L''_{\beta_1\beta_1} & -L''_{\beta_1\alpha_2} & -L''_{\beta_1\beta_2} & -L''_{\beta_1\theta} \\ -L''_{\alpha_2\alpha_1} & -L''_{\alpha_2\beta_1} & -L''_{\alpha_2\alpha_2} & -L''_{\alpha_2\beta_2} & -L''_{\alpha_2\theta} \\ -L''_{\beta_2\alpha_1} & -L''_{\beta_2\beta_1} & -L''_{\beta_2\alpha_2} & -L''_{\beta_2\beta_2} & -L''_{\beta_2\theta} \\ -L''_{\theta\alpha_1} & -L''_{\theta\beta_1} & -L''_{\theta\alpha_2} & -L''_{\theta\beta_2} & -L''_{\theta\theta} \end{bmatrix} \\
&= \begin{bmatrix} I_{\hat{\alpha}_1\hat{\alpha}_1} & I_{\hat{\alpha}_1\hat{\beta}_1} & I_{\hat{\alpha}_1\hat{\alpha}_2} & I_{\hat{\alpha}_1\hat{\beta}_2} & I_{\hat{\alpha}_1\hat{\theta}} \\ I_{\hat{\beta}_1\hat{\alpha}_1} & I_{\hat{\beta}_1\hat{\beta}_1} & I_{\hat{\beta}_1\hat{\alpha}_2} & I_{\hat{\beta}_1\hat{\beta}_2} & I_{\hat{\beta}_1\hat{\theta}} \\ I_{\hat{\alpha}_2\hat{\alpha}_1} & I_{\hat{\alpha}_2\hat{\beta}_1} & I_{\hat{\alpha}_2\hat{\alpha}_2} & I_{\hat{\alpha}_2\hat{\beta}_2} & I_{\hat{\alpha}_2\hat{\theta}} \\ I_{\hat{\beta}_2\hat{\alpha}_1} & I_{\hat{\beta}_2\hat{\beta}_1} & I_{\hat{\beta}_2\hat{\alpha}_2} & I_{\hat{\beta}_2\hat{\beta}_2} & I_{\hat{\beta}_2\hat{\theta}} \\ I_{\hat{\theta}\hat{\alpha}_1} & I_{\hat{\theta}\hat{\beta}_1} & I_{\hat{\theta}\hat{\alpha}_2} & I_{\hat{\theta}\hat{\beta}_2} & I_{\hat{\theta}\hat{\beta}_2} \end{bmatrix}
\end{aligned} \tag{5.1}
$$

An approximate 95% two side confidence intervals for $(\alpha_1, \beta_1, \alpha_2, \beta_2, \theta)$ are respectively

$$
\hat{\alpha}_i \pm Z_{0.025}\sqrt{I_{\hat{\alpha}_i\hat{\alpha}_i}}, \quad \hat{\beta}_i \pm Z_{0.025}\sqrt{I_{\hat{\beta}_i\hat{\beta}_i}}; \quad i = 1, 2, \quad \text{and} \quad \hat{\theta} \pm Z_{0.025}\sqrt{I_{\hat{\theta}\hat{\theta}}}
$$

6 Simulation Study

In this section; Monte Carlo simulation is done for comparison between estimation methods based on copula such as: MLE, IFM and SP. For estimating FGMBW distribution parameters by R language.

Simulation Algorithm Monte Carlo experiments were carried out based on the following data- generated form Weibull Distributions, where Y_1, Y_2 are distributed as Weibull with β_i shape parameters and α_i scale parameter, $i = 1, 2$ the values of the parameters $\alpha_1, \beta_1, \alpha_2, \beta_2$ and θ is chosen as the following cases for the random variables generating:

$$
\text{Case 1:} \quad \begin{cases} \left(\alpha_1 = 1.8, \quad \beta_1 = 3.5, \quad \alpha_2 = 1.5, \quad \beta_2 = 2.5, \quad \theta = 0.25\right) \\ \left(\alpha_1 = 1.8, \quad \beta_1 = 3.5, \quad \alpha_2 = 1.5, \quad \beta_2 = 2.5, \quad \theta = 0.75\right) \\ \left(\alpha_1 = 1.8, \quad \beta_1 = 3.5, \quad \alpha_2 = 1.5, \quad \beta_2 = 2.5, \quad \theta = -0.25\right) \\ \left(\alpha_1 = 1.8, \quad \beta_1 = 3.5, \quad \alpha_2 = 1.5, \quad \beta_2 = 2.5, \quad \theta = -0.75\right) \end{cases}
$$

$$
\text{Case 2:} \quad \begin{cases} \left(\alpha_1 = 1.5, \quad \beta_1 = 1.2, \quad \alpha_2 = 2.1, \quad \beta_2 = 2.8, \quad \theta = 0.25\right) \\ \left(\alpha_1 = 1.5, \quad \beta_1 = 1.2, \quad \alpha_2 = 2.1, \quad \beta_2 = 2.8, \quad \theta = 0.75\right) \end{cases}
$$

For different sample size $n = 30$, 50, 70, 100, 125 and 150. All computations are obtained based on the R language. The simulation methods are compared using the criteria of parameters estimation, the comparison is performed by calculate in the Bias, the MSE and the length of confidence interval (L.CI) for each method as following

$$Bias = \left(\hat{\delta} - \delta\right). \tag{6.1}$$

where $\hat{\delta}$ is the estimated value of δ.

$$MSE = Mean\left(\hat{\delta} - \delta\right)^2. \tag{6.2}$$

and

$$L.CI = Upper.CI - Lower.CI \tag{6.3}$$

We restricted the number of repeated-samples to 1000.

Based on Eqs. 1.7 and 1.8 for Spearman's and Kendall's correlation coefficient (Table 2).

On the basis of the results summarized in tables and figures, some conclusions can be drawn which are stated as follows: it is observed that as sample size increases and fixed vector value of δ, the Bias, MSE and Length of confidence interval of the estimates decreases in all the considered methods. In large sample size all of them are nearly equivalent, where the difference is less and there are no significant differences in Bias and MSE values for alternative methods and MLE method. The compare between parametric estimation and non-parametrically estimation have done. The parametric estimation methods are better than non-parametrically estimation method, when copula parameter is not high, approximately (−0.7:0.7). The SP method is discussed where, the marginal distributions are estimated non-parametrically by their sample empirical distributions and it estimated the copula parameter, this compare between parametric estimation and non-parametrically estimation, whenever the value of copula parameter is close to (−1 or 1), the efficiency will increase for SP method compared with other methods. IFM method is better than another methods, this is clear for copula parameter θ. It is noted that result, IFM method is the best method because it is a two steps of estimation, first, the marginal distribution parameters estimated and second the copula parameter is estimated, taking into consideration of previous parameter estimates of marginal distribution. That get more efficiency (Figs. 4, 5, Tables 3, 4, 5, 6, 7, 8).

On the basis of the results summarized in table, some conclusions can be drawn which are stated as follows: It is observed that as sample size increases for fixed vector values of δ the MSE of the estimates decreases in all the considered methods and for large size all of them are nearly equivalent but IFM performs better than another method when increase θ (parameter of copula). While θ decreases, the MLE is better than other methods Based on MSE term, but the SP method is discussed where, the marginal distributions are estimated non-parametrically by their sample empirical distributions and it estimated the parameter of copula, this compare between parametric estimation and non-parametrically estimation.

Table 2 Correlation spearman and kendall correlation of FGMBW distribution with various value of parameters

n	Case 1		Case 2		Case 3		Case 4		Case 5		Case 6	
	$\rho_{sperman}$	ρ_{Kendall}	$\rho_{sperman}$	ρ_{Kendall}	$\rho_{sperman}$	ρ_{Kendall}	$\rho_{sperman}$	ρ_{Kendall}	$\rho_{sperman}$	ρ_{Kendall}	$\rho_{sperman}$	ρ_{Kendall}
30	0.072	0.048	0.246	0.164	0.054	0.036	0.252	0.168	−0.227	−0.164	−0.074	−0.055
50	0.075	0.050	0.243	0.162	0.065	0.043	0.252	0.168	−0.229	−0.166	−0.073	−0.052
70	0.073	0.049	0.241	0.161	0.059	0.039	0.245	0.163	−0.236	−0.171	−0.081	−0.058
100	0.075	0.050	0.242	0.162	0.068	0.045	0.248	0.166	−0.226	−0.165	−0.081	−0.058
125	0.078	0.052	0.239	0.159	0.066	0.044	0.249	0.166	−0.229	−0.167	−0.080	−0.057
150	0.076	0.051	0.243	0.162	0.064	0.043	0.247	0.165	−0.227	−0.164	−0.075	−0.057

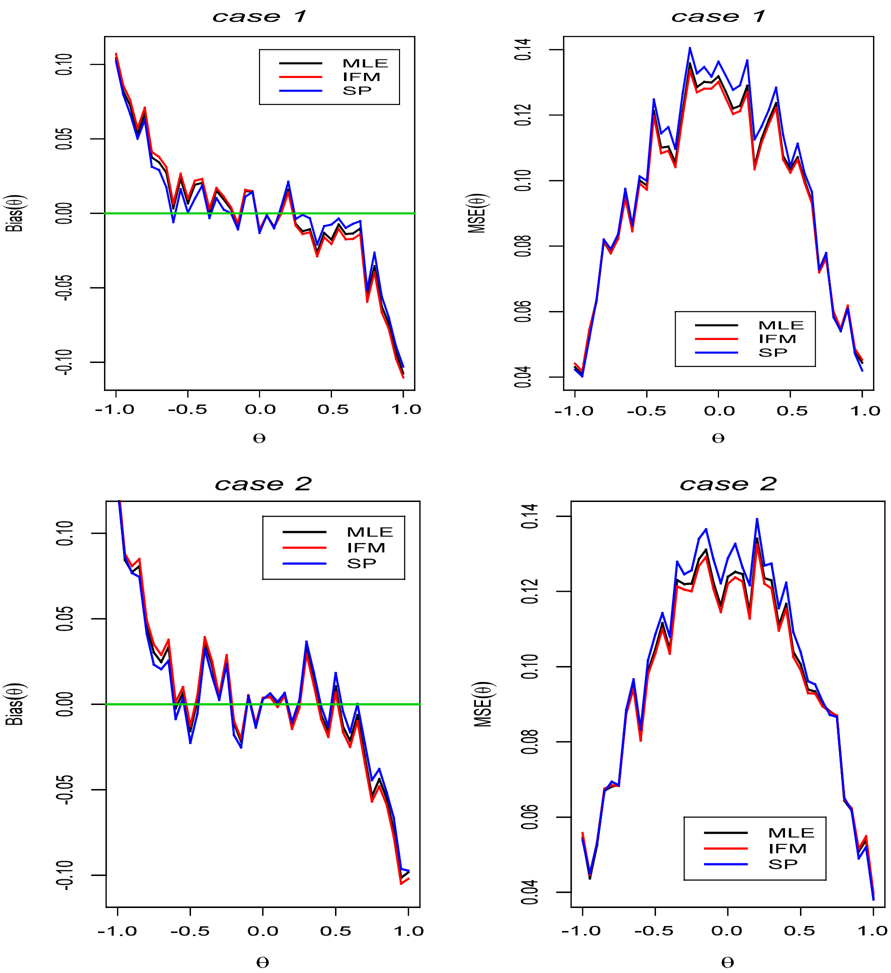

Fig. 4 Bias and MSE of the copula parameter estimate for different methods and different cases with variation of sample size

7 Application of Real Data

The data for 30 patients set from McGilchrist and Aisbett in [15]. Let Y_1 refers to first recurrence time and Y_2 to second recurrence time, as following Y_1 *is* (8, 23, 22, 447, 30, 24, 7, 511, 53, 15, 7, 141, 96, 149, 536, 17, 185, 292, 22, 15, 152, 402, 13, 39, 12, 113, 132, 34, 2, 130) and Y_2 *is* (16, 13, 28, 318, 12, 245, 9, 30, 196, 154, 333, 8, 38, 70, 25, 4, 117, 114, 159, 108, 362, 24, 66, 46, 40, 201, 156, 30, 25, 26). Elaal and Jarwan [3] discussed the estimation of the parameters of bivariate generalized exponential distribution for this data (Table 9).

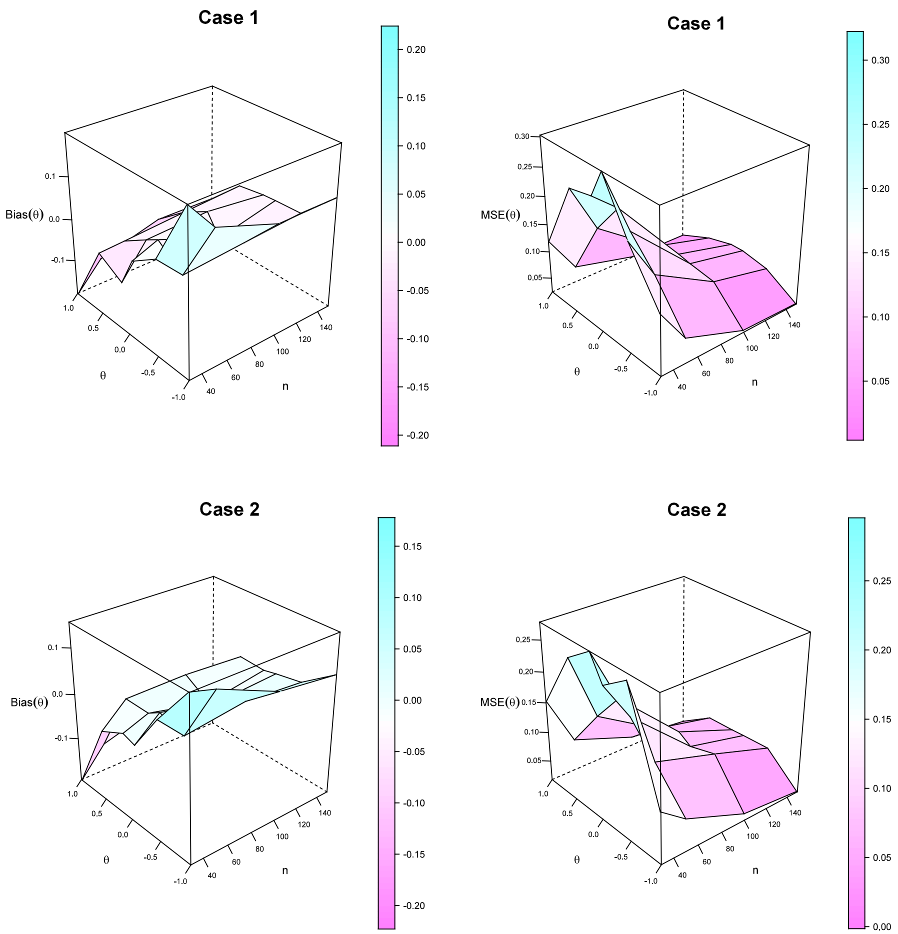

Fig. 5 The bias and MSE of copula parameter for FGMBW distribution with various value of the parameters and sample size

Genest et al. [7] introduced Multiplier bootstrap-based goodness-of-fit test. We use the concludes of Genest to fit of Farlie–Gumbel–Morgenstern (FGM) by R package then (Table 10)

This by using a parametric bootstrap N = 10,000 time and the empirical copula estimate.

Goodness of fit test one-sample Kolmogorov–Smirnov test (Table 11).

A comparison has been done between FGM bivariate Gamma (FGMBG), which was discussed by Kotz et al. [11], bivariate Marshall–Olkin Weibull (BMOW), which was discussed by Kundu and Dey [12] and FGM Bivariate Generalized Exponential (FGMBGE), which was discussed by Elaal and Jarwan [3].

In Table 12, it is observed that, the FGMBW model provides a better fit than the other tested models (FGMBG FGMBGE BMOW), because it has the smallest value

Table 3 Estimation of the parameters of FGMBW distribution: case 1.1

n			$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\alpha}_2$	$\hat{\beta}_2$	$\hat{\theta}$
$\alpha_1 = 1.8, \beta_1 = 3.5, \alpha_2 = 1.5, \beta_2 = 2.5, \theta = 0.25$							
30	MLE	Mean	1.8635	3.4813	1.5648	2.5062	0.3283
		Bias	0.0635	−0.0187	0.0648	0.0062	0.0783
		MSE	0.0834	0.1470	0.0557	0.1031	0.8086
		L.CI	1.1049	1.5018	0.8901	1.2589	3.5133
	IFM	Mean	1.8668	3.4831	1.5670	2.5050	0.2860
		Bias	0.0668	−0.0169	0.0670	0.0050	0.0360
		MSE	0.0820	0.1398	0.0556	0.1000	0.3996
		L.CI	1.0918	1.4649	0.8868	1.2401	2.4753
	SP	coef	–	–	–	–	0.2930
		Bias	–	–	–	–	0.0430
		MSE	–	–	–	–	0.4471
		L.CI	–	–	–	–	2.6171
50	MLE	Mean	1.8430	3.4963	1.5467	2.4993	0.2688
		Bias	0.0430	−0.0037	0.0467	−0.0007	0.0188
		MSE	0.0469	0.0860	0.0329	0.0623	0.2149
		L.CI	0.8325	1.1499	0.6876	0.9790	1.8167
	IFM	Mean	1.8442	3.4975	1.5472	2.4991	0.2634
		Bias	0.0442	−0.0025	0.0472	−0.0009	0.0134
		MSE	0.0469	0.0856	0.0330	0.0613	0.1991
		L.CI	0.8312	1.1472	0.6877	0.9713	1.7493
	SP	Mean	–	–	–	–	0.2678
		Bias	–	–	–	–	0.0178
		MSE	–	–	–	–	0.2148
		L.CI	–	–	–	–	1.8166
70	MLE	Mean	1.8336	3.5003	1.5330	2.4979	0.2591
		Bias	0.0336	0.0003	0.0330	−0.0021	0.0091
		MSE	0.0328	0.0598	0.0216	0.0434	0.1377
		L.CI	0.6977	0.9589	0.5618	0.8174	1.4549
	IFM	Mean	1.8342	3.5014	1.5334	2.4980	0.2564
		Bias	0.0342	0.0014	0.0334	−0.0020	0.0064
		MSE	0.0329	0.0601	0.0218	0.0430	0.1340
		L.CI	0.6985	0.9612	0.5640	0.8134	1.4356
	SP	Mean	–	–	–	–	0.2618
		Bias	–	–	–	–	0.0118
		MSE	–	–	–	–	0.1443
		L.CI	–	–	–	–	1.4892

Table 3 (continued)

n			$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\alpha}_2$	$\hat{\beta}_2$	$\hat{\theta}$
100	MLE	Mean	1.8196	3.5009	1.5254	2.5001	0.2543
		Bias	0.0196	0.0009	0.0254	0.0001	0.0043
		MSE	0.0217	0.0429	0.0160	0.0325	0.0926
		L.CI	0.5727	0.8126	0.4860	0.7071	1.1935
	IFM	Mean	1.8200	3.5016	1.5255	2.5001	0.2529
		Bias	0.0200	0.0016	0.0255	0.0001	0.0029
		MSE	0.0217	0.0426	0.0160	0.0324	0.0913
		L.CI	0.5720	0.8093	0.4862	0.7060	1.1851
	SP	Mean	–	–	–	–	0.2560
		Bias	–	–	–	–	0.0060
		MSE	–	–	–	–	0.0955
		L.CI	–	–	–	–	1.2117
125	MLE	Mean	1.8125	3.4981	1.5190	2.4981	0.2499
		Bias	0.0125	−0.0019	0.0190	−0.0019	−0.0001
		MSE	0.0167	0.0357	0.0118	0.0231	0.0714
		L.CI	0.5039	0.7407	0.4203	0.5967	1.0479
	IFM	Mean	1.8128	3.4988	1.5191	2.4983	0.2489
		Bias	0.0128	−0.0012	0.0191	−0.0017	−0.0011
		MSE	0.0167	0.0358	0.0119	0.0230	0.0706
		L.CI	0.5038	0.7416	0.4208	0.5951	1.0421
	SP	Mean	–	–	–	–	0.2506
		Bias	–	–	–	–	0.0006
		MSE	–	–	–	–	0.0737
		L.CI	–	–	–	–	1.0650
150	MLE	Mean	1.8091	3.4984	1.5164	2.5000	0.2544
		Bias	0.0091	−0.0016	0.0164	0.0000	0.0044
		MSE	0.0132	0.0296	0.0095	0.0206	0.0586
		L.CI	0.4496	0.6744	0.3767	0.5623	0.9492
	IFM	Mean	1.8091	3.4984	1.5163	2.4996	0.2535
		Bias	0.0091	−0.0016	0.0163	−0.0004	0.0035
		MSE	0.0133	0.0296	0.0095	0.0205	0.0582
		L.CI	0.4502	0.6742	0.3772	0.5611	0.9457
	SP	Mean	–	–	–	–	0.2548
		Bias	–	–	–	–	0.0048
		MSE	–	–	–	–	0.0603
		L.CI	–	–	–	–	0.9627

Table 4 Estimation of the parameters of FGMBW distribution: case 1.2

n			$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\alpha}_2$	$\hat{\beta}_2$	$\hat{\theta}$
$\alpha_1 = 1.8,\quad \beta_1 = 3.5,\quad \alpha_2 = 1.5,\quad \beta_2 = 2.5,\quad \theta = 0.75$							
30	MLE	Mean	1.8505	3.4805	1.5505	2.5004	0.9758
		Bias	0.0505	−0.0195	0.0505	0.0004	0.2258
		MSE	0.0749	0.1402	0.0490	0.1144	1.3055
		L.CI	1.0548	1.4664	0.8457	1.3268	4.3928
	IFM	Mean	1.8614	3.4864	1.5587	2.4986	0.8320
		Bias	0.0614	−0.0136	0.0587	−0.0014	0.0820
		MSE	0.0739	0.1266	0.0486	0.0928	0.4285
		L.CI	1.0384	1.3942	0.8336	1.1944	2.5470
	SP	coef	–	–	–	–	0.8763
		Bias	–	–	–	–	0.1263
		MSE	–	–	–	–	0.4759
		L.CI	–	–	–	–	2.6600
50	MLE	Mean	1.8395	3.4952	1.5435	2.4990	0.8408
		Bias	0.0395	−0.0048	0.0435	−0.0010	0.0908
		MSE	0.0469	0.0872	0.0328	0.0651	0.3629
		L.CI	0.8353	1.1583	0.6899	1.0006	2.3357
	IFM	Mean	1.8442	3.4975	1.5468	2.4997	0.8024
		Bias	0.0442	−0.0025	0.0468	−0.0003	0.0524
		MSE	0.0469	0.0856	0.0331	0.0622	0.2418
		L.CI	0.8312	1.1472	0.6896	0.9778	1.9177
	SP	Mean	–	–	–	–	0.8346
		Bias	–	–	–	–	0.0846
		MSE	–	–	–	–	0.2649
		L.CI	–	–	–	–	1.9911
70	MLE	Mean	1.8318	3.4994	1.5309	2.4989	0.7893
		Bias	0.0318	−0.0006	0.0309	−0.0011	0.0393
		MSE	0.0327	0.0591	0.0211	0.0443	0.1512
		L.CI	0.6976	0.9532	0.5566	0.8259	1.5172
	IFM	Mean	1.8342	3.5014	1.5327	2.4988	0.7755
		Bias	0.0342	0.0014	0.0327	−0.0012	0.0255
		MSE	0.0329	0.0601	0.0216	0.0438	0.1359
		L.CI	0.6985	0.9612	0.5614	0.8212	1.4424
	SP	Mean	–	–	–	–	0.7964
		Bias	–	–	–	–	0.0464
		MSE	–	–	–	–	0.1461
		L.CI	–	–	–	–	1.4881
100	MLE	Mean	1.8189	3.5006	1.5239	2.5000	0.7729
		Bias	0.0189	0.0006	0.0239	0.0000	0.0229
		MSE	0.0216	0.0427	0.0156	0.0330	0.0926
		L.CI	0.5717	0.8109	0.4809	0.7120	1.1900
	IFM	Mean	1.8200	3.5016	1.5250	2.5005	0.7651
		Bias	0.0200	0.0016	0.0250	0.0005	0.0151

Table 4 (continued)

n		$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\alpha}_2$	$\hat{\beta}_2$	$\hat{\theta}$
	MSE	0.0217	0.0426	0.0158	0.0330	0.0861
	L.CI	0.5720	0.8093	0.4826	0.7121	1.1493
SP	Mean	–	–	–	–	0.7790
	Bias	–	–	–	–	0.0290
	MSE	–	–	–	–	0.0919
	L.CI	–	–	–	–	1.1834
125 MLE	Mean	1.8119	3.4977	1.5179	2.4977	0.7625
	Bias	0.0119	−0.0023	0.0179	−0.0023	0.0125
	MSE	0.0165	0.0353	0.0115	0.0231	0.0672
	L.CI	0.5013	0.7371	0.4155	0.5963	1.0153
IFM	Mean	1.8128	3.4988	1.5186	2.4981	0.7574
	Bias	0.0128	−0.0012	0.0186	−0.0019	0.0074
	MSE	0.0167	0.0358	0.0117	0.0232	0.0645
	L.CI	0.5038	0.7416	0.4179	0.5969	0.9954
SP	Mean	–	–	–	–	0.7669
	Bias	–	–	–	–	0.0169
	MSE	–	–	–	–	0.0671
	L.CI	–	–	–	–	1.0134
150 MLE	Mean	1.8089	3.4983	1.5153	2.4998	0.7634
	Bias	0.0089	−0.0017	0.0153	−0.0002	0.0134
	MSE	0.0131	0.0294	0.0094	0.0204	0.0522
	L.CI	0.4476	0.6721	0.3749	0.5598	0.8944
IFM	Mean	1.8091	3.4984	1.5156	2.4992	0.7596
	Bias	0.0091	−0.0016	0.0156	−0.0008	0.0096
	MSE	0.0133	0.0296	0.0095	0.0205	0.0508
	L.CI	0.4502	0.6742	0.3775	0.5612	0.8829
SP	Mean	–	–	–	–	0.7672
	Bias	–	–	–	–	0.0172
	MSE	–	–	–	–	0.0530
	L.CI	–	–	–	–	0.9006

of L, AIC and BIC. The FGMBW distribution is a good alternative to bivariate several lifetime distributions for modeling non-negative real-valued data in application.

In Tables 13, it is observed that, the IFM method provides a better fit than the other tested methods, because it has the smallest value of stander deviation and L.CI for parameters of FGMBW distribution.

Table 5 Estimation of the parameters of FGMBW distribution: case 2.1

n			$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\alpha}_2$	$\hat{\beta}_2$	$\hat{\theta}$
$\beta_1 = 1.2,$	$\alpha_1 = 1.5,$	$\beta_2 = 2.8,$	$\alpha_2 = 2.1,$	$\theta = 0.25$			
30	MLE	Mean	1.5528	1.1940	2.1907	2.8002	0.3285
		Bias	0.0528	−0.0060	0.0907	0.0002	0.0785
		MSE	0.0579	0.0248	0.1092	0.0658	0.8049
		L.CI	0.9210	0.6174	1.2463	1.0063	3.5051
	IFM	Mean	1.5557	1.1947	2.1939	2.7994	0.2861
		Bias	0.0557	−0.0053	0.0939	−0.0006	0.0361
		MSE	0.0569	0.0236	0.1091	0.0638	0.3997
		L.CI	0.9098	0.6021	1.2417	0.9909	2.4755
	SP	coef	–	–	–	–	0.2930
		Bias	–	–	–	–	0.0430
		MSE	–	–	–	–	0.4471
		L.CI	–	–	–	–	2.6171
50	MLE	Mean	1.5358	1.1995	2.1654	2.7966	0.2688
		Bias	0.0358	−0.0005	0.0654	−0.0034	0.0188
		MSE	0.0326	0.0146	0.0645	0.0400	0.2150
		L.CI	0.6938	0.4736	0.9624	0.7841	1.8172
	IFM	Mean	1.5369	1.2000	2.1661	2.7965	0.2635
		Bias	0.0369	0.0000	0.0661	−0.0035	0.0135
		MSE	0.0326	0.0145	0.0646	0.0393	0.1991
		L.CI	0.6927	0.4724	0.9628	0.7777	1.7490
	SP	Mean	–	–	–	–	0.2678
		Bias	–	–	–	–	0.0178
		MSE	–	–	–	–	0.2148
		L.CI	–	–	–	–	1.8166
70	MLE	Mean	1.5280	1.2008	2.1462	2.7963	0.2591
		Bias	0.0280	0.0008	0.0462	−0.0037	0.0091
		MSE	0.0227	0.0101	0.0424	0.0278	0.1376
		L.CI	0.5811	0.3949	0.7866	0.6540	1.4542
	IFM	Mean	1.5285	1.2013	2.1467	2.7964	0.2565
		Bias	0.0285	0.0013	0.0467	−0.0036	0.0065
		MSE	0.0228	0.0102	0.0427	0.0275	0.1340
		L.CI	0.5821	0.3958	0.7896	0.6508	1.4354
	SP	Mean	–	–	–	–	0.2618
		Bias	–	–	–	–	0.0118
		MSE	–	–	–	–	0.1443
		L.CI	–	–	–	–	1.4892

Table 5 (continued)

n			$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\alpha}_2$	$\hat{\beta}_2$	$\hat{\theta}$
100	MLE	Mean	1.5164	1.2009	2.1355	2.7986	0.2544
		Bias	0.0164	0.0009	0.0355	−0.0014	0.0044
		MSE	0.0151	0.0073	0.0314	0.0208	0.0926
		L.CI	0.4773	0.3345	0.6805	0.5655	1.1932
	IFM	Mean	1.5167	1.2012	2.1357	2.7986	0.2529
		Bias	0.0167	0.0012	0.0357	−0.0014	0.0029
		MSE	0.0150	0.0072	0.0314	0.0207	0.0913
		L.CI	0.4767	0.3331	0.6807	0.5646	1.1851
	SP	Mean	–	–	–	–	0.2560
		Bias	–	–	–	–	0.0060
		MSE	–	–	–	–	0.0955
		L.CI	–	–	–	–	1.2117
125	MLE	Mean	1.5105	1.1996	2.1266	2.7975	0.2501
		Bias	0.0105	−0.0004	0.0266	−0.0025	0.0001
		MSE	0.0116	0.0060	0.0232	0.0148	0.0714
		L.CI	0.4198	0.3048	0.5883	0.4771	1.0477
	IFM	Mean	1.5107	1.1999	2.1267	2.7976	0.2488
		Bias	0.0107	−0.0001	0.0267	−0.0024	−0.0012
		MSE	0.0116	0.0061	0.0233	0.0147	0.0706
		L.CI	0.4199	0.3052	0.5891	0.4759	1.0424
	SP	Mean	–	–	–	–	0.2506
		Bias	–	–	–	–	0.0006
		MSE	–	–	–	–	0.0737
		L.CI	–	–	–	–	1.0650
150	MLE	Mean	1.5076	1.1997	2.1229	2.7991	0.2544
		Bias	0.0076	−0.0003	0.0229	−0.0009	0.0044
		MSE	0.0092	0.0050	0.0186	0.0131	0.0586
		L.CI	0.3748	0.2776	0.5272	0.4495	0.9489
	IFM	Mean	1.5076	1.1997	2.1228	2.7987	0.2536
		Bias	0.0076	−0.0003	0.0228	−0.0013	0.0036
		MSE	0.0092	0.0050	0.0186	0.0131	0.0582
		L.CI	0.3751	0.2774	0.5281	0.4487	0.9458
	SP	Mean	–	–	–	–	0.2548
		Bias	–	–	–	–	0.0048
		MSE	–	–	–	–	0.0603
		L.CI	–	–	–	–	0.9627

8 Conclusion

In this paper, we have proposed a FGMBW distribution based on FGM copula function. Moreover, we have the reliability functions for FGMBW distribution; therefore,

Table 6 Estimation of the parameters of FGMBW distribution: case 2.2

n			$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\alpha}_2$	$\hat{\beta}_2$	$\hat{\theta}$
$\beta_1 = 1.2, \quad \alpha_1 = 1.5, \quad \beta_2 = 2.8, \quad \alpha_2 = 2.1, \quad \theta = 0.75$							
30	MLE	Mean	1.5419	1.1936	2.1697	2.7952	0.9872
		Bias	0.0419	−0.0064	0.0697	−0.0048	0.2372
		MSE	0.0520	0.0236	0.0962	0.0723	1.5195
		L.CI	0.8788	0.6025	1.1856	1.0547	4.7442
	IFM	Mean	1.5512	1.1959	2.1822	2.7947	0.8320
		Bias	0.0512	−0.0041	0.0822	−0.0053	0.0820
		MSE	0.0513	0.0214	0.0953	0.0593	0.4285
		L.CI	0.8653	0.5736	1.1669	0.9551	2.5470
	SP	coef	–	–	–	–	0.8763
		Bias	–	–	–	–	0.1263
		MSE	–	–	–	–	0.4759
		L.CI	–	–	–	–	2.6600
50	MLE	Mean	1.5329	1.1990	2.1609	2.7962	0.8405
		Bias	0.0329	−0.0010	0.0609	−0.0038	0.0905
		MSE	0.0326	0.0148	0.0644	0.0418	0.3621
		L.CI	0.6961	0.4770	0.9659	0.8013	2.3332
	IFM	Mean	1.5369	1.2000	2.1656	2.7969	0.8024
		Bias	0.0369	0.0000	0.0656	−0.0031	0.0524
		MSE	0.0326	0.0145	0.0649	0.0399	0.2418
		L.CI	0.6927	0.4724	0.9655	0.7830	1.9175
	SP	Mean	–	–	–	–	0.8346
		Bias	–	–	–	–	0.0846
		MSE	–	–	–	–	0.2649
		L.CI	–	–	–	–	1.9911
70	MLE	Mean	1.5265	1.2005	2.1434	2.7972	0.7893
		Bias	0.0265	0.0005	0.0434	−0.0028	0.0393
		MSE	0.0227	0.0100	0.0413	0.0284	0.1511
		L.CI	0.5812	0.3924	0.7790	0.6611	1.5167
	IFM	Mean	1.5285	1.2013	2.1458	2.7970	0.7754
		Bias	0.0285	0.0013	0.0458	−0.0030	0.0254
		MSE	0.0228	0.0102	0.0423	0.0281	0.1359
		L.CI	0.5821	0.3958	0.7860	0.6573	1.4422
	SP	Mean	–	–	–	–	0.7964
		Bias	–	–	–	–	0.0464
		MSE	–	–	–	–	0.1461
		L.CI	–	–	–	–	1.4881

Table 6 (continued)

n			$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\alpha}_2$	$\hat{\beta}_2$	$\hat{\theta}$
100	MLE	Mean	1.5157	1.2007	2.1335	2.7985	0.7729
		Bias	0.0157	0.0007	0.0335	−0.0015	0.0229
		MSE	0.0150	0.0072	0.0306	0.0211	0.0926
		L.CI	0.4763	0.3338	0.6733	0.5695	1.1902
	IFM	Mean	1.5167	1.2012	2.1350	2.7989	0.7650
		Bias	0.0167	0.0012	0.0350	−0.0011	0.0150
		MSE	0.0150	0.0072	0.0309	0.0211	0.0862
		L.CI	0.4767	0.3331	0.6756	0.5695	1.1498
	SP	Mean	–	–	–	–	0.7790
		Bias	–	–	–	–	0.0290
		MSE	–	–	–	–	0.0919
		L.CI	–	–	–	–	1.1834
125	MLE	Mean	1.5099	1.1995	2.1249	2.7971	0.7626
		Bias	0.0099	−0.0005	0.0249	−0.0029	0.0126
		MSE	0.0114	0.0060	0.0226	0.0148	0.0672
		L.CI	0.4178	0.3032	0.5817	0.4773	1.0154
	IFM	Mean	1.5107	1.1999	2.1261	2.7974	0.7574
		Bias	0.0107	−0.0001	0.0261	−0.0026	0.0074
		MSE	0.0116	0.0061	0.0229	0.0148	0.0644
		L.CI	0.4199	0.3052	0.5851	0.4776	0.9952
	SP	Mean	–	–	–	–	0.7669
		Bias	–	–	–	–	0.0169
		MSE	–	–	–	–	0.0671
		L.CI	–	–	–	–	1.0134
150	MLE	Mean	1.5074	1.1997	2.1213	2.7988	0.7635
		Bias	0.0074	−0.0003	0.0213	−0.0012	0.0135
		MSE	0.0091	0.0050	0.0184	0.0130	0.0522
		L.CI	0.3730	0.2765	0.5249	0.4478	0.8946
	IFM	Mean	1.5076	1.1997	2.1218	2.7985	0.7596
		Bias	0.0076	−0.0003	0.0218	−0.0015	0.0096
		MSE	0.0092	0.0050	0.0186	0.0131	0.0508
		L.CI	0.3751	0.2774	0.5285	0.4490	0.8829
	SP	Mean	–	–	–	–	0.7672
		Bias	–	–	–	–	0.0172
		MSE	–	–	–	–	0.0530
		L.CI	–	–	–	–	0.9006

it can be used quite effectively in life testing data. Additionally, the new FGMBW model can be used as an alternative to any bivariate Weibull distribution; it might work better, where the marginal function of FGMBW distribution has the same basic distribution and has closed forms for product moment. A comparison between different estimation methods of the FGMBW distribution are concluded. The results

Table 7 Estimation of the parameters of FGMBW distribution: case 1.3

n			$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\alpha}_2$	$\hat{\beta}_2$	$\hat{\theta}$
$\alpha_1 = 1.8$,	$\beta_1 = 3.5$,	$\alpha_2 = 1.5$,	$\beta_2 = 2.5$,	$\theta = -0.75$			
30	MLE	Mean	1.8588	3.5082	1.5542	2.5024	−1.2236
		Bias	0.0588	0.0082	0.0542	0.0024	−0.4736
		MSE	0.0854	0.2446	0.0594	0.1407	1.9634
		L.CI	1.1228	1.9394	0.9315	1.4710	4.3645
	IFM	Mean	1.8668	3.4831	1.5691	2.5090	−0.8351
		Bias	0.0668	−0.0169	0.0691	0.0090	−0.0851
		MSE	0.0820	0.1398	0.0557	0.0979	0.4748
		L.CI	1.0918	1.4649	0.8851	1.2267	2.6817
	SP	coef	–	–	–	–	−0.8871
		Bias	–	–	–	–	−0.1371
		MSE	–	–	–	–	0.5806
		L.CI	–	–	–	–	2.9398
50	MLE	Mean	1.8527	3.5064	1.5494	2.5045	−0.8471
		Bias	0.0527	0.0064	0.0494	0.0045	−0.0971
		MSE	0.0496	0.0948	0.0353	0.0738	0.5402
		L.CI	0.8488	1.2074	0.7114	1.0654	2.8573
	IFM	Mean	1.8574	3.5086	1.5528	2.4993	−0.7934
		Bias	0.0574	0.0086	0.0528	−0.0007	−0.0434
		MSE	0.0487	0.0875	0.0353	0.0602	0.2447
		L.CI	0.8353	1.1596	0.7076	0.9626	1.9327
	SP	Mean	–	–	–	–	−0.8210
		Bias	–	–	–	–	−0.0710
		MSE	–	–	–	–	0.2658
		L.CI	–	–	–	–	2.0028
70	MLE	Mean	1.8233	3.4968	1.5284	2.4952	−0.7816
		Bias	0.0233	−0.0032	0.0284	−0.0048	−0.0316
		MSE	0.0306	0.0642	0.0221	0.0422	0.1526
		L.CI	0.6799	0.9933	0.5718	0.8056	1.5273
	IFM	Mean	1.8244	3.4941	1.5306	2.4979	−0.7673
		Bias	0.0244	−0.0059	0.0306	−0.0021	−0.0173
		MSE	0.0309	0.0654	0.0221	0.0422	0.1358
		L.CI	0.6824	1.0028	0.5702	0.8056	1.4436
	SP	Mean	–	–	–	–	−0.7889
		Bias	–	–	–	–	−0.0389
		MSE	–	–	–	–	0.1476
		L.CI	–	–	–	–	1.4992

Table 7 (continued)

n			$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\alpha}_2$	$\hat{\beta}_2$	$\hat{\theta}$
100	MLE	Mean	1.8253	3.4934	1.5183	2.4967	−0.7581
		Bias	0.0253	−0.0066	0.0183	−0.0033	−0.0081
		MSE	0.0201	0.0424	0.0154	0.0313	0.0893
		L.CI	0.5474	0.8076	0.4821	0.6939	1.1713
	IFM	Mean	1.8259	3.4930	1.5196	2.4977	−0.7517
		Bias	0.0259	−0.0070	0.0196	−0.0023	−0.0017
		MSE	0.0201	0.0424	0.0155	0.0314	0.0858
		L.CI	0.5471	0.8075	0.4819	0.6951	1.1486
	SP	Mean	–	–	–	–	−0.7666
		Bias	–	–	–	–	−0.0166
		MSE	–	–	–	–	0.0891
		L.CI	–	–	–	–	1.1690
125	MLE	Mean	1.8223	3.4932	1.5189	2.5000	−0.7608
		Bias	0.0223	−0.0068	0.0189	0.0000	−0.0108
		MSE	0.0174	0.0330	0.0117	0.0241	0.0648
		L.CI	0.5101	0.7125	0.4170	0.6092	0.9971
	IFM	Mean	1.8227	3.4939	1.5189	2.4995	−0.7561
		Bias	0.0227	−0.0061	0.0189	−0.0005	−0.0061
		MSE	0.0175	0.0330	0.0117	0.0246	0.0628
		L.CI	0.5107	0.7125	0.4169	0.6149	0.9822
	SP	Mean	–	–	–	–	−0.7670
		Bias	–	–	–	–	−0.0170
		MSE	–	–	–	–	0.0658
		L.CI	–	–	–	–	1.0036
150	MLE	Mean	1.8126	3.4897	1.5177	2.4983	−0.7572
		Bias	0.0126	−0.0103	0.0177	−0.0017	−0.0072
		MSE	0.0139	0.0274	0.0104	0.0202	0.0556
		L.CI	0.4595	0.6480	0.3933	0.5570	0.9242
	IFM	Mean	1.8137	3.4907	1.5176	2.4973	−0.7532
		Bias	0.0137	−0.0093	0.0176	−0.0027	−0.0032
		MSE	0.0141	0.0283	0.0104	0.0204	0.0538
		L.CI	0.4624	0.6584	0.3941	0.5596	0.9098
	SP	Mean	–	–	–	–	−0.7608
		Bias	–	–	–	–	−0.0108
		MSE	–	–	–	–	0.0550
		L.CI	–	–	–	–	0.9190

show that the best method of estimation is IFM method, whereas real data application show that MLE perform better than their counterparts. Hence, we can argue that IFM estimators and MLE are the best performing estimators for FGMBW distribution.

Table 8 Estimation of the parameters of FGMBW distribution: case 1.4

n			$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\alpha}_2$	$\hat{\beta}_2$	$\hat{\theta}$
$\alpha_1 = 1.8,$	$\beta_1 = 3.5,$	$\alpha_2 = 1.5,$	$\beta_2 = 2.5,$	$\theta = -0.25$			
30	MLE	Mean	1.8617	3.4955	1.5638	2.5119	−0.2501
		Bias	0.0617	−0.0045	0.0638	0.0119	−0.0001
		MSE	0.0838	0.3154	0.0567	0.1306	0.7260
		L.CI	1.1095	2.2024	0.8996	1.4167	2.9092
	IFM	Mean	1.8668	3.4831	1.5680	2.5068	−0.2536
		Bias	0.0668	−0.0169	0.0680	0.0068	−0.0036
		MSE	0.0820	0.1398	0.0554	0.0988	0.3649
		L.CI	1.0918	1.4649	0.8838	1.2325	2.3690
	SP	coef	–	–	–	–	−0.2744
		Bias	–	–	–	–	−0.0244
		MSE	–	–	–	–	0.4225
		L.CI	–	–	–	–	2.5475
50	MLE	Mean	1.8561	3.5073	1.5515	2.5036	−0.2715
		Bias	0.0561	0.0073	0.0515	0.0036	−0.0215
		MSE	0.0490	0.0899	0.0352	0.0707	0.5537
		L.CI	0.8402	1.1758	0.7078	1.0429	2.9171
	IFM	Mean	1.8574	3.5086	1.5529	2.5009	−0.2496
		Bias	0.0574	0.0086	0.0529	0.0009	0.0004
		MSE	0.0487	0.0875	0.0351	0.0612	0.1949
		L.CI	0.8353	1.1596	0.7054	0.9705	1.7315
	SP	Mean	–	–	–	–	−0.2621
		Bias	–	–	–	–	−0.0121
		MSE	–	–	–	–	0.2111
		L.CI	–	–	–	–	1.8015
70	MLE	Mean	1.8242	3.4948	1.5289	2.4956	−0.2529
		Bias	0.0242	−0.0052	0.0289	−0.0044	−0.0029
		MSE	0.0307	0.0652	0.0217	0.0409	0.1358
		L.CI	0.6810	1.0011	0.5665	0.7932	1.4450
	IFM	Mean	1.8244	3.4941	1.5293	2.4964	−0.2504
		Bias	0.0244	−0.0059	0.0293	−0.0036	−0.0004
		MSE	0.0309	0.0654	0.0216	0.0410	0.1328
		L.CI	0.6824	1.0028	0.5655	0.7943	1.4291
	SP	Mean	–	–	–	–	−0.2586
		Bias	–	–	–	–	−0.0086
		MSE	–	–	–	–	0.1399
		L.CI	–	–	–	–	1.4665

Table 8 (continued)

n			$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\alpha}_2$	$\hat{\beta}_2$	$\hat{\theta}$
100	MLE	Mean	1.8259	3.4931	1.5184	2.4965	−0.2492
		Bias	0.0259	−0.0069	0.0184	−0.0035	0.0008
		MSE	0.0202	0.0426	0.0155	0.0310	0.0889
		L.CI	0.5479	0.8095	0.4835	0.6906	1.1696
	IFM	Mean	1.8259	3.4930	1.5186	2.4967	−0.2480
		Bias	0.0259	−0.0070	0.0186	−0.0033	0.0020
		MSE	0.0201	0.0424	0.0155	0.0309	0.0879
		L.CI	0.5471	0.8075	0.4832	0.6895	1.1631
	SP	Mean	–	–	–	–	−0.2536
		Bias	–	–	–	–	−0.0036
		MSE	–	–	–	–	0.0910
		L.CI	–	–	–	–	1.1831
125	MLE	Mean	1.8229	3.4941	1.5195	2.4993	−0.2527
		Bias	0.0229	−0.0059	0.0195	−0.0007	−0.0027
		MSE	0.0175	0.0331	0.0114	0.0245	0.0715
		L.CI	0.5111	0.7135	0.4125	0.6137	1.0488
	IFM	Mean	1.8227	3.4939	1.5192	2.4990	−0.2517
		Bias	0.0227	−0.0061	0.0192	−0.0010	−0.0017
		MSE	0.0175	0.0330	0.0114	0.0245	0.0709
		L.CI	0.5107	0.7125	0.4111	0.6139	1.0446
	SP	Mean	–	–	–	–	−0.2553
		Bias	–	–	–	–	−0.0053
		MSE	–	–	–	–	0.0738
		L.CI	–	–	–	–	1.0650
150	MLE	Mean	1.8134	3.4903	1.5177	2.4969	−0.2474
		Bias	0.0134	−0.0097	0.0177	−0.0031	0.0026
		MSE	0.0140	0.0280	0.0106	0.0205	0.0587
		L.CI	0.4617	0.6557	0.3976	0.5611	0.9501
	IFM	Mean	1.8137	3.4907	1.5176	2.4964	−0.2467
		Bias	0.0137	−0.0093	0.0176	−0.0036	0.0033
		MSE	0.0141	0.0283	0.0106	0.0204	0.0582
		L.CI	0.4624	0.6584	0.3972	0.5600	0.9463
	SP	Mean	–	–	–	–	−0.2491
		Bias	–	–	–	–	0.0009
		MSE	–	–	–	–	0.0603
		L.CI	–	–	–	–	0.9635

Table 9 The correlation coefficient and test of correlation for real data

	Corr.	Test	p Value
Pearson’s	0.05105553	0.27051	0.7887
Kendall’s	0.1109827	0.85705	0.3914

Table 10 Goodness of fit test of FGM copula

	Statistic	$\hat{\theta}$	p Value
Anderson–Darling-type (Rn)	0.29031	0.46704	0.3936

Table 11 Goodness of fit test of Weibull distribution

	y_1	y_2
D	0.14343	0.14441
p Value	0.5678	0.5589

Table 12 The estimates parameters of bivariate distributions

	$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\alpha}_2$	$\hat{\beta}_2$	$\hat{\theta}$	− LL	AIC	BIC
FGMBW	0.75106	100.11993	0.92435	98.24665	0.34801	338.907	687.814	694.826
FGMBG	0.67780	175.52654	0.92321	107.75374	0.37959	339.492	688.984	695.986
FGMBGE	0.66607	0.00631	0.92584	0.00958	0.3780	339.545	689.090	696.106
BMOW	0.7363	104.2288	89.8322	93.5875	–	349.068	706.137	711.742

Table 13 The estimates and the corresponding stander deviation of parameters of FGMBW distribution

	α_1	β_1	α_2	β_2	θ
FGMBW					
ML					
Coef (SD)	0.7510 (0.1053)	100.119 (25.735)	0.9243 (0.1329)	98.2466 (20.769)	0.3480 (0.4126)
L.CI	0.4133	100.884	0.5212	81.4157	1.6173
IFM					
Coef (SD)	0.7502 (0.1053)	99.7960 (25.627)	0.9309 (0.1325)	95.2893 (19.725)	0.3990 (0.4106)
L.CI	0.4130	100.458	0.5195	77.3239	1.6095
SP					
Coef (SD)	–	–	–	–	0.4670 (0.5465)
L.CI					2.1423

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