Ann. Data. Sci. (2017) 4(2):227–248
DOI 10.1007/s40745-017-0101-8

Analysis of Burr Type-XII Distribution Under Step Stress Partially Accelerated Life Tests with Type-I and Adaptive Type-II Progressively Hybrid Censoring Schemes

M. Nassar[1] · **S. G. Nassr**[2] · **S. Dey**[3]

Received: 8 November 2016 / Revised: 4 February 2017 / Accepted: 11 February 2017 /
Published online: 4 March 2017

Abstract In this paper, we investigate the maximum likelihood estimation of the unknown parameters of the Burr Type-XII distribution and the acceleration factor based on two different progressively hybrid censoring schemes, namely, Type-I progressive hybrid censoring scheme (T-I PHCS) proposed by Kundu and Joarder (Comput Stat Data Anal 50:2509–2528, 2006) and adaptive Type-II progressive hybrid censoring scheme (AT-II PHCS) introduced by Ng et al. (Nav Res Logist 56:687–698, 2009) under step-stress partially accelerated life test model. The observed Fisher information matrix is obtained to construct an approximate confidence interval for the unknown parameters. The performances of the estimators of the model parameters using the above mentioned progressively hybrid censoring schemes are evaluated and compared in terms of the mean squared errors and relative errors through a Monte Carlo simulation study.

Keywords Burr Type-XII distribution · Adaptive Type-II progressive hybrid censoring · Progressive Type-I hybrid censoring scheme · Maximum likelihood estimation

✉ M. Nassar
mezo10011@gmail.com

1 Department of Statistics, Faculty of Commerce, Zagazig University, Zagazig, Egypt

2 Faculty of Business Administration and International Marketing, Sinai University, Sinai, Egypt

3 Department of Statistics, St. Anthony's College, Shillong, Meghalaya, India

1 Introduction

In reliability analysis, it is not easy to collect lifetimes on highly reliable products with very long lifetimes since very few or even no failures may occur within a limited testing time under normal operating conditions. To induce early failures an accelerated life test (ALT) or partially accelerated life test (PALT) is often used. If all test units are exposed to higher-than usual stress levels, then the test is called ALT. But if only some of them are run under severe condition then the test is called PALT. The information obtained from the test performed in the accelerated or partially accelerated environment is used to predict actual product performance in the usual environment. The stress can be applied in several ways. According to Nelson [26] the common methods are step-stress and constant stress. Under step-stress PALT (SSPALT), a test item is first run at normal (use) condition and, if it does not fail for a specified time, then it is run at accelerated condition until the test terminates. But the constant-stress PALT runs each item at either use condition or accelerated condition only, i.e. each unit is run at a constant-stress level until it fails or censors. As indicated by Lin et al. [24], there are many situations in life testing and reliability experiments in which units are lost or removed from test before failure. The experimenter may not obtain complete information on failure times for all test units. Data observed from such experiments are called censored data.

Very often in life testing and reliability studies, it is not easy to obtain sufficient failure data of highly reliable products within a limited testing time under normal operating conditions. Consequently, with conventional life-testing experiments under either Type-I or Type-II censoring, it is almost impossible to obtain adequate information about the failure time distribution and its associated parameters. To overcome these problems, accelerated life test (ALT) or partially accelerated life test (PALT) can be adopted to yield information about the lifetime distributions of products by inducing early or rapid failure of items with stronger stress than normal with lower cost and shorter period of time. In PALT, items are tested at both accelerated and use conditions. The stress can be applied in several ways, the common methods are constant-stress and step-stress. In constant-stress PALT, each unit runs at either used condition or accelerated condition only, i.e., each unit runs at a constant-stress level until it fails or censored. But under step-stress PALT (SS-PALT), if the items do not fail under normal used condition for a specified time, then we raise the stress on it until the items fail or the censoring time is reached. Many authors have considered SS-PALT, including Goel [18], DeGroot and Goel [16], Bai et al. [8], Abdel-Ghaly et al. [2], Abd-Elfattah [1], Ismail and Aly [20] and Ismail [21].

In life testing and reliability experiments there are numerous situations where the units are removed from the test before failure. Data obtained from such experiments are called censored data. The two most common censoring schemes, namely Type-I and Type-II censoring schemes are widely used in the life-testing and reliability studies. The combination of Type-I and Type-II censoring schemes called the hybrid censoring scheme proposed by Epstein [17] is quite common in life-testing or reliability experiments in which the experiment is terminated at a random time $\eta^* = min(x_{m:m:n}, \eta)$, where $x_{m:m:n}$ is the mth failure and $\eta \in (0, \infty)$ is a predetermined time. However, one of the limitations of these schemes is that they do not allow withdrawing units from the test at any time point other than the terminal point. To tackle this problem, a more general censoring scheme called the progressive Type-II censoring or progressively Type-II hybrid censoring schemes are used. In progressive Type-II censoring scheme n items are placed on a life test and the quantity m is a predetermined number of items to be observed. At the time of the first failure $x_{1:m:n}$, R_1 units are randomly removed from the remaining $n-1$ surviving units. Similarly, at the time of the second failure $x_{2:m:n}$, R_2 units of the remaining $n-1-R_1$ units are randomly removed and so on. At the time of the mth failure $x_{m:m:n}$ all the remaining $n-m-R_1-R_2-\cdots-R_{m-1}$ units are removed. For further details on Type-II progressive censoring and for its different advantages, the readers may refer to the excellent monograph of Balakrishnan and Aggarwala [13], Balakrishnan [9] and Balakrishnan and Cramer [11].

T-I PHCS was proposed by Kundu and Joarder [22] in which n items are placed on life test with a predetermined quantities, η, m and progressive censoring scheme $R_1, R_2, \ldots, R_m$. The experiment is terminated at random time $\eta^* = min(x_{m:m:n}, \eta)$. In this case, the experiment will terminate at $x_{m:m:n}$ if $x_{m:m:n} < \eta$, otherwise it will terminate at time η. The advantage of this censoring scheme is that the choice of η depends on how much maximum experimental time the experimenter can afford to continue. Moreover, the experimental time is bounded. The available data under T-I PHCS will be one of the following two forms

Case I: $x_{1:m:n} < \cdots < x_{m:m:n}$ if $x_{m:m:n} \leq \eta$
Case II: $x_{1:m:n} < \cdots < x_{J:m:n}$ if $x_{m:m:n} > \eta$

where J is the number of failures before time T. For further details see Balakrishnan and Kundu [12]. Based on T-I PHCS, few authors have made statistical inference on the SS-PALT, see for example, Ismail [21], Shi et al [29], Cai et al. [15] and Zhang et al. [32]. In T-I PHCS, the effective sample size is random and a few failures (or even equal to zero) would take place before the prefixed time, leading to low efficiency in the statistical inference procedure.

To overcome the drawback of the T-I PHCS, Ng et al. [27] proposed an AT-II PHCS, in which the effective number of failures m is predetermined and the progressive censoring scheme $R_1, R_2, \ldots, R_m$ is provided, but the values of some of the $R_i's$ may be changed accordingly during the experiment. In the AT-II PHCS, the experimental time is allowed to run over the (predetermined) threshold time η. If $(X_{m:m:m} < \eta)$, we will have a usual Type-II progressive censoring scheme with a pre-fixed progressive censoring scheme $R_1, R_2, \ldots, R_m$ and the experiment stops at time $X_{m:m:m}$. Otherwise, once the experimental time passes η, then we can terminate the experiment as soon as possible by setting $R_{J+1}, R_{J+2}, \ldots, R_{m-1} = 0$, i.e. if $X_{J:m:n} < \eta < X_{J+1:m:n}$, where $J+1 < m$ and $X_{J:m:n}$ is the Jth failure time occur before time η, we will not remove any surviving unit from the experiment until the effective number of failures m is reached and then all remaining items $R_m = \left(n - J - \sum_{i=1}^{J} R_i\right)$, are removed.

Several life testing studies based on AT-II PHCS have been carried out in the recent times, see for example, Lin et al. [24], Hemmati and Khorram [19], Mahmoud et al. [25], Ashour and Nassar [7], AL Sobhi and Soliman [6] and Zhang and Shi [33]. Recently, Ismail [21] studied the likelihood estimation of Weibull distribution parameters and the acceleration factor under step-stress partially accelerated life test models based on AT-II PHCS.

To the best our knowledge, there are hardly any studies related to SS-PALT by using both T-I PHCS and AT-II PHCS. The aim of this paper is to derive maximum likelihood estimators (MLEs) and the asymptotic confidence intervals of the unknown parameters of the Burr Type-XII distribution and acceleration factor when the data is coming from two types of censoring schemes: T-I PHCS and AT-II PHCS under SS-PALT. The rest of this paper is organized as follows: In Sect. 2, we describe the model. In Sect. 3, we obtain the maximum likelihood estimators and the corresponding asymptotic confidence intervals under SS-PALT with T-I PHCS. In Sect. 4, the maximum likelihood estimators and the corresponding asymptotic confidence intervals under SS-PALT with AT-II PHCS are provided. In Sect. 5, the method developed has been illustrated using simulated data from the proposed models with both T-I PHCS and AT-II PHCS. Some concluding remarks are made in Sect. 6.

Section 4 contains the simulation results that demonstrate and evaluate the performance of the estimators based on the proposed censoring schemes.

2 Model Description

Burr [14] introduced twelve different forms of cumulative distribution functions for modeling lifetime data or survival data. Of these twelve distribution functions, Burr Type-X and Burr Type-XII were extensively used by the researchers. A random variable X is said to has Burr Type-XII distribution with shape parameters c and k, denoted by Burr(c,k), if its probability density function (pdf) is given by

$$f(x; c, k) = ckx^{c-1}(1 + x^c)^{-(k+1)}, \quad x > 0, c, k > 0 \tag{1}$$

It is important to note that when $c = 1$, the Burr-XII reduces to the log-logistic distribution and the fact that it can be a good approximation to the Weibull distribution which is a limiting distribution of the Burr XII. Although, the Burr XII has a non-monotone failure rate similar to the log-normal distribution, it also has other properties that distinguish it from the log-normal which makes it a viable alternative in some situations (see [34]). For $c > 1$, the p.d.f. as Eq. (1) is unimodal and is L-shaped for $c \leq 1$. The Burr XII distribution has been recognized as a useful model for the analysis of lifetime data. Readers may refer to Rodriguez [28], Tadikamalla [31], Lewis [23], Ali Mousa [4], Ali Mousa and Jaheen [5], Soliman [30], Abdel-Hamid [3], among others for extensive reviews of the literature on Burr-XII distribution.

The survival function of Burr(c,k) distribution in (1) takes the form

$$S(x; c, k) = (1 + x^c)^{-k} \tag{2}$$

Under SS-PALT, the lifetime of the unit is given as follows

$$X = \begin{cases} T & \text{if } T \leq \tau \\ \tau + \beta(T - \tau) & \text{if } T > \tau \end{cases}$$

where T is is the lifetime of the unit under normal use condition, τ is the stress change time and β, $\beta > 1$ is the acceleration factor. The pdf of X under SS-PALT model can be given by

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ f_1(x) \equiv f(x; c, k) & \text{if } 0 < x < \tau \\ f_2(x) & \text{if } x > \tau \end{cases} \tag{3}$$

where $f_2(x)$ is given by

$$f_2(x) \equiv f_2(x; c, k, \beta) = ck\beta \left(\tau + \beta(x - \tau)\right)^{c-1} \left(1 + (\tau + \beta(x - \tau))^c\right)^{-(k+1)}, \tag{4}$$

and the corresponding survival function is

$$S_2(x) \equiv S_2\left(x; c, k, \beta\right) = \left(1 + (\tau + \beta(x - \tau))^c\right)^{-k} \tag{5}$$

Under SS-PALT with T-I PHCS, n units are put on a life test with progressive censoring scheme $R_1, R_2, \ldots, R_m$. Each unit is run under normal use condition, if it does not fail or removed up to time η, the accelerated condition is applied and the experiment is terminated at $\eta^* = min(x_{m:m:n}, \eta)$. If the mth progressively censored observed failures occurs before time η, the experiment terminated at this time $X_{m:m:n}$. Otherwise, the experiment will be stopped at time η, and all the remaining items $\left(n - \sum_{i=1}^{J} R_i - J\right)$ are removed. The main purpose of this scheme is to control the total time on test based on a predetermined time η . In this case the observed data will be

$$\begin{aligned} &Case\ I : x_{1:m:n} < \cdots < x_{m_u:m:n} \leq \tau < x_{m_u+1:m:n} < \cdots < x_{m:m:n} \text{ if } x_{m:m:n} \leq \eta \\ &Case\ II : x_{1:m:n} < \cdots < x_{J_u:m:n} \leq \tau < x_{J_u+1:m:n} < \cdots < x_{J:m:n} \text{ if } x_{m:m:n} > \eta \end{aligned}$$

where $\tau < \eta$ and m_u and J_u are the number of failed items at use condition for case I and case II, respectively. It is to be noted that when $x_m \leq \eta$ or $\eta \to \infty$, we will have the conventional progressive Type-II censoring scheme.

Similarly, based on SS-PALT with AT-II PHCS, n units are tested with progressive censoring scheme $R_1, R_2, \ldots, R_m$ under normal use condition, if any item out of n items does not fail up to time τ, it is put under accelerated condition and the experiment will be run until censoring time η, then we do not withdraw any items at all except for the time of the mth failure where all remaining surviving units are removed. Thus, the effectively applied scheme in this case is $R_1, \ldots, R_J, 0, 0, 0, R_m$ and the observed data takes the form

$$\begin{aligned} x_{1:m:n} < \cdots < x_{m_u:m:n} \leq \tau < x_{m_u+1:m:n} < \cdots \\ < x_{J:m:n} < \eta < x_{J+1:m:n} < \cdots < x_{m:m:n} \end{aligned}$$

If $\eta \to 0$, the AT-II PHCS reduces to the traditional Type-II censoring scheme. If $\eta \to \infty$, the AT-II PHCS will lead us to the conventional progressive Type-II censoring scheme. The statistical inference based on T-I PHCS and AT-II PHCS under SS-PALS will be described in the next sections.

3 Statistical Inference Based on T-I PHCS

In this section the MLEs of the unknown parameters of Burr(c,k) and the acceleration factor β and the corresponding approximate confidence intervals is obtained under SS-PALT with T-II PHCS. According to T-I PHCS, the Likelihood function under SS-PALT is given by

$$L_1 \propto \prod_{i=1}^{\zeta_u} f_1(x_i)\,[S_1(x_i)]^{R_i} \prod_{i=\zeta_u+1}^{\zeta} f_2(x_i)\,[S_2(x_i)]^{R_i}\,[S_2(\eta)]^{R_\zeta^*} \tag{6}$$

where $x_i = x_{i:m:n}$ for simplicity of notation, $\zeta = m$,$\zeta_u = m_u$ $R_\zeta^* = 0$ for case I and $\zeta = J$,$\zeta_u = J_u$ $R_\zeta^* = (n - J - \sum_{i=1}^{J} R_i)$ for case II. from (2), (3), (5) and (6), the likelihood function can be written as

$$L_1(c,k,\beta) \propto c^\zeta k^\zeta \beta^{\zeta_a} \prod_{i=1}^{\zeta_u}\left(\frac{x_i^{c-1}}{1+x_i^c}\right) \prod_{i=\zeta_u+1}^{\zeta}\left(\frac{\psi_i^{c-1}}{1+\psi_i^c}\right) exp\left(-k\phi(c,\beta,\underline{x})\right) \tag{7}$$

where $\underline{x} = x_1, ..., x_m, m_a = m - m_u, \zeta_a = \zeta - \zeta_u, \psi_i = \tau + \beta(x_i - \tau), i = 1, ..., m,$ $\psi_\eta = \tau + \beta(\eta - \tau)$, and

$$\phi(c,\beta,\underline{x}) = \sum_{i=1}^{\zeta_u}(1+R_i)ln(1+x_i^c) + \sum_{i=\zeta_u+1}^{\zeta}(1+R_i)ln(1+\psi_i^c) + R_\zeta^* ln\left(1+\psi_\eta^c\right)$$

The natural logarithm of the likelihood function, denoted by, $l_1 = ln(L_1(c,k,\beta))$ when $\zeta > 1$, is given by

$$l_1 = \zeta ln(ck) + \zeta_a ln(\beta) + (c-1)\left[\sum_{i=1}^{\zeta_u} ln(x_i) + \sum_{i=\zeta_u+1}^{\zeta} ln(\psi_i)\right] - \sum_{i=1}^{\zeta_u} ln\left(1+x_i^c\right) - \sum_{i=\zeta_u+1}^{\zeta} ln\left(1+\psi_i^c\right) - k\phi\left(c,\beta,\underline{x}\right) \tag{8}$$

Differentiating (8) with respect to c, β and k and equating each result to zero we get the likelihood equations as

$$\frac{\partial l_1}{\partial c} = \frac{\zeta}{c} + \sum_{i=1}^{\zeta_u}(ln(x_i) - \varphi_{1i}) + \sum_{i=\zeta_u+1}^{\zeta}(ln(\psi_i) - \varphi_{2i})$$

$$-k\left(\sum_{i=1}^{\zeta_u}(1+R_i)\varphi_{1i}+\sum_{i=\zeta_u+1}^{\zeta}(1+R_i)\varphi_{2i}+R_{\zeta}^{*}\varphi_{2\eta}\right)=0, \tag{9}$$

$$\frac{\partial l_1}{\partial \beta}=\frac{\zeta_a}{\beta}+(c-1)\sum_{i=\zeta_u+1}^{\zeta}(x_i-\tau)\psi_i^{-1}$$

$$-c(k+1)\sum_{i=\zeta_u+1}^{\zeta}\varphi_{3i}-ck\left(\sum_{i=\zeta_u+1}^{\zeta}R_i\varphi_{3i}+R_{\zeta}^{*}\varphi_{3\eta}\right)=0, \tag{10}$$

$$\frac{\partial l_1}{\partial k}=\frac{\zeta}{k}-\phi(c,\beta,\underline{x})=0. \tag{11}$$

where $\varphi_{1i}=x_i^c ln(x_i)(1+x_i^c)^{-1}$, $\varphi_{2i}=\psi_i^c ln(\psi_i)(1+\psi_i^c)^{-1}$, $\varphi_{2\eta}=\psi_\eta^c ln(\psi_\eta)(1+\psi_\eta^c)^{-1}$, $\varphi_{3i}=(x_i-\tau)\psi_i^{c-1}(1+\psi_i^c)^{-1}$ and $\varphi_{3\eta}=(\eta-\tau)\psi_\eta^{c-1}(1+\psi_\eta^c)^{-1}$. From (11), the MLE of k, denoted by $\hat{k}$ can be obtained as

$$\widehat{k}=\frac{\zeta}{\phi(c,\beta,\underline{x})} \tag{12}$$

substituting the value of $\hat{k}$ in (9) and (10), the MLEs of c and β, say, $\hat{c}$ and $\hat{\beta}$ are obtained by solving the following two nonlinear equations

$$\frac{\zeta}{\hat{c}}+\sum_{i=1}^{\zeta_u}(ln(x_i)-\varphi_{1i})+\sum_{i=\zeta_u+1}^{\zeta}(ln(\psi_i)-\varphi_{2i})$$

$$-\frac{\zeta}{\phi\left(\hat{c},\hat{\beta},\underline{x}\right)}\left(\sum_{i=1}^{\zeta_u}(1+R_i)\varphi_{1i}+\sum_{i=\zeta_u+1}^{\zeta}(1+R_i)\varphi_{2i}+R_{\zeta}^{*}\varphi_{2\eta}\right)=0, \tag{13}$$

$$\frac{\zeta_a}{\hat{\beta}}+(\hat{c}-1)\sum_{i=\zeta_u+1}^{\zeta}(x_i-\tau)\psi_i^{-1}-\hat{c}\left(\frac{\zeta}{\phi\left(\hat{c},\hat{\beta},\underline{x}\right)}+1\right)\sum_{i=\zeta_u+1}^{\zeta}\varphi_{3i}$$

$$-\frac{\hat{c}\zeta}{\phi\left(\hat{c},\hat{\beta},\underline{x}\right)}\left(\sum_{i=\zeta_u+1}^{\zeta}R_i\varphi_{3i}+R_{\zeta}^{*}\varphi_{3\eta}\right)=0, \tag{14}$$

it is be noted that Eqs. (13) and (14) have no explicit solutions. Therefore, the MLEs of c and β can be obtained numerically. Substitute the ML estimates of $\hat{c}$ and $\hat{\beta}$ in (12), the MLE of k can be obtained. In order to construct the approximate confidence intervals of c, β and k, we use the asymptotic normality theory of MLEs. The negative second partial derivatives of Eq. (8) with respect to c, β and k consists of the Fisher information matrix and given by

$$-\frac{\partial l_1^2}{\partial c^2}=\frac{\zeta}{c^2}+\sum_{i=1}^{\zeta_u}\varphi_{4i}+\sum_{i=\zeta_\zeta+1}^{\zeta}\varphi_{5i}$$

$$+k\left(\sum_{i=1}^{\zeta_u}(1+R_i)\varphi_{4i}+\sum_{i=\zeta_u+1}^{\zeta}(1+R_i)\varphi_{5i}+R_\zeta^*\varphi_{5\eta}\right),$$

$$-\frac{\partial l_1^2}{\partial\beta^2}=\frac{\zeta_a}{\beta^2}+(c-1)\sum_{i=\zeta_u+1}^{\zeta}(x_i-\tau)^2\psi_i^{-2}+c(k+1)\sum_{i=\zeta_u+1}^{\zeta}\varphi_{6i}$$

$$+ck\left(\sum_{i=\zeta_u+1}^{\zeta}R_i\varphi_{6i}+R_\zeta^*\varphi_{6\eta}\right),-\frac{\partial l_1^2}{\partial k^2}=\frac{\zeta}{k^2},$$

$$-\frac{\partial l_1^2}{\partial c\partial\beta}=-\frac{\partial l_1^2}{\partial\beta\partial c}=-\sum_{i=\zeta_u+1}^{\zeta}(x_i-\tau)\psi_i^{-1}+\sum_{i=\zeta_u+1}^{\zeta}\varphi_{7i}$$

$$+k\left(\sum_{i=\zeta_u+1}^{\zeta}R_i\varphi_{7i}+R_\zeta^*\varphi_{7\eta}\right),$$

$$-\frac{\partial l_1^2}{\partial c\partial k}=-\frac{\partial l_1^2}{\partial k\partial c}=\sum_{i=1}^{\zeta_u}(1+R_i)\varphi_{1i}+\sum_{i=\zeta_u+1}^{\zeta}(1+R_i)\varphi_{2i}+R_\zeta^*\varphi_{2\eta},$$

$$-\frac{\partial l_1^2}{\partial\beta\partial k}=c\sum_{i=\zeta_u+1}^{\zeta}\varphi_{3i}+c\left(\sum_{i=\zeta_u+1}^{\zeta}R_i\varphi_{3i}+R_\zeta^*\varphi_{3\eta}\right).$$

where $\varphi_{4i}=x_i^c ln(x_i)^2(1+x_i^c)^{-2}$, $\varphi_{5i}=\psi_i^c ln(\psi_i)^2(1+\psi_i^c)^{-2}$, $\varphi_{5\eta}=\psi_\eta^c ln(\psi_\eta^2)(1+\psi_\eta^c)^{-1}[1-\psi_\eta^c(1+\psi_\eta^c)^{-1}]$, $\varphi_{6i}=(x_i-\tau)^2\psi_i^{c-2}(1+\psi_i^c)^{-2}[(c-1)(1+\psi_i^c)-c\psi_i^c]$, $\varphi_{6\eta}=(x_\eta-\tau)^2\psi_\eta^{c-2}(1+\psi_\eta^c)^{-2}[(c-1)(1+\psi_\eta^c)-c\psi_\eta^c]$, $\varphi_{7i}=(x_i-\tau)\psi_i^{c-1}(1+\psi_i^c+cln(\psi_i))(1+\psi_i^c)^{-2}$, and $\varphi_{7\eta}=(x_\eta-\tau)\psi_\eta^{c-1}(1+\psi_\eta^c+cln(\psi_\eta))(1+\psi_\eta^c)^{-2}$.

The asymptotic variance-covariance matrix is obtained by inverting the Fisher information matrix as follows

$$\begin{pmatrix}-\frac{\partial l_1^2}{\partial c^2} & -\frac{\partial l_1^2}{\partial c\partial\beta} & -\frac{\partial l_1^2}{\partial c\partial k}\\ -\frac{\partial l_1^2}{\partial\beta\partial c} & -\frac{\partial l_1^2}{\partial\beta^2} & -\frac{\partial l_1^2}{\partial\beta\partial k}\\ -\frac{\partial l_1^2}{\partial k\partial c} & -\frac{\partial l_1^2}{\partial k\partial\beta} & -\frac{\partial l_1^2}{\partial k^2}\end{pmatrix}^{-1}_{c=\hat{c},\beta=\hat{\beta},k=\hat{k}}=\begin{pmatrix}var(\hat{c}) & cov(\hat{c},\hat{\beta}) & cov(\hat{c},\hat{k})\\ cov(\hat{\beta},\hat{c}) & var(\hat{\beta}) & var(\hat{\beta},\hat{k})\\ cov(\hat{k},\hat{c}) & cov(\hat{k},\hat{\beta}) & var(\hat{k}),\end{pmatrix} \quad (15)$$

Thus, the approximate confidence intervals of c, β and k, are respectively, given by

$$\hat{c}\pm z_{\alpha/2}\sqrt{var(\hat{c})},\ \hat{\beta}\pm z_{\alpha/2}\sqrt{var(\hat{\beta})},\ \text{and}\ \hat{k}\pm,\ z_{\alpha/2}\sqrt{var(\hat{k})}. \quad (16)$$

where $z_{\alpha/2}$ is the upper $\alpha/2$ percentile of a standard normal distribution.

4 Statistical Inference Based on AT-II PHCS

The Likelihood function under SS-PALT based on AT-II PHCS is given by

$$L_2 \propto \prod_{i=1}^{m_u} f_1(x_i)\,[S_1(x_i)]^{R_i} \prod_{i=m_u+1}^{m} f_2(x_i) \prod_{i=m_u+1}^{J} [S_2(x_i)]^{R_i}\,[S_2(x_m)]^{R_m} \tag{17}$$

from (2), (3), (5) and (17), the likelihood function can be written as

$$L_2(c,k,\beta) \propto c^m k^m \beta^{m_a} \prod_{i=1}^{m_u}\left(\frac{x_i^{c-1}}{1+x_i^c}\right) \prod_{i=m_u+1}^{m}\left(\frac{\psi_i^{c-1}}{1+\psi_i^c}\right) exp\left(-k\omega\left(c,\beta,\underline{x}\right)\right) \tag{18}$$

where

$$\omega\left(c,\beta,\underline{x}\right) = \sum_{i=1}^{m_u}(1+R_i)ln\left(1+x_i^c\right) + \sum_{i=m_u+1}^{m} ln\left(1+\psi_i^c\right) + \sum_{i=m_u+1}^{J} R_i ln\left(1+\psi_i^c\right) + R_m ln\left(1+\psi_m^c\right)$$

the natural logarithm of the likelihood function, $l_2 = ln(L_2(c,k,\beta))$ is given by

$$l_2 = mln(ck) + m_a ln(\beta) + (c-1)\left[\sum_{i=1}^{m_u} ln(x_i) + \sum_{i=m_u+1}^{m} ln(\psi_i)\right] - \sum_{i=1}^{m_u} ln\left(1+x_i^c\right) - \sum_{i=m_u+1}^{m} ln\left(1+\psi_i^c\right) - k\omega\left(c,\beta,\underline{x}\right) \tag{19}$$

from (19), the likelihood equations of c, β and k can be expressed as

$$\frac{\partial l_2}{\partial c} = \frac{m}{c} + \sum_{i=1}^{m_u} ln(x_i) + \sum_{i=m_u+1}^{m} ln(\psi_i) - \sum_{i=1}^{m_u}\varphi_{1i} - \sum_{i=m_u+1}^{m}\varphi_{2i} - k\omega_1(c,\beta,\underline{x}) = 0, \tag{20}$$

$$\frac{\partial l_2}{\partial \beta} = \frac{m_a}{\beta} + (c-1)\sum_{i=m_u+1}^{m}(x_i-\tau)\psi_i^{-1} - c(k+1)\sum_{i=m_u+1}^{m}\varphi_{3i} - ck\left(\sum_{i=m_u+1}^{J} R_i\varphi_{3i} + R_m\varphi_{3m}\right) = 0, \tag{21}$$

$$\frac{\partial l_1}{\partial k} = \frac{m}{k} - \omega(c,\beta,\underline{x}) = 0. \tag{22}$$

where φ_{1i}, φ_{2i}, φ_{3i} as defined in the previous section,and

$$\omega_1(c,\beta,\underline{x}) = \sum_{i=1}^{m_u}(1+R_i)\varphi_{1i} + \sum_{i=m_u+1}^{m}\varphi_{2i} + \sum_{i=m_u+1}^{J} R_i\varphi_{2i} + R_m\varphi_{2m}.$$

After equating (22) by zero, the MLE of the parameter k can be obtained as follows

$$\widehat{k} = \frac{m}{\omega(c, \beta, \underline{x})} \tag{23}$$

substituting the value of $\widehat{k}$ given by (23) in (20) and (21), yields two nonlinear equations in $\hat{c}$ and $\hat{\beta}$ as follows

$$\frac{m}{\hat{c}} + \sum_{i=1}^{m_u} ln(x_i) + \sum_{i=m_u+1}^{m} ln(\psi_i) - \sum_{i=1}^{m_u} \varphi_{1i} - \sum_{i=m_u+1}^{m} \varphi_{2i} - \frac{m\omega_1(\hat{c}, \hat{\beta}, \underline{x})}{\omega(\hat{c}, \hat{\beta}, \underline{x})} = 0, \tag{24}$$

$$\frac{m_a}{\hat{\beta}} + (\hat{c} - 1) \sum_{i=m_u+1}^{m} (x_i - \tau)\psi_i^{-1} - \hat{c}\left(\frac{m}{\omega(\hat{c}, \hat{\beta}, \underline{x})} + 1\right) \sum_{i=m_u+1}^{m} \varphi_{3i}$$
$$- \frac{\hat{c}m}{\omega(\hat{c}, \hat{\beta}, \underline{x})}\left(\sum_{i=m_u+1}^{J} R_i\varphi_{3i} + R_m\varphi_{3m}\right) = 0, \tag{25}$$

it is observed that there is no closed form solution for (24) and (25). Therefore, an iterative procedure such as Newton-Raphson method can be used to find the ML estimates of $\hat{c}$ and $\hat{\beta}$. Substitute the ML estimates $\hat{c}$ and $\hat{\beta}$ in (12), the MLE of k can be obtained. The elements of the Fisher information matrix in this case can be expressed as follows

$$-\frac{\partial l_2^2}{\partial c^2} = \frac{m}{c^2} + \sum_{i=1}^{m_u} \varphi_{4i} + \sum_{i=m_u+1}^{m} \varphi_{5i} + k\omega_2(c, \beta, \underline{x}),$$
$$-\frac{\partial l_2^2}{\partial \beta^2} = \frac{m_a}{\beta^2} + (c - 1) \sum_{i=m_u+1}^{m} (x_i - \tau)^2\psi_i^{-2} + c(k + 1) \sum_{i=m_u+1}^{m} \varphi_{6i}$$
$$+ ck\left(\sum_{i=m_u+1}^{J} R_i\varphi_{6i} + R_m\varphi_{6m}\right),$$
$$-\frac{\partial l_2^2}{\partial k^2} = \frac{m}{k^2},$$
$$-\frac{\partial l_2^2}{\partial c\partial \beta} = -\frac{\partial l_1^2}{\partial \beta\partial c} = -\sum_{i=m_u+1}^{m} (x_i - \tau)\psi_i^{-1} + \sum_{i=m_u+1}^{m} \varphi_{7i} + k\omega_3(c, \beta, \underline{x}),$$
$$-\frac{\partial l_2^2}{\partial c\partial k} = -\frac{\partial l_1^2}{\partial k\partial c} = \omega_1(c, \beta, \underline{x}),$$
$$-\frac{\partial l_1^2}{\partial \beta\partial k} = -\frac{\partial l_1^2}{\partial k\partial \beta} = c \sum_{i=m_u+1}^{m} \varphi_{3i} + c\left(\sum_{i=m_u+1}^{J} R_i\varphi_{3i} + R_m\varphi_{3m}\right).$$

where $\varphi_{4i}, \varphi_{5i}, \varphi_{6i}, \varphi_{7i}$, as defined before and

$$\omega_2(c, \beta, \underline{x}) = \sum_{i=1}^{m_u}(1 + R_i)\varphi_{4i} + \sum_{i=m_u+1}^{m} \varphi_{5i} + \sum_{i=m_u+1}^{J} R_i\varphi_{5i} + R_m\varphi_{5m},$$

and

$$\omega_3(c, \beta, \underline{x}) = \sum_{i=m_u+1}^{m} \varphi_{7i} + \sum_{i=m_u+1}^{J} R_i\varphi_{7i} + R_m\varphi_{7m}.$$

The asymptotic variance-covariance matrix and the approximate confidence intervals of c, β and k can be obtained as in (15) and (16).

5 Simulation Study

In this section, a simulation study is carried out to compare the performance of the estimators based on different sampling schemes and the corresponding approximate confidence intervals in terms of mean squared errors (MSEs), relative errors (REs), where RE=$\sqrt{MSE}/\theta$, $\theta = c, \beta, k$, and confidence lengths (CLs) discussed in the previous sections. We perform a simulation study to illustrate the statistical behaviour of the estimators by considering $(n, m) = (30, 10), (45, 15), (60, 10)$ and $(60, 20)$ and different values of τ, η and by choosing ($c = 3$, $\beta = 1.2$ and $k = 1.5$), ($c = 2$, $\beta = 2$ and $k = 2$) based on T-I PHC and AT-II PHC schemes. We consider the following three progressive censoring schemes

- Scheme 1: $R_1 = \cdots = R_{m-1} = 0 \textit{ and } R_m = n - m.$
- Scheme 2: $R_1 = \cdots = R_{m-1} = 1 \textit{ and } R_m = n - 2m + 1.$
- Scheme 3: $R_1 = \cdots = R_m = \frac{n-m}{m}$.

It is observed that scheme 1 describes the case of Type-II censoring scheme, where $n - m$ units are removed from the experiment at the time of the mth failure, while scheme 2 and scheme 3 is the usual Type-II progressive censoring scheme with censoring. For each setting, we replicate the process 1000 times and obtain average estimates, MSEs, REs and CLs.The simulation results are reported in Tables 1, 2, 3 and 4. The simulation study is performed as follows

1. Determine the values of $n, m, R_i's, \tau, \eta$ and the values of parameters c, β and k.
2. Generate progressive Type-II censored sample from Burr Type-XII distribution using the method proposed by Balakrishnan and Sandhu [10], by setting $X = ((1 - U)^{\frac{-1}{k}} - 1)^{\frac{1}{c}}$, if $x \leq \tau$, and $X = \frac{((1-U)^{\frac{-1}{k}}-1)^{\frac{1}{c}}-\tau}{\beta} + \tau$, if $x > \tau$, where U represents a uniform random variable from [0, 1].Based SSPALT with T-I PHCS the experiment is terminated at $\eta^* = min(x_m, \eta)$.
3. Under SSPALT with AT-II PHCS, we generate the data by following the same procedure in step 2,but if $\eta < x_m$, we generate an additional Type-II censored sample of size $m - J - 1$ from truncation distribution $f_2(x)/[1 - F_2(x_{J+1})]$. Hence, $X = \frac{(V(1-U)^{\frac{-1}{k}}-1)^{\frac{1}{c}}-\tau}{\beta} + \tau$, where $V = [1 + (\tau + \beta(x_{J+1} - \tau))^c]^{-k}$. In this case, the experiment stops at x_m.

4. From the ordered observations obtained in step 2 and step 3, obtain the MLEs of the unknown parameters and the corresponding approximate confidence intervals.
5. Repeat steps 2–4, 1000 times.
6. Obtain the average number of observed failures before the predetermined time η, denoted by J_A, in the case of T-I PHCS.
7. Obtain the average values of estimates, MSEs, REs and CLs.

Based on the results shown in Tables 1, 2, 3 and 4, we see that in all cases the MLEs of c, β and k based on AT-II PHCS always give smaller MSEs and REs in compare to those based on T-I PHCS. For fixed n and η, when m increases, the MSEs and REs of the MLEs based on AT-II PHCS decrease in all cases, this is also true for T-I PHCS when η is large. For fixed n and m, when η increases the MSEs and REs of the MLEs based on T-I PHCS decrease in all cases, but there have been no such changes under the AT-II PHCS, because the number of failures is predetermined and no additional observed failures when η increase. As a matter of fact, based on T-I PHCS for small $\eta's$, the difference between MSEs and REs of the two schemes is more sensible, but when η increases, the observed number of failures increase and this difference will be smaller or equal to zero, see Table 4, the case of $(n, m) = (60, 10)$ under schemes 1 and 2. Therefore, considering the MSEs and REs of MLEs, the AT-II PHCS scheme will be a preferable scheme in order to estimate the unknown parameters with a higher efficiency when the time of the experiment is not the major concern.

Comparing the three censoring schemes (Sch), the results of Tables 1, 2, 3 and 4 show that Sch 3 of AT-II PHC and T-I PHCS gives the smallest values of MSEs and REs in most cases. Based on AT-II PHCS, for fixed m and η, the MSEs and REs decrease with the increase in the sample size n for parameters c and k for schemes 1, 2 and 3, while the MSEs and REs of parameter β increase along with n in the schemes 1, 2 and 3. For fixed m and η, when n increases, we do not observe a specific pattern in the MSEs and REs in the case of T-I PHCS. Also, the simulation results show that the MLEs of the parameters (Par) based AT-II PHCS perform better than the MLEs based on T-I PHCS in terms of CLs. Based on T-I PHCS, when the predetermined time η increases, the CLs decrease in all cases and the difference between CLs of the two schemes decrease. It is because of the fact that, when the η increase the two schemes approaches to the conventional progressive Type-II censoring scheme, i.e. $m = J_A$.

6 Conclusion

In this paper, we have discussed the maximum likelihood estimators of the unknown parameters and the approximate confidence intervals of the Burr Type-II distribution and the acceleration factor under SS-PALT when the data are coming from two different types of progressively hybrid censoring schemes which are Type-I progressive hybrid censoring scheme and adaptive Type-II progressive hybrid censoring scheme. The MLEs of the model parameters are obtained numerically using the Newton-Raphson method and their performances are evaluated and discussed in terms of mean squared errors (MSEs) and relative errors (REs). The efficiency of the MLEs are compared using a simulation study. The results of the simulation study suggests that the MLEs based on AT-II PHCS perform better than those under T-I PHCS in terms of MSEs

Table 1 Average values of MLEs, MSEs, RE, CLs and J_A for $c = 3$, $\beta = 1.2$, $k = 1.5$, $\tau = 0.25$ and $\eta = 0.4$

(n, m)	Sch	Par	AT-II PHCS				T-I PHCS				
			MLEs	MSEs	RE	CLs	MLEs	MSEs	RE	CLs	J_A
(30, 10)	1	c	2.4017	0.4165	0.215	1.7930	2.8317	0.6468	0.268	2.7926	3.2650
		β	1.1700	0.0227	0.126	0.4526	1.1011	0.1189	0.287	1.2162	
		k	0.8472	0.7384	0.573	1.7452	1.5706	0.7867	0.591	3.4183	
	2	c	2.5054	0.4251	0.217	1.7792	2.8354	0.6016	0.259	2.7685	3.0880
		β	1.1735	0.0170	0.109	0.4682	1.1094	0.1239	0.293	1.1041	
		k	0.9369	0.6886	0.553	1.7427	1.5972	0.8896	0.629	3.4589	
	3	c	2.5777	0.4073	0.213	1.7891	2.8451	0.4237	0.217	2.7346	2.9420
		β	1.1757	0.0126	0.094	0.4921	1.1250	0.1231	0.292	0.9426	
		k	1.0006	0.6527	0.539	1.6825	1.6389	1.0239	0.675	3.4945	
(45, 15)	1	c	2.2310	0.2722	0.174	1.4374	3.0397	0.7190	0.283	3.0421	4.9540
		β	1.1993	0.0189	0.115	0.1764	1.1973	0.1040	0.269	1.3930	
		k	0.6948	0.4438	0.444	1.5635	1.5653	0.7482	0.577	3.4983	
	2	c	2.3163	0.2715	0.174	1.4268	3.0406	0.6069	0.260	3.3643	4.6670
		β	1.2094	0.0199	0.118	0.1794	1.1976	0.1108	0.277	1.2843	
		k	0.7737	0.4898	0.467	1.5505	1.5813	0.6925	0.555	3.8721	
	3	c	2.4121	0.2695	0.173	1.4535	3.0389	0.4802	0.231	3.7785	4.3990
		β	1.2044	0.0128	0.094	0.1845	1.1989	0.1149	0.282	1.0201	
		k	0.8511	0.5227	0.482	1.4299	1.5965	0.6056	0.519	4.3643	
(60, 10)	1	c	2.5421	0.2847	0.178	1.8851	3.1617	0.6543	0.270	2.7494	6.3920
		β	1.1834	0.0771	0.231	0.1168	1.3035	0.1260	0.296	0.5906	
		k	1.0789	0.4395	0.442	1.7564	1.5436	0.9046	0.634	3.0300	
	2	c	2.5003	0.2676	0.172	1.8548	3.1512	0.6508	0.269	2.9373	6.0700
		β	1.1735	0.0616	0.207	0.1186	1.2972	0.1224	0.292	0.6245	

Table 1 continued

(n, m)	Sch	Par	AT-II PHCS				T-I PHCS				
			MLEs	MSEs	RE	CLs	MLEs	MSEs	RE	CLs	J_A
		k	1.0289	0.4362	0.440	1.7010	1.5394	0.9236	0.641	3.2468	
	3	c	2.5428	0.2596	0.170	1.7381	3.1580	0.4928	0.234	2.7404	4.9410
		β	1.2274	0.1419	0.314	0.1195	1.3175	1.6831	1.081	0.5847	
		k	1.0196	0.5173	0.479	1.5795	1.5856	0.7737	0.586	3.2348	
(60, 20)	1	c	2.1535	0.2603	0.170	1.2273	3.1610	0.8356	0.305	2.7046	6.4930
		β	1.2392	0.0259	0.134	0.1110	1.2928	0.1089	0.275	1.3422	
		k	0.6223	0.3054	0.368	1.3602	1.5152	0.8349	0.609	2.9702	
	2	c	2.2590	0.2487	0.166	1.2136	3.1623	0.6550	0.270	2.7867	6.1430
		β	1.2375	0.0212	0.121	0.1112	1.2970	0.1103	0.277	1.0439	
		k	0.6951	0.3306	0.383	1.2060	1.5282	0.7402	0.574	3.0918	
	3	c	2.3599	0.2485	0.166	1.2196	3.1649	0.5048	0.237	2.7678	5.8170
		β	1.2247	0.0119	0.091	0.1108	1.3019	0.1158	0.284	0.7795	
		k	0.7725	0.3642	0.402	1.1069	1.5422	0.6432	0.535	3.1208	

Table 2 Average values of MLEs, MSEs, RE, CLs and J_A for $c = 3$, $\beta = 1.2$, $k = 1.5$, $\tau = 0.25$ and $\eta = 0.7$

(n, m)	Sch	Par	AT-II PHCS				T-I PHCS				
			MLEs	MSEs	RE	CLs	MLEs	MSEs	RE	CLs	J_A
(30, 10)	1	c	3.3393	0.5906	0.256	1.9781	3.3411	0.6031	0.259	1.9972	9.8790
		β	1.2551	0.0323	0.150	0.1879	1.2533	0.0326	0.150	0.2122	
		k	1.6946	0.4221	0.433	1.7862	1.6950	0.4245	0.434	1.7933	
	2	c	3.3073	0.5898	0.256	1.9483	3.2851	0.6006	0.258	2.0154	9.4530
		β	1.2454	0.0261	0.135	0.1920	1.2373	0.0277	0.139	0.2006	
		k	1.6884	0.4341	0.439	1.7432	1.6778	0.4369	0.441	1.7917	
	3	c	3.3084	0.4070	0.213	1.9069	3.2393	0.4146	0.215	2.1015	8.3150
		β	1.2359	0.0185	0.113	0.1821	1.2207	0.0252	0.132	0.1931	
		k	1.7089	0.4290	0.437	1.7062	1.6705	0.4381	0.441	1.9084	
(45, 15)	1	c	3.2597	0.4277	0.218	1.6060	3.2610	0.4314	0.219	1.6102	14.9490
		β	1.2479	0.0207	0.120	0.1428	1.2475	0.0207	0.120	0.1527	
		k	1.6332	0.1975	0.296	1.4337	1.6339	0.1989	0.297	1.4382	
	2	c	3.1911	0.4008	0.211	1.5709	3.2161	0.4081	0.213	1.6139	14.4670
		β	1.2427	0.0168	0.108	0.1434	1.2353	0.0172	0.109	0.1873	
		k	1.6061	0.2100	0.306	1.4074	1.6183	0.2314	0.321	1.4261	
	3	c	3.1846	0.3548	0.199	1.5291	3.1681	0.3711	0.203	1.6880	12.5370
		β	1.2335	0.0110	0.087	0.1420	1.2242	0.0155	0.104	0.1592	
		k	1.6221	0.2098	0.305	1.3657	1.5980	0.2338	0.322	1.5233	
(60, 10)	1	c	3.3557	0.4574	0.225	1.8430	3.3552	0.4582	0.226	1.8433	9.9980
		β	1.3733	0.1024	0.267	0.0928	1.3732	0.1024	0.267	0.0929	
		k	1.6816	0.3771	0.409	1.9433	1.6815	0.3773	0.409	1.9434	
	2	c	3.3492	0.4528	0.224	1.8399	3.3487	0.4530	0.224	1.8409	9.9940

Table 2 continued

(n, m)	Sch	Par	AT-II PHCS				T-I PHCS				
			MLEs	MSEs	RE	CLs	MLEs	MSEs	RE	CLs	J_A
		β	1.3648	0.0953	0.257	0.0924	1.3647	0.0953	0.257	0.0928	
		k	1.6819	0.3768	0.409	1.9259	1.6817	0.3772	0.409	1.9268	
	3	c	3.2331	0.3900	0.208	1.7660	3.2603	0.4287	0.218	1.8990	9.0630
		β	1.3148	0.0594	0.203	0.0995	1.3071	0.0632	0.209	0.1490	
		k	1.6356	0.3866	0.415	1.7972	1.6489	0.4280	0.436	1.9526	
(60, 20)	1	c	3.2006	0.2926	0.180	1.3873	3.2049	0.2931	0.180	1.3903	19.9560
		β	1.2442	0.0155	0.104	0.1221	1.2437	0.0155	0.104	0.1267	
		k	1.5802	0.1508	0.259	1.2224	1.5818	0.1535	0.261	1.2226	
	2	c	3.1397	0.2725	0.174	1.3565	3.1719	0.2975	0.182	1.3903	19.4030
		β	1.2423	0.0123	0.092	0.1185	1.2340	0.0134	0.096	0.1705	
		k	1.5530	0.1585	0.265	1.2008	1.5712	0.1855	0.287	1.2134	
	3	c	3.0975	0.2507	0.167	1.3104	3.1439	0.2551	0.168	1.4610	16.7090
		β	1.2325	0.0077	0.073	0.1142	1.2284	0.0125	0.093	0.1388	
		k	1.5399	0.1612	0.268	1.1483	1.5564	0.2029	0.300	1.3031	

Table 3 Average values of MLEs, MSEs, RE, CLs and J_A for $c = 2$, $\beta = 2$, $k = 2$, $\tau = 0.2$ and $\eta = 0.3$

(n, m)	Sch	Par	AT-II PHCS				T-I PHCS				
			MLEs	MSEs	RE	CLs	MLEs	MSEs	RE	CLs	J_A
(30, 10)	1	c	2.0200	0.2514	0.251	1.4114	2.1828	0.2559	0.253	1.7308	7.4640
		β	2.2488	0.2463	0.248	0.6132	2.3988	0.4167	0.323	1.3001	
		k	1.8468	0.6158	0.392	2.0462	2.0056	0.8815	0.469	2.5064	
	2	c	2.0308	0.2591	0.255	1.4323	2.1684	0.2640	0.257	1.8796	6.6230
		β	2.2106	0.1774	0.211	0.6271	2.4005	0.4181	0.323	1.3477	
		k	1.8270	0.5883	0.384	2.0386	1.9755	0.7757	0.440	2.7289	
	3	c	2.0744	0.1273	0.178	1.4022	2.1756	0.1456	0.191	1.9333	5.8650
		β	2.2339	0.1710	0.207	0.6263	2.4660	0.5183	0.360	1.1663	
		k	1.9404	0.6417	0.401	1.9737	2.0110	0.7535	0.434	2.8988	
(45, 15)	1	c	1.8596	0.2240	0.237	1.1413	2.1428	0.2330	0.241	1.3592	11.3230
		β	2.2372	0.2842	0.267	0.4419	2.3690	0.3356	0.290	1.6203	
		k	1.5930	0.3516	0.296	1.6533	1.9170	0.8209	0.453	1.9700	
	2	c	1.8359	0.1270	0.178	1.2063	2.1416	0.1477	0.192	1.4284	9.9970
		β	2.1855	0.1333	0.183	0.4242	2.4032	0.3845	0.310	2.0608	
		k	1.5245	0.3417	0.292	1.7471	1.9058	0.7689	0.438	2.1065	
	3	c	1.9430	0.0990	0.157	1.1288	2.1494	0.1306	0.181	1.4786	8.8430
		β	2.1938	0.1032	0.161	0.4090	2.4701	0.4756	0.345	1.0797	
		k	1.7157	0.3844	0.310	1.5507	1.9363	0.6185	0.393	2.2475	
(60, 10)	1	c	2.1533	0.1039	0.161	1.2353	2.1606	0.1068	0.163	1.2467	9.9520
		β	2.8428	1.2239	0.553	0.2779	2.8410	1.2318	0.555	0.3073	
		k	2.0615	0.4783	0.346	2.1786	2.0646	0.4882	0.349	2.1943	
	2	c	2.1454	0.0937	0.153	1.2368	2.1574	0.1145	0.169	1.2615	9.8930
		β	2.8046	1.1216	0.530	0.2614	2.7959	1.1628	0.539	0.2829	

Table 3 continued

(n, m)	Sch	Par	AT-II PHCS				T-I PHCS				
			MLEs	MSEs	RE	CLs	MLEs	MSEs	RE	CLs	J_A
		k	2.0532	0.4800	0.346	2.1538	2.0568	0.4933	0.351	2.1857	
	3	c	2.0493	0.0802	0.142	1.2366	2.1186	0.0958	0.155	1.4039	8.3500
		β	2.5461	0.6199	0.394	0.2783	2.6897	0.8414	0.459	0.5620	
		k	1.8991	0.4892	0.350	1.9843	1.9600	0.6316	0.397	2.3357	
(60, 20)	1	c	1.8679	0.0920	0.152	1.0310	2.1215	0.1051	0.162	1.1589	15.2330
		β	2.1364	0.0875	0.148	0.3440	2.3519	0.2758	0.263	1.2410	
		k	1.5645	0.2607	0.255	1.4626	1.8951	0.6581	0.406	1.6816	
	2	c	1.8982	0.0857	0.146	1.0203	2.1203	0.0949	0.154	1.2099	13.3640
		β	2.1360	0.0659	0.128	0.3225	2.3909	0.3275	0.286	1.0280	
		k	1.5883	0.2766	0.263	1.3838	1.8877	0.5223	0.361	1.7948	
	3	c	1.8791	0.0859	0.147	0.9560	2.1354	0.0927	0.152	1.2747	11.8780
		β	2.2015	0.1164	0.171	0.2919	2.4694	0.4018	0.317	0.9850	
		k	1.6112	0.3331	0.289	1.2916	1.9367	0.6394	0.400	1.9410	

Table 4 Average values of MLEs, MSEs, RE, CLs and J_A for $c = 2$, $\beta = 2$, $k = 2$, $\tau = 0.3$ and $\eta = 0.45$

(n, m)	Sch	Par	AT-II PHCS				T-I PHCS				
			MLEs	MSEs	RE	CLs	MLEs	MSEs	RE	CLs	J_A
(30, 10)	1	c	2.2767	0.2241	0.237	1.4566	2.2766	0.2342	0.242	1.4567	9.9980
		β	2.4228	0.4228	0.325	0.5645	2.4227	0.4228	0.325	0.5646	
		k	2.1389	0.4883	0.349	2.0549	2.1388	0.4884	0.349	2.0550	
	2	c	2.2558	0.2450	0.247	1.4481	2.2560	0.2470	0.248	1.4542	9.9350
		β	2.3763	0.3414	0.292	0.5956	2.3728	0.3426	0.293	0.6338	
		k	2.1292	0.4922	0.351	2.0233	2.1283	0.4922	0.351	2.0262	
	3	c	2.2093	0.2152	0.232	1.4020	2.2049	0.2216	0.235	1.4797	9.0810
		β	2.3299	0.2631	0.256	0.6526	2.3261	0.2631	0.256	0.6886	
		k	2.0967	0.5113	0.358	1.9455	2.0667	0.5145	0.359	2.0497	
(45, 15)	1	c	2.1898	0.1345	0.183	1.1724	2.1895	0.1346	0.183	1.1727	14.9940
		β	2.3337	0.2325	0.241	0.4139	2.3336	0.2325	0.241	0.4153	
		k	2.0442	0.2836	0.266	1.6494	2.0440	0.2840	0.266	1.6500	
	2	c	2.1732	0.1293	0.180	1.1632	2.1729	0.1307	0.181	1.1689	14.9250
		β	2.3019	0.1866	0.216	0.4175	2.2978	0.1907	0.218	0.4246	
		k	2.0356	0.2859	0.267	1.6137	2.0340	0.2873	0.268	1.6174	
	3	c	2.1470	0.1150	0.170	1.1308	2.1384	0.1215	0.174	1.1907	13.6110
		β	2.2709	0.1457	0.191	0.4359	2.2768	0.1523	0.195	0.4951	
		k	2.0226	0.2905	0.269	1.5650	1.9837	0.3012	0.274	1.6439	
(60, 10)	1	c	2.1776	0.1046	0.162	1.2519	2.1776	0.1046	0.162	1.2519	10.0000
		β	2.8512	1.2312	0.555	0.2716	2.8512	1.2312	0.555	0.2716	
		k	2.0429	0.4245	0.326	2.1750	2.0429	0.4245	0.326	2.1750	
	2	c	2.1767	0.1047	0.162	1.2576	2.1767	0.1047	0.162	1.2576	10.0000
		β	2.8106	1.1338	0.532	0.2541	2.8106	1.1338	0.532	0.2541	

Table 4 continued

(n, m)	Sch	Par	AT-II PHCS				T-I PHCS				
			MLEs	MSEs	RE	CLs	MLEs	MSEs	RE	CLs	J_A
		k	2.0386	0.4200	0.324	2.1554	2.0386	0.4200	0.324	2.1554	
	3	c	2.1537	0.1005	0.159	1.2480	2.1492	0.1014	0.159	1.2562	9.9080
		β	2.6219	0.6854	0.414	0.2651	2.6206	0.6858	0.414	0.2673	
		k	2.0052	0.3994	0.316	2.0154	1.9965	0.4124	0.414	2.0247	
(60, 20)	1	c	2.1498	0.0897	0.150	1.0089	2.1499	0.0897	0.150	1.0089	19.9960
		β	2.2910	0.1613	0.201	0.3222	2.2910	0.1614	0.201	0.3245	
		k	2.0032	0.2104	0.229	1.4156	2.0033	0.2105	0.229	1.4164	
	2	c	2.1353	0.0867	0.147	1.0019	2.1366	0.0889	0.149	1.0038	19.9540
		β	2.2659	0.1299	0.180	0.3067	2.2624	0.1332	0.182	0.3257	
		k	1.9947	0.2112	0.230	1.3827	1.9951	0.2136	0.231	1.3832	
	3	c	2.0995	0.0773	0.139	0.9690	2.1064	0.0872	0.148	1.0238	18.1410
		β	2.2437	0.1002	0.158	0.3140	2.2512	0.1107	0.166	0.3476	
		k	1.9595	0.2214	0.235	1.3258	1.9440	0.2431	0.247	1.4096	

and REs, because a large number of failures can be observed when AT-II PHCS is applied. Also, the results of the simulation study suggests that the MLEs based AT-II PHCS perform better than the MLEs based on T-I PHCS in terms of CLs. In general, if the experimental time and the number of failed items in the experiment are not the major concern, then the AT-II PHCS scheme is recommended in order to obtain better estimates of the parameters. In contrast, if one wants to have a shorter experimental time and/or allow only a few experimental units damaged during the experiment, then the T-I PHCS scheme shall be a reasonable alternative to achieve the goal.

Acknowledgements The authors would like to thank the reviewers and the editors for their comments which helped improve the paper.

References

1. Abd-Elfattah AM, Hassan AS, Nassr S (2008) Estimation in step-stress partially accelerated life tests for the Burr type XII distribution using type I censoring. Stat Methodol 5:502–514
2. Abdel-Ghaly AA, El-Khodary EH, Ismail AA (2003) Estimation and optimal design in step partially accelerated life tests for the compound Pareto distribution using type II censoring. Proceeding of the 15th annual conference on statistics and computer modeling in human and social sciences. Cairo University, Egypt, Faculty of Economics and Political Science, pp 35–47
3. Abdel-Hamid AH (2009) Constant-partially accelerated life tests for Burr XII distribution with progressive type II censoring. Comput Stat Data Anal 53:2511–2523
4. Ali Mousa MAM (1995) Empirical Bayes estimators for the Burr type XII accelerated life testing model based on type-2 censored data. J Statist Comput Simul 52:95–103
5. Ali Mousa MAM, Jaheen ZF (2002) Statistical inference for the Burr model based on progressively censored data. Comput Math Appl 43(10–11):1441–1449
6. Sobhi MMAL, Soliman AA (2015) Estimation for the exponentiated Weibull model with adaptive Type-II progressive censored schemes. Appl Math Modell 40(2):1180–1192
7. Ashour SK, Nassar MMA (2014) Analysis of generalized exponential distribution under adaptive Type-II progressive hybrid censored competing risks data. Int J Adv Stat Probab 2:108–113
8. Bai DS, Chung SW, Chun YR (1993) Optimal design of partially accelerated life tests for the lognormal distribution under type I censoring. Reliab Eng Syst Saf 40:85–92
9. Balakrishnan N (2007) Progressive censoring methodology: an appraisal (with discussions). Test 16:211–296
10. Balakrishnan N, Sandhu RA (1995) A simple simulation algorithm for generating progressively type-II censored samples. Am Statis 49:229–230
11. Balakrishnan N, Cramer E (2014) The art of progressive censoring: censoring: applications to reliability and quality. Birkhauser, New York
12. Balakrishnan N, Kundu D (2013) Hybrid censoring: models, inferential results and applications. Comput Stat Data Anal 57(1):166–209
13. Balakrishnan N, Aggarwala R (2000) Progressive censoring: theory, methods and applications. Birkhuser, Boston
14. Burr IW (1942) Cumulative frequency distribution. Ann Math. Stat 13:215–231
15. Cai J, Shi Y, Liu B (2015) Bayesian analysis for Burr-XII masked system in step-stress partially accelerated life test under Type-I progressive hybrid censoring. Int J Reliab Qual Saf Eng 22(4): 1550018–1550034
16. DeGroot MH, Goel PK (1979) Bayesian and optimal design in partially accelerated life testing. Nav Res Logist Q 16(2):223–235
17. Epstein B (1954) Truncated life-test in exponential case. Ann Math Stat 25:555–564
18. Goel PK (1971) Some estimation problems in the study of tampered random variables. Technical Report No. 50, Department of Statistics, Carnegie-Mellon University, Pittspurgh
19. Hemmati F, Khorram S (2013) Statistical analysis of the log-normal distribution under type-II progressive hybrid censoring schemes. Commun Stat Simul Comput 42:52–75

20. Ismail AA, Aly HM (2010) Optimal planning of failure-step stress partially accelerated life test under Type-II censoring. J Stat Comput Simul 80(12):1335–1348
21. Ismail AA (2014) Inference for a step-stress partially accelerated life test model with an adaptive Type-II progressively hybrid censored data from Weibull distribution. J Comput Appl Math 260: 533–542
22. Kundu D, Joarder A (2006) Analysis of type-II progressively hybrid censored data. Comput Stat Data Anal 50:2509–2528
23. Lewis AW (1981) The Burr distribution as a general parametric family in survivorship and reliability theory applications. Ph.D. Thesis, Department of Biostatistics, University of North Carolina
24. Lin CT, Ng HKT, Chan PS (2009) Statistical inference of type-II progressively hybrid censored data with Weibull lifetimes. Commun Stat Theory Methods 38:1710–1729
25. Mahmoud MAW, Soliman AA, Abd Ellah AH, El-Sagheer RM (2013) Estimation of generalized Pareto under an adaptive type-II progressive censoring. Intell Inf Manag 5:73–83
26. Nelson W (1982) Applied life data analysis. Wiley, NewYork
27. Ng HKT, Kundu D, Chan PS (2009) Statistical analysis of exponential lifetimes under an adaptive Type-II progressively censoring scheme. Nav Res Logist 56:687–698
28. Rodriguez RN (1977) A guide to the Burr type XII distributions. Biometrika 64(1):129–134
29. Shi Y, Shi X (2015) Estimation and optimal plan in step-stress partially accelerated life test model with progressive hybrid censored data from pareto distribution. J Phys Sci 20:53–62
30. Soliman AA (2005) Estimation of parameters of life from progressively censored data using Burr-XII model. IEEE Trans Reliab 54:34–42
31. Tadikamalla PR (1980) A look at the Burr and related distributions. Int Stat Rev 48:337–344
32. Zhang C, Shi Y, Wu M (2015) Statistical inference for competing risks model in step-stress partially accelerated life tests with progressively Type-I hybrid censored Weibull life data. J Comput Appl Math. doi:10.1016/j.cam.2015.11.002
33. Zhang C, Shi Y (2016) Estimation of the extended Weibull parameters and acceleration factors in the step-stress accelerated life tests under an adaptive progressively hybrid censoring data. J Stat Comput Simul. doi:10.1080/00949655.2016.1166366
34. Zimmer WJ, Keats JB, Wang FK (1998) The Burr XII distribution in reliability analysis. J Qual Technol 30:386–394