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Corrigendum to "Harmonic analysis and statistics of the first Galois cohomology group"

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Abstract

In our earlier paper, there are special cases in which the main theorem could not hold for reasons related to the Grunwald–Wang theorem. We correct the statement and its proof, and we include a short discussion of the added hypothesis of "viability" needed to make our theorem true.

The authors regret to report that our paper [2] has an error in the main result, [2, Theorem 1.1]. The corollary [2, Corollary 1.2] is correct as stated, but the proof requires some revisiting to justify that fact. The error is in converting from the Dirichlet series to an asymptotic: we originally claimed that the main term is never canceled out in the finite sum of Euler products that arise from a periodic w function. However, this is not always true. The Poisson summation technique in [2, Theorem 2.3] and the other results on w functions not involving asymptotics are not affected.

This theorem states a nonzero asymptotic for any family of local conditions $\mathcal{L}$ which are Frobenian and for which L_p is closed under translation by $H^1_{ur}(K_p, T)$ for all but finitely many places of K, with no restrictions on the remaining finitely many places. However, it is known that not every local condition can occur at all by the Grunwald–Wang theorem, contradicting the statement of [2, Theorem 1.1].

In fact, Grunwald made the same error in the original statement of the Grunwald–Wang theorem, which stood until Wang produced a counterexample. Wang's counterexample translates to our setting to give the following counterexample to Theorem 1.1 as originally stated:

Example 0.1 It is known by the Grunwald–Wang theorem that there exist no C_8-extensions of $\mathbb{Q}$ in which 2 is totally inert.

Thus, if C_8 carries the trivial Galois action and $\mathcal{L}$ is defined by

$$L_p = \begin{cases} \{\text{totally inert coclasses}\} & p = 2 \\ H^1(\mathbb{Q}_p, C_8) & \text{else,} \end{cases}$$

Then $H^1_{\mathcal{L}}(\mathbb{Q}, C_8) = \emptyset$.

Wood deals with a similar issue in [10], which corresponds to the case that T carries the trivial Galois action and the invariant "inv" is the conductor (or some other fair invariant). Wood proves that, as long as at least one extension with the prescribed local conditions exists, then the asymptotic growth rate is as expected.

We make similar definitions to Wood to distinguish these cases:

Definition 0.2 Let $\mathcal{L} = (L_p)$ be a family of subsets $L_p \subseteq H^1(K_p, T)$ and S a finite set of places such that for all $p \notin S$, $H^1_{ur}(K_p, T) \subseteq L_p$.

We say $\phi = (\phi_p) \in \prod_{p \in S} L_p$ is **viable** if there exists some $f \in H^1_{\mathcal{L}}(K, T)$ for which $f|_{G_{K_p}} = \phi_p$ for each $p \in S$. Otherwise we say ϕ is **inviable**.

Likewise, we say $\mathcal{L}$ is **viable** if $\prod_{p \in S} L_p$ contains at least one viable coclass; otherwise we say $\mathcal{L}$ is **inviable**.

From this definition, it is clear that $\mathcal{L}$ is viable if and only if $H^1_{\mathcal{L}}(K, T) \neq \emptyset$. Example 0.1 is then an example of an inviable family of local conditions.

When L_p is the whole $H^1(K_p, T)$ at almost all places, the search for a global coclass f satisfying the finitely many given local conditions is called *weak approximation* or the *Grunwald problem* in the literature [3–6,9]. In particular, it is known that there is a finite set S_0 of places, depending only on T, such that if local conditions are imposed only at a finite set of places disjoint from S_0, the Grunwald problem is solvable.

Just as in [10], when $\mathcal{L}$ is viable we get the full asymptotic growth rate. Below is the corrected version of our main theorem:

Theorem 0.3 (Corrected version of [2, Theorem 1.1]) *Let T be a finite Galois K-module, and $\mathcal{L} = (L_p)_p$ and* inv *a family of local conditions and admissible ordering respectively. Suppose*

(a) $\mathcal{L}$ and inv *are Frobenian.*
(b) For all but finitely many places p, L_p is a union of cosets of $H^1_{ur}(K_p, T)$ containing the identity coset.
(c) $\mathcal{L}$ is viable.

Then

$$|H^1_{\mathcal{L}}(K, T; X)| \sim c_{\text{inv}}(K, \mathcal{L}) X^{1/a_{\text{inv}}(\mathcal{L})} (\log X)^{b_{\text{inv}}(\mathcal{L})-1}$$

for explicit positive integers $a_{\text{inv}}(\mathcal{L})$ and $b_{\text{inv}}(\mathcal{L})$, and an explicit positive real number $c_{\text{inv}}(K, \mathcal{L})$.

The original [2, Theorem 1.1] will remain true as long as $\mathcal{L}$ contains at least one *viable* coclass (i.e., $\mathcal{L}$ gives a nonempty Selmer set). The viable coclasses form a subgroup of $H^1(\mathbb{A}_K, T)$, explaining why this issue does not occur in the original version of the asymptotic Wiles theorem in [1].

To summarize, the corrected theorem decomposes the counting result into two cases:

1. $\mathcal{L}$ is viable, that is $H^1_{\mathcal{L}}(K,T) \neq \emptyset$. In this case, the original asymptotic

$$|H^1_{\mathcal{L}}(K,T;X)| \sim c_{\text{inv}}(K,\mathcal{L})X^{1/a_{\text{inv}}(\mathcal{L})}(\log X)^{b_{\text{inv}}(\mathcal{L})-1}$$

still holds, with the a- and b-invariants unchanged from the original statement.

2. $\mathcal{L}$ is inviable, that is $H^1_{\mathcal{L}}(K,T) = \emptyset$. In this case, one trivially has

$$|H^1_{\mathcal{L}}(K,T;X)| = 0.$$

We divide this corrigendum into three sections. In the first section, we discuss viability and explain how we will test for it in the proof of Theorem 0.3. In the case considered in [10], only certain local behaviors above 2 or ∞ could result in inviable families of local conditions. However, in our general setting, inviability is a (slightly) more common phenomenon. We give an introductory discussion of this phenomenon, and provide an example demonstrating how inviability can occur.

We prove Theorem 0.3 in the second section and simultaneously prove an equivalent description of viability. We will refer to the original proof for many of the unchanged details, but we take care to highlight and expand on the error in the original proof. We originally made the following implicit assumption:

If $g \in H^1(K,T^*)$ *is nonzero, then* $g(\text{Fr}_p) \neq 0$ *for infinitely many places* p.

This, however, is not true. Counterexamples to this statement are the source of inviability.

In the third and final section, we justify that the proof of [2, Corollary 1.2] still works when viability is taken into consideration.

1 Viability

As stated in the introduction, viability is equivalent to $H^1_{\mathcal{L}}(K,T) \neq \emptyset$. With essentially no work, we can prove that a wide class of families are viable:

Lemma 1.1 *If* $\mathcal{L} = (L_p)$ *is a family of local conditions such that* $0 \in L_p$ *for every place* p, *then* $\mathcal{L}$ *is viable.*

The proof is immediate, as these conditions imply $0 \in H^1_{\mathcal{L}}(K,T)$.

In general, determining which conditions are viable is tantamount to studying the image of the map

$$H^1(K,T) \to \prod_{p\in S}{}' H^1(K_p,T).$$

By Poitou-Tate duality, this is closely related to the kernel of the map

$$H^1(K,T^*) \to \prod_{p\notin S}{}' H^1(K_p,T^*). \tag{1}$$

These maps are studied extensively, in various cases, in Neukirch–Schmidt–Wingberg [7]. In the course of proving Theorem 0.3, we will also prove the following classification of viability:

Theorem 1.2 *Let* $\mathcal{L} = (L_p)$ *be a Frobenian family of local conditions such that for all but finitely many places,* $H^1_{ur}(K_p,T) \subseteq L_p$ *and* L_p *is a union of cosets of* $H^1_{ur}(K,T)$.

Define the ***viability testing subgroup***

$$V_{\mathcal{L}} = \{f \in H^1(K, T^*) : f_p \in L_p^{\perp} \text{ for all but finitely many } p\}.$$

Its orthogonal complement $V_{\mathcal{L}}^{\perp} \subseteq H^1(\mathbb{A}_K, T)$ *has the property that the system* $\mathcal{L} = (L_p)$ *of local conditions is viable if and only if*

$$V_{\mathcal{L}}^{\perp} \cap \prod_p{}' L_p \neq \emptyset,$$

where the product is the restricted product with respect to $H^1_{ur}(K_p, T)$*.*

The viability testing subgroup can alternatively be written as a union of dual Selmer sets

$$V_{\mathcal{L}} = \bigcup_S H^1_{S\mathcal{L}^{\perp}}(K, T^*),$$

where the union is over all finite sets of places S, and $S\mathcal{L}^{\perp}$ is the family of local conditions

$$\begin{cases} H^1(K_p, T^*) & p \in S \\ L_p^{\perp} & p \notin S. \end{cases}$$

If L_p is larger, then we generally expect $V_{\mathcal{L}}$ to be smaller. The proof of Theorem 1.2 separates naturally into two cases depending on the value of $a_{\mathrm{inv}}(\mathcal{L})$:

1. Suppose $a_{\mathrm{inv}}(\mathcal{L}) = \infty$, which by [1, Definition 4.6] is equivalent to $L_p = H^1_{ur}(K, T)$ for all but finitely many places p. This corresponds to the case that $H^1_{\mathcal{L}}(K, T)$ is finite, which follows from the Greenberg-Wiles identity. Then

$$\begin{aligned} V_{\mathcal{L}} &= \{f \in H^1(K, T^*) : f_p \in H^1_{ur}(K_p, T^*) \text{ for all but finitely many } p\} \\ &= H^1(K, T^*). \end{aligned}$$

   Poitou–Tate duality states that $H^1(K, T^*)$ exactly annihilates $i_T(H^1(K, T)) \subset H^1(\mathbb{A}_k, T)$, so it immediately follows that

$$i_T(H^1(K, T)) = H^1(K, T^*)^{\perp} \cap \prod_p{}' L_p.$$

   One side is nonempty if and only if the other is, concluding the proof in this case.
2. Suppose $a_{\mathrm{inv}}(\mathcal{L}) < \infty$. Equivalently, $H^1_{ur}(K, T) \subsetneq L_p$ for a positive proportion of places. This corresponds the the case that $H^1_{\mathcal{L}}(K, T)$ is infinite, and will be proven in the course of proving Theorem 0.3.

We discuss a few features of viability, but we do not give a full treatment of the structure of $V_{\mathcal{L}}$. Such work would certainly be interesting, but would go well beyond the bounds of a corrigendum.

1.1 Places that witness inviability

Viability of local conditions is well studied in the case that T has the trivial action. In fact, a complete classification of viable local restrictions when T has the trivial action is given in [7,10].

One of the more useful features of the trivial action case is that inviability is only witnessed by places above 2 and ∞. This helps make viability statements more digestible, and generally speaking number theorists are happy to accept that 2 and ∞ have unique problems not seen by odd primes. However, it turns out that for general Galois modules odd primes *can* be the source of inviable local conditions. See [3–6,9] for some examples when $L_p = H^1(K_p, T)$ is as large as possible for all but finitely many places.

In the greatest generality, inviability can get particularly bad. There is no guarantee that only finitely many places witness inviability and it is possible for $V_{\mathcal{L}}$ to be an infinite group. For example, in the case that $L_p = H^1_{ur}(K_p, T)$ for all but finitely many p one finds that

$$\begin{aligned} V_{\mathcal{L}} &= \{f \in H^1(K, T^*) : f \text{ unramified at all but finitely many places}\} \\ &= H^1(K, T^*). \end{aligned}$$

When $V_{\mathcal{L}}$ is too large, $V_{\mathcal{L}}^{\perp}$ becomes very small and makes the intersection in Theorem 1.2 more likely to be empty.

While we do not propose to give a full treatment of viability in this corrigendum, we do consider a case of interest where $V_{\mathcal{L}}$ is necessarily small.

Lemma 1.3 *Let T^* be a Galois module over K and $\mathcal{L}$ be a family of local conditions such that L_p generates $H^1(K_p, T)$ for all but finitely many places. Then the viability testing subgroup is given by*

$$V_{\mathcal{L}} = \{f \in H^1(K, T^*) : f_p = 0 \textit{ for all but finitely many } p\}.$$

Moreover, $V_{\mathcal{L}}$ satisfies the following properties:

(a) $V_{\mathcal{L}}$ is a finite group,
(b) $\mathrm{res}^F_K(V_{\mathcal{L}}) = \{0\} \subseteq H^1(F, T^*)$, where F is the field of definition of T^* (that is, the smallest field for which G_F acts trivially on T^*);
(c) For each $f \in V_{\mathcal{L}}$ and $p \notin S_0$, $f_p = 0$, where S_0 is the finite set of places at which T^* is ramified.

Lemma 1.3 generalizes the work in [3] by allowing infinitely many local restrictions. When T has the trivial action, it is known that S_0 can be shrunk further to the set of places above 2 and ∞ only. In fact, a complete classification of viable local restrictions when T has the trivial action is given in [7,10]. Unfortunately, this feature is specific to the trivial action case. The S_0 given in Lemma 1.3(c) cannot be shrunk in general as discussed in [3].

In Example 1.4 we give examples where inviability is witnessed by any odd prime (i.e., cases for which S_0 necessarily contains an odd prime).

Proof of Lemma 1.3 Part (a) follows from part (b). Because $V_{\mathcal{L}} \subseteq \ker(\mathrm{res}^F_K)$, we have embedded $V_{\mathcal{L}}$ as a subgroup of $H^1(\mathrm{Gal}(F/K), T^*)$ by the inflation-restriction sequence. Thus, $V_{\mathcal{L}}$ is finite.

We now prove (b). Let $f \in V_{\mathcal{L}}$. By a convenient viewpoint on Galois cohomology (see [8], Proposition 4.21), we can view f as a homomorphism

$$\sigma_f : G_K \to T^* \rtimes \mathrm{Aut}(T^*)$$

whose projection on the second factor agrees with the G_K-action on T^*. That is, $\sigma_f(g) = (f(g), \phi_{T^*}(g)) \in T^* \rtimes \mathrm{Aut}(T^*)$. Let $E \supset F$ be the fixed field of the kernel of σ_f. Note that E is a Galois extension of K containing F, which is also Galois, and we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Gal}(E/K) & \overset{\sigma_f}{\hookrightarrow} & T^* \rtimes \mathrm{Aut}(T^*) \\ \downarrow & & \downarrow \\ \mathrm{Gal}(F/K) & \overset{\phi_{T^*}}{\hookrightarrow} & \mathrm{Aut}(T^*). \end{array}$$

We claim that $E = F$. If not, let $g \in \mathrm{Gal}(E/F) \subseteq \mathrm{Gal}(E/K)$ be a nonidentity element. By the Chebotarev density theorem, there exist infinitely many p such that $\mathrm{Fr}_p(E/K) = g$. But for such p, $f_p(\mathrm{Fr}_p) = \sigma_f(\mathrm{Fr}_p) = \mathrm{Fr}_p(E/K) \neq 0 \in T^*$. This means f_p is a nonzero cocycle. Additionally, $\mathrm{Fr}_p(E/K) \in \mathrm{Gal}(E/F)$ implies $\mathrm{Fr}_p(F/K) = 1$. Thus, for an infinite set of these places unramified in F, the action of G_{K_p} on T^* is trivial. This implies the set of coboundaries is trivial, so $f_p \neq 0$ as a coclass in $H^1(K_p, T^*)$ for infinitely many places. This contradicts $f \in V_{\mathcal{L}}$, and so proves (b).

Lastly we prove (c). Let p be a place unramified in F/K with $g = \mathrm{Fr}_p(E/K)$. For unramified infinite places, we trivially have $H^1(\mathrm{Gal}(F_\infty/K_\infty), T^*) = 0$, so it suffices to consider finite places. For any other place ℓ with $\mathrm{Fr}_\ell(E/K) = \mathrm{Fr}_p(E/K) = g$, the local action of $T^*$is the same at p and ℓ so that $H^1(K_p, T^*) \cong H^1(K_\ell, T^*)$. Moreover, $\sigma_f(\mathrm{Fr}_p)$ and $\sigma_f(\mathrm{Fr}_\ell)$ are equal up to conjugation. Thus,

$$\begin{aligned} f_p(\mathrm{Fr}_p) &= \sigma_f(\mathrm{Fr}_p)\phi_{T^*}(g)^{-1} \\ &= h\sigma_f(\mathrm{Fr}_\ell)h^{-1}\phi_{T^*}(g)^{-1} \\ &= h\sigma_f(\mathrm{Fr}_\ell)h^{-1}\sigma_f(\mathrm{Fr}_\ell)f_\ell(\mathrm{Fr}_\ell). \end{aligned}$$

This implies f_p and f_ℓ are equivalent (up to coboundaries) under the isomorphism of local cohomology groups. By the Chebotarev density theorem, there are infinitely many choices for ℓ with $\mathrm{Fr}_\ell(E/K) = g$. Thus, if f_p is not a coboundary, the f_ℓ is not a coboundary for infinitely many ℓ and it follows that $f \notin V_{\mathcal{L}}$. Then (c) follows from the contrapositive. □

1.2 Example of inviability at odd places

A complete classification of viability for general T and $\mathcal{L}$ is outside the scope of this paper. Instead, we provide the following example to demonstrate that inviability can be witnessed by any place when we allow nontrivial actions, which generalizes Example 0.1.

Example 1.4 We will construct an inviable local restriction at an odd prime over $\mathbb{Q}$. The precise example we prove is given below, although with a little more work this construction can be expanded to create a number of other examples.

Let us use the following notation. If $a, b \in \mathbb{Q}^\times$, denote by $T(a, b)$ the Galois module whose underlying group is $\mathbb{Z}/8\mathbb{Z}$ and such that $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ permutes the four generators $1, 3, 5, 7 \in \mathbb{Z}/8\mathbb{Z}$ in the same manner as the ordered pairs

$$(\sqrt{a}, \sqrt{b}), \quad (-\sqrt{a}, -\sqrt{b}), \quad (\sqrt{a}, -\sqrt{b}), \quad (-\sqrt{a}, \sqrt{b}).$$

Observe that $T(a, b)$ has field of definition $\mathbb{Q}(\sqrt{a}, \sqrt{b})$. For example, $\mu_8 = T(-1, 2)$, from which we find that the Tate dual of $T(a, b)$ is

$$\mathrm{Hom}(T(a, b), \mu_8) \cong T(-a, 2b).$$

Proposition 1.5 *Let ℓ be an odd prime and consider the Galois module $T = T(-\ell, 2b)$ where b is a quadratic non-residue modulo ℓ. Let S_0 be a finite set of places including 2, ℓ,* *and all places dividing b. Then there exists a $\phi \in \prod_{p \in S_0} H^1(\mathbb{Q}_p, T)$ for which the family of local conditions $\mathcal{L}$ on $H^1(\mathbb{Q}, T)$ given by*

$$L_p = \begin{cases} H^1(K_p, T) & p \notin S_0 \\ \{\phi_p\} & p \in S_0 \end{cases}$$

is inviable.

Proof These examples are constructed by finding an element in $V_{\mathcal{L}}$ which is nontrivial at the place ℓ, which forces $V_{\mathcal{L}}^{\perp} \cap H^1(\mathbb{Q}_\ell, T) \neq H^1(\mathbb{Q}_\ell, T)$. It then suffices to take any $\phi \in H^1(\mathbb{Q}_\ell, T)$ outside of $V_{\mathcal{L}}^{\perp}$.

Take $f \in H^1((\mathbb{Z}/8\mathbb{Z})^\times, \mathbb{Z}/8\mathbb{Z})$ given by the equivalence class of the crossed homomorphism

$$f(n) = \begin{cases} 0 & n = 1, 7 \\ 4 & n = 3, 5. \end{cases}$$

We check this is a crossed homomorphism. Notice that since $\mathrm{im}(f) = 4\mathbb{Z}/8\mathbb{Z}$, we have

$$(n-1)f(m) = 0$$

for all $n, m \in (\mathbb{Z}/8\mathbb{Z})^\times$, which rearranging implies that

$$nf(m) = f(m).$$

Thus it suffices to prove that $f : (\mathbb{Z}/8\mathbb{Z})^\times \to 4\mathbb{Z}/8\mathbb{Z}$ is a homomorphism, which is evident since $\{1, 7\} \subset (\mathbb{Z}/8\mathbb{Z})^\times$ is a subgroup of index 2.

The cocycle f has the following nice properties:

(i) f is not a coboundary. This is because all coboundaries are of the form $n \mapsto na - a = (n-1)a$. However, the equations $(7-1)a = 0$ and $(3-1)a = 4$ are not simultaneously solvable modulo 8.

(ii) The image of f under restriction to any cyclic subgroup *is* a coboundary. This is trivial for $f|_{\langle 1 \rangle} = 0$ and $f|_{\langle 7 \rangle} = 0$. Meanwhile,

$$\begin{aligned} f|_{\langle 3 \rangle}(m) &= \begin{cases} 0 & m = 1 \\ 4 & m = 3 \end{cases} \\ &= 2m - 2 \end{aligned}$$

and

$$\begin{aligned} f|_{\langle 5 \rangle}(m) &= \begin{cases} 0 & m = 1 \\ 4 & m = 5 \end{cases} \\ &= 1m - 1. \end{aligned}$$

We now inflate f along the Galois action on $T^* = T(\ell, b)$ to an element $\inf(f) \in H^1(\mathbb{Q}, T^*)$, which we claim is a nontrivial element of the viability testing subgroup $V_{\mathcal{L}} \leq H^1(\mathbb{Q}, T^*)$. Consider that any place p unramified in the field $F = \mathbb{Q}(\sqrt{\ell}, \sqrt{b})$ of definition of T^*, as well as any infinite place p, has cyclic decomposition group in F. Let $C < (\mathbb{Z}/8\mathbb{Z})^\times$ be the cyclic image of D_p under the isomorphism with $\mathrm{Gal}(F/\mathbb{Q})$. We have

$|C| = 1$ or 2. Highlighting the fact that the action of D_p on T^* factors through the quotient $G_{\mathbb{Q}} \twoheadrightarrow (\mathbb{Z}/8\mathbb{Z})^\times$ as the quotient $D_p \twoheadrightarrow C$, we produce a commutative diagram

$$\begin{array}{ccc} H^1((\mathbb{Z}/8\mathbb{Z})^\times, T^*) & \xrightarrow{\text{inf}} & H^1(\mathbb{Q}, T^*) \\ \downarrow{\scriptstyle \text{res}} & & \downarrow{\scriptstyle \text{res}_{D_p}} \\ H^1(C, T^*) & \xrightarrow{\text{inf}} & H^1(D_p, T^*). \end{array}$$

By a diagram chase, we find that $\text{res}(f) = 0$ implies that $\text{res}_{D_p} \inf(f) = 0$ at all places unramified in $F/\mathbb{Q}$. This proves that $\inf(f) \in V_{\mathcal{L}}$.

Meanwhile, by construction, ℓ is ramified in $F/\mathbb{Q}$ with $D_\ell = \text{Gal}(F/\mathbb{Q}) \cong (\mathbb{Z}/8\mathbb{Z})^\times$, so $\text{res}_{D_\ell} \inf(f) \neq 0$ because f is not a coboundary. Thus $\inf(f)$ is nonzero at at least one place, namely ℓ; in particular, $f \in V_{\mathcal{L}} \setminus \{0\}$. Let S_0 be the finite set of places at which $\inf(f) \neq 0$ (taking 2, ℓ, and all primes dividing b is sufficient). Take

$$\phi \in \prod_{p \in S_0} H^1(\mathbb{Q}_p, T)$$

that is not orthogonal to $\text{res}_{S_0}(\inf f)$. Then no global coclass $g \in H^1(\mathbb{Q}, T)$ can reduce to ϕ at every place in S_0, or it would have a globally nontrivial pairing with $\inf f$. □

2 The corrected proof

The framework of the proof of Theorem 0.3 is by and large the same. We still utilize the periodic function w as in [2, Proposition 4.1]. The decomposition of the Dirichlet series

$$\sum_{f \in H^1(K,T)} w(f) = \frac{|H^0(K,T)|}{|H^0(K,T^*)|} \sum_{g \in H^1(K,T^*)} \hat{w}(g)$$

proceeds without change. Each of the following facts is unchanged from the original proof:

- The Fourier transform $\hat{w}$ has finite support contained in the compact $Y_S^\perp$, because w is periodic. Recall that

  $$Y_S = \prod_{p \notin S} H^1_{ur}(K_p, T)$$

  so that

  $$Y_S^\perp = \prod_{p \in S} H^1(K_p, T^*) \times \prod_{p \notin S} H^1_{ur}(K_p, T^*).$$

- The summand $\hat{w}(0)$ is given by a Frobenian Euler product

  $$\hat{w}(0) = \prod_p \left(\frac{1}{|H^0(K_p, T)|} \sum_{f \in L_p} \mathcal{N}_{K/\mathbb{Q}}(\text{inv}(f))^{-s} \right)$$

  with meromorphic continuation and an explicit pole at $1/a_{\text{inv}}(\mathcal{L})$ following the work in [1].
- Each $\hat{w}(g)$ is also a Frobenian Euler product bounded above by $\hat{w}(0)$ in absolute value. Thus, the work in [1] gives a meromorphic continuation to the left of the line $\text{Re}(s) = 1/a_{\text{inv}}(\mathcal{L})$ and for which the order of the pole at $1/a_{\text{inv}}(\mathcal{L})$ is less than or equal to the order of the pole of $\hat{w}(0)$.

- Applying a Tauberian theorem to each summand $\hat{w}(g)$ individually produces an asymptotic main term for each summand. The asymptotic corresponding to $\hat{w}(0)$ is given by

  $$c(0)X^{1/a_{\mathrm{inv}}(\mathcal{L})}(\log X)^{b_{\mathrm{inv}}(K,\mathcal{L})-1},$$

  while the asymptotic for the finitely many other $\hat{w}(g)$ is of the same or smaller order of magnitude.

It suffices to prove that the asymptotic main terms do not cancel, or equivalently that the poles of $\hat{w}(g)$ at $s = 1/a_{\mathrm{inv}}(\mathcal{L})$ do not cancel. This is the place where we made an error in the original proof. We do this in three steps:

1. Partitioning the sum of $\hat{w}(g)$ by cosets of the viability testing subgroup. This step is new, and will prove the "only if" direction of Theorem 1.2.
2. Proving Theorem 0.3 in the case that inv is given by the product of ramified primes outside S and the minimal index classes generate T. This section applies our original argument for bounding the order of the pole of $\hat{w}(g)$ in the case that $g \notin V_{\mathcal{L}}$ (i.e., where the argument actually works). The "if" direction of Theorem 1.2 follows from these cases.
3. Proving the remaining cases of Theorem 0.3. The primary ideas are the same as in our original proof; however, we take greater care to ensure that the induced family of local conditions on T' can also be chosen to be viable.

Remark It is likely that step 2 can be done for any "fair" counting function, with an appropriate analog of the notion of a fair counting function defined in [10]. This step produces a leading term given by a single Euler product, and some notion of fairness will likely preserve this property. Meanwhile, as demonstrated for the discriminant ordering in [10], some orderings do not produce a single Euler product for the leading constant and necessarily require the work in step 3.

2.1 Partitioning by cosets of the viability testing subgroup

We know that w is periodic with respect to Y_S in the setting of Theorem 0.3, which means $\hat{w}$ is supported on the compact subgroup $Y_S^{\perp}$. We concluded the sum of $\hat{w}(g)$ is finite because $Y_S^{\perp} \cap H^1(K, T^*)$ is finite. We partition the sum

$$\sum_{g\in H^1(K,T^*)} \hat{w}(g) = \sum_{g\in R}\left(\sum_{v\in Y_S^{\perp}\cap V_{\mathcal{L}}} \hat{w}(g+v)\right),$$

where R is a set of representatives for the cosets $(Y_S^{\perp} \cap H^1(K, T^*))/(Y_S^{\perp} \cap V_{\mathcal{L}})$.

For each $g \in R$, we evaluate the sum of Fourier transforms

$$\sum_{v\in Y_S^{\perp}\cap V_{\mathcal{L}}} \hat{w}(g+v)$$
$$= \sum_{v\in Y_S^{\perp}\cap V_{\mathcal{L}}} \prod_p \left(\frac{1}{|H^0(K_p, T)|}\sum_{f\in L_p} \langle f, (g+v)|_{G_{K_p}}\rangle \mathcal{N}_{K/\mathbb{Q}}(\mathrm{inv}(f))^{-s}\right).$$

Each element $v \in V_{\mathcal{L}}$ has $v|_{G_{K_p}} \in L_p^{\perp}$ for all but finitely many places, so for each of those places $\langle f, (g+v)|_{G_{K_p}}\rangle = \langle f, g|_{G_{K_p}}\rangle$ is independent of v. Take $\widetilde{S}$ to be the set of places defined by

$$\widetilde{S} = S \cup \bigcup_{v \in Y_S^{\perp} \cap V_{\mathcal{L}}} \{p : v|_{G_{K_p}} \notin L_p^{\perp}\},$$

so that $\widetilde{S}$ is precisely the set of places for which the Euler factor depends on v. We factor these places out so that

$$\sum_{v \in Y_S^{\perp} \cap V_{\mathcal{L}}} \hat{w}(g+v) = \sum_{v \in Y_S^{\perp} \cap V_{\mathcal{L}}} \prod_{p \in \widetilde{S}} \left(\frac{1}{|H^0(K_p, T)|} \sum_{f \in L_p} \langle f, (g+v)|_{G_{K_p}} \rangle \mathcal{N}_{K/\mathbb{Q}}(\mathrm{inv}(f))^{-s} \right) \times \prod_{p \notin \widetilde{S}} \left(\frac{1}{|H^0(K_p, T)|} \sum_{f \in L_p} \langle f, g|_{G_{K_p}} \rangle \mathcal{N}_{K/\mathbb{Q}}(\mathrm{inv}(f))^{-s} \right).$$

For simplicity, we write the second Euler product as

$$\frac{1}{\prod_{p \in \widetilde{S}} \hat{w}_p(g)} \hat{w}(g).$$

Next, we simplify the first Euler product. The number of places contributed to $\widetilde{S}$ by each individual v is finite since $v \in V_{\mathcal{L}}$. Meanwhile, the set of exceptional places S is finite by definition, and

$$Y_S^{\perp} \cap V_{\mathcal{L}} \subset Y_S^{\perp} \cap H^1(K, T^*)$$

is finite by $Y_S^{\perp}$ compact and $H^1(K, T^*)$ discrete. Thus $\widetilde{S}$ is a finite set.

Let $\mathrm{res}_{\widetilde{S}} : H^1(K, T^*) \to \prod_{p \in \widetilde{S}} H^1(K_p, T^*)$ be the product of restriction maps to places $p \in \widetilde{S}$. We multiply the Euler factors over $\widetilde{S}$ together to get

$$\left(\prod_{p \in \widetilde{S}} \frac{1}{|H^0(K_p, T)|} \right) \sum_{f \in \prod_{p \in \widetilde{S}} L_p} \langle f, \mathrm{res}_{\widetilde{S}}(g+v) \rangle \mathcal{N}_{K/\mathbb{Q}}(\mathrm{inv}(f))^{-s},$$

Moving the summation over v all the way to the inside gives an inner sum

$$\sum_{v \in Y_S^{\perp} \cap V_{\mathcal{L}}} \langle f, \mathrm{res}_{\widetilde{S}}(g) + \mathrm{res}_{\widetilde{S}}(v) \rangle = \langle f, \mathrm{res}_{\widetilde{S}}(g) \rangle \sum_{v \in Y_S^{\perp} \cap V_{\mathcal{L}}} \langle f, \mathrm{res}_{\widetilde{S}}(v) \rangle = \begin{cases} \langle f, \mathrm{res}_{\widetilde{S}}(g) \rangle |Y_S^{\perp} \cap V_{\mathcal{L}}| & f \in \mathrm{res}_{\widetilde{S}}(Y_S + V_{\mathcal{L}}^{\perp}) \\ 0 & \text{else.} \end{cases}$$

Remark Here we see that the sum over v of the Euler factors at $\widetilde{S}$ plays the same role as the sum over $\mathcal{E}(C)$ in [10]. Whether this sum is zero or not detects inviability.

Putting it all together, we conclude that

$$\sum_{v \in V_{\mathcal{L}}} \hat{w}(g+v) = |Y_S^{\perp} \cap V_{\mathcal{L}}| \left(\sum_{f \in \mathrm{res}_{\widetilde{S}}(Y_S + V_{\mathcal{L}}^{\perp}) \cap \prod_{p \in \widetilde{S}} L_p} \langle f, g \rangle \mathcal{N}_{K/\mathbb{Q}}(\mathrm{inv}(f))^{-s} \right) \times \left(\prod_{p \in \widetilde{S}} \frac{1}{\hat{w}_p(g) |H^0(K_p, T)|} \right) \hat{w}(g).$$

Consider that

$$\mathrm{res}_{\widetilde{S}}(Y_S + V_{\mathcal{L}}^{\perp}) \cap \prod_{p \in \widetilde{S}} L_p = \emptyset$$

if and only if

$$(Y_S + V_{\mathcal{L}}^{\perp}) \cap \prod_{p} L_p = \emptyset$$

by pulling back along $\mathrm{res}_{\widetilde{S}}$. By assumption, $\prod_p L_p$ is closed under translation by Y_S, so this set is nonempty if and only if

$$V_{\mathcal{L}}^{\perp} \cap \prod_{p} L_p = \emptyset.$$

As this fact is independent of the representative g, emptiness of this set implies $\mathcal{L}$ is inviable and proves the "only if" direction of Theorem 1.2.

2.2 Ordering by the product of the ramified primes

We consider the specific ordering

$$\mathrm{ram}_S(f) = \prod_{\substack{p \notin S \\ f|_{I_p} \neq 1}} \mathcal{N}_{K/\mathbb{Q}}(p).$$

This is an example of a "fair" counting function given in [10], and we can prove Theorem 0.3 for this ordering with similar strength to Wood's results. If $a_{\mathrm{ram}_S}(\mathcal{L}) = \infty$, Theorem 0.3 follows from bounding the counting function by a finite Selmer group in the Greenberg-Wiles identity. Otherwise, by definition $a_{\mathrm{ram}_S}(\mathcal{L}) = 1$ as primes can only occur in ram_S with exponent 1.

Suppose $g \in Y_S^{\perp} \setminus V_{\mathcal{L}}$, so that $g|_{G_{K_p}} \notin L_p^{\perp}$ for infinitely many places and g is unramified at all places $p \notin S$. The Chebotarev density theorem and $\mathcal{L}$ being Frobenian implies that $g|_{G_{K_p}} \notin L_p^{\perp}$ for a positive proportion of places. This implies that, for a positive proportion of places p, there exists an $f \in L_p$ such that $\langle f, \mathrm{res}_p(g)\rangle \neq 1$. However, the fact that $g \in H^1_{ur}(K_p, T^*)$ for these places implies that $f \notin H^1_{ur}(K_p, T)$ as these subgroups annihilate each other. Because all ramified coclasses have weight $1 = a_{\mathrm{ram}_S}(\mathcal{L})$ under the ordering ram_S, we have proven the following:

For a positive proportion of places, there exists an $f \in L_p^{[1]}$ such that $\langle f, \mathrm{res}_p(g)\rangle \neq 1$ (where, as in [1], we set $L_p^{[m]} = \{f_p \in L_p : \nu_p(\mathrm{inv}(f_p)) = m\}$). Using this fact, we will prove that the order of the singularity of $\hat{w}(g)$ is strictly less than for $\hat{w}(0)$.

The order of the singularity is given by the b-invariant, which is the average value of coefficients of $|p|^{-a_{\mathrm{ram}_S}(\mathcal{L})s} = |p|^{-s}$ for $|p| \leq x$ as $x \to \infty$ (see [1, Definition 4.6]). This coefficient for $\hat{w}(g)$ is given by

$$\sum_{f \in L_p^{[1]}} \langle f, \mathrm{res}_p(g)\rangle |p|^{-s} \leq \sum_{f \in L_p^{[1]}} |p|^{-s}.$$

By our choice of g, this inequality is necessarily strict for a positive proportion of places. Taking the average of both sides over $|p| \leq x$ as $x \to \infty$, this implies

$$\begin{matrix}\text{order of singularity of} \\ \hat{w}(g) \text{ at } s = 1/a_{\mathrm{ram}_S}(\mathcal{L})\end{matrix} < b_{\mathrm{ram}_S}(K, T) = \begin{matrix}\text{order of singularity of} \\ \hat{w}(0) \text{ at } s = 1/a_{\mathrm{ram}_S}(\mathcal{L}).\end{matrix}$$

In particular, this implies that for any $g \in Y_S^{\perp} \setminus V_{\mathcal{L}}$, the singularity of $\hat{w}(g)$ does not cancel with the singularity of $\hat{w}(0)$. Since $\hat{w}$ has finite support in $Y_S^{\perp}$, the sum over all $g \notin V_{\mathcal{L}}$

cannot cancel with the singularity of $\hat{w}(0)$ either. Thus, the order of the singularity at $s = 1/a_{\mathrm{ram}_S}(\mathcal{L})$ is determined by those g in the identity coset $V_{\mathcal{L}}$, which is given by

$$\sum_{v \in Y_S^\perp \cap V_{\mathcal{L}}} \hat{w}(v) = |Y_S^\perp \cap V_{\mathcal{L}}| \left(\sum_{f \in \mathrm{res}_{\widetilde{S}}(Y_S + V_{\mathcal{L}}^\perp) \cap \prod_{p \in \widetilde{S}} L_p} \mathcal{N}_{K/\mathbb{Q}}(\mathrm{inv}(f))^{-s} \right) \times \left(\prod_{p \in \widetilde{S}} \frac{1}{\hat{w}_p(0)|H^0(K_p, T)|} \right) \hat{w}(0).$$

All factors are positive at $s = 1/a_{\mathrm{ram}_S}(\mathcal{L}) = 1$, so the order of this singularity agrees with that of $\hat{w}(0)$.

Applying a Tauberian theorem, this proves the asymptotic in Theorem 0.3 as long as

(a) $\mathrm{inv} = \mathrm{ram}_S$, and
(b) $V_{\mathcal{L}}^\perp \cap \prod L_p \neq \emptyset$

The nonzero asymptotic in particular shows that (b) implies viability, proving the remaining direction of Theorem 1.2.

2.3 Extending to the general case

The final step for proving Theorem 0.3 is the case that

- $\mathrm{inv} \neq \mathrm{ram}_S$, and
- $\mathcal{L}$ is viable (or equivalently $V_{\mathcal{L}}^\perp \cap \prod_{p \in S} L_p \neq \emptyset$, as we have now proven Theorem 1.2).

This step proceeds similarly to the original proof, where we produce a lower bound with the desired order of magnitude in order to prove that the main term cannot be canceled out.

We are assuming that $\mathcal{L}$ is viable, so we choose some $f_0 \in H^1_{\mathcal{L}}(K, T)$. Define $\mathcal{L}(f_0) = (L(f_0)_p)$ to be the family of local conditions given by

$$L(f_0)_p = \begin{cases} \{0\} & p \in S \\ L_p - f_0|_{G_{K_p}} & p \notin S. \end{cases}$$

For $p \notin S$, L_p is a union of cosets of $H^1_{ur}(K_p, T)$. This property is preserved in the construction of $L(f_0)_p$, and $f_0 \in H^1_{\mathcal{L}}(K_p, T)$ implies $f_0|_{G_{K_p}} \in L_p$ so that $0 \in L(f_0)_p$. Thus the corresponding w function is still Y_S-periodic. $\mathcal{L}(f_0)$ is certainly viable, as $0 \in L(f_0)_p$ for all places p.

The map $x \mapsto x + f_0$ gives an injection

$$H^1_{\mathcal{L}(f_0)}(K, T) \hookrightarrow H^1_{\mathcal{L}}(K, T).$$

This map preserves the p-part of the invariant, inv, at every place unramified in f_0. As a result, the invariant can change by at most a bounded amount. This implies that there exists some constant $C_1 > 0$ depending only on f_0 such that

$$H^1_{\mathcal{L}}(K, T; X) \geq H^1_{\mathcal{L}(f_0)}(K, T; C_1 X).$$

Thus it suffices to prove $H^1_{\mathcal{L}(f_0)}(K, T; X)$ has the expected order of magnitude.

Define the subfamily $\mathcal{L}(f_0)^{\min} = (L(f_0)_p^{\min})$ of $\mathcal{L}(f_0)$ to be the family of local conditions

$$L(f_0)_p^{\min} = \begin{cases} L(f_0)_p & p \in S \\ L(f_0)_p^{[a_{\mathrm{inv}}(\mathcal{L})]} \cup H^1_{ur}(K,T) & p \notin S \end{cases}$$

Once again, for all but finitely many places $L(f_0)_p^{[a_{\mathrm{inv}}(\mathcal{L})]}$ is a union of cosets of $H^1_{ur}(K_p,T)$. One still has $0 \in H^1_{\mathcal{L}(f_0)^{[a_{\mathrm{inv}}(\mathcal{L})]}}(K,T)$, so this family is viable as well. Thus, we have a containment of nonempty Selmer sets

$$H^1_{\mathcal{L}(f_0)^{[a_{\mathrm{inv}}(\mathcal{L})]}}(K,T) \subseteq H^1_{\mathcal{L}(f_0)}(K,T).$$

By construction, it follows that for any $f \in H^1_{\mathcal{L}(f_0)^{[a_{\mathrm{inv}}(\mathcal{L})]}}(K,T)$ the invariant is given by

$$\begin{aligned} \mathrm{inv}(f) &= \prod_{p\in S} p^{\nu_p(\mathrm{inv}(f))} \cdot \prod_{\substack{p\notin S \\ f|_{I_p}\neq 1}} p^{a_{\mathrm{inv}}(\mathcal{L})} \\ &\geq C_2 \mathrm{ram}_S(f)^{a_{\mathrm{inv}}(\mathcal{L})}. \end{aligned}$$

for some positive constant C_2 determined by the minimal values of $\nu_p\mathrm{inv}$ on the finite sets $L(f_0)_p^{[a_{\mathrm{inv}}(\mathcal{L})]}$ for the finitely many $p \in S$.

By combining the two embeddings, we produce a lower bound

$$|H^1_{\mathrm{inv},\mathcal{L}}(K,T;X)| \geq \left| H^1_{\mathrm{ram}_S,\mathcal{L}(f_0)^{[a_{\mathrm{inv}}(\mathcal{L})]}}\left(K,T;\left(C_1C_2^{-1}X\right)^{1/a_{\mathrm{inv}}(\mathcal{L})}\right)\right|.$$

Step 2 applies to the lower bound, so we can produce the order of magnitude. One sees directly from the definitions that

$$a_{\mathrm{ram}_S}(\mathcal{L}(f_0)^{[a_{\mathrm{inv}}(\mathcal{L})]}) = 1 \quad \text{and} \quad b_{\mathrm{ram}_S}(K,\mathcal{L}(f_0)^{[a_{\mathrm{inv}}(\mathcal{L})]}) = b_{\mathrm{inv}}(K,\mathcal{L}),$$

as "inv" at all but finitely many places is preserved under both embeddings of Selmer sets and $L(f_0)_p^{[a_{\mathrm{inv}}(\mathcal{L})]}$ only contains elements of index $a_{\mathrm{inv}}(\mathcal{L})$ or unramified elements (i.e., index ∞). Much of this reasoning proceeds without change[1] from the original proof, just without restricting to a submodule T'. Thus, we produce a lower bound of the desired magnitude proving that the main terms do not cancel.

3 Notes on [2, Corollary 1.2]

The authors had the incidental foresight to state [2, Corollary 1.2](i) as applying to the cases where "$\mathcal{L}$ and inv satisfy the hypotheses of [2, Theorem 1.1]". As long as we amend that statement to be "…satisfy the hypotheses of Theorem 0.3", the Corollary is still correct. The new statement, for reference, would read

Corollary 3.1 (Corollary 1.2 of [2]) *Let T be an abelian normal subgroup of a finite group G, and $\pi : \mathrm{Gal}(\overline{K}/K) \to G$ a homomorphism with $T\pi(\mathrm{Gal}(\overline{K}/K)) = G$ (or equivalently π surjects onto G/T), defining a Galois action on T by $x.t = \pi(x)t\pi(x)^{-1}$.*

(i) If $\mathcal{L}$ and inv *satisfy the hypotheses of Theorem 0.3 and S is the set of irregular places, then the limit*

$$\lim_{X\to\infty} \frac{\left|\{f \in H^1_{\mathcal{L}}(K,T;X) : f * \pi \text{ surjective}\}\right|}{|H^1_{\mathcal{L}}(K,T;X)|}$$

[1] There is also a typo in the inflation-restriction sequence: the $H^1(I_p,T)$ should be $H^1(I_p,T)^{G_{K_p}}$. This does not affect the actual proof.

converges, where $(f * \pi)(x) = f(x)\pi(x)$ *is understood to apply to a representative of* f *in* $Z^1(K, T)$*. Moreover, the limit is*

- *(a) positive if* π *is surjective,*
- *(b) positive if* $T = \langle f_p(I_p) : f_p \in L_p, p \notin S\rangle$*, and*
- *(c) equal to* 1 *if* $T = \langle f_p(I_p) : f_p \in L_p^{[a_{\mathrm{inv}}(\mathcal{L})]}, p \notin S\rangle$ *where* $L_p^{[m]} = \{f \in L_p : v_p(\mathrm{inv}(f)) = m\}$*.*

(ii) If $G \subset S_n$ *is a transitive representation of* G*, then the invariant*

$$\mathrm{disc}_\pi(f) = \mathrm{disc}(f * \pi)$$

satisfies the hypotheses of Theorem 0.3, where $\mathrm{disc} : \mathrm{Hom}(\mathrm{Gal}(\overline{K}/K), S_n) \to I_K$ *is the discriminant on the associated étale algebra of degree n.*

Indeed, it is reasonable to only expect [2, Corollary 1.2](i) to be true when $\mathcal{L}$ is viable. If $\mathcal{L}$ were inviable, the denominator in the limit would be identically zero and the entire statement would be ill-founded. We address each part of the proof below:

(i) The proof of the existence of the limit remains unchanged. Indeed, the inclusion–exclusion argument is not affected by viability. At worst, some more terms than expected are zero.
The proofs of (a), (b), and (c) are referred to [1]. The extra reasoning we need to provide here is that viability implies the existence of a *surjective* coclass, not just any coclass.

- (a) In this case, [1] bounds the numerator below by

$$|H^1_{\mathcal{L}_\pi}(K, T; X)|,$$

where the family $\mathcal{L}_\pi$ is given by

$$(L_\pi)_p = \begin{cases} \{0\} & p \in U \\ L_p & \text{else} \end{cases}$$

and U is chosen to be a set of places unramified in π such that $\{\pi(\mathrm{Fr}_p) : p \in U\} = G$. Without loss of generality, U can be chosen disjoint from $\widetilde{S}$, so that Theorem 1.2 implies the viability of $\mathcal{L}_\pi$. Theorem 0.3 then gives an asymptotic lower bound.

- (b) Part (c) with the product of ramified primes invariant implies that there exists some f for which $f * \pi$ is surjective. Replacing π with $f * \pi$ to define the action, we then apply part (a). Any reference to viability need only be made in parts (a) and (c).
- (c) In this case, it is proven that all but the leading term of the inclusion–exclusion tend to zero in the limit. Thus, the numerator and denominator are necessarily asymptotic to each other. As it was not required to reference viability, this part implies that viability implies the existence of a surjective coclass.

(ii) This part does not involve checking asymptotics, so the original proof stands without error.

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