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Tight Hilbert polynomial and F-rational local rings

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Dedicated to the memory of Professor Shiro Goto

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Abstract

Let $(R, \mathfrak{m})$ be a Noetherian local ring of prime characteristic p and Q be an $\mathfrak{m}$-primary parameter ideal. We give criteria for F-rationality of R using the tight Hilbert function $H^*_Q(n) = \ell(R/(Q^n)^*)$ and the coefficient $e^*_1(Q)$ of the tight Hilbert polynomial $P^*_Q(n) = \sum_{i=0}^{d}(-1)^i e^*_i(Q)\binom{n+d-1-i}{d-i}$. We obtain a lower bound for the tight Hilbert function of Q for equidimensional excellent local rings that generalizes a result of Goto and Nakamura. We show that if $\dim R = 2$, the Hochster–Huneke graph of R is connected and this lower bound is achieved, then R is F-rational. Craig Huneke asked if the F-rationality of unmixed local rings may be characterized by the vanishing of $e^*_1(Q)$. We construct examples to show that without additional conditions, this is not possible. Let R be an excellent, reduced, equidimensional Noetherian local ring and Q be generated by parameter test elements. We find formulas for $e^*_1(Q), e^*_2(Q), \ldots, e^*_d(Q)$ in terms of Hilbert coefficients of Q, lengths of local cohomology modules of R, and the length of the tight closure of the zero submodule of $H^d_{\mathfrak{m}}(R)$. Using these, we prove: R is F-rational $\iff$ $e^*_1(Q) = e_1(Q)$ $\iff$ depth $R \geq 2$ and $e^*_1(Q) = 0$.

Keywords: Tight Hilbert polynomial, F-rational rings, Parameter test elements, d-Sequences, Local cohomology

1 Introduction

The theory of tight closure created by Hochster and Huneke in the 1980's introduced several types of local rings such as F-regular, weakly F-regular, F-rational and F-injective local rings, see for example [7,8,24]. It is well known that the Hilbert coefficients can be used to characterize regular, Cohen–Macaulay and Buchsbaum local rings. It is natural to expect that F-singularities could be characterized using a certain kind of Hilbert polynomial that involves the tight closure of ideals. The first step in this direction was taken by Shiro Goto and Y. Nakamura. In response to a conjecture of Watanabe and Yoshida [26], Goto and Nakamura [6] proved the following interesting characterization of F-rational local rings. The length of an R-module M is denoted by $\ell_R(M)$. The tight closure of an ideal I is denoted by I^*, see Sect. 2 for definitions.

Theorem 1.1 *Goto–Nakamura, 2001 Suppose R has prime characteristic and it is an equidimensional local ring of dimension d. Suppose that R is a homomorphic image of a Cohen–Macaulay local ring. Then,*

1. $e_0(Q) \geq \ell_R(R/Q^*)$ *for every* $\mathfrak{m}$*-primary parameter ideal Q in R.*
2. *If* $\dim R/\mathfrak{p} = d$ *for all* $\mathfrak{p} \in \operatorname{Ass}(R)$*, and* $e_0(Q) = \ell_R(R/Q^*)$ *for some parameter ideal Q in R, then R is a Cohen–Macaulay F-rational local ring.*

For a recent treatment of Goto–Nakamura theorem, see [14]. Since Q^* is contained in the integral closure $\overline{Q}$ of Q, $e_0(Q) = e_0^*(Q)$. Therefore, the F-rationality of R is a consequence of the equality $e_0^*(Q) = \ell(R/Q^*)$ for rings mentioned in (2) above. This was an indication that F-singularities could be characterized in terms of the *tight Hilbert function* $H_Q^*(n) = \ell(R/(Q^n)^*)$. Let I be an $\mathfrak{m}$-primary ideal of R and R be analytically unramified, i.e., the $\mathfrak{m}$-adic completion $\hat{R}$ is reduced. By a theorem of Rees [19], $H_I^*(n)$ is given by a polynomial $P_I^*(n)$ for large n. We call it the *tight Hilbert polynomial of* I and write it as

$$P_I^*(n) = \sum_{i=0}^{d} (-1)^i e_i^*(I) \binom{n+d-1-i}{d-i}.$$

The coefficient $e_0^*(I)$ is the multiplicity $e_0(I)$ of I. The other coefficients $e_i^*(I) \in \mathbb{Z}$ are called the *tight Hilbert coefficients* of I. The tight Hilbert polynomial was introduced in [4] where it was proved that an analytically unramified Cohen–Macaulay local ring R having prime characteristic is F-rational if and only if $e_1^*(Q) = 0$ for some ideal Q generated by a system of parameters of R. This paper is motivated by the following question of Craig Huneke

Question 1.2 *Is it true that an unmixed Noetherian local ring R is F-rational if and only if for some ideal Q of R generated by a system of parameters,* $e_1^*(Q) = 0$*?*

We provide a negative answer to Question 1.2, see Proposition 5.3. We show that F-rationality can be characterized by the vanishing of $e_1^*(Q)$ where Q is an ideal generated by parameter test elements which form a system of parameters of R where R is reduced, excellent and equidimensional local Noetherian ring, see Corollary 4.6.

This paper is organized as follows. In Sect. 2, we review the necessary background material related to tight closure of ideals, test ideals, F-rational local rings, excellent rings and the tight closure of the zero submodule of $H^d_{\mathfrak{m}}(R)$. In Sect. 3, we generalize the result of Goto–Nakamura [Theorem 1.1 (1)] for equidimensional excellent local rings by proving a lower bound for the tight Hilbert function.

Theorem 1.3 *Let* $(R, \mathfrak{m})$ *be an equidimensional excellent local ring of prime characteristic p and Q be an ideal generated by a system of parameters for R. Then, for all* $n \geq 0$*,*

$$\ell(R/(Q^{n+1})^*) \geq \ell(R/Q^*) \binom{n+d}{d}.$$

Corollary 1.4 *Let* $(R, \mathfrak{m})$ *be a reduced equidimensional excellent local ring of prime characteristic p and Q be an ideal generated by a system of parameters for R. Then,*

$$e_0(Q) \geq \ell(R/Q^*).$$

In the next result, we show that if equality holds for some n in Theorem 1.3, then R is F-rational which can be considered as a generalization of Goto–Nakamura result [Theorem 1.1 (2)] under additional hypothesis.

Theorem 1.5 *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d and prime characteristic p. Let (S, n) be a Cohen–Macaulay local ring of dimension d and $Q(R)$ be the total quotient ring of R such that $R \subseteq S \subseteq Q(R)$ and S is a finite R-module. Let Q be an ideal of R generated by a system of parameters. Suppose that for some fixed $n \geq 0$,*

$$\ell(R/(Q^{n+1})^*) = e_0(Q)\binom{n+d}{d}.$$

Then, $R = S$. In particular, R is F-rational.

If $d = 2$ and the Hochster–Huneke graph of R, denoted by $\mathcal{G}(R)$, is connected, then we can take S in the above theorem to be the S_2-ification of R and obtain the following

Corollary 1.6 *Let $(R, \mathfrak{m})$ be a Noetherian local ring with $\dim(R/\mathfrak{p}) = 2$ for all $\mathfrak{p} \in \operatorname{Ass} R$ of prime characteristic p such that $\mathcal{G}(R)$ is connected. If for an ideal Q generated by a system of parameters for R and for some $n \geq 0$,*

$$\ell(R/(Q^{n+1})^*) = e_0(Q)\binom{n+2}{2},$$

then R is F-rational.

Let $(R, \mathfrak{m})$ be a d-dimensional local Noetherian ring and I be an $\mathfrak{m}$-primary ideal. Then, the *Hilbert function* of I is defined as $H_I(n) = \ell(R/I^n)$. For large n, it coincides with a polynomial of degree d called the *Hilbert polynomial* of I and it is written as

$$P_I(n) = e_0(I)\binom{n+d-1}{d} - e_1(I)\binom{n+d-2}{d-1} + \cdots + (-1)^d e_d(I).$$

If R is analytically unramified then by a Theorem of Rees [19], the *normal Hilbert function* of an $\mathfrak{m}$-primary ideal I, namely $\overline{H_I}(n) = \ell(R/\overline{I^n})$ coincides with a polynomial of degree d for large n. This polynomial is called the *normal Hilbert polynomial of* I and is given by

$$\overline{P_I}(n) = e_0(I)\binom{n+d-1}{d} - \overline{e_1}(I)\binom{n+d-2}{d-1} + \cdots + (-1)^d \overline{e_d}(I).$$

In [17], M. Moralés, N. V. Trung and O. Villamayor characterized regular local rings in terms of the equality $\overline{e_1}(Q) = e_1(Q)$ for a parameter ideal Q of an excellent analytically unramified local ring. It is worth noting that this result was proved in [15] by replacing the excellence hypothesis of R with its unmixedness. In Sect. 4, we find an analogous characterization for F-rational local rings as a consequence of explicit formulas for the tight Hilbert coefficients in terms of the lengths of local cohomology modules $H^j_{\mathfrak{m}}(R)$ for $0 \leq j \leq d-1$, $e_i(Q)$ for $0 \leq i \leq d$ and $\ell(0^*_{H^d_{\mathfrak{m}}(R)})$.

Theorem 1.7 *Let $(R, \mathfrak{m})$ be an excellent reduced equidimensional local ring of prime characteristic p and dimension $d \geq 2$. Let $x_1, x_2, \ldots, x_d$ be parameter test elements and $Q = (x_1, x_2, \ldots, x_d)$ be $\mathfrak{m}$-primary. Then,*

(1) $e_1^*(Q) = e_0(Q) - \ell(R/Q^*) + e_1(Q)$ *and* $e_j^*(Q) = e_j(Q) + e_{j-1}(Q)$ *for all* $2 \leq j \leq d$,

(2) $e_1^*(Q) = \sum_{i=2}^{d-1}\binom{d-2}{i-2}\ell(H^i_{\mathfrak{m}}(R)) + \ell(0^*_{H^d_{\mathfrak{m}}(R)})$,

(3) $e_i^*(Q) = (-1)^{i-1}\left[\sum_{j=0}^{d-i}\binom{d-i-1}{j-2}\ell(H^j_{\mathfrak{m}}(R)) + \ell(H^{d-i+1}_{\mathfrak{m}}(R))\right]$ *for* $i = 2, \ldots, d-1$

and

(4) $e_d^*(Q) = (-1)^{d-1}\ell(H^1_{\mathfrak{m}}(R))$.

Corollary 1.8 *Let $(R, \mathfrak{m})$ be an excellent reduced equidimensional local ring of prime characteristic p and dimension $d \geq 2$. Let $x_1, x_2, \ldots, x_d$ be parameter test elements and $Q = (x_1, x_2, \ldots, x_d)$ be $\mathfrak{m}$-primary. Then, the following are equivalent.*

(i) *R is F-rational*
(ii) $e_1^*(Q) = e_1(Q)$
(iii) $e_1^*(Q) = 0$ *and* $\operatorname{depth} R \geq 2$.

In Sect. 5, we construct examples to illustrate some of the above results.

1.1 Notation and conventions

All the rings in this paper are commutative Noetherian rings with multiplicative identity 1. We use $(R, \mathfrak{m}, k)$ to denote local ring R with unique maximal ideal $\mathfrak{m}$ and the residue field $k := R/\mathfrak{m}$. For basic results on Cohen–Macaulay rings, excellent rings, tight closure, Hilbert functions and multiplicity, we refer the reader to [3, 16].

2 Preliminaries

In this section, we set up some notation and recall results needed in later sections.

2.1 Background on tight closure

Let R be a commutative ring and I be an ideal of R. An element $x \in R$ is said to be *integral* over I if

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0$$

for some $a_i \in I^i$ for $1 \leq i \leq n$. The *integral closure* of I, denoted by $\overline{I}$, is the collection of all elements that are integral over I.

Let R be a Noetherian ring of prime characteristic p and R° denote the subset of R consisting of all elements which are not in any minimal prime ideal of R. For $I = (x_1, \ldots, x_n)$, let $I^{[p^e]} = (x_1^{p^e}, \ldots, x_n^{p^e})$. The *tight closure* of I, denoted by I^*, is the set of all elements x for which there exists some $c \in R^\circ$ such that $cx^{p^e} \in I^{[p^e]}$ for all $p^e >> 0$. An ideal I is said to be *tightly closed* if $I = I^*$. For any ideal I, we have $I \subseteq I^* \subseteq \overline{I}$.

Definition 2.1 The *test ideal* of R, denoted by $\tau(R)$, is the ideal generated by elements $c \in R$ which satisfies any of the following equivalent conditions.

(i) $cx^q \in I^{[q]}$ for all $q = p^0, p^1, p^2, \ldots,$ whenever $x \in I^*$ for any ideal I of R.
(ii) $cx \in I$ whenever $x \in I^*$ for any ideal I of R.

An element of $\tau(R) \cap R^\circ$ is called a *test element*.

A Noetherian ring R is said to be *weakly F-regular* if every ideal of R is tightly closed. Note that the test ideal of R is the unit ideal if and only if R is weakly F-regular. Recall that a *parameter ideal of height n* is an ideal of height n generated by n elements. For excellent local equidimensional rings, parameter ideals are those generated by a part of a system of parameters for R [23].

Definition 2.2 The *parameter test ideal* of R, denoted by $\tau_{\mathrm{par}}(R)$, is the ideal generated by $c \in R$ such that $cI^* \subset I$ for all parameter ideals I of R (equivalently, $cx^q \in I^{[q]}$ for all $q = p^e$, $e = 0, 1, 2, \ldots$). An element of $\tau_{\mathrm{par}}(R) \cap R^\circ$ is called a *parameter test element.*

Definition 2.3 A Noetherian ring R is called *F-rational* if all parameter ideals are tightly closed.

Let $(R, \mathfrak{m})$ be a d-dimensional local Noetherian ring and $x_1, \ldots, x_d$ be a system of parameters. Then, the local cohomology module $H^d_{\mathfrak{m}}(R)$ can be expressed as the dth cohomology of the Čech complex with respect to $x := x_1, \ldots, x_d$ since $H^d_{\mathfrak{m}}(R) \cong H^d_I(R)$, where $I = (x_1, \ldots, x_d)$. Any element of $H^d_{\mathfrak{m}}(R)$ can be represented as $\eta := \left[\frac{r}{x_1^i x_2^i \ldots x_d^i}\right]$. Let R be a ring of characteristic $p > 0$. The Frobenius map $F : R \to R$ defined by $F(r) = r^p$ naturally induces an action called the Frobenius action on $H^d_{\mathfrak{m}}(R)$ which takes an element $\eta = \left[\frac{r}{(x_1 x_2 \ldots x_d)^i}\right]$ to $F(\eta) = \left[\frac{r^p}{(x_1 x_2 \ldots x_d)^{ip}}\right]$. Similarly, the eth iteration of the Frobenius map $F^e : R \to R$ defined as $F^e(r) = r^{p^e}$ induces a similar action on $H^d_{\mathfrak{m}}(R)$.

Definition 2.4 Let $(R, \mathfrak{m})$ be a Noetherian local ring of characteristic p. Then,

$$0^*_{H^d_{\mathfrak{m}}(R)} = \{\eta \in H^d_{\mathfrak{m}}(R) : \exists\, c \in R^\circ \text{ such that } cF^e(\eta) = 0 \text{ for all } e >> 0\}.$$

We record a result from [22] which reveals the interplay of tight closure of the zero submodule of $H^d_{\mathfrak{m}}(R)$ with tight closure of ideal generated by a system of parameters of R.

Theorem 2.5 [22, Proposition 3.3(i)] *Let $(R, \mathfrak{m})$ be an excellent equidimensional local ring of dimension d, and let $x_1, \ldots, x_d$ be a system of parameters. Then, any $z \in (x_1, \ldots, x_d)^*$ uniquely determines an element $\eta = \left[\frac{z}{x_1 x_2 \ldots x_d}\right] \in 0^*_{H^d_{\mathfrak{m}}(R)}$. Conversely, if $\eta = \left[\frac{z}{x_1 x_2 \ldots x_d}\right] \in 0^*_{H^d_{\mathfrak{m}}(R)}$, then $z \in (x_1, \ldots, x_d)^*$.*

Remark 2.6 Note that if R is Cohen–Macaulay, $\eta = \left[\frac{z}{x_1 x_2 \ldots x_d}\right] \in 0^*_{H^d_{\mathfrak{m}}(R)}$ and $\eta = 0$ if and only if $z \in (x_1, \ldots, x_d)$. Therefore Theorem 2.5 implies that an excellent Cohen–Macaulay local ring $(R, \mathfrak{m})$ of dimension d is F-rational if and only if $0^*_{H^d_{\mathfrak{m}}(R)} = 0$.

2.2 Excellent rings

Very often, results in this paper and many results for tight closure assume that the given local ring is excellent. We shall use the following properties of excellent rings frequently.

(1) Let $(R, \mathfrak{m})$ be an excellent local ring with $\mathfrak{m}$-adic completion $\hat{R}$ and I be an $\mathfrak{m}$-primary ideal. Then, $I^*\hat{R} = (I\hat{R})^*$ [3, Proposition 10.3.18].
(2) Any excellent reduced local ring is analytically unramified [16, Theorem 70].
(3) Test elements exist in reduced excellent local rings [8, Theorem 6.1 (a)].
(4) If R is excellent, then it is a homomorphic image of Cohen–Macaulay ring [12, Corollary 1.2].

3 The tight Hilbert function and F-rationality of *R*

In this section, we give a generalization of Goto–Nakamura results [Theorem 1.1] for equidimensional excellent local rings. We provide a lower bound for tight Hilbert function and show that when the lower bound is achieved, then the ring is F-rational under some additional conditions on R. Let us first prove a crucial lemma required for this purpose.

Lemma 3.1 follows from [9, Theorem 8.20]. However, we are giving a simpler proof of Lemma 3.1(b). We thank the referee for giving us a clear proof of the next lemma.

Lemma 3.1 *Let $(R, \mathfrak{m})$ be an equidimensional excellent local ring of prime characteristic p and Q be an $\mathfrak{m}$-primary parameter ideal.*

(a) *Then, for all $n \geq 0$ we have $Q^n \cap (Q^{n+1})^* = Q^n Q^*$.*
(b) *Q^n/Q^nQ^* is a free R/Q^*-module of rank $\binom{n+d-1}{d-1}$, where $d = \dim R$.*

Proof (b) We note that Q^n is a R-module generated by monomials of degree n in $x_1, \ldots, x_d$ which form minimal generators of Q^n since $x_1, \ldots, x_d$ are analytically independent [18, Theorem 5]. Let $A = \mathbb{F}_p[x_1, \ldots, x_d]$ be the polynomial subring of R generated by $x_1, \ldots, x_d$. Set $q = (x_1, \ldots, x_d)A$. Let $m_1, \ldots, m_t$ be monomials in the x_i of degree n that form a minimal generating set of the finite R/Q^*-module Q^n/Q^nQ^* (since any monomial of greater degree will sit in $Q^{n+1} \subseteq Q^nQ^*$). Suppose we have $u_i \in R$ such that $z = \sum_{i=1}^t u_i m_i \in Q^nQ^*$. To show that the R/Q^*-module Q^n/Q^nQ^* is free, we must show that each $u_i \in Q^*$. For each $1 \leq i \leq t$, set $J_i := (m_1, \ldots, \widehat{m_i}, \ldots, m_t)A$. Then, since $Q^nQ^* \subseteq (Q^{n+1})^*$, we have $u_i m_i \in (Q^{n+1})^* + J_iR = (q^{n+1}R)^* + J_iR \subseteq ((q^{n+1} + J_i)R)^*$. Thus, $u_i \in ((q^{n+1} + J_i)R)^* :_R m_i \subseteq (((q^{n+1} + J_i) :_A m_i)R)^*$ by [2, Theorem 2.3]. But it is easy to see in the polynomial ring A that $(q^{n+1} + J_i) :_A m_i \subseteq q$. Thus, $u_i \in (qR)^* = Q^*$. □

Theorem 3.2 *Let $(R, \mathfrak{m})$ be an equidimensional excellent local ring of prime characteristic p and Q be an ideal generated by a system of parameters for R. Then, for all $n \geq 0$,*

$$\ell(R/(Q^{n+1})^*) \geq \ell(R/Q^*)\binom{n+d}{d}.$$

Proof We have

$$\ell(R/(Q^{n+1})^*) = \sum_{k=0}^{n} \ell((Q^k)^*/(Q^{k+1})^*).$$

For each k, we have

$$\ell\left(\frac{(Q^k)^*}{(Q^{k+1})^*}\right) \geq \ell\left(\frac{Q^k + (Q^{k+1})^*}{(Q^{k+1})^*}\right) = \ell\left(\frac{Q^k}{Q^k \cap (Q^{k+1})^*}\right) = \ell\left(\frac{Q^k}{Q^kQ^*}\right).$$

Since Q^k is minimally generated over R by $\binom{k+d-1}{d-1}$ generators, the base-changed module $Q^k/(Q^kQ^*)$ is also generated over R/Q^* by $\binom{k+d-1}{d-1}$ generators. As it must be free on these generators by Lemma 3.1,

$$\ell((Q^k)^*/(Q^{k+1})^*) \geq \ell(Q^k/Q^kQ^*) = \ell(R/Q^*)\binom{k+d-1}{d-1}.$$

Therefore,

$$\ell(R/(Q^{n+1})^*) \geq \ell(R/Q^*)\sum_{k=0}^{n}\binom{k+d-1}{d-1} = \ell(R/Q^*)\binom{n+d}{d}.$$

The proof is complete. □

Corollary 3.3 *Let $(R, \mathfrak{m})$ be a reduced equidimensional excellent local ring of prime characteristic p and Q be an ideal generated by a system of parameters for R. Then,*

$$e_0(Q) \geq \ell(R/Q^*).$$

Proof Since R is analytically unramified, by using Theorem 3.2 for $n >> 0$ we have,

$$\left[e_0(Q) - \ell(R/Q^*)\right]\binom{n+d}{d} - e_1^*(Q)\binom{n+d-1}{d-1} + \cdots + (-1)^d e_d^*(Q) \geq 0.$$

Therefore, $e_0(Q) \geq \ell(R/Q^*)$. □

The following lemma provides equivalent conditions for F-rationality of Cohen–Macaulay rings.

Lemma 3.4 *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of prime characteristic p. Let Q be an ideal of R generated by a system of parameters. Then, the following are equivalent.*

(a) $Q^* = Q$,
(b) $(Q^n)^* = Q^n$ *for all* $n \geq 1$,
(c) $(Q^n)^* = Q^n$ *for some* $n \geq 1$.

Proof (a) $\Longrightarrow$ (b). Observe that, using [4, Proposition 4.2], $Q^n \cap (Q^{n+1})^* = Q^*Q^n$ for all $n \geq 1$. Let $Q^* = Q$. Apply induction on n. The $n = 1$ case is an assumption. Suppose that $(Q^n)^* = Q^n$ for $n = 1, 2, \ldots, r$. As $(Q^{r+1})^* \subset (Q^r)^* = Q^r$, we have

$$(Q^{r+1})^* = (Q^{r+1})^* \cap Q^r = Q^*Q^r = Q^{r+1}.$$

By induction $(Q^n)^* = Q^n$ for all $n \geq 1$.

(b) $\Longrightarrow$ (c). This is clear.

(c) $\Longrightarrow$ (a). Let $(Q^n)^* = Q^n$ for some $n \geq 1$. Therefore, $Q^n = Q^{n-1} \cap (Q^n)^* = Q^*Q^{n-1}$. Hence, $Q^* \subseteq Q^n : Q^{n-1} = Q$. Therefore, $Q^* = Q$. □

Theorem 3.5 *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d and prime characteristic p. Let (S, n) be a Cohen–Macaulay local ring of dimension d and $Q(R)$ be the total quotient ring of R such that $R \subseteq S \subseteq Q(R)$ and S is a finite R-module. Let Q be an ideal of R generated by a system of parameters. Suppose that for some fixed $n \geq 0$,*

$$\ell(R/(Q^{n+1})^*) = e_0(Q)\binom{n+d}{d}.$$

Then, $R = S$. In particular R is F-rational.

Proof Using [3, Proposition 10.1.5], we get $(Q^nS)^* \cap R \subseteq (Q^n)^*$. Let $f = [S/\mathfrak{n} : R/\mathfrak{m}]$. Then, we obtain the following

$$\ell_R(R/(Q^{n+1})^*) \leq \ell_R(R/(Q^{n+1}S)^* \cap R) \leq \ell_R(S/(Q^{n+1}S)^*) \leq \ell_R(S/Q^{n+1}S), \tag{1}$$

$$\ell_R(S/Q^{n+1}S) = f\,\ell_S(S/(Q^{n+1}S)) = fe_0(QS)\binom{n+d}{d} = e_0(Q)\binom{n+d}{d}. \tag{2}$$

Therefore, if $\ell(R/(Q^{n+1})^*) = e_0(Q)\binom{n+d}{d}$, then $(Q^{n+1}S)^* = (Q^{n+1}S)$. As S is Cohen–Macaulay, using Lemma 3.4 it follows that $(QS)^* = QS$ and therefore, S is F-rational. Now, consider the exact sequence of finite R-modules

$$0 \to R \to S \to C \to 0,$$

where $C = S/R$. From (1) and (2), it follows that $(Q^{n+1})^* = (Q^{n+1}S)^* \cap R = Q^{n+1}S \cap R$. Tensor this sequence with R/Q^{n+1} to get the exact sequence of R-modules

$$0 \to R/(Q^{n+1})^* \to S/Q^{n+1}S \to C/Q^{n+1}C \to 0.$$

As $\ell(R/(Q^{n+1})^*) = e_0(Q)\binom{n+d}{d}$, using (1) and (2), we get $\ell_R(R/(Q^{n+1})^*) = \ell_R(S/Q^{n+1}S)$ which yields $C = Q^{n+1}C$. By Nakayama's lemma, $C = 0$. This means $R = S$. In particular, R is F-rational. □

We discuss a relationship of $e_1^*(Q)$ with S_2-ification. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of dimension d. We recall a few facts about S_2-ification of R from [10].

Definition 3.6 (1) We say that R is *equidimensional* if $\dim R/\mathfrak{p} = d$ for all minimal primes $\mathfrak{p}$ of R. If R is equidimensional and it has no embedded associated primes, then R is called *unmixed*.

(2) Let $(R, \mathfrak{m})$ be an equidimensional local ring of dimension d. The *Hochster–Huneke graph* $\mathcal{G}(R)$ is a graph where the vertices are the minimal prime ideals of R and the edges are the pairs of prime ideals (P_1, P_2) with $\operatorname{ht}(P_1 + P_2) = 1$.

(3) Let $(R, \mathfrak{m}, k)$ be an equidimensional and unmixed local ring. We say that a ring S is an *S_2-ification* of R if

(i) S lies between R and its total quotient ring,

(ii) S is module-finite over R and is S_2 as an R-module, and

(iii) for every element $s \in S \setminus R$, the ideal $D(s) := \{r \in R : rs \in R\}$ has height at least two.

If R is S_2, then $\mathcal{G}(R)$ is connected. Moreover, $\mathcal{G}(R)$ is connected if and only if the S_2-ification of R is local [10, Theorem 3.6].

Corollary 3.7 *Let $(R, \mathfrak{m})$ be a Noetherian local ring with $\dim(R/\mathfrak{p}) = 2$ for all $\mathfrak{p} \in \operatorname{Ass} R$ of prime characteristic p such that $\mathcal{G}(R)$ is connected. If for an ideal Q generated by a system of parameters for R and for some $n \geq 0$,*

$$\ell(R/(Q^{n+1})^*) = e_0(Q)\binom{n+2}{2},$$

then R is F-rational.

Proof By the result above, the S_2-ification S of R is a Cohen–Macaulay local ring that is a finite R-module. □

4 On the equality $e_1^*(Q) = e_1(Q)$ and F-rational local rings

In [17], M. Moralés, N. V. Trung and O. Villamayor proved the following characterization of regular local rings.

Theorem 4.1 [17, Theorem 1,2] *Let $(R, \mathfrak{m})$ be an analytically unramified excellent local domain and I be an $\mathfrak{m}$-primary parameter ideal. If $\overline{e}_1(I) = e_1(I)$, then R is a regular and $\overline{I^n} = I^n$ for all n.*

In this section, we find explicit formulas for the tight Hilbert coefficients of an ideal Q generated by system of parameters that are parameter test elements, in terms of the lengths of local cohomology modules $H^j_{\mathfrak{m}}(R)$ for $0 \leq j \leq d-1$, $e_i(Q)$ for $0 \leq i \leq d$ and $\ell(0^*_{H^d_{\mathfrak{m}}(R)})$. We use these formulas to characterize F-rationality of the ring in terms of the equality $e_1^*(Q) = e_1(Q)$ and also in terms of vanishing of $e_1^*(Q)$ under the condition that depth $R \geq 2$.

Let $(R, \mathfrak{m})$ be a local ring of dimension d and I be any $\mathfrak{m}$-primary parameter ideal of R. It is well known that $\ell(R/I) \geq e_0(I)$. Moreover, R is Cohen–Macaulay if and only if $\ell(R/I) = e_0(I)$ for some (and hence for all) I. Recall that R is called *Buchsbaum* if $\ell(R/I) - e_0(I)$ is independent of the choice of I.

Definition 4.2 Let $(R, \mathfrak{m})$ be a d-dimensional Noetherian local ring. An $\mathfrak{m}$-primary parameter ideal I is said to be *standard* if

$$\ell(R/I) - e_0(I) = \sum_{i=0}^{d-1} \binom{d-1}{i} \ell(H^i_{\mathfrak{m}}(R)).$$

The following result due to Linquan Ma and Pham Hung Quy plays a crucial role for proving a characterization of F-rationality in terms of vanishing of $e_1^*(Q)$ for $\mathfrak{m}$-primary parameter ideals generated by parameter test elements.

Theorem 4.3 [13, Theorem 4.3] *Let $(R, \mathfrak{m})$ be an excellent equidimensional local ring such that $\tau_{\text{par}}(R)$ is $\mathfrak{m}$-primary. Let Q be an ideal generated by a system of parameters contained in $\tau_{\text{par}}(R)$. Then we have*

$$\ell(Q^*/Q) = \sum_{i=0}^{d-1} \binom{d}{i} \ell(H^i_{\mathfrak{m}}(R)) + \ell\left(0^*_{H^d_{\mathfrak{m}}(R)}\right).$$

Remark 4.4 (i) If Q is an ideal generated by a system of parameters of R consisting of parameter test elements, then it is a standard system of parameters of R [11, Remark 5.11] and [21, Proposition 3.8].

(ii) If Q is generated by a standard system of parameters, then the Hilbert polynomial, in fact Hilbert function of Q can be found in [20, Corollary 3.2], [25, Corollary 4.2], [5, Theorem 7], etc. For $n \geq 0$,

$$\ell(R/Q^n) = \sum_{i=0}^{d} (-1)^i e_i(Q) \binom{n+d-1-i}{d-i}, \text{ where}$$

$$e_i(Q) = (-1)^i \sum_{j=0}^{d-i} \binom{d-i-1}{j-1} \ell(H^j_{\mathfrak{m}}(R)) \text{ for all } i = 1, 2, \ldots, d.$$

(iii) If $x_1, \ldots, x_d \in \tau_{\text{par}}(R)$ and $Q = (x_1, \ldots, x_d)$ is $\mathfrak{m}$-primary in $(R, \mathfrak{m})$, then $Q \subseteq \tau_{\text{par}}(R)$ and taking radicals on both sides, we obtain $\mathfrak{m} \subseteq \text{rad}(\tau_{\text{par}}(R))$ which implies that $\tau_{\text{par}}(R)$ is either $\mathfrak{m}$-primary or R.

Theorem 4.5 *Let $(R, \mathfrak{m})$ be an excellent reduced equidimensional local ring of prime characteristic p and dimension $d \geq 2$. Let $x_1, x_2, \ldots, x_d$ be parameter test elements and $Q = (x_1, x_2, \ldots, x_d)$ be $\mathfrak{m}$-primary. Then,*

(1) $e_1^*(Q) = e_0(Q) - \ell(R/Q^*) + e_1(Q)$ *and* $e_j^*(Q) = e_j(Q) + e_{j-1}(Q)$ *for all* $2 \leq j \leq d$,

(2) $e_1^*(Q) = \sum_{i=2}^{d-1} \binom{d-2}{i-2} \ell(H^i_{\mathfrak{m}}(R)) + \ell\left(0^*_{H^d_{\mathfrak{m}}(R)}\right)$,

(3) $e_i^*(Q) = (-1)^{i-1} \left[\sum_{j=0}^{d-i} \binom{d-i-1}{j-2} \ell(H^j_{\mathfrak{m}}(R)) + \ell(H^{d-i+1}_{\mathfrak{m}}(R))\right]$ *for* $i = 2, \ldots, d$.

Proof (1) By Lemma 3.1, Q^n/Q^nQ^* is a free R/Q^*-module of rank $\binom{n+d-1}{d-1}$ for all $n \geq 1$ and by [1, Lemma 3.1], $(Q^{n+1})^* = Q^nQ^*$ for all $n \geq 1$. Hence,

$$\ell(Q^n/Q^nQ^*) = \ell(Q^n/(Q^{n+1})^*) = \ell(R/Q^*)\binom{n+d-1}{d-1}.$$

Thus $\ell(R/(Q^{n+1})^*) = \ell(R/Q^n) + \ell(R/Q^*)\binom{n+d-1}{d-1}$ for all $n \geq 1$. By Remark 4.4(ii), the tight Hilbert function of Q is given by

$$\begin{aligned}
H_Q^*(n) &= e_0(Q)\binom{n+d-2}{d} - e_1(Q)\binom{n+d-3}{d-1} + \cdots + (-1)^d e_d(Q)\\
&\quad + \ell(R/Q^*)\binom{n+d-2}{d-1}\\
&= \sum_{i=0}^{d} e_i(Q)(-1)^i\binom{n+d-2-i}{d-i} + \ell(R/Q^*)\binom{n+d-2}{d-1}\\
&= \sum_{i=0}^{d} e_i(Q)(-1)^i\left[\binom{n+d-1-i}{d-i} - \binom{n+d-2-i}{d-1-i}\right]\\
&\quad + \ell(R/Q^*)\binom{n+d-2}{d-1}\\
&= e_0(Q)\binom{n+d-1}{d} - [e_0(Q) - \ell(R/Q^*) + e_1(Q)]\binom{n+d-2}{d-1}\\
&\quad + \sum_{i=2}^{d}(-1)^i[e_i(Q) + e_{i-1}(Q)]\binom{n+d-i-1}{d-i}.
\end{aligned}$$

Equating like terms on both sides, we obtain the desired formulas.

(2) From (1), we have $e_1^*(Q) = e_0(Q) - \ell(R/Q^*) + e_1(Q)$. On the other hand, since Q is standard, using Remark 4.4(iii) and Theorem 4.3 we have

$$\begin{aligned}
\ell(R/Q^*) &= \ell(R/Q) - \sum_{i=0}^{d-1}\binom{d}{i}\ell\big(H_{\mathfrak{m}}^i(R)\big) - \ell\left(0^*_{H_{\mathfrak{m}}^d(R)}\right)\\
&= e_0(Q) + \sum_{i=0}^{d-1}\binom{d-1}{i}\ell\big(H_{\mathfrak{m}}^i(R)\big) - \sum_{i=0}^{d-1}\binom{d}{i}\ell\big(H_{\mathfrak{m}}^i(R)\big) - \ell\left(0^*_{H_{\mathfrak{m}}^d(R)}\right)\\
&= e_0(Q) - \sum_{i=1}^{d-1}\binom{d-1}{i-1}\ell\big(H_{\mathfrak{m}}^i(R)\big) - \ell\left(0^*_{H_{\mathfrak{m}}^d(R)}\right),
\end{aligned}$$

where the second equality above follows from Remark 4.4(i). Hence,

$$e_1^*(Q) = \sum_{i=1}^{d-1}\binom{d-1}{i-1}\ell\big(H_{\mathfrak{m}}^i(R)\big) + \ell\left(0^*_{H_{\mathfrak{m}}^d(R)}\right) + e_1(Q). \tag{3}$$

Furthermore by Remark 4.4(ii), it follows that

$$\begin{aligned}
e_1^*(Q) &= \sum_{i=1}^{d-1}\binom{d-1}{i-1}\ell\big(H_{\mathfrak{m}}^i(R)\big) + \ell\left(0^*_{H_{\mathfrak{m}}^d(R)}\right) - \sum_{j=0}^{d-1}\binom{d-2}{j-1}\ell\big(H_{\mathfrak{m}}^j(R)\big)\\
&= \sum_{i=1}^{d-1}\left[\binom{d-1}{i-1} - \binom{d-2}{i-1}\right]\ell\big(H_{\mathfrak{m}}^i(R)\big) + \ell\left(0^*_{H_{\mathfrak{m}}^d(R)}\right)\\
&= \sum_{i=1}^{d-1}\binom{d-2}{i-2}\ell\big(H_{\mathfrak{m}}^i(R)\big) + \ell\left(0^*_{H_{\mathfrak{m}}^d(R)}\right)\\
&= \sum_{i=2}^{d-1}\binom{d-2}{i-2}\ell\big(H_{\mathfrak{m}}^i(R)\big) + \ell\left(0^*_{H_{\mathfrak{m}}^d(R)}\right).
\end{aligned}$$

(3) Using Remark 4.4(i)–(ii), we obtain

$$\ell(R/Q^n) = \sum_{i=0}^{d} \binom{n+d-1-i}{d-i}(-1)^i e_i(Q) \text{ for all } n \geq 1,$$

$$(-1)^i e_i(Q) = \sum_{j=0}^{d-i} \binom{d-i-1}{j-1} \ell\big(H^j_{\mathfrak{m}}(R)\big) \text{ for all } i = 1, 2, \ldots, d,$$

$$\ell(R/Q) - e_0(Q) = \sum_{j=0}^{d-1} \binom{d-1}{j} \ell\big(H^j_{\mathfrak{m}}(R)\big)$$

$$e_d(Q) = (-1)^d \ell\big(H^0_{\mathfrak{m}}(R)\big).$$

In the formulas above, we follow the convention $\binom{n}{-1} = 1$ if $n = -1$ and $\binom{n}{-1} = 0$ if $n \neq -1$. By the above formulas and the fact that R is reduced and equidimensional,

$$e^*_d(Q) = e_d(Q) + e_{d-1}(Q) = (-1)^{d-1} \ell\big(H^1_{\mathfrak{m}}(R)\big).$$

Next, we find the formulas for $e^*_i(Q)$ where $i = 2, 3, \ldots, d$ in terms of the lengths of the local cohomology modules. Put $h^j = \ell(H^j_{\mathfrak{m}}(R))$.

$$\begin{aligned} e^*_i(Q) &= e_i(Q) + e_{i-1}(Q) \\ &= (-1)^i \sum_{j=0}^{d-i} \binom{d-i-1}{j-1} h^j + (-1)^{i-1} \left[\sum_{j=0}^{d-i} \binom{d-i}{j-1} h^j + h^{d-i+1} \right] \\ &= (-1)^{i-1} \left[\sum_{j=0}^{d-i} \binom{d-i-1}{j-2} h^j + h^{d-i+1} \right]. \end{aligned}$$

□

In the dim 1 case, Question 1.2 has an affirmative answer. Let $(R, \mathfrak{m})$ be a one-dimensional analytically unramified local ring and $I = (a)$ be $\mathfrak{m}$-primary. Since R is reduced and $\dim R = 1$, R is Cohen–Macaulay. Let

$$P^*_I(n) = e(I)n - e^*_1(I).$$

If $e^*_1(I) = 0$, then R is F-rational. Let $(b) \subseteq \mathfrak{m}$ be a minimal reduction of $\mathfrak{m}$. By Briançon-Skoda Theorem, $\overline{(b)} = (b)^*$. As R is F-rational, $(b)^* = (b)$. Thus, $(b) = \overline{(b)} = \mathfrak{m}$. Hence, R is a regular local ring. In the case, $\dim R \geq 2$, we have answered Huneke's question with some additional hypothesis which can be derived as a consequence of Theorem 4.5.

Corollary 4.6 *Let $(R, \mathfrak{m})$ be an excellent reduced equidimensional local ring of prime characteristic p and dimension $d \geq 2$. Let $x_1, x_2, \ldots, x_d$ be parameter test elements and $Q = (x_1, x_2, \ldots, x_d)$ be $\mathfrak{m}$-primary. Then, the following are equivalent.*

(i) *R is F-rational,*
(ii) *$e^*_1(Q) = e_1(Q)$,*
(iii) *$e^*_1(Q) = 0$ and* depth $R \geq 2$.

Proof (i) $\Longleftrightarrow$ (ii): If R is F-rational, then R is Cohen–Macaulay. Therefore, $Q^n = (Q^n)^*$ for all $n \geq 1$ [4, Corollary 4.3]. Hence, $\ell(R/(Q^{n+1})^*) = e_0(Q)\binom{n+d}{d}$ for all $n \geq 0$ which implies that $e^*_1(Q) = e_1(Q) = 0$.

Conversely, let $e_1^*(Q) = e_1(Q)$. Using Theorem 4.5(1), $e_0(Q) = \ell(R/Q^*)$. As R is unmixed, by [6], R is F-rational.

(i) $\iff$ (iii): If R is F-rational, then it is Cohen–Macaulay so that (iii) holds. Conversely, let $e_1^*(Q) = 0$ and $\operatorname{depth} R \geq 2$. By Theorem 4.5(2), it follows that $0^*_{H^d_{\mathfrak{m}}(R)} = 0$ and $H^i_{\mathfrak{m}}(R) = 0$ for $2 \leq i \leq d-1$. As $\operatorname{depth} R \geq 2$, $H^0_{\mathfrak{m}}(R) = H^1_{\mathfrak{m}}(R) = 0$. Hence, R is Cohen–Macaulay ring with $0^*_{H^d_{\mathfrak{m}}(R)} = 0$. By Remark 2.6, it follows that R is F-rational. □

5 A counterexample to Huneke's question

We provide a negative answer to Huneke's question by constructing examples of unmixed local rings in which $e_1^*(Q) = 0$ for an ideal Q generated by a system of parameters, but R is not F-rational. The next proposition gives a class of examples where $0^*_{H^d_{\mathfrak{m}}(R)}$ vanishes.

Proposition 5.1 *Let $(R, \mathfrak{m})$ be an equidimensional reduced local ring of dimension d, and $\operatorname{Ass} R = \{P_1, P_2\}$. Suppose R/P_1 and R/P_2 are both F-rational and $\dim R/(P_1+P_2) \leq d-2$. Then, $0^*_{H^d_{\mathfrak{m}}(R)} = 0$.*

Proof Consider the long exact sequence of local cohomology arising from the following short exact sequence.

$$0 \to R \to R/P_1 \oplus R/P_2 \to R/(P_1+P_2) \to 0.$$

Since $\dim R/(P_1+P_2) \leq d-2$, it follows that $H^i_{\mathfrak{m}}(R/(P_1+P_2)) = 0$ for $i = d-1, d$. This implies that $H^d_{\mathfrak{m}}(R) \cong H^d_{\mathfrak{m}}(R/P_1) \oplus H^d_{\mathfrak{m}}(R/P_2)$. Clearly, $0^*_{H^d_{\mathfrak{m}}(R)} \cong 0^*_{H^d_{\mathfrak{m}}(R/P_1)} \oplus 0^*_{H^d_{\mathfrak{m}}(R/P_2)}$. Since R/P_i is F-rational for $i = 1, 2$, we have $0^*_{H^d_{\mathfrak{m}}(R/P_i)} = 0$ which implies that $0^*_{H^d_{\mathfrak{m}}(R)} = 0$. □

Lemma 5.2 *Let $(R, \mathfrak{m})$ be an equidimensional reduced local ring of dimension d, and $\operatorname{Ass} R = \{P_1, P_2\}$. Then, for any $\mathfrak{m}$-primary parameter ideal Q in R,*

$$e_0(Q) = e_0\big((Q+P_1)/P_1\big) + e_0\big((Q+P_2)/P_2\big).$$

Proof Since R is reduced, $\ell_{R_{P_i}}(R_{P_i}) = 1$ for $i = 1, 2$. By the associativity formula for multiplicity, we get

$$\begin{aligned} e_0(Q) &= e_0\big((Q+P_1)/P_1\big)\ell(R_{P_1}) + e_0(Q+P_2/P_2)\ell(R_{P_2}) \\ &= e_0\big((Q+P_1)/P_1\big) + e_0\big((Q+P_2)/P_2\big). \end{aligned}$$

□

Proposition 5.3 *Let $(R, \mathfrak{m})$ be an equidimensional reduced local ring of dimension d and prime characteristic p with $\operatorname{Ass} R = \{P_1, P_2\}$. Suppose R/P_1 and R/P_2 are both F-rational and $\dim R/(P_1+P_2) \leq d-2$. Then, R is not Cohen–Macaulay and for any ideal generated by a system of parameters Q, we have $e_1^*(Q) = 0$.*

Proof Since R/P_i is F-rational, we have $(Q^{n+1}R/P_i)^* = (Q^{n+1}+P_i)/P_i$ for $i = 1, 2$. Using [7, Proposition 6.25(a)], we have $(Q^{n+1})^* + P_i = Q^{n+1} + P_i$ for all $i = 1, 2$. Thus, $(Q^{n+1})^* \subseteq (Q^{n+1}+P_1) \cap (Q^{n+1}+P_2)$. Moreover, $x \in (Q^{n+1})^*$ if and only if the image of x in R/P_i is contained in $(Q^{n+1}R/P_i)^* = (Q^{n+1}+P_i)/P_i$ for $i = 1, 2$. Hence, $(Q^{n+1})^* = (Q^{n+1}+P_1) \cap (Q^{n+1}+P_2)$. Therefore, we have the short exact sequence

$$0 \to R/(Q^{n+1})^* \to R/(Q^{n+1}+P_1) \oplus R/(Q^{n+1}+P_2) \to R/(Q^{n+1}+P_1+P_2) \to 0$$

for all $n \geq 0$. Thus, we have

$$\begin{aligned}\ell(R/(Q^{n+1})^*) &= \ell\big(R/(Q^{n+1}+P_1)\big) + \ell\big(R/(Q^{n+1}+P_2)\big) - \ell\big(R/(Q^{n+1}+P_1+P_2)\big)\\ &= \big[e_0\big((Q+P_1)/P_1\big) + e_0\big((Q+P_2)/P_2\big)\big]\binom{n+d}{d}\\ &\quad - \ell\big(R/(Q^{n+1}+P_1+P_2)\big)\\ &= e_0(Q)\binom{n+d}{d} - \ell(R/(Q^{n+1}+P_1+P_2)),\end{aligned}$$

where the last equality follows from Lemma 5.2.

Since $\ell(R/(Q^{n+1}+P_1+P_2))$ is a polynomial of degree atmost $d-2$, $e_1^*(Q)=0$ for all Q. Consider the short exact sequence of R-modules

$$0 \to R \to R/P_1 \oplus R/P_2 \to R/(P_1+P_2) \to 0.$$

Since $\operatorname{depth}(R/P_1 \oplus R/P_2) = d > \dim(R/(P_1+P_2))$, by the depth Lemma depth $R \leq d-1$. Hence, R is not Cohen–Macaulay. □

We construct an example to show that the condition depth $R \geq 2$ in Corollary 4.6 is not superfluous for characterization of F-rationality in terms of vanishing of $e_1^*(Q)$.

Example 5.4 Let $S = \mathbb{F}_p[|X,Y,Z,W|]$ and $R = \frac{S}{I \cap J}$, where $I=(X,Y)$ and $J=(Z,W)$. Let the lower case letters denote images of the upper case letters. Put $\mathfrak{m}=(x,y,z,w)$. Let $a=x+z$, $b=y+w$. Then, a,b is a system of parameters. Set $Q=(a,b)$. Since R is Buchsbaum

$$\ell\left(\frac{R}{Q}\right) - e_0(Q) = \sum_{i=0}^{d-1}\binom{d-1}{i}\ell(H^i_{\mathfrak{m}}(R)) = \ell(H^1_{\mathfrak{m}}(R)) = 1,$$

Note that $H^1_{\mathfrak{m}}(R) \cong H^0_{\mathfrak{m}}(R/\mathfrak{m}) \cong R/\mathfrak{m}$. Using $e_i(Q) = (-1)^i \sum_{j=0}^{d-i}\binom{d-i-1}{j-1}\ell(H^j_{\mathfrak{m}}(R))$, we get $e_1(Q) = -\ell(H^1_{\mathfrak{m}}(R)) = -1$, $e_2(Q)=0$. Since R is Buchsbaum and $0^*_{H^d_{\mathfrak{m}}}(R)=0$, it follows that $\tau_{\text{par}}(R)=\mathfrak{m}$. Thus, by Theorem 4.5(1), $e_2^*(Q) = e_2(Q)+e_1(Q) = -1$ and $e_1^*(Q)=0$ by Proposition 5.3. Therefore,

$$P_Q^*(n) = 2\binom{n+1}{2} - 1.$$

Example 5.5 We construct a complete local domain of dimension 2 that is not F-rational, but there exists an ideal Q generated by a system of parameters Q for which $e_1^*(Q)=0$. Let k be a field of prime characteristic $p \geq 3$ and $R = k[[x^4, x^3y, xy^3, y^4]]$. We have the S_2-ification of R is the local ring $S = k[[x^4, x^3y, x^2y^2, xy^3, y^4]]$. We have $C := S/R \cong k$, so that $\ell(C/JC)=1$ for any $\mathfrak{m}$-primary ideal J of R. Let Q be any $\mathfrak{m}$-primary ideal parameter ideal of R. Consider the short exact sequence,

$$0 \to R/(Q^{n+1})^* \to S/(Q^{n+1}S)^* \to C \to 0.$$

We have

$$\ell(R/(Q^{n+1})^*) = \ell(S/(Q^{n+1})^*S) - 1.$$

Since S is F-regular,

$$\ell(R/(Q^{n+1})^*) = e_0(Q)\binom{n+2}{2} - 1$$

for all $n \geq 1$. Since $S/\mathfrak{n} \cong R/\mathfrak{m}$, $e_0(Q) = e_0(QS)$. Hence, $e_1^*(Q) = 0$.

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Data availability

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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