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Stability for quantitative photoacoustic tomography revisited

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Abstract

This paper deals with the issue of stability in determining the absorption and the diffusion coefficients in quantitative photoacoustic imaging. We establish a global conditional Hölder stability inequality from the knowledge of two internal data obtained from optical waves, generated by two point sources in a region where the optical coefficients are known.

Keywords: Elliptic equations, Diffusion coefficient, Absorption coefficient, Stability inequality, Multiwave imaging

Mathematics Subject Classification: 35R30

1 Introduction

Throughout this text, $n \ge 3$ is a fixed integer. If $0 < \beta \le 1$, we denote by $C^{0,\beta}(\mathbb{R}^n)$ the vector space of bounded continuous functions f on $\mathbb{R}^n$ satisfying

$$[f]_\beta = \sup\left\{\frac{|f(x)-f(y)|}{|x-y|^\beta};\ x, y \in \mathbb{R}^n,\ x \ne y\right\} < \infty.$$

$C^{0,\beta}(\mathbb{R}^n)$ is then a Banach space when it is endowed with its natural norm

$$\|f\|_{C^{0,\beta}(\mathbb{R}^n)} = \|f\|_{L^\infty(\mathbb{R}^n)} + [f]_\beta.$$

Define $C^{1,\beta}(\mathbb{R}^n)$ as the vector space of functions f from $C^{0,\beta}(\mathbb{R}^n)$ so that $\partial_j f \in C^{0,\beta}(\mathbb{R}^n)$, $1 \le j \le n$. The vector space $C^{1,\beta}(\mathbb{R}^n)$ equipped with the norm

$$\|f\|_{C^{1,\beta}(\mathbb{R}^n)} = \|f\|_{C^{0,\beta}(\mathbb{R}^n)} + \sum_{j=1}^n \|\partial_j f\|_{C^{0,\beta}(\mathbb{R}^n)}$$

is a Banach space.

The data in this paper consist in $\xi_1, \xi_2 \in \mathbb{R}^n$, $\Omega \Subset \mathbb{R}^n \setminus \{\xi_1, \xi_2\}$ a $C^{1,1}$ bounded domain with boundary Γ, $0 < \alpha < 1$, $0 < \theta < \alpha$, $\lambda > 1$ and $\kappa > 1$. For notational convenience, the set of data will denoted by $\mathfrak{D}$. That is

$$\mathfrak{D} = (n, \xi_1, \xi_2, \Omega, \alpha, \theta, \lambda, \kappa).$$

Denote by $\mathcal{D}(\lambda, \kappa)$ the set of couples $(a, b) \in C^{1,1}(\mathbb{R}^n) \times C^{0,1}(\mathbb{R}^n)$ satisfying

$$\lambda^{-1} \le a \quad\text{and}\quad \|a\|_{C^{1,1}(\mathbb{R}^n)} \le \lambda, \tag{1.1}$$

$$\kappa^{-1} \le b \quad\text{and}\quad \|b\|_{C^{0,1}(\mathbb{R}^n)} \le \kappa. \tag{1.2}$$

Define further the elliptic operator $L_{a,b}$ acting as follows

$$L_{a,b}u(x) = -\mathrm{div}(a(x)\nabla u(x)) + b(x)u(x). \tag{1.3}$$

We show in Sect. 2 that if $(a, b) \in \mathcal{D}(\lambda, \kappa)$, then the operator $L_{a,b}$ admits a unique fundamental solution $G_{a,b}$ satisfying, where $\xi \in \mathbb{R}^n$,

$$G_{a,b}(\cdot, \xi) \in C^{2,\alpha}_{loc}(\mathbb{R}^n \setminus \{\xi\}), \quad L_{a,b}G_{a,b}(\cdot, \xi) = 0 \text{ in } \mathbb{R}^n \setminus \{\xi\},$$

and, for any $f \in C_0^\infty(\mathbb{R}^n)$,

$$u = \int_{\mathbb{R}^n} G_{a,b}(\cdot, \xi) f(\xi) d\xi$$

belongs to $H^2(\mathbb{R}^n)$ and it is the unique solution of $L_{a,b}u = f$.

We deal in the present work with the problem of reconstructing $(a, b) \in \mathcal{D}(\lambda, \kappa)$ from energies generated by two point sources located at ξ_1 and ξ_2. Precisely, if $u_j(a, b) = G_{a,b}(\cdot, \xi_j)$, $j = 1, 2$, we want to determine (a, b) from the internal measurements

$$v_j(a, b) = bu_j(a, b) \quad \text{in } \Omega, \quad j = 1, 2.$$

This inverse problem is related to photoacoustic tomography (PAT) where optical energy absorption causes thermoelastic expansion of the tissue, which in turn generates a pressure wave [25]. This acoustic signal is measured by transducers distributed on the boundary of the sample, and it is used for imaging optical properties of the sample. The internal data $v_1(a, b)$ and $v_2(a, b)$ are obtained by performing a first step consisting in a linear initial to boundary inverse problem for the acoustic wave equation. Therefore, the inverse problem that arises from this first inversion is to determine the diffusion coefficient a and the absorption coefficient b from the internal data $v_1(a, b)$ and $v_2(a, b)$ that are proportional to the local absorbed optical energy inside the sample. This inverse problem is known in the literature as quantitative photoacoustic tomography [1–4,7,8,11,21].

Photoacoustic imaging provides in theory images of optical contrasts and ultrasound resolution [25]. Indeed, the resolution is mainly due to the small wavelength of acoustic waves, while the contrast is somehow related to the sensitivity of optical waves to absorption and scattering properties of the medium in the diffusive regime. However, in practice, it has been observed in various experiments that the imaging depth, i.e., the maximal depth of the medium at which structures can be resolved at expected resolution, of (PAT) is still fairly limited, usually on the order of millimeters. This is mainly due to the fact that optical waves are significantly attenuated by absorption and scattering. In fact the generated optical signal decays very fast in the depth direction. This is indeed a well-known faced issue in optical tomography [24]. In most physicists works dealing with quantitative (PAT), the absorption $b > 0$ is assumed to be constant and the optical wave is simplified to Ce^{-bz}, as a function of the depth z, which is known as the Beer–Lambert–Bouguer law [12]. Recently in [22], assuming that medium is layered, the authors derived a stability estimate that shows that the reconstruction of the optical coefficients is stable in the region close to the optical illumination source and deteriorates exponentially far away.

Stability inequalities for this inverse problem were first obtained in [7,8] under a strong non-degeneracy assumption. Later in [1], the authors improved these results by removing the non-degeneracy assumption for well-chosen boundary conditions (Definition 2.3).

Assuming that the optical waves are generated by two point sources δ_{ξ_i}, $i = 1, 2$, we aim to derive a stability estimate for the recovery of the optical coefficients from internal data.

We point out that taking the optical wave generated by a point source outside the sample seems to be more realistic than assuming a boundary condition.

In the statement of Theorem 1, $C = C(\mathfrak{D}) > 0$ and $0 < \gamma = \gamma(\mathfrak{D}) < 1$ are constants.

Theorem 1 *For any* $(a, b), (\tilde{a}, \tilde{b}) \in \mathcal{D}(\lambda, \kappa)$ *satisfying* $(a, b) = (\tilde{a}, \tilde{b})$ *on* Γ, *we have*

$$\|a - \tilde{a}\|_{C^{1,\alpha}(\overline{\Omega})} + \|b - \tilde{b}\|_{C^{0,\alpha}(\overline{\Omega})} \le C \left(\|v_1 - \tilde{v}_1\|_{C(\overline{\Omega})} + \|v_2 - \tilde{v}_2\|_{C(\overline{\Omega})} \right)^{\gamma}.$$

The rest of this text is organized as follows. In Sect. 2, we construct a fundamental solution and give its regularity induced by that of the coefficients of the operator under consideration. We derive pointwise lower and upper bounds for the fundamental solution that are of interest themselves. These bounds show how the optical signal decays fast in the depth direction. We also establish in this section a lower bound of the local L^2-norm of the gradient of the quotient of two fundamental solutions near one of the point sources. This is the key point for establishing our stability inequality. This last result is then used in Sect. 3 to obtain a uniform polynomial lower bound of the local L^2-norm of the gradient in a given region. This polynomial lower bound is obtained in two steps. In the first step, we derive, via a three-ball inequality for the gradient, a uniform lower bound of negative exponential type. We use then in the second step an argument based on the so-called frequency function in order to improve this lower bound. In the last section, we prove our main theorem following the known method consisting in reducing the original problem to the stability of an inverse conductivity problem.

2 Fundamental solutions

2.1 Constructing fundamental solutions

In this subsection, we construct a fundamental solution of divergence form elliptic operators. Since our construction relies on heat kernel estimates, we first recall some known results.

Consider the parabolic operator $P_{a,b}$ acting as follows:

$$P_{a,b}u(x, t) = -L_{a,b}u(x, t) - \partial_t u(x, t)$$

and set

$$Q = \{(x, t, \xi, \tau) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R};\ \tau < t\}.$$

Recall that a fundamental solution of the operator $P_{a,b}$ is a function $E_{a,b} \in C^{2,1}(Q)$ verifying $P_{a,b}E = 0$ in Q and, for every $f \in C_0^\infty(\mathbb{R}^n)$,

$$\lim_{t \downarrow \tau} \int_{\mathbb{R}^n} E_{a,b}(x, t, \xi, \tau) f(\xi) d\xi = f(x), \quad x \in \mathbb{R}^n.$$

The classical results in the monographs by A. Friedman [14], O. A. Ladyzenskaja, V. A. Solonnikov and N.N Ural'ceva [20] show that $P_{a,b}$ admits a nonnegative fundamental solution when $(a, b) \in \mathcal{D}(\lambda, \kappa)$.

It is worth mentioning that if $a = c$, for some constant $c > 0$, and $b = 0$, then the fundamental solution $E_{c,0}$ is explicitly given by

$$E_{c,0}(x, t, \xi, \tau) = \frac{1}{[4\pi c(t - \tau)]^{n/2}} e^{-\frac{|x-\xi|^2}{4c(t-\tau)}}, \quad (x, t, \xi, \tau) \in Q.$$

Examining carefully the proof of the two-sided Gaussian bounds in [13], we see that these bounds remain valid whenever $a \in C^{1,1}(\mathbb{R}^n)$ satisfies

$$\lambda^{-1} \le a \quad \text{and} \quad \|a\|_{C^{1,1}(\mathbb{R}^n)} \le \lambda. \tag{2.1}$$

More precisely, we have the following theorem in which

$$\mathcal{E}_c(x,t) = \frac{c}{t^{n/2}} e^{-\frac{|x|^2}{ct}}, \quad x \in \mathbb{R}^n,\ t > 0,\ c > 0.$$

Theorem 2 *There exists a constant $c = c(n,\lambda) > 1$ so that, for any $a \in C^{1,1}(\mathbb{R}^n)$ satisfying* (2.1)*, we have*

$$\mathcal{E}_{c^{-1}}(x-\xi, t-\tau) \le E_{a,0}(x,t;\xi,\tau) \le \mathcal{E}_c(x-\xi,t-\tau), \tag{2.2}$$

for all $(x,t,\xi,\tau) \in Q$.

The relationship between $\mathcal{E}_c$ and $E_{c,0}$ is given by the formula

$$\mathcal{E}_c(x-\xi,t-\tau) = \frac{(\pi c)^{n/2+1}}{\pi} E_{c/4,0}(x,t,\xi,\tau), \quad (x,t,\xi,\tau) \in Q. \tag{2.3}$$

The following comparison principle will be useful in the sequel.

Lemma 1 *Let $(a,b_1), (a,b_2) \in \mathcal{D}(\lambda,\kappa)$ so that $b_1 \le b_2$. Then, $E_{a,b_2} \le E_{a,b_1}$.*

Proof Pick $0 \le f \in C_0^\infty(\mathbb{R}^n)$. Let u be the solution of the initial value problem

$$P_{a,b_1}u(x,t) = 0 \ \in \mathbb{R}^n \times \{t > \tau\}, \quad u(x,\tau) = f.$$

We have

$$u(x,t) = \int_{\mathbb{R}^n} E_{a,b_1}(x,t;\xi,\tau) f(\xi)d\xi \ge 0. \tag{2.4}$$

On the other hand, as $P_{a,b_1}u(x,t) = 0$ can be rewritten as

$$P_{a,b_2}u(x,t) = [b_1(x) - b_2(x)]u(x,t),$$

We obtain

$$\begin{aligned} u(x,t) = &\int_{\mathbb{R}^n} E_{a,b_2}(x,t;\xi,\tau) f(\xi)d\xi \\ &- \int_\tau^t \int_{\mathbb{R}^n} E_{a,b_2}(x,t;\xi,s)[b_1(\xi) - b_2(\xi)]u(\xi,s)d\xi\, ds. \end{aligned} \tag{2.5}$$

Combining (2.4) and (2.5), we get

$$\int_{\mathbb{R}^n} E_{a,b_2}(x,t;\xi,\tau) f(\xi)d\xi \le \int_{\mathbb{R}^n} E_{a,b_1}(x,t;\xi,\tau) f(\xi)d\xi,$$

which yields in a straightforward manner the expected inequality. □

Consider, for $(a,b) \in \mathcal{D}(\lambda,\kappa)$, the unbounded operator $A_{a,b} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ defined

$$A_{a,b} = -L_{a,b}, \quad D(A_{a,b}) = H^2(\mathbb{R}^n).$$

It is well known that $A_{a,b}$ generates an analytic semigroup $e^{tA_{a,b}}$. Therefore in light of [6, Theorem 4 on p. 30, Theorem 18 on p. 44 and the proof in the beginning of Sect. 1.4.2 on page 35] $k_{a,b}(t,x;\xi)$, the Schwarz kernel of $e^{tA_{a,b}}$ is Hölder continuous with respect to x and ξ and satisfies

$$|k_{a,b}(t,x,\xi)| \le e^{-\delta t}\mathcal{E}_c(x-\xi,t) \tag{2.6}$$

and, for $|h| \leq \sqrt{t} + |x-\xi|$,

$$|k_{a,b}(t,x+h,\xi) - k_{a,b}(t,x,\xi)| \leq e^{-\delta t}\left(\frac{|h|}{\sqrt{t}+|x-\xi|}\right)^{\eta} \mathcal{E}_c(x-\xi,t), \tag{2.7}$$

$$|k_{a,b}(t,x,\xi+h) - k_{a,b}(t,x,\xi)| \leq e^{-\delta t}\left(\frac{|h|}{\sqrt{t}+|x-\xi|}\right)^{\eta} \mathcal{E}_c(x-\xi,t), \tag{2.8}$$

where $c = c(n,\lambda,\kappa) > 0$ and $\delta = \delta(n,\lambda,\kappa) > 0$ and $\eta > 0$ are constants.

From the uniqueness of solutions of the Cauchy problem

$$u'(t) = A_{a,b}u(t),\ t > 0, \quad u(0) = f \in C_0^\infty(\mathbb{R}^n), \tag{2.9}$$

we deduce in a straightforward manner that $k_{a,b}(t,x;\xi) = E_{a,b}(x,t;\xi,0)$.

Prior to giving the construction of the fundamental solution for the variable coefficients operators, we state a result for operators with constant coefficients. This result is proved in "Appendix A" section.

Lemma 2 *Let $\mu > 0$ and $\nu > 0$ be two constants. Then, the fundamental solution for the operator $-\mu\Delta + \nu$ is given by $G_{\mu,\nu}(x,\xi) = \mathcal{G}_{\mu,\nu}(x-\xi)$, $x, \xi \in \mathbb{R}^n$, with*

$$\mathcal{G}_{\mu,\nu}(x) = (2\pi\mu)^{-n/2}(\sqrt{\nu\mu}/|x|)^{n/2-1}K_{n/2-1}(\sqrt{\nu}|x|/\sqrt{\mu}), \quad x \in \mathbb{R}^n.$$

Here, $K_{n/2-1}$ is the usual modified Bessel function of second kind. Moreover, the following two-sided inequality holds

$$C^{-1}\frac{e^{-\sqrt{\nu}|x|/\sqrt{\mu}}}{|x|^{n-2}} \leq \mathcal{G}_{\mu,\nu}(x) \leq C\frac{e^{-\sqrt{\nu}|x|/(2\sqrt{\mu})}}{|x|^{n-2}}, \quad x \in \mathbb{R}^n, \tag{2.10}$$

for some constant $C = C(n,\mu,\nu) > 1$.

The main result of this section is the following theorem.

Theorem 3 *Let $(a,b) \in \mathcal{D}(\lambda,\kappa)$. Then, there exists a unique function $G_{a,b}$ satisfying $G_{a,b}(\cdot,\xi) \in C(\mathbb{R}^n \setminus \{\xi\})$, $\xi \in \mathbb{R}^n$, $G_{a,b}(x,\cdot) \in C(\mathbb{R}^n \setminus \{x\})$, $x \in \mathbb{R}^n$, and*

(i) $L_{a,b}G_{a,b}(\cdot,\xi) = 0$ in $\mathscr{D}'(\mathbb{R}^n \setminus \{\xi\})$, $\xi \in \mathbb{R}^n$,
(ii) For any $f \in C_0^\infty(\mathbb{R}^n)$,

$$u(x) = \int_{\mathbb{R}^n} G_{a,b}(x,\xi)f(\xi)d\xi$$

belongs to $H^2(\mathbb{R}^n)$ and it is the unique solution of $L_{a,b}u = f$,
(iii) There exist two constants $c = c(n,\lambda) > 1$ and $C = C(n,\lambda,\kappa) > 1$ so that

$$C^{-1}\frac{e^{-2\sqrt{c\kappa}|x-\xi|}}{|x-\xi|^{n-2}} \leq G_{a,b}(x,\xi) \leq C\frac{e^{-\frac{|x-\xi|}{\sqrt{c\kappa}}}}{|x-\xi|^{n-2}}. \tag{2.11}$$

Proof Pick $s \geq 1$ arbitrary and let $f \in C_0^\infty(\mathbb{R}^n)$. Applying Hölder's inequality, we find

$$\int_{\mathbb{R}^n} k_{a,b}(t,x,\xi)|f(\xi)|d\xi \leq \|k_{a,b}(t,x,\cdot)\|_{L^s(\mathbb{R}^n)}\|f\|_{L^{s'}(\mathbb{R}^n)},$$

where s' is the conjugate exponent of s.

But, according to (2.6),

$$\|k_{a,b}(t,x,\cdot)\|^s_{L^s(\mathbb{R}^n)} \leq \left(\frac{c}{t^{n/2}}\right)^s \int_{\mathbb{R}^n} e^{-\frac{s|x-\xi|^2}{ct}}d\xi.$$

Next, making the change of variable $\xi = (\sqrt{ct/s})\eta + x$, we get

$$\|k_{a,b}(t,x,\cdot)\|^s_{L^s(\mathbb{R}^n)} \leq \left(\frac{c}{t^{n/2}}\right)^s \left(\frac{ct}{s}\right)^{n/2} \int_{\mathbb{R}^n} e^{-|\eta|^2} d\eta.$$

Hence,

$$\|k_{a,b}(t,x,\cdot)\|_{L^s(\mathbb{R}^n)} \leq t^{n(1/s-1)/2} C_s,$$

with

$$C_s = c\left(\frac{c}{s}\right)^{n/2} \left(\int_{\mathbb{R}^n} e^{-|\eta|^2} d\eta\right)^{1/s}.$$

We get, by choosing $1 \leq s < \frac{n}{n-2} < \tilde{s}$,

$$\begin{aligned}
&\int_0^{+\infty} \int_{\mathbb{R}^n} k_{a,b}(t,x,\xi)|f(\xi)|d\xi\, dt \\
&\quad = \int_0^1 \int_{\mathbb{R}^n} k_{a,b}(t,x,\xi)|f(\xi)|d\xi\, dt + \int_1^{+\infty} \int_{\mathbb{R}^n} k_{a,b}(t,x,\xi)|f(\xi)|d\xi\, dt \\
&\quad \leq C_s \|f\|_{L^{s'}(\mathbb{R}^n)} \int_0^1 t^{\frac{n}{2}(1/s-1)} dt + C_{\tilde{s}} \|f\|_{L^{\tilde{s}'}(\mathbb{R}^n)} \int_1^{+\infty} t^{\frac{n}{2}(1/\tilde{s}-1)} dt.
\end{aligned}$$

In light of Fubini's theorem, we obtain

$$\int_0^{+\infty} \int_{\mathbb{R}^n} k_{a,b}(t,x,\xi) f(\xi) d\xi\, dt = \int_{\mathbb{R}^n} \left(\int_0^{+\infty} k_{a,b}(t,x,\xi) dt\right) f(\xi) d\xi. \tag{2.12}$$

Define $G_{a,b}$ as follows

$$G_{a,b}(x,\xi) = \int_0^{+\infty} k_{a,b}(t,x,\xi) dt.$$

Then, (2.12) takes the form

$$\int_0^{+\infty} \int_{\mathbb{R}^n} k_{a,b}(t,x,\xi) f(\xi) d\xi\, dt = \int_{\mathbb{R}^n} G_{a,b}(x,\xi) f(\xi) d\xi. \tag{2.13}$$

Noting that $A_{a,b}$ is invertible, we obtain

$$\begin{aligned}
-A_{a,b}^{-1} f(x) &= \left(\int_0^{+\infty} e^{tA_{a,b}} f dt\right)(x) \\
&= \int_0^{+\infty} \int_{\mathbb{R}^n} k_{a,b}(t,x,\xi) f(\xi) d\xi\, dt, \quad x \in \mathbb{R}^n.
\end{aligned}$$

This and (2.13) entail

$$-A_{a,b}^{-1} f(x) = \int_{\mathbb{R}^n} G_{a,b}(x,\xi) f(\xi) d\xi, \quad x \in \mathbb{R}^n.$$

In other words, u defined by

$$u(x) = \int_{\mathbb{R}^n} G_{a,b}(x,\xi) f(\xi) d\xi, \quad x \in \mathbb{R}^n$$

belongs to $H^2(\mathbb{R}^n)$ and satisfies $L_{a,b} u = f$.

Since, for $x \neq \xi$,

$$\int_0^{+\infty} \frac{1}{t^{n/2}} e^{-\frac{|x-\xi|^2}{ct}} dt = \left(c^{n/2-1} \int_0^{+\infty} \tau^{n/2-2} e^{-\tau} d\tau\right) \frac{1}{|x-\xi|^{n-2}},$$

we get in light of (2.7)

$$|G_{a,b}(x+h,\xi)-G_{a,b}(x,\xi)|\le \frac{C}{|x-\xi|^{n-2+\eta}}|h|^{\eta},\quad x\neq\xi,\ |h|\le|x-\xi|,$$

where $C=C(n,\lambda,\kappa)$ is a constant. In particular, $G_{a,b}(\cdot,\xi)\in C(\mathbb{R}^n\setminus\{\xi\})$. Similarly, using (2.8) instead of (2.7), we obtain $G_{a,b}(x,\cdot)\in C(\mathbb{R}^n\setminus\{x\})$. More specifically, we have

$$|G_{a,b}(x,\xi+h)-G_{a,b}(x,\xi)|\le \frac{C}{|x-\xi|^{n-2+\eta}}|h|^{\eta},\quad x\neq\xi,\ |h|\le|x-\xi|. \tag{2.14}$$

Let $\xi\in\mathbb{R}^n$ and $\omega\Subset\mathbb{R}^n\setminus\{\xi\}$, and pick $g\in C_0^\infty(\omega)$. Then, set

$$w_{a,b}(y)=\int_\omega G_{a,b}(x,y)g(x)dx,\quad y\in B(\xi,\mathrm{dist}(\xi,\overline{\omega})/2).$$

It follows from (2.14) that, for $y\in B(\xi,\mathrm{dist}(\xi,\overline{\omega}))$ and $|h|<\mathrm{dist}(y,\overline{\omega})$, we have

$$|w_{a,b}(y+h)-w_{a,b}(y)|\le \frac{C}{\mathrm{dist}(y,\overline{\omega})^{n-2+\eta}}|h|^{\eta}.$$

Therefore, $w_{a,b}\in C(B(\xi,\mathrm{dist}(\xi,\overline{\omega})/2)$.

Let $\mathcal{M}(\mathbb{R}^n)$ be the space of bounded measures on $\mathbb{R}^n$. Pick a sequence (f_k) of a positive functions of $C_0^\infty(\mathbb{R}^n)$ converging in $\mathcal{M}(\mathbb{R}^n)$ to δ_ξ and let $u_k=-A_{a,b}^{-1}f_k$. In that case, according to Fubini's theorem, we have

$$\begin{aligned}\int_\omega u_k(x)g(x)dx&=\int_\omega\int_{\mathbb{R}^n}G_{a,b}(x,y)g(x)f_k(y)dy\,dx\\&=\int_{\mathbb{R}^n}w_{a,b}(y)f_k(y)dy\longrightarrow w_{a,b}(\xi)=\int_\omega G_{a,b}(x,\xi)g(x)dx,\end{aligned}$$

where we used that $\mathrm{supp} f_k\subset B(\xi,\mathrm{dist}(\xi,\overline{\omega})/2)$, provided that k is sufficiently large. That is we proved that u_k converges to $G_{a,b}(\cdot,\xi)$ weakly in $L^2_{loc}(\mathbb{R}^n\setminus\{\xi\})$ (think to the fact that $C_0^\infty(\omega)$ is dense in $L^2(\omega)$).

Now, as $L_{a,b}u_k=f_k$, we find $L_{a,b}G_{a,b}(\cdot,\xi)=0$ in $\mathbb{R}^n\setminus\{\xi\}$ in the distributional sense.

The uniqueness of $G_{a,b}$ follows from that of u and, as $\kappa^{-1}\le b\le\kappa$, we deduce from Lemma 1 that

$$E_{a,\kappa}(x,t,\xi,0)\le E_{a,b}(x,t,\xi,0)\le E_{a,\kappa^{-1}}(x,t,\xi,0).$$

But a simple change of variable shows that

$$E_{a,\kappa^{-1}}(x,t,\xi,0)=e^{-\kappa^{-1}t}E_{a,0}(x,t,\xi,0) \tag{2.15}$$

and

$$E_{a,\kappa}(x,t,\xi,0)=e^{-\kappa t}E_{a,0}(x,t,\xi,0). \tag{2.16}$$

Therefore, from Theorem 2 and identity (2.3), there exists a constant $c=c(n,\lambda)>1$ so that

$$\begin{aligned}e^{-\kappa t}\frac{(\pi c^{-1})^{n/2+1}}{\pi}E_{c^{-1}/4,0}(x,t,\xi,0)&\le E_{a,b}(x,t,\xi,0)\\&\le e^{-\kappa^{-1}t}\frac{(\pi c)^{n/2+1}}{\pi}E_{c/4,0}(x,t,\xi,0),\end{aligned}$$

which, combined with identities (2.15) and (2.16), gives

$$\frac{(\pi c^{-1})^{n/2+1}}{\pi} E_{c^{-1}/4,\kappa}(x,t,\xi,0) \leq E_{a,b}(x,t,\xi,0) \leq \frac{(\pi c)^{n/2+1}}{\pi} E_{c/4,\kappa^{-1}}(x,t,\xi,0).$$

From the uniqueness of $G_{a,b}$, we obtain by integrating over $(0,+\infty)$, with respect to t, each member of the above inequalities

$$\frac{(\pi c^{-1})^{n/2+1}}{\pi} G_{c^{-1}/4,\kappa}(x,\xi) \leq G_{a,b}(x,\xi) \leq \frac{(\pi c)^{n/2+1}}{\pi} G_{c/4,\kappa^{-1}}(x,\xi).$$

These two-sided inequalities together with (2.10) yield in a straightforward manner (2.11). □

The function $G_{a,b}$ given by the previous theorem is usually called a fundamental solution of the operator $L_{a,b}$.

2.2 Regularity of fundamental solutions

Let $\xi \in \mathbb{R}^n$ and $\mathcal{O} \Subset \mathcal{O}' \Subset \mathbb{R}^n \setminus \{\xi\}$ with $\mathcal{O}'$ of class $C^{1,1}$. As $G_{a,b}(\cdot,\xi) \in C(\partial\mathcal{O}')$, we get from [17, Theorem 6.18, page 106] (interior Hölder regularity) that $G_{a,b}(\cdot,\xi)$ belongs to $C^{2,\alpha}(\overline{\mathcal{O}})$.

Proposition 1 *There exist $C = C(n,\lambda,\kappa,\alpha)$ and $\nu = \nu(\alpha) > 2$ so that, for any $\xi \in \mathbb{R}^n$ and $\mathcal{O} \Subset \mathbb{R}^n \setminus \{\xi\}$, we have*

$$\|G_{a,b}(\cdot,\xi)\|_{C^{2,\alpha}(\overline{\mathcal{O}})} \leq C\Lambda(\mathbf{d}+\varrho)^{\nu} \max\left(\varrho^{-(2+\alpha)},1\right)\varrho^{-n+2}. \tag{2.17}$$

Here, $\varrho = dist\left(\xi,\overline{\mathcal{O}}\right)$, $\mathbf{d} = diam(\mathcal{O})$ and

$$\Lambda(h) = [1+2h+2h^2+h^3]\lambda, \quad h>0.$$

The proof of this proposition is based the following lemma consisting in an adaptation of the usual interior Schauder estimates. The proof of this technical lemma will be given in "Appendix A" section.

Lemma 3 *There exist two constants $C = C(n,\alpha)$ and $\nu = \nu(\alpha) > 1$ with the property that, for any bounded subset $\mathcal{Q}$ of $\mathbb{R}^n$, $\delta > 0$ so that $\mathcal{Q}_\delta = \{x \in \mathcal{Q};\ dist(x,\partial\mathcal{Q}) > \delta\} \neq \emptyset$, $w \in C^{2,\alpha}(\mathcal{Q}) \cap C\left(\overline{\mathcal{Q}}\right)$ satisfying $L_{a,b}w = 0$ in $\mathcal{Q}$ and $\mathcal{Q}' \subset \mathcal{Q}_\delta$, we have*

$$\|w\|_{C^{2,\alpha}\left(\overline{\mathcal{Q}'}\right)} \leq C\max\left(\delta^{-(2+\alpha)},1\right)\Lambda(\mathbf{d})^{\nu}\|w\|_{C(\overline{\mathcal{Q}})}, \tag{2.18}$$

where Λ is as in Proposition 1 and $\mathbf{d} = diam(\mathcal{Q})$.

Proof of Proposition 1 We get, by applying Lemma 3 with $\mathcal{Q}' = \mathcal{O}$, $\delta = \varrho/2$ and $\mathcal{Q} = \left\{x \in \mathbb{R}^n;\ \text{dist}\left(x,\overline{\mathcal{O}}\right) < \varrho/2\right\}$,

$$\|G_{a,b}(\cdot,\xi)\|_{C^{2,\alpha}(\overline{\mathcal{O}})} \leq C\Lambda(\mathbf{d}+\varrho)^{\nu} \max\left(\delta^{-(2+\alpha)},1\right)\|G_{a,b}(\cdot,\xi)\|_{C(\overline{\mathcal{Q}})}.$$

This and (2.11) yield

$$\|G_{a,b}(\cdot,\xi)\|_{C^{2,\alpha}(\overline{\mathcal{O}})} \leq C\Lambda(\mathbf{d}+\varrho)^{\nu} \max\left(\delta^{-(2+\alpha)},1\right)\varrho^{-n+2}e^{-\varrho/\sqrt{c\kappa}}, \tag{2.19}$$

with $C = C(n,\lambda,\kappa,\alpha)$ and $c = c(n,\lambda)$. It is then clear that (2.19) implies (2.17). □

The preceding proposition together with Lemma 15 enables us to state the following corollary.

Corollary 1 *There exist $C = C(n, \lambda, \kappa, \alpha, \theta)$ and $\nu = \nu(\alpha) > 1$ so that, for any $\xi \in \mathbb{R}^n$ and $\mathcal{O} \Subset \mathbb{R}^n \setminus \{\xi\}$, we have*

$$\begin{aligned}&\|G_{a,b}(\cdot,\xi)\|_{H^{2+\theta}(\mathcal{O})}\\ &\quad\le C\Lambda(\mathbf{d}+\varrho)^{\nu}\max\left(\mathbf{d}^{n/2},\mathbf{d}^{n/2+\alpha-\theta}\right)\max\left(\varrho^{-(2+\alpha)},1\right)\varrho^{-n+2},\end{aligned}\tag{2.20}$$

where $\varrho = \mathrm{dist}\left(\xi, \overline{\mathcal{O}}\right)$, $\mathbf{d} = \mathrm{diam}(\mathcal{O})$.

Corollary 2 *There exist $C = C(n, \lambda, \kappa, \alpha)$ and $c = c(n, \lambda, \kappa, \alpha)$ so that, for any $\xi_1, \xi_2 \in \mathbb{R}^n$ and $\mathcal{O} \Subset \mathbb{R}^n \setminus \{\xi_1, \xi_2\}$, we have*

$$\left\|\frac{G_{a,b}(\cdot,\xi_2)}{G_{a,b}(\cdot,\xi_1)}\right\|_{C^{2,\alpha}(\overline{\mathcal{O}})} \le Ce^{c(\mathbf{d}+\varrho_+)}\left(1+\max\left(\varrho_-^{-(2+\alpha)},1\right)\varrho_-^{-n+2}\right)^4,\tag{2.21}$$

where $\varrho_- = \min\left(\mathrm{dist}\left(\xi_1, \mathcal{O}\right), \mathrm{dist}\left(\xi_2, \mathcal{O}\right)\right)$ and $\varrho_+ = \max\left(\mathrm{dist}\left(\xi_1, \mathcal{O}\right), \mathrm{dist}\left(\xi_2, \mathcal{O}\right)\right)$.

Proof In this proof $C = C(n, \lambda, \kappa, \alpha)$, $c = c(n, \lambda, \kappa, \alpha)$ and $\nu = \nu(\alpha) > 2$ are generic constants.

From Proposition 1, we have

$$\|G_{a,b}(\cdot,\xi_j)\|_{C^{2,\alpha}(\overline{\mathcal{O}})} \le C\Lambda(\mathbf{d}+\varrho_+)^{\nu}\max\left(\varrho_-^{-(2+\alpha)},1\right)\varrho_-^{-n+2},\quad j=1,2.\tag{2.22}$$

Let $C_0 \ge 1$ end $c_0 \ge 1$ be the constants in (2.11) and fix $0 < \delta_0 \le 1$. Then, the first inequality in (2.11) gives

$$\frac{1}{G_{a,b}(\cdot,\xi_1)} \le C_0\left(\mathbf{d}+\varrho_+\right)^{n-2}e^{2\sqrt{c_0\kappa}(\mathbf{d}+\varrho_+)}.$$

This inequality together with Lemma 14 in "Appendix A" yields

$$\left\|\frac{1}{G_{a,b}(\cdot,\xi_1)}\right\|_{C^{2,\alpha}(\overline{\mathcal{O}})} \le Ce^{c(\mathbf{d}+\varrho_+)}\left(1+\|G_{a,b}(\cdot,\xi_1)\|_{C^{2,\alpha}(\overline{\mathcal{O}})}\right)^3.\tag{2.23}$$

Then in light of (2.22) and (2.23), we get in a straightforward manner

$$\left\|\frac{G_{a,b}(\cdot,\xi_2)}{G_{a,b}(\cdot,\xi_1)}\right\|_{C^{2,\alpha}(\overline{\mathcal{O}})} \le Ce^{c(\mathbf{d}+\varrho_+)}\left(1+(1+\mathbf{d})^{\nu}\max\left(\varrho_-^{-(2+\alpha)},1\right)\varrho_-^{-n+2}\right)^4,$$

and hence

$$\left\|\frac{G_{a,b}(\cdot,\xi_2)}{G_{a,b}(\cdot,\xi_1)}\right\|_{C^{2,\alpha}(\overline{\mathcal{O}})} \le Ce^{c(\mathbf{d}+\varrho_+)}\left(1+\max\left(\varrho_-^{-(2+\alpha)},1\right)\varrho_-^{-n+2}\right)^4.$$

This is the expected inequality. □

This corollary combined with Lemma 15 yields the following result.

Corollary 3 *There exist $C = C(n, \lambda, \kappa, \alpha, \theta)$ and $c = c(n, \lambda, \kappa, \alpha, \theta)$ so that, for any $\xi_1, \xi_2 \in \mathbb{R}^n$ and $\mathcal{O} \Subset \mathbb{R}^n \setminus \{\xi_1, \xi_2\}$, we have*

$$\left\|\frac{G_{a,b}(\cdot,\xi_2)}{G_{a,b}(\cdot,\xi_1)}\right\|_{H^{2+\theta}(\mathcal{O})} \le Ce^{c(\mathbf{d}+\varrho_+)}\left(1+\max\left(\varrho_-^{-(2+\alpha)},1\right)\varrho_-^{-n+2}\right)^4.\tag{2.24}$$

Here, $\varrho_\pm$ is the same as in Corollary 2.

2.3 Gradient estimate of the quotient of two fundamental solutions

The following result uses the singularity of the Green function near the location of the point source.

Lemma 4 *There exist* $x^* \in B(\xi_2, |\xi_1-\xi_2|/2)\setminus\{\xi_2\}$, $C=(n,\lambda,\kappa,|\xi_1-\xi_2|)>0$ *and* $\rho=\rho(n,\lambda,\kappa,|\xi_1-\xi_2|)>0$ *so that* $\overline{B}(x^*,\rho)\subset B(\xi_2,|\xi_1-\xi_2|/2)\setminus\{\xi_2\}$ *and*

$$C \le \left\| \nabla\left(\frac{G_{a,b}(\cdot,\xi_2)}{G_{a,b}(\cdot,\xi_1)}\right)\right\|_{L^2(B(x^*,\rho))}.$$

Proof We set for notational convenience $w=G_{a,b}(\cdot,\xi_2)/G_{a,b}(\cdot,\xi_1)$. In light of Theorem 3, we obtain by straightforward computations the following two-sided inequality

$$\frac{C^{-1}}{|x-\xi_2|^{n-2}} \le w(x) \le \frac{C}{|x-\xi_2|^{n-2}},\quad x\in B(\xi_2,|\xi_1-\xi_2|/2)\setminus\{\xi_2\}. \tag{2.25}$$

Here and until the end of this proof $C=C(n,\lambda,\kappa,|\xi_1-\xi_2|)$ is a generic constant.

Set $\tilde t=|\xi_1-\xi_2|/4$ and define

$$\varphi(t,\theta)=w(\xi_2+t\theta),\quad (t,\theta)\in(0,\tilde t]\times\mathbb{S}^{n-1}.$$

According to Corollary 2, $\varphi\in C^{2,\alpha}_{\mathrm{loc}}((0,\tilde t]\times\mathbb{S}^{n-1})$ and hence

$$\varphi(\tilde t,\theta)-\varphi(t,\theta)=\int_t^{\tilde t}\nabla w(\xi_2+s\theta)\cdot\theta ds,$$

which in turn gives

$$\begin{aligned}|\varphi(\tilde t,\theta)-\varphi(t,\theta)|^2 &\le (\tilde t-t)\int_t^{\tilde t}\left|\nabla w(\xi_2+s\theta)\right|^2 ds\\ &\le \tilde t\int_t^{\tilde t}\left|\nabla w(\xi_2+s\theta)\right|^2 ds\\ &\le \tilde t\int_t^{\tilde t}\frac{s^{n-1}}{t^{n-1}}\left|\nabla w(\xi_2+s\theta)\right|^2 ds,\quad (t,\theta)\in(0,\tilde t]\times\mathbb{S}^{n-1}.\end{aligned}$$

Whence, where $t\in(0,\tilde t]$,

$$t^{n-1}\int_{\mathbb{S}^{n-1}}|\varphi(\tilde t,\theta)-\varphi(t,\theta)|^2 d\theta \le \tilde t\int_{\mathscr{C}_t}\left|\nabla w(x)\right|^2 dx. \tag{2.26}$$

Here,

$$\mathscr{C}_t=\left\{x\in\mathbb{R}^n;\ t<|x-\xi_2|<\tilde t\right\}.$$

On the other hand, inequalities (2.25) imply, where $(t,\theta)\in(0,\tilde t]\times\mathbb{S}^{n-1}$,

$$\frac{C^{-1}}{t^{n-2}}\le\varphi(t,\theta)\le\frac{C}{t^{n-2}}.$$

Let us then choose $t_0\le\tilde t$ sufficiently small in such a way that

$$\frac{C^{-1}}{t^{n-2}}-\frac{C}{\tilde t^{n-2}}>0,\quad t\in(0,t_0].$$

Therefore, for $(t,\theta)\in(0,t_0]\times\mathbb{S}^{n-1}$, we have

$$\left(\frac{C^{-1}}{t^{n-2}}-\frac{C}{\tilde t^{n-2}}\right)^2\le|\varphi(\tilde t,\theta)-\varphi(t,\theta)|^2. \tag{2.27}$$

We then obtain by combining inequalities (2.26) and (2.27)

$$|\mathbb{S}^{n-1}|\left(\frac{C^{-1}}{t^{n-2}}-\frac{C}{\tilde{t}^{n-2}}\right)^2 \le \tilde{t}\int_{\mathscr{C}_t}\left|\nabla w(x)\right|^2 dx, \quad t\in(0,t_0].$$

We have in particular

$$C\le \int_{\mathscr{C}_{t_0}}\left|\nabla w(x)\right|^2 dx.$$

Let $\rho = t_0/4$. Then, it is straightforward to check that, for any $x\in\overline{\mathscr{C}_{t_0}}$,

$$\overline{B}(x,\rho)\subset\{y\in\mathbb{R}^n;\ 3t_0/4\le|y-\xi_2|\le 5\tilde{t}/4\}\subset B(\xi_2,|\xi_1-\xi_2|/2)\setminus\{\xi_2\}.$$

Since $\overline{\mathscr{C}_{t_0}}$ is compact, we find a positive integer $N=N(\lambda,\kappa,|\xi_1-\xi_2|)$ and $x_j\in\overline{\mathscr{C}_{t_0}}$, $j=1,\cdots,N$, so that

$$\overline{\mathscr{C}_{t_0}}\subset\bigcup_{j=1}^N B(x_j,\rho).$$

Hence,

$$C\le\int_{\cup_{j=1}^N B(x_j,\rho)}\left|\nabla w(x)\right|^2 dx.$$

Pick then $x^*\in\{x_j,\ 1\le j\le N\}$ in such a way that

$$\int_{B(x^*,\rho)}\left|\nabla w(x)\right|^2 dx=\max_{1\le j\le N}\int_{B(x_j,\rho)}\left|\nabla w(x)\right|^2 dx.$$

Therefore,

$$C\le\int_{B(x^*,\rho)}\left|\nabla w(x)\right|^2 dx.$$

This finishes the proof. □

3 Uniform lower bound for the gradient

Let $\mathcal{O}$ be a Lipschitz bounded domain of $\mathbb{R}^n$ and $\sigma\in C^{0,1}(\overline{\mathcal{O}})$ satisfying

$$\varkappa^{-1}\le\sigma \quad\text{and}\quad \|\sigma\|_{C^{0,1}(\overline{\mathcal{O}})}\le\varkappa, \tag{3.1}$$

for some fixed constant $\varkappa>1$.

We prove in this section a polynomial lower bound of the local L^2-norm of the gradient of solutions of

$$L_\sigma u=\mathrm{div}(\sigma\nabla u)=0\quad\text{in }\mathcal{O}.$$

In a first step, we establish, via a three-ball inequality for the gradient, a uniform lower bound of negative exponential type. We use then in a second step an argument based on the so-called frequency function in order to improve this lower bound.

3.1 Preliminary lower bound

We need hereafter the following three-ball inequality for the gradient.

Theorem 4 *Let $0<k<\ell<m$ be real. There exist two constants $C=C(n,\varkappa,k,\ell,m)>0$ and $0<\gamma=\gamma(n,\varkappa,k,\ell,m)<1$ so that, for any $v\in H^1(\mathcal{O})$ satisfying $L_\sigma v=0$, $y\in\mathcal{O}$ and $0<r<dist(y,\partial\mathcal{O})/m$, we have*

$$C\|\nabla v\|_{L^2(B(y,\ell r))}\le\|\nabla v\|^{\gamma}_{L^2(B(y,kr))}\|\nabla v\|^{1-\gamma}_{L^2(B(y,mr))}.$$

A proof of this theorem can be found in [9] or [10].

Define the geometric distance d_g^D on the bounded domain D of $\mathbb{R}^n$ by

$$d_g^D(x,y) = \inf\left\{\ell(\psi);\ \psi : [0,1] \to D \text{ Lipschitz path joining } x \text{ to } y\right\},$$

where

$$\ell(\psi) = \int_0^1 |\dot\psi(t)|dt$$

is the length of ψ.

Note that according to Rademacher's theorem any Lipschitz continuous function $\psi : [0,1] \to D$ is almost everywhere differentiable with $|\dot\psi(t)| \le k$ a.e. $t \in [0,1]$, where k is the Lipschitz constant of ψ.

Lemma 5 *Let D be a bounded Lipschitz domain of $\mathbb{R}^n$. Then, $d_g^D \in L^\infty(D\times D)$ and there exists a constant $\mathfrak{c}_D > 0$ so that*

$$|x-y| \le d_g^D(x,y) \le \mathfrak{c}_D|x-y|, \quad x,y \in D. \tag{3.2}$$

We refer to [23, Lemma A3] for a proof.

In this subsection, we use the following notations

$$\mathcal{O}^\delta = \{x \in \mathcal{O};\ \mathrm{dist}(x, \partial\mathcal{O}) > \delta\}$$

and

$$\chi(\mathcal{O}) = \sup\{\delta > 0;\ \mathcal{O}^\delta \ne \emptyset\}.$$

Define

$$\begin{aligned}\mathscr{S}(\mathcal{O}, x_0, M, \eta, \delta) = \big\{&u \in H^1(\mathcal{O});\ L_\sigma u = 0 \text{ in } \mathcal{O},\\ &\|\nabla u\|_{L^2(\mathcal{O})} \le M,\ \|\nabla u\|_{L^2(B(x_0,\delta))} \ge \eta\big\},\end{aligned} \tag{3.3}$$

with $\delta \in (0, \chi(\mathcal{O})/3)$, $x_0 \in \mathcal{O}^{3\delta}$, $\eta > 0$ and $M \ge 1$ satisfying $\eta < M$.

Lemma 6 *There exist two constants $c = c(n,\varkappa) \ge 1$ and $0 < \gamma = \gamma(n,\varkappa) < 1$ so that, for any $u \in \mathscr{S}(\mathcal{O}, x_0, M, \eta, \delta)$ and $x \in \mathcal{O}^{3\delta}$, we have*

$$e^{-[\ln(cM/\eta)/\gamma]e^{[2n|\ln\gamma|]\mathfrak{c}|x-x_0|/\delta}} \le \|\nabla u\|_{L^2(B(x,\delta))}, \tag{3.4}$$

where $\mathfrak{c} = \mathfrak{c}_\mathcal{O}$ is as in Lemma 5.

Proof Pick $u \in \mathscr{S}(\mathcal{O}, x_0, M, \eta, \delta)$. Let $x \in \mathcal{O}^{3\delta}$ and $\psi : [0,1] \to \mathcal{O}$ be a Lipschitz path joining $x = \psi(0)$ to $x_0 = \psi(1)$, so that $\ell(\psi) \le 2d_g(x_0,x)$. Here and henceforth, for simplicity convenience, we use $d_g(x_0,x)$ instead of $d_g^\mathcal{O}(x_0,x)$.

Let $t_0 = 0$ and $t_{k+1} = \inf\{t \in [t_k,1];\ \psi(t) \notin B(\psi(t_k),\delta)\}$, $k \ge 0$. We claim that there exists an integer $N \ge 1$ verifying $\psi(1) \in B(\psi(t_N),\delta)$. If not, we would have $\psi(1) \notin B(\psi(t_k),\delta)$ for any $k \ge 0$. As the sequence (t_k) is non-decreasing and bounded from above by 1, it converges to $\hat t \le 1$. In particular, there exists an integer $k_0 \ge 1$ so that $\psi(t_k) \in B\left(\psi(\hat t), \delta/2\right)$, $k \ge k_0$. But this contradicts the fact that $\left|\psi(t_{k+1}) - \psi(t_k)\right| \ge \delta$, $k \ge 0$.

Let us check that $N \le N_0$, where $N_0 = N_0(n, |x-x_0|, \mathfrak{c}, \delta)$. Pick $1 \le j \le n$ so that

$$\max_{1\le i\le n}\left|\psi_i(t_{k+1}) - \psi_i(t_k)\right| = \left|\psi_j(t_{k+1}) - \psi_j(t_k)\right|,$$

where ψ_i is the ith component of ψ. Then,

$$\delta \le n\left|\psi_j(t_{k+1})-\psi_j(t_k)\right| = n\left|\int_{t_k}^{t_{k+1}}\dot{\psi}_j(t)dt\right| \le n\int_{t_k}^{t_{k+1}}|\dot{\psi}(t)|dt.$$

Consequently, where $t_{N+1}=1$,

$$(N+1)\delta \le n\sum_{k=0}^{N}\int_{t_k}^{t_{k+1}}|\dot{\psi}(t)|dt = n\ell(\psi) \le 2nd_g(x_0,x) \le 2n\mathfrak{c}|x-x_0|.$$

Therefore,

$$N \le N_0 = \left[\frac{2n\mathfrak{c}|x-x_0|}{\delta}\right].$$

Let $y_0=x$ and $y_k=\psi(t_k)$, $1\le k\le N$. If $|z-y_{k+1}|<\delta$, then $|z-y_k|\le|z-y_{k+1}|+|y_{k+1}-y_k|<2\delta$. In other words, $B(y_{k+1},\delta)\subset B(y_k,2\delta)$. We get from Theorem 4

$$\|\nabla u\|_{L^2(B(y_j,2\delta))} \le C\|\nabla u\|^{1-\gamma}_{L^2(B(y_j,3\delta))}\|\nabla u\|^{\gamma}_{L^2(B(y_j,\delta))},\quad 0\le j\le N, \tag{3.5}$$

for some constants $C=C(n,\varkappa)>0$ and $0<\gamma=\gamma(n,\varkappa)<1$.

Set $I_j=\|\nabla u\|_{L^2(B(y_j,\delta))}$, $0\le j\le N$ and $I_{N+1}=\|\nabla u\|_{L^2(B(x_0,\delta))}$. Since $B(y_{j+1},\delta)\subset B(y_j,2\delta)$, $1\le j\le N-1$, estimate (3.5) implies

$$I_{j+1} \le CM^{1-\gamma}I_j^{\gamma},\quad 0\le j\le N. \tag{3.6}$$

Let $C_1=C^{1+\gamma+\dots+\gamma^{N+1}}$ and $\beta=\gamma^{N+1}$. Then, by a simple induction argument, estimate (3.6) yields

$$I_{N+1} \le C_1M^{1-\beta}I_0^{\beta}. \tag{3.7}$$

Without loss of generality, we assume in the sequel that $C\ge 1$ in (3.6). Using that $N\le N_0$, we have

$$\beta \ge \beta_0 = \gamma^{N_0+1},$$
$$C_1 \le C^{\frac{1}{1-\gamma}},$$
$$\left(\frac{I_0}{M}\right)^{\beta} \le \left(\frac{I_0}{M}\right)^{\beta_0}.$$

These estimates in (3.7) give

$$\frac{I_{N+1}}{M} \le C^{\frac{1}{1-\gamma}}\left(\frac{I_0}{M}\right)^{\gamma^{N_0+1}},$$

from which we deduce that

$$\|\nabla u\|_{L^2(B(x_0,\delta))} \le C^{\frac{1}{1-\gamma}}M^{1-\gamma^{N_0+1}}\|\nabla u\|^{\gamma^{N_0+1}}_{L^2(B(x,\delta))}.$$

But $M\ge 1$. Whence

$$\eta \le \|\nabla u\|_{L^2(B(x_0,\delta))} \le C^{\frac{1}{1-\gamma}}M\|\nabla u\|^{\gamma^{N_0+1}}_{L^2(B(x,\delta))}.$$

The expected inequality follows readily from this last estimate. □

3.2 An estimate for the frequency function

Some tools in the present section are borrowed from [15,16,19]. Let $u \in H^1(\mathcal{O})$ and $\sigma \in C^{0,1}(\overline{\mathcal{O}})$ satisfying the bounds (3.1). We recall that the usual frequency function, relative to the operator L_σ, associated with u is defined by

$$N(u)(x_0, r) = \frac{rD(u)(x_0, r)}{H(u)(x_0, r)},$$

provided that $B(x_0, r) \Subset \mathcal{O}$, with

$$D(u)(x_0, r) = \int_{B(x_0,r)} \sigma(x)|\nabla u(x)|^2 dx,$$
$$H(u)(x_0, r) = \int_{\partial B(x_0,r)} \sigma(x)u^2(x)dS(x).$$

Define also

$$K(u)(x_0, r) = \int_{B(x_0,r)} \sigma(x)u^2(x)dx.$$

Prior to studying the properties of the frequency function, we prove some preliminary results. Fix $u \in H^2(\mathcal{O})$ so that $L_\sigma u = 0$ in $\mathcal{O}$ and, for simplicity convenience, we drop in the sequel the dependence on u of N, D, H and K.

Lemma 7 *For $x_0 \in \mathcal{O}^\delta$ and $0 < r < \delta$, we have*

$$\partial_r H(x_0, r) = \frac{n-1}{r} H(x_0, r) + \tilde{H}(x_0, r) + 2D(x_0, r), \tag{3.8}$$
$$\partial_r D(x_0, r) = \frac{n-2}{r} D(x_0, r) + \frac{1}{r}\tilde{D}(x_0, r) + 2\hat{H}(x_0, r). \tag{3.9}$$

Here,

$$\tilde{H}(x_0, r) = \int_{\partial B(x_0,r)} u^2 \nabla\sigma(x) \cdot \nu(x) dS(x),$$
$$\hat{H}(x_0, r) = \int_{\partial B(x_0,r)} \sigma(x)(\partial_\nu u(x))^2 dS(x),$$
$$\tilde{D}(x_0, r) = \int_{B(x_0,r)} |\nabla u(x)|^2 \nabla\sigma(x) \cdot (x - x_0) dx.$$

Proof Pick $x_0 \in \mathcal{O}^\delta$ and $0 < r < \delta$. A simple change of variable yields

$$H(x_0, r) = \int_{\partial B(0,1)} \sigma(x_0 + ry)u^2(x_0 + ry)r^{n-1} dS(y).$$

Hence,

$$\begin{aligned}\partial_r H(x_0,r) &= \frac{n-1}{r}H(x_0,r) + \int_{\partial B(0,1)} \nabla(\sigma u^2)(x_0+ry)\cdot y r^{n-1} dS(y)\\ &= \frac{n-1}{r}H(x_0,r) + \int_{\partial B(0,1)} u^2(x_0+ry)\nabla\sigma(x_0+ry)\cdot y r^{n-1} dS(y)\\ &\quad + \int_{\partial B(0,1)} \sigma(x_0+ry)\nabla(u^2)(x_0+ry)\cdot y r^{n-1} dS(y)\\ &= \frac{n-1}{r}H(x_0,r) + \int_{\partial B(x_0,r)} u^2(x)\nabla\sigma(x)\cdot \nu(x) dS(x)\\ &\quad + \int_{\partial B(x_0,r)} \sigma(x)\nabla(u^2)(x)\cdot \nu(x) dS(x)\\ &= \frac{n-1}{r}H(x_0,r) + \tilde{H}(x_0,r) + \int_{\partial B(x_0,r)} \sigma(x)\nabla(u^2)(x)\cdot \nu(x) dS(x).\end{aligned}$$

Identity (3.8) will follow if we prove

$$2D(x_0,r) = \int_{\partial B(x_0,r)} \sigma(x)\nabla(u^2)(x)\cdot\nu(x)dS(x). \tag{3.10}$$

To this end, we observe that $\mathrm{div}(\sigma\nabla u)=0$ implies

$$\mathrm{div}(\sigma\nabla(u^2)) = 2u\,\mathrm{div}(\sigma\nabla u) + 2\sigma|\nabla u|^2 = 2\sigma|\nabla u|^2.$$

We then get by applying the divergence theorem

$$\begin{aligned}2D(x_0,r) &= \int_{B(x_0,r)} \mathrm{div}(\sigma(x)\nabla(u^2)(x))dx \\ &= \int_{\partial B(x_0,r)} \sigma(x)\nabla(u^2)(x)\cdot\nu(x)dS(x).\end{aligned} \tag{3.11}$$

This proves (3.10).

By a change of variable, we have

$$D(x_0,r) = \int_0^r \int_{\partial B(0,1)} \sigma(x_0+ty)|\nabla u(x_0+ty)|^2 t^{n-1} dS(y)\, dt.$$

Hence,

$$\begin{aligned}\partial_r D(x_0,r) &= \int_{\partial B(0,1)} \sigma(x_0+ry)|\nabla u(x_0+ry)|^2 r^{n-1} dS(y)\\ &= \int_{\partial B(x_0,r)} \sigma(x)|\nabla u(x)|^2 dS(x)\\ &= \frac{1}{r}\int_{\partial B(x_0,r)} \sigma(x)|\nabla u(x)|^2 (x-x_0)\cdot\nu(x)dS(x).\end{aligned}$$

An application of the divergence theorem then gives

$$\partial_r D(x_0,r) = \frac{1}{r}\int_{B(x_0,r)} \mathrm{div}(\sigma(x)|\nabla u(x)|^2(x-x_0))dx.$$

Therefore,

$$\begin{aligned}\partial_r D(x_0,r) &= \frac{1}{r}\int_{B(x_0,r)} |\nabla u(x)|^2 \mathrm{div}(\sigma(x)(x-x_0))dx\\ &\quad + \frac{1}{r}\int_{B(x_0,r)} \sigma(x)(x-x_0)\cdot\nabla(|\nabla u(x)|^2)dx\end{aligned}$$

implying

$$\partial_r D(x_0, r) = \frac{n}{r} D(x_0, r) + \frac{1}{r}\tilde{D}(x_0, r) + \frac{1}{r}\int_{B(x_0,r)} \sigma(x)(x - x_0)\cdot\nabla(|\nabla u(x)|^2)dx. \tag{3.12}$$

On the other hand,

$$\begin{aligned}\int_{B(x_0,r)} \sigma(x)(x_j - x_{0,j})\partial_j(\partial_i u(x))^2 dx &= 2\int_{B(x_0,r)} \sigma(x)(x_j - x_{0,j})\partial^2_{ij} u\partial_i u(x)dx\\ &= -2\int_{B(x_0,r)} \partial_i\left[\partial_i u(x)\sigma(x)(x_j - x_{0,j})\right]\partial_j u(x)dx\\ &\quad + 2\int_{\partial B(x_0,r)} \sigma(x)\partial_i u(x)(x_j - x_{0,j})\partial_j u(x)\nu_i(x)dS(x)\\ &= -2\int_{B(x_0,r)} \partial^2_{ii} u(x)\sigma(x)(x_j - x_{0,j})\partial_j u(x)dx\\ &\quad - 2\int_{B(x_0,r)} \partial_i u(x)\partial_j u(x)\partial_i\left[\sigma(x)(x_j - x_{0,j})\right]dx\\ &\quad + 2\int_{\partial B(x_0,r)} \sigma(x)\partial_i u(x)(x_j - x_{0,j})\partial_j u(x)\nu_i(x)dS(x).\end{aligned}$$

Thus, taking into account that $\sigma\Delta u = -\nabla\sigma\cdot\nabla u$,

$$\begin{aligned}\int_{B(x_0,r)} \sigma(x)(x - x_0)\cdot\nabla(|\nabla u(x)|^2)dx &= -2\int_{B(x_0,r)} \sigma(x)|\nabla u(x)|^2 dx\\ &\quad + 2r\int_{\partial B(x_0,r)} \sigma(x)(\partial_\nu u(x))^2 dS(x).\end{aligned}$$

This identity in (3.12) yields

$$\partial_r D(x_0, r) = \frac{n-2}{r} D(x_0, r) + \frac{1}{r}\tilde{D}(x_0, r) + 2\hat{H}(x_0, r).$$

That is we proved (3.9). □

Lemma 8 *We have*

$$K(x_0, r) \le re^{r\varkappa^2} H(x_0, r), \quad x_0 \in \mathcal{O}^\delta,\ 0 < r < \delta.$$

Proof Taking into account that $H(x_0, r) \ge 0$ and $D(x_0, r) \ge 0$, we obtain from identity (3.8)

$$\begin{aligned}\partial_r H(x_0, r) &\ge \int_{\partial B(x_0,r)} \partial_\nu\sigma(x)u^2(x)dS(x)\\ &\ge \int_{\partial B(x_0,r)} \frac{\partial_\nu\sigma(x)}{\sigma(x)}\sigma(x)u^2(x)dS(x) \ge -\varkappa^2 H(x_0, r).\end{aligned}$$

Consequently, $r \to e^{r\varkappa^2} H(x_0, r)$ is non-decreasing and then

$$\begin{aligned}\int_0^r H(x_0, t)dt &\le \int_0^r e^{t\varkappa^2} H(x_0, t)dt\\ &\le \int_0^r e^{r\varkappa^2} H(x_0, r)dt \le re^{r\varkappa^2} H(x_0, r).\end{aligned}$$

As

$$K(x_0, r) = \int_0^r H(x_0, t)dt,$$

We end up getting

$$K(x_0, r) \leq re^{r\varkappa^2} H(x_0, r).$$

This completes the proof. □

Now, straightforward computations yield, for $x_0 \in \mathcal{O}^\delta$ and $0 < r < \delta$,

$$\frac{\partial_r N(x_0, r)}{N(x_0, r)} = \frac{1}{r} + \frac{\partial_r D(x_0, r)}{D(x_0, r)} - \frac{\partial_r H(x_0, r)}{H(x_0, r)}. \tag{3.13}$$

Lemma 9 *For $x_0 \in \mathcal{O}^\delta$ and $0 < r < \delta$, we have*

$$N(x_0, r) \leq e^{2\varkappa^2 \delta} N(x_0, \delta).$$

Proof We have from formulas (3.8) and (3.9) and identity (3.13)

$$\begin{aligned}
\frac{\partial_r N(x_0, r)}{N(x_0, r)} &= \frac{\tilde{D}(x_0, r)}{rD(x_0, r)} - \frac{\tilde{H}(x_0, r)}{H(x_0, r)} + 2\frac{\hat{H}(x_0, r)}{D(x_0, r)} - 2\frac{D(x_0, r)}{H(x_0, r)} \\
&= \frac{\tilde{D}(x_0, r)}{rD(x_0, r)} - \frac{\tilde{H}(x_0, r)}{H(x_0, r)} + 2\frac{\hat{H}(x_0, r)H(x_0, r) - D(x_0, r)^2}{D(x_0, r)H(x_0, r)}.
\end{aligned} \tag{3.14}$$

But from (3.11), we have

$$D(x_0, r) = \int_{\partial B(x_0, r)} \sigma(x)u(x)\partial_\nu u(x)dS(x).$$

Then, we find by applying Cauchy–Schwarz's inequality

$$D(x_0, r)^2 \leq \left(\int_{\partial B(x_0, r)} \sigma(x)u^2(x)dS(x)\right)\left(\int_{\partial B(x_0, r)} \sigma(x)(\partial_\nu u)^2(x)dS(x)\right).$$

That is

$$D^2(x_0, r) \leq H(x_0, r)\hat{H}(x_0, r). \tag{3.15}$$

This and (3.14) lead

$$\frac{\partial_r N(x_0, r)}{N(x_0, r)} \geq \frac{\tilde{D}(x_0, r)}{rD(x_0, r)} - \frac{\tilde{H}(x_0, r)}{H(x_0, r)}. \tag{3.16}$$

On the other hand

$$\left|\tilde{H}(x_0, r)\right| \leq \varkappa \|\nabla a\|_\infty H(x_0, r) \leq \varkappa^2 H(x_0, r), \tag{3.17}$$

and similarly

$$\left|\tilde{D}(x_0, r)\right| \leq \varkappa^2 r D(x_0, r). \tag{3.18}$$

In light of (3.16), (3.17) and (3.18), we derive

$$\frac{\partial_r N(x_0, r)}{N(x_0, r)} \geq -2\varkappa^2,$$

that is to say

$$\partial_r (e^{2\varkappa^2 r} N(x_0, r)) \geq 0.$$

Consequently,

$$N(x_0, r) \leq e^{2\varkappa^2(\delta - r)} N(x_0, \delta) \leq e^{2\varkappa^2 \delta} N(x_0, \delta),$$

as expected. □

3.3 Polynomial lower bound

Lemma 10 *There exist a universal constant* ϖ *and two constants* $c = c(n,\varkappa) > 0$ *and* $0 < \gamma = \gamma(n,\varkappa) < 1$ *so that if*

$$\mathcal{C}_0(h) = M\varpi\varkappa^4(1+\mathbf{d})\delta^{-1}e^{3\varkappa^2\delta+[2\ln(cM/\eta)/\gamma]e^{[6n|\ln\gamma|]\mathfrak{c}h}}, \quad h>0,$$

then

$$\|N(u)(x,\cdot)\|_{L^\infty(0,\delta)} \le \mathcal{C}_0(|x-x_0|/\delta),$$

for any $u \in \mathscr{S}(\mathcal{O}, x_0, M, \eta, \delta/3)$, *where* $\mathfrak{c} = \mathfrak{c}_{\mathcal{O}}$ *is as in Lemma 5.*

Proof Pick $x \in \mathcal{O}^\delta$. Then, from Lemma 6

$$\|\nabla u\|_{L^2(B(x,\delta/3))} \ge e^{-[\ln(cM/\eta)/\gamma]e^{[6n|\ln\gamma|]\mathfrak{c}|x-x_0|/\delta}},$$

for some constant $c = c(n,\varkappa)$ and $0<\gamma=\gamma(n,\varkappa))<1$.

On the other hand, we establish in a quite classical manner the following Caccioppoli's inequality

$$\|\nabla u\|^2_{L^2(B(x,\delta/3))} \le \frac{\varpi\varkappa^2(1+\mathbf{d})}{\delta^2}\|u\|^2_{L^2(B(x,\delta))},$$

where ϖ is a universal constant. Therefore,

$$\|u\|^2_{L^2(B(x,\delta))} \ge \tilde{\mathcal{C}}_0(|x-x_0|/\delta), \tag{3.19}$$

where

$$\tilde{\mathcal{C}}_0(h) = \frac{\delta^2}{\varpi\varkappa^2(1+\mathbf{d})}e^{-[2\ln(cM/\eta)/\gamma]e^{[6n|\ln\gamma|\mathfrak{c}]h}}, \quad h>0. \tag{3.20}$$

Since $K(u)(x,\delta) \ge \varkappa^{-1}\|u\|^2_{L^2(B(x,\delta))}$, we find

$$K(u)(x,\delta) \ge \frac{\delta^2}{\varpi\varkappa^3(1+\mathbf{d})}e^{-[2\ln(cM/\eta)/\gamma]e^{[6n|\ln\gamma|]\mathfrak{c}|x-x_0|/\delta}}. \tag{3.21}$$

In light of Lemma 8, we derive from (3.21)

$$H(u)(x,\delta) \ge \frac{\delta e^{-\varkappa^2\delta}}{\varpi\varkappa^3(1+\mathbf{d})}e^{-[2\ln(cM/\eta)/\gamma]e^{[6n|\ln\gamma|]\mathfrak{c}|x-x_0|/\delta}}. \tag{3.22}$$

In light of Lemma 9, we get

$$N(x,r) \le \varkappa e^{2\varkappa^2\delta}\frac{\|\nabla u\|_{L^2(\mathcal{O})}}{H(u)(x,\delta)}, \quad 0<r<\delta,$$

This inequality and (3.22) give, where $c = c(n,\varkappa)$ is a constant,

$$N(x,r) \le M\varpi\varkappa^4(1+\mathbf{d})\delta^{-1}e^{3\varkappa^2\delta+[2\ln(cM/\eta)/\gamma]e^{[6n|\ln\gamma|]\mathfrak{c}|x-x_0|/\delta}}, \quad 0<r<\delta,$$

which is the expected inequality. □

Proposition 2 *Let* $\mathcal{C}_0$ *be as in Lemma 10,* $\tilde{\mathcal{C}}_0$ *as in (3.20) and set*

$$\mathcal{C}_1(h) = 2\mathcal{C}_0(h) + n, \quad h>0, \tag{3.23}$$

$$\tilde{\mathcal{C}}_2(h) = \varkappa^{-2}e^{-\varkappa^2\delta}\tilde{\mathcal{C}}_0(h), \quad h>0. \tag{3.24}$$

If $u \in \mathscr{S}(\mathcal{O}, x_0, M, \eta, \delta/3)$, *then*

$$\tilde{\mathcal{C}}_2(|x-x_0|/\delta)\left(\frac{r}{\delta}\right)^{\mathcal{C}_1(|x-x_0|/\delta)} \le \|u\|^2_{L^2(B(x,r))}, \quad x\in\mathcal{O}^\delta,\ 0<r<\delta.$$

Proof Observing that, where $H = H(u)$,

$$\partial_r\left(\ln \frac{H(x,r)}{r^{n-1}}\right) = \frac{\partial_r H(x,r)}{H(x,r)} - \frac{n-1}{r},$$

We get from Lemma 10, (3.8) and the fact that $|\tilde{H}(x,r)| \le \varkappa^2 H(x,r)$,

$$\partial_r\left(\ln \frac{H(x,r)}{r^{n-1}}\right) \le \varkappa^2 + \frac{2N(x,r)}{r} \le \varkappa^2 + \frac{2\mathcal{C}_0(|x-x_0|/\delta)}{r}, \quad 0 < r < \delta,$$

Thus,

$$\int_{sr}^{s\delta} \partial_t\left(\ln \frac{H(x,t)}{t^{n-1}}\right) dt = \ln \frac{H(x,s\delta)r^{n-1}}{H(x,sr)\delta^{n-1}} \le \varkappa^2(\delta - r)s + 2\mathcal{C}_0(|x-x_0|/\delta)\ln\frac{\delta}{r},$$

for $0 < s < 1$ and $0 < r < \delta$. Hence,

$$H(x,s\delta) \le e^{\varkappa^2\delta}\left(\frac{\delta}{r}\right)^{\mathcal{C}_1(|x-x_0|/\delta)-1} H(x,sr),$$

and then

$$\begin{aligned}
\|u\|^2_{L^2(B(x,\delta))} &\le \varkappa\delta\int_0^1 H(x,s\delta)ds \\
&\le \varkappa\delta e^{\varkappa^2\delta}\left(\frac{\delta}{r}\right)^{\mathcal{C}_1(|x-x_0|/\delta)-1}\int_0^1 H(x,rs)ds \\
&\le \varkappa^2 e^{\varkappa^2\delta}\left(\frac{\delta}{r}\right)^{\mathcal{C}_1(|x-x_0|/\delta)} \|u\|^2_{L^2(B(x,r))}.
\end{aligned}$$

Combined with (3.19), this estimate yields in a straightforward manner

$$\varkappa^{-2}e^{-\varkappa^2\delta}\tilde{\mathcal{C}}_0(|x-x_0|/\delta)\left(\frac{r}{\delta}\right)^{\mathcal{C}_1(|x-x_0|/\delta)} \le \|u\|^2_{L^2(B(x,r))}.$$

This is the expected inequality. □

For a bounded domain D, we denote the first nonzero eigenvalue of the Laplace–Neumann operator on D by $\mu_2(D)$. Since $\mu_2(B(x_0,r)) = \mu_2(B(0,1))/r^2$, we get by applying Poincaré–Wirtinger's inequality

$$\begin{aligned}
\|w - \{w\}\|^2_{L^2(B(x,r))} &\le \frac{1}{\mu_2(B(x,r))}\|\nabla w\|^2_{L^2(B(x,r))} \\
&\le \frac{r^2}{\mu_2(B(0,1))}\|\nabla w\|^2_{L^2(B(x,r))},
\end{aligned} \tag{3.25}$$

for any $w \in H^1(B(x,r))$, where $\{w\} = \frac{1}{|B(x,r)|}\int_{B(x,r)} w(x)dx$.

Noting that $\mathscr{S}(\mathcal{O}, x_0, M, \eta, \delta/3)$ is invariant under the transformation $u \to u - \{u\}$, we can state the following consequence of Proposition 2

Corollary 4 *With the notations of Proposition 2, if $u \in \mathscr{S}(\mathcal{O}, x_0, M, \eta, \delta/3)$, then*

$$\mathcal{C}_2(|x-x_0|/\delta)\left(\frac{r}{\delta}\right)^{\mathcal{C}_1(|x-x_0|/\delta)} \le \|\nabla u\|^2_{L^2(B(x,r))}, \quad x \in \mathcal{O}^\delta,\ 0 < r < \delta,$$

with

$$\mathcal{C}_2(h) = \mu_2(B(0,1))\delta^{-2}\tilde{\mathcal{C}}_2(h), \quad h > 0, \tag{3.26}$$

with $\tilde{\mathcal{C}}_2$ as in Proposition 2.

It is important to remark that the argument we used to obtain Corollary 4 from Proposition 2 is no longer valid if we substitute L_σ by L_σ plus a multiplication operator by a function σ_0.

The following consequence of the preceding corollary will be useful in the proof of Theorem 1.

Lemma 11 *Let $\omega \Subset \mathcal{O}$ and set $\delta = dist(\omega, \partial\mathcal{O})$. Let $u \in \mathscr{S}(\mathcal{O}, x_0, M, \eta, \delta/3)$ and $f \in C^{0,\alpha}(\overline{\mathcal{O}})$. Then, we have*

$$\|f\|_{L^\infty(\omega)} \le \hat{\mathcal{C}}_3 \|f\|^{1-\hat{\mu}}_{C^{0,\alpha}(\overline{\mathcal{O}})} \|f|\nabla u|^2\|^{\hat{\mu}}_{L^1(\mathcal{O})}, \tag{3.27}$$

with

$$\hat{\mu} = \frac{\alpha}{\max_{x\in\overline{\mathcal{O}}} \mathcal{C}_1(|x-x_0|/\delta) + \alpha},$$
$$\hat{\mathcal{C}}_3 = \max\left(2\delta^\alpha(\max\left(1, (\hat{\mathcal{C}}_2\delta^\alpha)^{-1}\right), \max\left(1, M^2\right)(\hat{\mathcal{C}}_2\delta^\alpha)^{-1}\right),$$

where $\hat{\mathcal{C}}_2 = \max_{x\in\overline{\mathcal{O}}} \mathcal{C}_2(|x-x_0|/\delta)$ with $\mathcal{C}_2$ being as in Corollary 4.

Proof By homogeneity, it is enough to consider those functions $f \in C^{0,\alpha}(\overline{\mathcal{O}})$ satisfying $\|f\|_{C^{0,\alpha}(\overline{\mathcal{O}})} = 1$. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be, respectively, as in (3.23) and (3.26). Let $u \in \mathscr{S}(\mathcal{O}, x_0, M, \eta, \delta/3)$ and $f \in C^{0,\alpha}(\overline{\mathcal{O}})$ satisfying $\|f\|_{C^{0,\alpha}(\overline{\mathcal{O}})} = 1$. Pick then $x \in \overline{\omega}$. From Corollary 4, we have

$$\mathcal{C}_2(|x-x_0|/\delta)\left(\frac{r}{\delta}\right)^{\mathcal{C}_1(|x-x_0|/\delta)} \le \|\nabla u\|^2_{L^2(B(x,r))}, \quad 0 < r < \delta. \tag{3.28}$$

On the other hand, it is straightforward to check that

$$|f(x)| \le |f(y)| + r^\alpha, \quad y \in B(x,r).$$

Whence

$$|f(x)| \int_{B(x,r)} |\nabla u(y)|^2 dy \le \int_{B(x,r)} |f(y)||\nabla u(y)|^2 dy + r^\alpha \int_{B(x,r)} |\nabla u(y)|^2 dy.$$

That is we have

$$|f(x)|\|\nabla u\|^2_{L^2(B(x,r)} \le \|f|\nabla u|^2\|_{L^1(B(x,r))} + r^\alpha \|\nabla u\|^2_{L^2(B(x,r))}.$$

Since u is non-constant, by the unique continuation property, we have $\|\nabla u\|^2_{L^2(B(x,r))} \ne 0$, $0 < r < \delta$. Therefore,

$$|f(x)| \le \frac{\|f|\nabla u|^2\|_{L^1(B(x,r))}}{\|\nabla u\|^2_{L^2(B(x,r))}} + r^\alpha, \quad 0 < r < \delta.$$

This and (3.28) entail

$$|f(x)| \le \mathcal{C}_2(|x-x_0|/\delta)^{-1}\left(\frac{\delta}{r}\right)^{\mathcal{C}_1(|x-x_0|)} \|f|\nabla u|^2\|_{L^1(B(x,r))} + r^\alpha, \quad 0 < r < \delta.$$

Hence,

$$|f(x)| \le \mathcal{C}_2(|x-x_0|/\delta)^{-1}\left(\frac{1}{s}\right)^{\mathcal{C}_1(|x-x_0|)} \|f|\nabla u|^2\|_{L^1(\mathcal{O})} + \delta^\alpha s^\alpha, \quad 0 < s < 1.$$

In consequence,

$$\|f\|_{L^\infty(\omega)} \le \hat{\mathcal{C}}_2 \left(\frac{1}{s}\right)^{\hat{\alpha}} \|f|\nabla u|^2\|_{L^1(\mathcal{O})} + \delta^\alpha s^\alpha, \quad 0 < s < 1,$$

where $\hat{\alpha} = \max_{x\in\overline{\mathcal{O}}} \mathcal{C}_1(|x-x_0|/\delta)$. The expected inequality follows by minimizing the right-hand side of the last inequality, with respect to s. □

4 Proof of Theorem 1

Pick $(a,b), (\tilde{a},\tilde{b}) \in \mathcal{D}(\lambda,\kappa)$ and let $u_j = G_{a,b}(\cdot,\xi_j)$ and $\tilde{u}_j = G_{\tilde{a},\tilde{b}}(\cdot,\xi_j)$, $j=1,2$. By simple computations we can check that $w = u_2/u_1$ is the solution of the equation

$$\mathrm{div}(\sigma\nabla w) = 0 \quad \text{in } \mathbb{R}^n \setminus \{\xi_1,\xi_2\},$$

with

$$\sigma = a u_1^2 = \frac{a v_1^2}{b^2}.$$

Similarly, $\tilde{w} = \tilde{u}_2/\tilde{u}_1$ is the solution of the equation

$$\mathrm{div}(\tilde{\sigma}\nabla \tilde{w}) = 0 \quad \text{in } \mathbb{R}^n \setminus \{\xi_1,\xi_2\},$$

with

$$\tilde{\sigma} = \tilde{a}\tilde{u}_1^2 = \frac{\tilde{a}\tilde{v}_1^2}{\tilde{b}^2}.$$

We know from Lemma 4 that there exist $x^* \in B(\xi_2, |\xi_1-\xi_2|/2)\setminus\{\xi_2\}$, $\eta_0 = \eta_0(n,\lambda,\kappa,|\xi_1-\xi_2|) > 0$ and $\rho = \rho(n,\lambda,\kappa,|\xi_1-\xi_2|) > 0$ so that $\overline{B}(x^*,\rho) \subset B(\xi_2,|\xi_1-\xi_2|/2)\setminus\{\xi_2\}$ and

$$\eta_0 \le \|\nabla w\|_{L^2(B(x^*,\rho))}. \tag{4.1}$$

Fix then a bounded domain $\mathcal{Q}$ of $\mathbb{R}^n\setminus\{\xi_1,\xi_2\}$ is such a way that $\Omega \cup B(x^*,\rho) \Subset \mathcal{Q}$, and set

$$\delta = \mathrm{dist}(\Omega\cup B(x^*,\rho), \partial\mathcal{Q}).$$

In the rest of this proof, $\mathbf{d} = \mathrm{diam}(\mathcal{Q})$. According to Corollary 3

$$\|\nabla w\|_{L^2(\mathcal{Q})} \le M = Ce^{c(\mathbf{d}+\varrho_+)}\left(1+\max\left(\varrho_-^{-(2+\alpha)},1\right)\varrho_-^{-n+2}\right)^4, \tag{4.2}$$

with $C = C(n,\lambda,\kappa,\alpha,\theta)$ and $c = c(n,\lambda,\kappa,\alpha,\theta)$, $\varrho_- = \min\left(\mathrm{dist}\left(\xi_1,\mathcal{Q}\right),\mathrm{dist}\left(\xi_2,\mathcal{Q}\right)\right)$ and $\varrho_+ = \max\left(\mathrm{dist}\left(\xi_1,\mathcal{Q}\right),\mathrm{dist}\left(\xi_2,\mathcal{Q}\right)\right)$.

Now, since

$$\|\sigma\|_{C^{0,1}(\overline{\mathcal{Q}})} \le \|a\|_{C^{0,1}(\overline{\mathcal{Q}})}\|u_1\|^2_{C^{0,1}(\overline{\mathcal{Q}})},$$

we get, similarly to the end of the proof of Corollary 3, from [17, Lemma 6.35, page 135]

$$\|\sigma\|_{C^{0,1}(\overline{\mathcal{Q}})} \le C\|a\|_{C^{0,1}(\overline{\mathcal{Q}})}\|u_1\|^2_{C^{2,\alpha}(\overline{\mathcal{Q}})},$$

where $C = C(n,\lambda,\kappa,\mathbf{d},\xi_1,\xi_2) > 0$ is a constant. This inequality together with Proposition 1 yields

$$\|\sigma\|_{C^{0,1}(\overline{\mathcal{Q}})} \le C, \tag{4.3}$$

for some constant $C = C(n,\lambda,\kappa,\mathbf{d},\xi_1,\xi_2) > 0$.

On the other hand, we have from (2.11)

$$C^{-1}\min_{x\in\overline{\mathcal{Q}}} \frac{e^{-2\sqrt{c\kappa}|x-\xi_1|}}{|x-\xi_1|^{n-2}} \le u_1, \quad \text{in } \overline{\mathcal{Q}}, \tag{4.4}$$

with constants $c = c(n, \lambda) > 0$ and $C = C(n, \lambda, \kappa) > 0$.

We get by combining (4.3) and (4.4) that there exists $\varkappa = \varkappa(n, \lambda, \kappa, \alpha, \Omega, \xi_1, \xi_2) > 1$ so that

$$\varkappa^{-1} \leq \sigma \quad \text{and} \quad \|\sigma\|_{C^{0,1}(\overline{Q})} \leq \varkappa.$$

Next, if $\rho \leq \delta/3$, then (4.1) implies obviously

$$\eta_0 \leq \|\nabla w\|_{L^2(B(x_0,\delta/3))}, \tag{4.5}$$

with η_0 as in (4.1). When $\rho > \delta/3$, we can use the three-ball inequality in Theorem 4 in order to get

$$\tilde{C}\|\nabla w\|_{L^2(B(x^*,\rho))} \leq \|\nabla w\|^s_{L^2(B(x_0,\delta/3))}\|\nabla w\|^{1-s}_{L^2(B(x^*,\rho+\delta/3))},$$

where $\tilde{C} = \tilde{C}(n, \lambda, \kappa, \Omega, \xi_1, \xi_2)$ and $0 < s = s(n, \lambda, \kappa, \Omega, \xi_1, \xi_2) < 1$ are constants. Whence

$$(\tilde{C}\eta_0)^{1/s} M^{(s-1)/s} \leq \|\nabla w\|_{L^2(B(x_0,\delta/3))}. \tag{4.6}$$

In light of (4.2), (4.5) and (4.6), we can infer that, for some constant $\eta = \eta(n, \lambda, \kappa, \Omega, \xi_1, \xi_2) > 0$, $w \in \mathscr{S}(\mathcal{Q}, x^*, M, \eta, \delta/3)$, where M is as in (4.2) and $\mathscr{S}(\mathcal{Q}, x^*, M, \eta, \delta/3)$ is defined in (3.3).

Lemma 12 *We have*

$$C\|(\sigma - \tilde{\sigma})|\nabla w|^2\|_{L^1(\Omega)} \leq \|w - \tilde{w}\|^{\theta/(2+\theta)}_{L^2(\Omega)} + \|\sigma - \tilde{\sigma}\|_{L^\infty(\Gamma)}, \tag{4.7}$$

where $C = C(n, \lambda, \kappa, \Omega, \alpha, \theta, \xi_1, \xi_2) > 0$ is a constant.

Proof Clearly, if $\zeta = \sigma - \tilde{\sigma}$ and $u = w - \tilde{w}$, then

$$\operatorname{div}(\tilde{\sigma}\nabla u) = \operatorname{div}(\zeta\nabla w).$$

Recall that sgn_0 is the sign function defined on $\mathbb{R}$ by: $\operatorname{sgn}_0(t) = -1$ if $t < 1$, $\operatorname{sgn}_0(0) = 0$ and $\operatorname{sgn}_0(t) = 1$ if $t > 0$. Since

$$\begin{aligned}\operatorname{div}(|\zeta|\nabla w) &= \nabla|\zeta| \cdot \nabla w + |\zeta|\Delta w \\ &= \operatorname{sgn}_0(\zeta)\nabla\zeta \cdot \nabla w + \operatorname{sgn}_0(\zeta)\zeta\Delta w \\ &= \operatorname{sgn}_0(\zeta)\operatorname{div}(\zeta\nabla w) = \operatorname{sgn}_0(\zeta)\operatorname{div}(\tilde{\sigma}\nabla u),\end{aligned}$$

we get by integrating by parts

$$\begin{aligned}\int_\Omega |\zeta||\nabla w|^2 dx &= -\int_\Omega \operatorname{div}(|\zeta|\nabla w) w dx + \int_\Gamma |\zeta| w \partial_\nu w dS(x) \\ &= -\int_\Omega \operatorname{sgn}_0(\zeta)\operatorname{div}(\tilde{\sigma}\nabla u) w dx + \int_\Gamma |\zeta| w \partial_\nu w dS(x).\end{aligned} \tag{4.8}$$

Thus,

$$\int_\Omega |\zeta||\nabla w|^2 dx \leq C\left(\|u\|_{H^2(\Omega)} + \|\zeta\|_{L^\infty(\Gamma)}\right).$$

Thus, the following interpolation inequality

$$\|u\|_{H^2(\Omega)} \leq c_\Omega \|u\|^{\theta/(2+\theta)}_{L^2(\Omega)} \|u\|^{2/(2+\theta)}_{H^{2+\theta}(\Omega)}$$

and Corollary 3 give (4.7). □

We have from (3.27) in Lemma 11

$$\|\tilde\sigma-\sigma\|_{C(\overline\Omega)}\le \hat{\mathcal{C}}_3\|\tilde\sigma-\sigma\|_{C^{0,\alpha}(\overline\Omega)}^{1-\hat\mu}\|(\sigma-\tilde\sigma)|\nabla w|^2\|_{L^1(\Omega)}^{\hat\mu},$$

from which we obtain

$$\|\tilde\sigma-\sigma\|_{C(\overline\Omega)}\le \hat{\mathcal{C}}_3\max\left(1,\|\tilde\sigma-\sigma\|_{C^{0,\alpha}(\overline\Omega)}\right)\|(\sigma-\tilde\sigma)|\nabla w|^2\|_{L^1(\Omega)}^{\hat\mu}.$$

Combined with Proposition 1, this inequality gives

$$\|\tilde\sigma-\sigma\|_{C(\overline\Omega)}\le C\|(\sigma-\tilde\sigma)|\nabla w|^2\|_{L^1(\Omega)}^{\hat\mu}.$$

Here and henceforward, $C=C(n,\lambda,\kappa,\Omega,\alpha,\theta,\xi_1,\xi_2)>0$ is a generic constant.

Therefore, we obtain in light of Lemma 12

$$\|\tilde\sigma-\sigma\|_{C(\overline\Omega)}\le C\left(\|w-\tilde w\|_{L^2(\Omega)}^{\theta/(2+\theta)}+\|\sigma-\tilde\sigma\|_{C(\Gamma)}\right)^{\hat\mu}.$$

Since $\tilde a=a$ and $\tilde b=b$ on Γ and regarding the regularity of u_i and $\tilde u_i$, $i=1,2$, we finally get

$$\|\tilde\sigma-\sigma\|_{C(\overline\Omega)}\le C\left(\|v_1-\tilde v_1\|_{C(\overline\Omega)}+\|v_2-\tilde v_2\|_{C(\overline\Omega)}\right)^{\hat\mu_0}, \tag{4.9}$$

with

$$\hat\mu_0=\frac{\theta\hat\mu}{2+\theta}.$$

The following lemma will be used in sequel.

Lemma 13 *We have*

$$\|u_1^{-1}-\tilde u_1^{-1}\|_{C^{2,\alpha}(\overline\Omega)}\le C\left(\|v_1-\tilde v_1\|_{C(\overline\Omega)}+\|v_2-\tilde v_2\|_{C(\overline\Omega)}\right)^{\hat\mu_1}, \tag{4.10}$$

where $0<\hat\mu_1=\hat\mu_1(n,\Omega,\lambda,\kappa,\alpha,\theta,\xi_1,\xi_2)<1$ *and* $C=C(n,\Omega,\lambda,\kappa,\alpha,\theta,\xi_1,\xi_2)>0$ *are constants.*

Proof In this proof $C=C(n,\Omega,\lambda,\kappa,\alpha,\theta,\xi_1,\xi_2)>0$ is a generic constant. It is not hard to check that

$$-\operatorname{div}(\sigma\nabla u_1^{-1})=v_1\quad\text{in }\Omega,$$
$$-\operatorname{div}(\tilde\sigma\nabla\tilde u_1^{-1})=\tilde v_1\quad\text{in }\Omega.$$

Hence,

$$-\operatorname{div}(\sigma\nabla(u_1^{-1}-\tilde u_1^{-1}))=(v_1-\tilde v_1)+\operatorname{div}((\sigma-\tilde\sigma)\nabla\widetilde{u}_1^{-1})\quad\text{in }\Omega.$$

By the usual Hölder a priori estimate (see [17, Theorem 6.6, page 98])

$$C\|u_1^{-1}-\tilde u_1^{-1}\|_{C^{2,\alpha}(\overline\Omega)}\le\|v_1-\tilde v_1\|_{C^{0,\alpha}(\overline\Omega)}$$
$$+\|\operatorname{div}((\sigma-\tilde\sigma)\nabla\widetilde{u}_1^{-1})\|_{C^{0,\alpha}(\overline\Omega)}+\|u_1^{-1}-\tilde u_1^{-1}\|_{C^{0,\alpha}(\Gamma)}.$$

Consequently,

$$\|u_1^{-1}-\tilde u_1^{-1}\|_{C^{2,\alpha}(\overline\Omega)}\le C\left(\|v_1-\tilde v_1\|_{C^{0,\alpha}(\overline\Omega)}+\|\sigma-\tilde\sigma\|_{C^{1,\alpha}(\overline\Omega)}\right), \tag{4.11}$$

where we used

$$\left\|u_1^{-1}-\tilde u_1^{-1}\right\|_{C^{0,\alpha}(\Gamma)}=\left\|b\left(v_1^{-1}-\tilde v_1^{-1}\right)\right\|_{C^{0,\alpha}(\Gamma)}.$$

On the other hand, since

$$\|\sigma - \tilde{\sigma}\|_{C^{1,1}(\overline{\Omega})} \le C, \quad \|v_1 - \tilde{v}_1\|_{C^{1,\alpha}(\overline{\Omega})} \le C$$

and Ω is $C^{1,1}$, we get again from the interpolation inequality in [17, Lemma 6.35, page 135]

$$\|\sigma - \tilde{\sigma}\|_{C^{1,\alpha}(\overline{\Omega})} \le C\|\sigma - \tilde{\sigma}\|^{\tau}_{C(\overline{\Omega})}, \quad \|v_1 - \tilde{v}_1\|_{C^{0,\alpha}(\overline{\Omega})} \le C\|v_1 - \tilde{v}_1\|^{\tau}_{C(\overline{\Omega})}, \tag{4.12}$$

where $0 < \tau = \tau(\Omega, \alpha) < 1$ is a constant. Inequality (4.12) in (4.11) yields

$$\left\|u_1^{-1} - \tilde{u}_1^{-1}\right\|_{C^{2,\alpha}(\overline{\Omega})} \le C\left(\|v_1 - \tilde{v}_1\|^{\tau}_{C(\overline{\Omega})} + \|\sigma - \tilde{\sigma}\|^{\tau}_{C(\overline{\Omega})}\right). \tag{4.13}$$

On the other hand, we have from (4.9)

$$\|\tilde{\sigma} - \sigma\|_{C(\overline{\Omega})} \le C\left(\|v_1 - \tilde{v}_1\|_{C(\overline{\Omega})} + \|v_2 - \tilde{v}_2\|_{C(\overline{\Omega})}\right)^{\hat{\mu}_0}. \tag{4.14}$$

Whence, we get in light of inequalities (4.13) and (4.14), where $\hat{\mu}_1 = \tau\hat{\mu}_0$,

$$\left\|u_1^{-1} - \tilde{u}_1^{-1}\right\|_{C^{2,\alpha}(\overline{\Omega})} \le C\left(\|v_1 - \tilde{v}_1\|_{C(\overline{\Omega})} + \|v_2 - \tilde{v}_2\|_{C(\overline{\Omega})}\right)^{\hat{\mu}_1}.$$

This is the expected inequality. □

Also, since

$$\|\sigma - \tilde{\sigma}\|_{C^{1,1}(\overline{\Omega})} \le C, \quad \|v_1 - \tilde{v}_1\|_{C^{2,\alpha}(\overline{\Omega})} \le C,$$

we can proceed as in the preceding proof to get

$$\|\sigma - \tilde{\sigma}\|_{C^{1,\alpha}(\overline{\Omega})} \le C\|\sigma - \tilde{\sigma}\|^{\tau}_{C(\overline{\Omega})}, \quad \|v_1 - \tilde{v}_1\|_{C^{1,\alpha}(\overline{\Omega})} \le C\|v_1 - \tilde{v}_1\|^{\tau}_{C(\overline{\Omega})}, \tag{4.15}$$

the constant $0 < \tau = \tau(\Omega, \alpha) < 1$. But

$$\begin{aligned} a - \tilde{a} = \sigma u_1^{-2} - \tilde{\sigma}\tilde{u}_1{}^{-2} &= (\sigma - \tilde{\sigma})\, u_1^{-2} + \tilde{\sigma}\left(u_1^{-2} - \tilde{u}_1^{-2}\right) \\ &= (\sigma - \tilde{\sigma})\, u_1^{-2} + \tilde{\sigma}\left(u_1^{-1} + \tilde{u}_1^{-1}\right)\left(u_1^{-1} - \tilde{u}_1^{-1}\right). \end{aligned}$$

Hence,

$$\|a - \tilde{a}\|_{C^{1,\alpha}(\overline{\Omega})} \le C\left(\|u_1^{-1} - \tilde{u}_1^{-1}\|_{C^{1,\alpha}(\overline{\Omega})} + \|\sigma - \tilde{\sigma}\|_{C^{1,\alpha}(\overline{\Omega})}\right). \tag{4.16}$$

This inequality together with (4.9), (4.10) and (4.15) implies

$$\|a - \tilde{a}\|_{C^{1,\alpha}(\overline{\Omega})} \le C\left(\|v_1 - \tilde{v}_1\|_{C(\overline{\Omega})} + \|v_2 - \tilde{v}_2\|_{C(\overline{\Omega})}\right)^{\hat{\mu}_1}. \tag{4.17}$$

We proceed similarly for $b - \tilde{b}$. Since

$$b - \tilde{b} = v_1 u_1^{-1} - \tilde{v}_1\tilde{u}_1^{-1} = (v_1 - \tilde{v}_1)u_1^{-1} + \tilde{v}_1(u_1^{-1} - \tilde{u}_1^{-1}),$$

we have

$$\|b - \tilde{b}\|_{C^{0,\alpha}(\overline{\Omega})} \le C\left(\|v_1 - \tilde{v}_1\|_{C(\overline{\Omega})} + \|v_2 - \tilde{v}_2\|_{C(\overline{\Omega})}\right)^{\hat{\mu}_1}. \tag{4.18}$$

The expected inequality follows by putting together (4.17) and (4.18).

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Appendix A: Proof of technical lemmas

Proof of Lemma 2 In this proof, $C = C(n, \mu, \nu) > 1$ is a generic constant.

It is well known that $G_{1,\nu}$, $\nu > 0$, the fundamental solution of the operator $-\Delta + \nu$, is given by $G_{1,\nu}(x, \xi) = \mathcal{G}_{1,\nu}(x - \xi)$, $x, \xi \in \mathbb{R}^n$, with

$$\mathcal{G}_{1,\nu}(x) = (2\pi)^{-n/2}(\sqrt{\nu}/|x|)^{n/2-1}K_{n/2-1}(\sqrt{\nu}|x|).$$

In the particular case $n = 3$, we have $K_{1/2}(z) = \sqrt{\pi/(2z)}e^{-z}$ and therefore

$$\mathcal{G}_{1,\nu}(x) = \frac{e^{-\sqrt{\nu}|x|}}{4\pi|x|}.$$

Let $f \in C_0^\infty(\mathbb{R}^n)$, $\mu > 0$ and $\nu > 0$ be two constants, and denote by u the solution of the equation

$$(-\mu\Delta + \nu)u = f \quad \text{in } \mathbb{R}^n.$$

Then,

$$u(x) = \int_{\mathbb{R}^n} G_{\mu,\nu}(x, \xi)f(\xi)d\xi, \quad x \in \mathbb{R}^n. \tag{A.1}$$

We remark that $v(x) = u(\sqrt{\mu}x)$, $x \in \mathbb{R}^n$ satisfies $(-\Delta + \nu)v = f(\sqrt{\mu}\,\cdot)$. Whence

$$\begin{aligned} u(\sqrt{\mu}x) = v(x) &= \int_{\mathbb{R}^n} \mathcal{G}_{1,\kappa}(x - \xi)f(\sqrt{\mu}\xi)d\xi \\ &= \mu^{-n/2}\int_{\mathbb{R}^n} \mathcal{G}_{1,\nu}(x - \xi/\sqrt{\mu})f(\xi)d\xi, \quad x \in \mathbb{R}^n. \end{aligned}$$

Hence,

$$u(x) = \mu^{-n/2}\int_{\mathbb{R}^n} \mathcal{G}_{1,\nu}((x - \xi)/\sqrt{\mu})f(\xi)d\xi, \quad x \in \mathbb{R}^n. \tag{A.2}$$

Comparing (A.1) and (A.2), we find

$$G_{\mu,\nu}(x, \xi) = \mu^{-n/2}\mathcal{G}_{1,\nu}((x - \xi)/\sqrt{\mu}), \quad x, \xi \in \mathbb{R}^n.$$

Consequently, $G_{\mu,\nu}(x, \xi) = \mathcal{G}_{\mu,\nu}(x - \xi)$ with

$$\mathcal{G}_{\mu,\nu}(x) = (2\pi\mu)^{-n/2}(\sqrt{\nu\mu}/|x|)^{n/2-1}K_{n/2-1}(\sqrt{\nu}|x|/\sqrt{\mu}), \quad x \in \mathbb{R}^n. \tag{A.3}$$

By the usual asymptotic formula for modified Bessel functions of the second kind (see for instance [5, 9.7.2, page 378]), we have, when $|x| \to \infty$,

$$K_{n/2-1}(\sqrt{\nu}|x|/\sqrt{\mu}) = \left(\frac{\pi\sqrt{\mu}}{2\sqrt{\nu}|x|}\right)^{1/2} e^{-\sqrt{\nu}|x|/\sqrt{\mu}}\,(1 + O(1/|x|)),$$

where $O(1/|x|)$ only depends on n, μ and ν.

Consequently, there exists $R = R(n, \mu, \nu) > 0$ so that

$$C^{-1}\frac{e^{-\sqrt{\nu}|x|/\sqrt{\mu}}}{|x|^{1/2}} \le K_{n/2-1}(\sqrt{\nu}|x|/\sqrt{\mu}) \le C\frac{e^{-\sqrt{\nu}|x|/\sqrt{\mu}}}{|x|^{1/2}}, \quad |x| \ge R. \tag{A.4}$$

Substituting if necessary R by $\max(R, 1)$, we have

$$\frac{1}{|x|^{n/2-1}} \le \frac{1}{|x|^{1/2}}, \quad |x| \ge R. \tag{A.5}$$

Moreover, we have

$$\frac{e^{-\sqrt{\nu}|x|/\sqrt{\mu}}}{|x|^{1/2}} = \left[|x|^{(n-3)/2}e^{-\sqrt{\nu}|x|/(2\sqrt{\mu})}\right]\frac{e^{-\sqrt{\nu}|x|/(2\sqrt{\mu})}}{|x|^{n/2-1}}, \quad |x| \ge R.$$

Since the function $x \to |x|^{(n-3)/2} e^{-\sqrt{\nu}|x|/(2\sqrt{\mu})}$ is bounded in $\mathbb{R}^n$, we deduce

$$\frac{e^{-\sqrt{\nu}|x|/\sqrt{\mu}}}{|x|^{1/2}} \leq C \frac{e^{-\sqrt{\nu}|x|/(2\sqrt{\mu})}}{|x|^{n/2-1}}, \quad |x| \geq R. \tag{A.6}$$

Using (A.5) and (A.6) in (A.4) in order to obtain

$$C^{-1} \frac{e^{-\sqrt{\nu}|x|/\sqrt{\mu}}}{|x|^{n/2-1}} \leq K_{n/2-1}(\sqrt{\nu}|x|/\sqrt{\mu}) \leq C \frac{e^{-\sqrt{\nu}|x|/(2\sqrt{\mu})}}{|x|^{n/2-1}}, \quad |x| \geq R. \tag{A.7}$$

We now establish a similar estimate when $|x| \to 0$. To this end, we recall that according to formula [5, 9.6.9, p. 375] we have

$$K_{n/2-1}(\rho) \sim \frac{1}{2}\Gamma(n/2-1)\left(\frac{2}{\rho}\right)^{n/2-1} \quad \text{as } \rho \to 0,$$

from which we deduce in a straightforward manner that there exists $0 < r \leq R$ so that

$$C^{-1} \frac{e^{-\sqrt{\nu}|x|/\sqrt{\mu}}}{|x|^{n/2-1}} \leq K_{n/2-1}(\sqrt{\nu}|x|/\sqrt{\mu}) \leq C \frac{e^{-\sqrt{\nu}|x|/(2\sqrt{\nu})}}{|x|^{n/2-1}}, \quad |x| \leq r. \tag{A.8}$$

The expected two-sided inequality (2.10) follows by combining (A.4), (A.7) and (A.8). □

Proof of Lemma 3 Let $\mathcal{Q}$ be an open subset of $\mathbb{R}^n$, set $d = \mathrm{diam}(\mathcal{Q})$, $d_x = \mathrm{dist}(x, \partial\mathcal{Q})$ and $d_{x,y} = \min(d_x, d_y)$.

We introduce the following weighted Hölder semi-norms and Hölder norms, where $\sigma \in \mathbb{R}$, $0 < \gamma \leq 1$, and k is nonnegative integer,

$$[w]^{(\sigma)}_{k,0;\mathcal{Q}} = [w]^{(\sigma)}_{k,\mathcal{Q}} = \sup_{x\in\mathcal{Q},\ |\beta|=k} d_x^{k+\sigma} |\partial^\beta w(x)|,$$

$$[w]^{(\sigma)}_{k,\gamma;\mathcal{Q}} = \sup_{x,y\in\mathcal{Q},\ |\beta|=k} d_{x,y}^{k+\gamma+\sigma} \frac{|\partial^\beta w(y) - \partial^\beta w(x)|}{|y-x|^\gamma},$$

$$|w|^{(\sigma)}_{k;\mathcal{Q}} = \sum_{j=0}^{k} [w]^{(\sigma)}_{j;\mathcal{Q}},$$

$$|w|^{(\sigma)}_{k,\gamma;\mathcal{Q}} = |w|^{(\sigma)}_{k;\mathcal{Q}} + [w]^{(\sigma)}_{k,\gamma;\mathcal{Q}}.$$

In terms of these notations, we have

$$|a|^{(0)}_{0,\alpha;\mathcal{Q}} = \sup_{x\in\mathcal{Q}} |a(x)| + \sup_{x,y\in\mathcal{Q}} d_{x,y}^{\alpha} \frac{|a(y)-a(x)|}{|y-x|^\alpha} \leq (1+\mathbf{d})\lambda,$$

$$|\partial_j a|^{(1)}_{0,\alpha;\mathcal{Q}} = \sup_{x\in\mathcal{Q}} d_x |\partial_j a(x)| + \sup_{x,y\in\mathcal{O}} d_{x,y}^{1+\alpha} \frac{|\partial_j a(y)-\partial_j a(x)|}{|y-x|^\alpha} \leq (\mathbf{d}+\mathbf{d}^2)\lambda,$$

$$|b|^{(2)}_{0,\alpha;\mathcal{Q}} = \sup_{x\in\mathcal{O}} d_x^2 |b(x)| + \sup_{x,y\in\mathcal{Q}} d_{x,y}^{2+\alpha} \frac{|b(y)-b(x)|}{|y-x|^\alpha} \leq (\mathbf{d}^2+\mathbf{d}^3)\lambda.$$

In consequence,

$$|a|^{(0)}_{0,\alpha;\mathcal{Q}} + |\partial_j a|^{(1)}_{0,\alpha;\mathcal{Q}} + |b|^{(2)}_{0,\alpha;\mathcal{Q}} \leq \Lambda(\mathbf{d}) = \left[1 + 2\mathbf{d} + 2\mathbf{d}^2 + \mathbf{d}^3\right]\lambda. \tag{A.9}$$

Following [17], we define also

$$[w]^*_{k,0;\mathcal{Q}} = [w]^*_{k,\mathcal{O}} = \sup_{x\in\mathcal{Q},\ |\beta|=k} d_x^k |\partial^\beta w(x)|,$$

$$[w]^*_{k,\gamma;\mathcal{Q}} = \sup_{x,y\in\mathcal{Q},\ |\beta|=k} d_{x,y}^{k+\alpha} \frac{|\partial^\beta w(y) - \partial^\beta w(x)|}{|y-x|^\gamma},$$

$$|w|^*_{k;\mathcal{Q}} = \sum_{j=0}^{k} [w]^*_{j;\mathcal{Q}},$$

$$|w|^*_{k,\gamma;\mathcal{Q}} = |w|^*_{k;\mathcal{Q}} + [w]^*_{k,\gamma;\mathcal{O}}.$$

From [17, Lemma 6.32, page 130] and its proof, we have the following interpolation inequalities: Suppose that j and k, nonnegative integers, and $0 \le \beta, \gamma \le 1$ are so that $j+\beta < k+\gamma$. Then, there exist $C = C(n,\alpha,\beta) > 0$ and $\vartheta = \vartheta(\alpha,\beta)$ so that, for any $w \in C^{k,\alpha}(\mathcal{Q})$ and $\epsilon > 0$, we have

$$[w]^*_{j,\beta;\mathcal{Q}} \le C\epsilon^{-\vartheta}|w|_{0;\mathcal{Q}} + \epsilon[w]^*_{k,\gamma;\mathcal{Q}}, \tag{A.10}$$

$$|w|^*_{j,\beta;\mathcal{Q}} \le C\epsilon^{-\vartheta}|w|_{0;\mathcal{Q}} + \epsilon[w]^*_{k,\gamma;\mathcal{Q}}. \tag{A.11}$$

Here, $|w|_{0;\mathcal{Q}} = \sup_{x\in\mathcal{Q}} |w(x)|$.

Checking carefully the proof of interior Schauder estimates in [17, Theorem 6.2, page 90], we get, taking into account inequalities (A.9)-(A.11), the following result: There exist a constant $C = C(n) > 0$ and $\tau = \tau(\alpha)$ so that, for any $0 < \mu \le 1/2$ and $w \in C^{k,\alpha}(\mathcal{Q})$ satisfying $L_{a,b}w = 0$ in $\mathcal{Q}$, we have

$$[w]^*_{2,\alpha,\mathcal{Q}} \le C\Lambda(\mathbf{d})\left(\mu^{-\tau}|w|_{0;\mathcal{Q}} + \mu^\alpha[w]^*_{2,\alpha,\mathcal{Q}}\right). \tag{A.12}$$

Substituting in (A.12) C by $\max(C, 2^{\alpha-1})$, we may assume in (A.12) that $C = C(n,\alpha) \ge 2^{\alpha-1}$. Bearing in mind that $\Lambda(\mathbf{d}) > 1$, we can take in (A.12), $\mu = (2C\Lambda(\mathbf{d}))^{-1/\alpha}$. We find

$$[w]^*_{2,\alpha,\mathcal{Q}} \le C\Lambda(\mathbf{d})^\varkappa |w|_{0;\mathcal{Q}}, \tag{A.13}$$

for some constants $C = C(n,\alpha) > 0$ and $\varkappa = \varkappa(\alpha) > 1$.

Using again interpolation inequalities (A.10) and (A.11), we deduce that

$$|w|^*_{2,\alpha,\mathcal{Q}} \le C\Lambda(\mathbf{d})^\varkappa |w|_{0;\mathcal{Q}}. \tag{A.14}$$

Let $\delta > 0$ be so that $\mathcal{Q}_\delta = \{x \in \mathcal{Q};\ \mathrm{dist}(x, \partial\mathcal{Q}) > \delta\}$ is non-empty. If $\mathcal{Q}'$ is an open subset of $\mathcal{Q}_\delta$, then (A.14) yields in a straightforward manner

$$\|w\|_{C^{2,\alpha}\left(\overline{\mathcal{Q}'}\right)} \le C\max\left(\delta^{-(2+\alpha)}, 1\right)\Lambda(\mathbf{d})^\varkappa |w|_{0;\mathcal{Q}}.$$

This is the expected inequality. □

Lemma 14 *Let K be a compact subset of $\mathbb{R}^n$ and $f \in C^{2,\alpha}(K)$ satisfying $\min_K |f| \ge c_- > 0$. Then,*

$$\|1/f\|_{C^{2,\alpha}(K)} \le Cc_+^4\left(1 + \|f\|_{C^{2,\alpha}(K)}\right)^3, \tag{A.15}$$

where $c_+ = \max(1, c_-^{-1})$ and $C = C(\mathrm{diam}(K))$ is a constant.

Proof Let $x, y \in K$. Using $|1/f|_{0;K} \le c_+$ and the following identities

$$\frac{1}{f^2(y)} - \frac{1}{f^2(x)} = \left(\frac{1}{f(x)f^2(y)} + \frac{1}{f(x)^2 f(y)}\right)(f(x) - f(y)),$$
$$\frac{1}{f^3(y)} - \frac{1}{f^3(x)} = \left(\frac{1}{f(x)f^3(y)} + \frac{1}{f^2(x)f^2(y)} + \frac{1}{f(x)^3 f(y)}\right)(f(x) - f(y)),$$

we easily get

$$[1/f^j]_{\alpha;K} \le 3c_+^4 [f]_{\alpha;K}, \quad j = 2, 3. \tag{A.16}$$

Also, we have

$$\begin{aligned}\frac{\partial_i f(y)\partial_j f(x)}{f^3(y)} - \frac{\partial_i f(y)\partial_j f(x)}{f^3(x)} &= \frac{\partial_i f(y)}{f^3(y)}(\partial_j f(y) - \partial_j f(x)) \\ &+ \frac{\partial_j f(x)}{f^3(y)}(\partial_i f(y) - \partial_i f(x)) \\ &+ \left(\frac{1}{f^3(y)} - \frac{1}{f^3(x)}\right)(\partial_i f(y)\partial_j f(x)).\end{aligned}$$

In light of (A.16), this identity yields

$$\begin{aligned}\left[\partial_i f \partial_j f / f^3\right]_{\alpha;K} \le c_+^4 \big([\partial_i f]_{\alpha;K}|\partial_j f|_{0;K} \\ +[\partial_j f]_{\alpha;K}|\partial_i f|_{0;K} + [f]_{\alpha;K}|\partial_i f|_{0;K}|\partial_j f|_{0;K}\big).\end{aligned} \tag{A.17}$$

On the other hand, since

$$\frac{\partial_{ij}^2 f(y)}{f^2(y)} - \frac{\partial_{ij}^2 f(x)}{f^2(x)} = \frac{1}{f^2(y)}\left(\partial_{ij}^2 f(y) - \partial_{ij}^2 f(x)\right) + \left(\frac{1}{f^2(y)} - \frac{1}{f^2(y)}\right)\partial_{ij}^2 f(x),$$

we find, by using again (A.16),

$$\left[\partial_{ij}^2 f / f^2\right]_{\alpha;K} \le 3c_+^4 \left(\left[\partial_{ij}^2 f\right]_{\alpha;K} + [f]_{\alpha;K}\left|\partial_{ij}^2 f\right|_{0,K}\right). \tag{A.18}$$

Inequalities (A.17), (A.18), the identity $\partial_{ij}^2(1/f) = 2\partial_i f \partial_j f / f^3 - \partial_{ij}^2 f / f^2$ and the interpolation inequality [17, Lemma 6.35, p. 135] (by proceeding as in Corollary 2) imply

$$[\partial_{ij}^2(1/f)]_{\alpha,K} \le Cc_+^4 \left(1 + \|f\|_{C^{2,\alpha}(K)}\right)^3, \tag{A.19}$$

where $C = C(\text{diam}(K))$ is a constant.

The other terms for $1/f$ appearing in the norms $\|\cdot\|_{C^{2,\alpha}(K)}$ can be estimated similarly to the semi-norm in (A.19). Inequality (A.15) then follows. □

Recall that $0 < \theta < \alpha < 1$.

Lemma 15 *$C^{2,\alpha}(\overline{\mathcal{O}})$ is continuously embedded in $H^{2+\theta}(\mathcal{O})$. Furthermore, there exists $C = C(n, \alpha - \theta)$ so that, for any $w \in C^{2,\alpha}(\overline{\mathcal{O}})$, we have*

$$\|w\|_{H^{2+\theta}(\mathcal{O})} \le C \max\left(\mathbf{d}^{n/2}, \mathbf{d}^{n/2+\alpha-\theta}\right)\|w\|_{C^{2,\alpha}(\overline{\mathcal{O}})}, \tag{A.20}$$

where $\mathbf{d} = \text{diam}(\mathcal{O})$.

Proof Let $w \in C^{2,\alpha}(\overline{\mathcal{O}})$ and, for fixed $1 \leq i, j \leq n$, set $g = \partial^2_{ij} w$. Then,

$$\int_{\mathcal{O}}\int_{\mathcal{O}} \frac{|g(x)-g(y)|^2}{|x-y|^{n+2\theta}}dxdy \leq [g]^2_{\alpha;\mathcal{O}} \int_{\mathcal{O}}\int_{\mathcal{O}} \frac{1}{|x-y|^{n-2(\alpha-\theta)}}dxdy.$$

In light of [10, Lemma A3, p. 246], this inequality yields

$$\int_{\mathcal{O}}\int_{\mathcal{O}} \frac{|g(x)-g(y)|^2}{|x-y|^{n+2\theta}}dxdy \leq \frac{|\mathbb{S}^{n-1}||\mathcal{O}|\mathbf{d}^{2(\alpha-\theta)}}{2(\alpha-\theta)}[g]^2_{\alpha;\mathcal{O}},$$

But $|\mathcal{O}| \leq |B(0,\mathbf{d})|$. Hence,

$$\int_{\mathcal{O}}\int_{\mathcal{O}} \frac{|g(x)-g(y)|^2}{|x-y|^{n+2\theta}}dxdy \leq \frac{|\mathbb{S}^{n-1}|^2\mathbf{d}^{n+2(\alpha-\theta)}}{2(\alpha-\theta)}[g]^2_{\alpha;\mathcal{O}}. \tag{A.21}$$

Using (A.21) and the inequality

$$\|h\|^2_{L^2(\mathcal{O})} \leq |\mathbb{S}^{n-1}|\mathbf{d}^n|h|_{0,\mathcal{O}}, \quad h \in C(\overline{\mathcal{O}}),$$

we get from the definition of the norm of H^s-spaces in [18, formula (1.3.2.2), page 17]

$$\|w\|_{H^{2+\theta}(\mathcal{O})} \leq C \max\left(\mathbf{d}^{n/2}, \mathbf{d}^{n/2+\alpha-\theta}\right)\|w\|_{C^{2,\alpha}(\overline{\mathcal{O}})},$$

for some constant $C = C(n, \alpha-\theta) > 0$. This is the expected inequality □

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