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A perturbation problem for transmission eigenvalues

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Abstract

In this paper, we consider a perturbation problem for real transmission eigenvalues. Real transmission eigenvalues are of particular interest in inverse scattering theory. They can be determined from scattering data and are related to injectivity of the related scattering operators. The goal of this paper is to provide examples of existence of real transmission eigenvalues for inhomogeneities whose refractive index does not satisfy the assumptions for which the (non-self-adjoint) transmission eigenvalue problem is understood. Such "irregular media" are obtained as perturbations of an inhomogeneity for which the existence of real transmission eigenvalues is known. Our perturbation approach uses an application of a version of the implicit function theorem to an appropriate function in the vicinity of an unperturbed real transmission eigenvalue. Several examples of interesting spherical perturbations of spherically symmetric media are included. Partial results are obtained for general media based on our perturbation approach.

Keywords: Perturbation theory, Spectral problems, Transmission eigenvalues, Scattering theory for inhomogeneous media, Non-scattering waves

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1 Introduction

The set of scattering poles, otherwise known as scattering resonances, is intrinsic to scattering theory [9]. At a scattering pole, there is a nonzero scattered field in the absence of the incident field. On the flip side of this characterization of the scattering poles, one could ask if there are frequencies for which there exists an incident field that doesn't scatterer by the scattering object. The answer to this question for inhomogeneous media leads to the concept of transmission eigenvalues [12] (see [4] for a dual characterization between scattering poles and transmission eigenvalues). The transmission eigenvalue problem has a deceptively simple formulation, namely the existence of nontrivial solutions to two elliptic PDEs in a bounded domain (one governs the wave propagation in the scattering medium and the other in the background that occupies the support of the medium) that share the same Cauchy data on the boundary, but presents a perplexing mathematical structure. In particular, it is a non-self-adjoint eigenvalue problem for a non-strongly elliptic operator, hence the investigation of its spectral properties becomes challenging. Roughly, its spectral properties are understood under a one-sign assumption on the contrasts in the media (i.e.,

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the difference of the respective coefficients in each of the equations) near the boundary [19,20]. Owing to the non-self-adjoint nature, complex transmission eigenvalues can exist [13], [14], [23]. However, the set of real transmission eigenvalues play an essential role in inverse scattering theory [2]. Real transmission eigenvalues are related to injectivity of the scattering operator and they can be determined from scattering data [3], hence they provide information on the refractive index of the media. The existence of *real* transmission eigenvalues in general is hard to prove, unless restrictive assumptions in the refractive index are imposed see [5] for general media and Colton and Kress [12] for spherically stratified media. The goal of this paper is to provide examples of existence of real transmission eigenvalues for media with refractive index that does not satisfy these assumptions.

In order to be more specific, let us formulate the scattering problem under consideration. Consider the scattering of an incident wave v of monochromatic radiation with frequency ω, which satisfies the Helmholtz equation

$$\Delta v + k^2 v = 0 \quad \text{in } \mathbb{R}^d, d = 2, 3 \tag{1}$$

by an inhomogeneity supported in the bounded region D with the refractive index n being a bounded real-valued function such that $\sup(n - 1) = \overline{D}$. Here k is referred to as the wave number and it is proportional to the frequency ω. The total field u is decomposed as $u = u^s + v$ where the scattered field $u^s \in H^2_{loc}(\mathbb{R}^d)$ satisfies

$$\Delta u^s + k^2 n u^s = -k^2(n - 1)v, \quad \text{in } \mathbb{R}^d, \tag{2}$$

together with the outgoing Sommerfeld radiation condition

$$\lim_{r \to \infty} r^{\frac{d-1}{2}} \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0, \tag{3}$$

which holds uniformly with respect to $\hat{x} := x/|x|, r = |x|$ [12]. Now, k is a non-scattering wave number if the scattered field u^s corresponding to the incident field v defined as above is in $H^2_0(D)$, i.e., is zero outside D. Non-scattering wave numbers are a subset of the real transmission eigenvalues, i.e., values of $k > 0$ such that there exists nonzero $v \in L^2(D)$ and $u^s \in H^2_0(D)$ such that

$$\Delta v + k^2 v = 0 \quad \text{and} \quad \Delta u^s + k^2 n u^s = -k^2(n - 1)v, \quad \text{both in } D$$
$$\text{where} \quad H^2_0(D) := \left\{ u \in H^2(D) : u = \frac{\partial u}{\partial \nu} = 0 \right\}.$$

A real transmission eigenvalue $k > 0$ is a non-scattering wave number if the part v of the corresponding eigenfunction can be extended to be a solution of the Helmholtz equation on all of $\mathbb{R}^d$. Most recent results on necessary conditions for a real transmission eigenvalue to be non-scattering wave number, or equivalently, by negation, sufficient conditions for it not to be non-scattering wave number can be found in [8,21]. For general media (D, n) with D Lipschitz and n a bounded function, the existence of (an infinite discrete set of) real transmission eigenvalues is proven only under the assumption that the contrast $n-1$ is one sign uniformly in D, i.e., either $n - 1 \geq \alpha > 0$ or $1 - n \geq \alpha > 0$ a.e. in D [2,5]. For spherical symmetric media, i.e., when $D := B_a(0)$ is a ball of radius a centered at the origin and the refractive index $n(r)$ is a radial function, the existence of real transmission eigenvalues is known for $n \in C^2[0, a]$ with only the additional assumption that $\frac{1}{a} \int_0^a \sqrt{n(\rho)} \, d\rho \neq 1$.

The goal of this paper is to provide examples of existence of real transmission eigenvalues for classes of refractive index that do not satisfy the above conditions. We use a perturbation method based on the implicit function theorem, hence the "irregular" refractive index is a perturbation of a refractive index for which a real transmission eigenvalue is known to exist. More specifically, for spherically symmetric media, we show that if a C^2 spherical refractive index n is perturbed to a nearby refractive index n_ϵ in the weak*-L^∞ sense, then for sufficiently small ϵ there exists a real transmission eigenvalue for n_ϵ in the vicinity of a real transmission eigenvalue for n. We provide significant representative examples of three types of perturbations: (1) refractive index with sign-changing contrast up to the boundary obtained as an L^∞ perturbation, (2) discontinuous refractive index obtained as an L^1 perturbation of small volume (radially thin shells) and (3) highly oscillating radially periodic discontinuous refractive index obtained as a weak*-L^∞ perturbation (such materials are used to build super-resolution spherical lenses). Note that for such examples, the contrast may be everywhere large and change sign. In all these examples we prove the existence of non-scattering wave numbers since by construction the part v of the transmission eigenfunction is an entire solution of the Helmholtz equation (we refer the reader to Cakoni and Vogelius [8], Salo and Shahgholian [21] and Vogelius and Xiao [24] for existence of non-scattering wave numbers for analytic domain and analytic refractive index).

For general media, our perturbation method provides only partial results. In order to restore some structure for the transmission eigenvalue problem when the contrast assumption is not satisfied, now we are faced with perturbing the zero eigenvalue of a compact self-adjoint operator, which brings up a number of new difficulties. More specifically, in the general case, we give a condition on the unperturbed problem which, if satisfied, guarantees existence of approximate transmission eigenvalues under perturbation, where by the approximate transmission eigenvalues we mean transmission eigenvalues for the same problem projected onto finite-dimensional subspaces of any sufficiently large dimension.

The paper is organized as follows. In the next section, we introduce a different equivalent formulation of the transmission eigenvalue problem and define a function of two variables, namely of the wave number k and the perturbation parameter ϵ, whose zero for a fixed ϵ yields a transmission eigenvalue for the perturbed inhomogeneity. Section 3 includes the analysis for spherically symmetric perturbations of spherically symmetric media, and contains interesting examples of the existence of transmission eigenvalue not covered by the existing literature. Section 4 provides future directions on generalizing our perturbation approach to arbitrary inhomogeneous media.

2 The transmission eigenvalue problem

Let us formulate precisely the above transmission eigenvalue problem in terms of v and $u := u^s + v$. Let $D \subset \mathbb{R}^d$, $d = 2, 3$ be an open and simply connected region with Lipschitz boundary ∂D. We assume that $n \in L^\infty(D)$ such that $n(x) \geq \eta > 0$ for almost all $x \in \overline{D}$. The *transmission eigenvalue problem* reads: Find $k \in \mathbb{C}$ such that there exists nontrivial u and v satisfying

$$\Delta u + k^2 n u = 0 \quad \text{in } D,$$
$$\Delta v + k^2 v = 0 \quad \text{in } D,$$

$$u = v \quad \text{on } \partial D,$$
$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} \quad \text{on } \partial D.$$

Such values of k are called *transmission eigenvalues* [2]. We are concerned with real transmission eigenvalues since, as discussed in the introduction, they contain non-scattering wave numbers and are the only transmission eigenvalues that can be measured from scattering data. We introduce a different equivalent formulation of the transmission eigenvalue problem which we use in our analysis. To this end, let us call $T_q^k : L^2(\partial D) \to L^2(\partial D)$ the Neumann-to-Dirichlet operator mapping

$$\varphi \mapsto U|_{\partial D}$$

where U solves

$$\Delta U + k^2 q U = 0 \quad \text{in } D, \qquad \frac{\partial U}{\partial \nu} = \varphi \quad \text{on } \partial D, \tag{4}$$

where we of course assume that k^2 is not a corresponding Neumann eigenvalue. The operator T_q^k is obviously self-adjoint and compact since the solution w of (4) is at least in $H^1(D)$ and hence $w|_{\partial D} \in H^{1/2}(\partial D)$ which is compactly embedded in $L^2(D)$. Then, if $k > 0$ is such that there is a nonzero $\varphi \in L^2(\partial D)$ in the kernel of the operator $\mathbb{T}_0^k := T_n^k - T_1^k$, i.e.,

$$\mathbb{T}_0^k \varphi = T_n^k \varphi - T_1^k \varphi = 0,$$

then this k is a transmission eigenvalue. Conversely, if $k > 0$ is a transmission eigenvalues and the corresponding eigenfunction v is sufficiently regular then $\partial v / \partial \nu$ is in the kernel of $\mathbb{T}_0^k$.

The following proposition is used in Sect. 4.

Proposition 2.1 *Assume $q > 0$ uniformly in D and consider an interval $(a, b) \subset \mathbb{R}$ such that for all $k \in (a, b)$, k^2 is not a Neumann eigenvalue for (4) with $\varphi = 0$. Let $\mathcal{C} := \{z \in \mathbb{C} : \Re(z) \in (a, b)\}$. Then, $T_q^k : \mathcal{C} \to \mathcal{L}(L^2(\partial D))$ is analytic in k.*

Note that $\mathcal{L}(L^2(\partial D))$ denotes the Banach space of bounded linear operators on $L^2(\partial D)$.

Proof If U_φ is a solution of (4) then equivalently U_φ solves

$$(B - zK)U_\varphi = \ell$$

where $z := (k^2 - 1)$, B and K are defined via Riesz representation theorem by

$$(BU, V)_{H^1(D)} := \int_D \left(\nabla U \cdot \nabla \overline{V} + q U \overline{V} \right) dx, \qquad \forall U, V \in H^1(D),$$

$$(KU, V)_{H^1(D)} := \int_D q U \overline{v} \, dx, \qquad \forall U, V \in H^1(D),$$

and $\ell \in H^1(D)$ is the Riesz representative given by

$$(\ell, V)_{H^1(D)} := - \int_{\partial D} \varphi \overline{V} ds, \quad \forall u, v \in H^1(D),$$

where we have used the trace theorem to prove the continuity of the sesquilinear form on the right-hand side. From Lax–Milgram we have that B^{-1} exists, and from the compact embedding of $H^1(D)$ into $L^2(D)$ we have that K is a compact operator. From the assumptions on k we have that $1/z$ is not an eigenvalue of the compact operator $B^{-1}K$ in the respective region, which means that $(B - zK)$ depends analytically on z and so does its inverse $(B - zK)^{-1}$. This implies that the solution U_φ is analytic in k. Finally, since $T_q : \varphi \mapsto U_\varphi|_{\partial D}$ the statement of the proposition follows. $\qquad \square$

From the above, for given (D, n) the operator $\mathbb{T}_0^k : L^2(\partial D) \to L^2(\partial D)$ is compact, self-adjoint and depends analytically on k. Hence $\mathbb{T}_0^k$ has an infinite sequence of real (positive and negative) eigenvalues $\{\Lambda_j(0, k)\}_{j \in \mathbb{N}}$ with 0 the only accumulation point. Thus, we have

$$\mathbb{T}_0^k \varphi_j - \Lambda_j(0, k)\varphi_j = 0 \tag{5}$$

snf if $\Lambda_j(0, k) = 0$ is an eigenvalue then k is a transmission eigenvalue, and conversely if k is a transmission eigenvalue with $\partial v / \partial \nu$ in $L^2(\partial D)$ then $\Lambda(0, k) = 0$. We would like to use perturbation techniques to prove the existence of real transmission eigenvalues provided that for the base problem with D and n, real transmission eigenvalues with a regular eigenfunction v exist.

Assume that for the base problem corresponding to the inhomogeneity (n, D) there exists a real transmission eigenvalue k_0 for which $\Lambda(0, k_0) = 0$. We consider n_ϵ, a one-parametric family of perturbations of n with parameter $\epsilon > 0$, such that n_ϵ converges to n in some sense (to become precise later). Let us call

$$\mathbb{T}_\epsilon^k := T_{n_\epsilon}^k - T_1^k$$

the difference of two Neumann-to-Dirichlet operators corresponding to (n_ϵ, D). The eigenvalues of the compact and self-adjoint operator $\mathbb{T}_\epsilon^k$ are $\Lambda_j(\epsilon, k)$, and again k is a transmission eigenvalue corresponding to the inhomogeneity (D, n_ϵ) if $\Lambda(\epsilon, k) = 0$ is an eigenvalue of $\mathbb{T}_\epsilon^k$. We consider perturbations such that the family of operators $\mathbb{T}_\epsilon^k$ for $\epsilon \in (-\delta, \delta)$ and $k \in (k_0 - \alpha, k_0 + \alpha)$ is continuous in ϵ and analytic k, i.e., the mapping

$$\epsilon, k \in (-\delta, \delta) \times (k_0 - \alpha, k_0 + \alpha) \mapsto \mathbb{T}_\epsilon^k \in \mathcal{L}(L^2(\partial D))$$

is continuous in ϵ and analytic in k. Here, $k_0 > 0$ is a transmission eigenvalue of the unperturbed problem and $\alpha > 0$ is chosen sufficiently small such that there are no other transmission eigenvalues of unperturbed problem in $(k_0 - \alpha, k_0 + \alpha)$. In addition of course, we exclude Neumann eigenvalues for the homogeneous version of (4) with $q := n$ and $q := 1$. Note that the base problem with the refractive index n corresponds to $\epsilon = 0$. Therefore, the *main assumption on the base unperturbed problem is that (n, D) is such that the transmission eigenvalues are discrete with $+\infty$ the only accumulation point.* This is the case if ∂D is Lipschitz, $n \in L^\infty(D)$, and $n - 1$ is one sign in a neighborhood of ∂D [2,17,22].

We want to use a version of the implicit function theorem (which is stated in Appendix A) applied to $\Lambda(\epsilon, k)$ in order to show that the perturbed problem for sufficiently small $\epsilon > 0$ has a transmission eigenvalue. In other words, there is a $k := k(\epsilon)$ such that $\Lambda(\epsilon, k(\epsilon)) = 0$. The goal is to prove the existence of real transmission eigenvalues for inhomogeneities (D, n) with refractive index that violates assumptions for which the transmission eigenvalue problem is not understood, such as the contrast $n - 1$ is not one sign in D [5], or is not sufficiently smooth in a neighborhood of ∂D [20,23]. We believe that Theorem 4.1 in Appendix A requires minimal assumptions on our perturbation problem to yield a proof of existence of real transmission eigenvalues for the case when the transmission eigenvalue problem lacks a good structure to apply standard perturbation theory for eigenvalue problems. However, the challenge one has to deal with in this perturbation problem is that we must work with the function $\Lambda(\epsilon, k)$ in a neighborhood of $\epsilon = 0$ and $k = k_0$, where $\Lambda(0, k_0) = 0$, which is an accumulation point for the eigenvalues of the self-adjoint compact operator $\mathbb{T}_0^{k_0}$. This complicates the choice of a particular eigenvalue of $\mathbb{T}_\epsilon^k$ in order to

define a continuous function $\Lambda(\epsilon, k)$ in the neighborhood $(-\delta, \delta) \times (k_0 - \alpha, k_0 + \alpha)$. As a proof of concept, next we consider spherically symmetric inhomogeneities which allows for an explicit definition of $\Lambda(\epsilon, k)$ where we apply the implicit function theorem.

3 Spherically symmetric media

We are interested here in spherically symmetric media in $\mathbb{R}^3$ (to fix our presentation we consider the 3-dimensional case, but the 2-dimensional case can be handled exactly in the same way), i.e., when the inhomogeneity D is the unit ball $B := \{x \in \mathbb{R}^3 : |x| < 1\}$ and $n(r) \geq n > 0$ is $n \in L^\infty(0, 1)$. The existence of transmission eigenvalues in this case has previously been proved under the assumption that $n(r)$ is in C^2 since the approach makes use of Liouville's transformation which involves second derivatives on n [12]. The goal here is to show examples of existence of real transmission eigenvalues for spherically symmetric $L^\infty(B)$ refractive index such that $n(r) - 1$ is not one sign uniformly in D (the case of one sign contrast uniformly in D is covered by the general case discussed in [5]).

The transmission problem for spherically stratified media reads

$$\Delta w + k^2 n(r) w = 0 \quad \text{in} \ \ B, \tag{6}$$

$$\Delta v + k^2 v = 0 \quad \text{in} \ \ B, \tag{7}$$

$$w = v \quad \text{in} \ \ \partial B, \tag{8}$$

$$\frac{\partial w}{\partial r} = \frac{\partial v}{\partial r} \quad \text{on} \ \ \partial B. \tag{9}$$

Introducing spherical coordinates (r, θ, φ) we look for solutions of (6)–(9) in the form

$$v(r, \hat{x}) = a_\ell j_\ell(kr) Y_\ell^m(\hat{x}),$$

$$w(r, \hat{x}) = b_\ell y_\ell(r) Y_\ell^m(\hat{x}),$$

where $\hat{x} \in S^2$ is the angular variable (S^2 denotes the unit sphere), $Y_\ell^m(\hat{x})$, $\ell = 0, 1, 2, \ldots$, $m = -\ell, \ldots, \ell$ are spherical harmonics which form a complete orthogonal system in $L^2(S^2)$, j_ℓ are the spherical Bessel functions, a_ℓ and b_ℓ are constants, and $y_\ell := y_\ell(r; k, n)$ is a solution of

$$y'' + \frac{2}{r} y' + \left(k^2 n(r) - \frac{\ell(\ell+1)}{r^2} \right) y = 0 \tag{10}$$

for $r > 0$ such that $y_\ell(r)$ behaves like $j_\ell(kr)$ as $r \to 0$, i.e.,

$$\lim_{r \to 0} r^{-\ell} y_\ell(r) = \frac{\sqrt{\pi} k^\ell}{2^{\ell+1} \Gamma(\ell + 3/2)}.$$

From Colton [10, pp. 45–50], we can represent $y_\ell(r)$ in the form

$$y_\ell(r) = j_\ell(kr) + \int_0^r G(r, s, k) j_\ell(ks) ds, \tag{11}$$

where $G(r, s, k)$ satisfies the Goursat problem

$$r^2 \left[\frac{\partial^2 G}{\partial r^2} + \frac{2}{r} \frac{\partial G}{\partial r} + k^2 n(r) G \right] = s^2 \left[\frac{\partial^2 G}{\partial s^2} + \frac{2}{s} \frac{\partial G}{\partial s} + k^2 G \right],$$

$$G(r, r, k) = \frac{k^2}{2r} \int_0^r \rho \left(1 - n(\rho) \right) d\rho,$$

$$G(r, s, k) = O\left((rs)^{1/2} \right),$$

for $0 < s \leq r < a$. It is shown in [10,11] that G can be solved by successive approximation for $n \in L^\infty(B)$. We do not show here these standard calculations for solving the Goursat

problem and refer the reader to Section 2 in [11] for the details. Estimates in [11, Section 2] in addition show that

$$\sup_{r,s\in[0,1]} |G(r,s,k)| \leq C_1 \|n\|_{L^\infty(0,1)}, \qquad \sup_{r,s\in[0,1]} \left| \frac{d}{dr} G(r,s,k) \right| \leq C_2 \|n\|_{L^\infty(0,1)}, \tag{12}$$

where the positive constants C_1 and C_2 are independent of n. Hence, (11) and (12) imply that the solution $y_\ell \in H^1(0,1)$ satisfies

$$\sup_{r\in[0,1]} |y_\ell(r)| \leq c_1 \|n\|_{L^\infty(0,1)} \quad \text{and} \quad \sup_{r\in[0,1]} |y_\ell'(r)| \leq c_2 \|n\|_{L^\infty(0,1)}, \tag{13}$$

with positive constants c_1 and c_2 independent of n. In particular, one can see that the point value $y_\ell(1) := y_\ell(1,k)$ is an entire function of k.

Next for a fixed integer $\ell \geq 0$, we consider the following problem for the radial function $y_\epsilon(r) := y_{\ell,\epsilon}(r)$:

$$y_\epsilon'' + \frac{2}{r} y_\epsilon' + \left(k^2 n_\epsilon(r) - \frac{\ell(\ell+1)}{r^2} \right) y_\epsilon = 0 \tag{14}$$

on $(0,1)$ with the condition at the origin

$$\lim_{r\to 0} r^{-\ell} y_\epsilon(r) = \frac{\sqrt{\pi} k^\ell}{2^{\ell+1} \Gamma(\ell+3/2)}, \tag{15}$$

which is perturbed off of the smooth $n(r) \in C^2$ background problem

$$y_0'' + \frac{2}{r} y_0' + \left(k^2 n - \frac{\ell(\ell+1)}{r^2} \right) y_0 = 0 \tag{16}$$

on $(0,1)$ with the same condition at the origin

$$\lim_{r\to 0} r^{-\ell} y_0(r) = \frac{\sqrt{\pi} k^\ell}{2^{\ell+1} \Gamma(l+3/2)}. \tag{17}$$

Proposition 3.1 *Assume that $\ell \geq 0$, and that the n_ϵ are uniformly bounded in L^∞ and that n_ϵ converges weak* L^∞ to n. Then, the solutions y_ϵ to (14), (15) converge strongly in L^2 and pointwise to y_0, the solution to (16), (17). Furthermore, the point value of $y_\epsilon'(1)$ converges to $y_0'(1)$ as $\epsilon \to 0$.*

Proof From the assumption of L^∞ boundedness of n_ϵ and (13), we have that y_ϵ and y_ϵ' are both uniformly bounded on $(0,1)$ by the L^∞-norm of n_ϵ. This implies that y_ϵ is a bounded sequence in $H^1(0,1)$, and hence is pre-compact in $L^2(0,1)$. Consider the difference $z_\epsilon = y_\epsilon - y_0$. We calculate that z_ϵ solves

$$z_\epsilon'' + \frac{2}{r} z_\epsilon' + \left(k^2 n - \frac{\ell(\ell+1)}{r^2} \right) z_\epsilon = -k^2 (n_\epsilon - n) y_\epsilon \tag{18}$$

on $(0,1)$ with the homogeneous condition at the origin

$$\lim_{r\to 0} r^{-\ell} z_\epsilon(r) = 0. \tag{19}$$

This is a generalized Bessel equation with a right-hand side

$$g_\epsilon = -k^2 (n_\epsilon - n) y_\epsilon,$$

so that we can apply variation of parameters. The homogeneous solutions are spanned by the analogues of j_ℓ and Y_ℓ, which in the constant n case are spherical Bessel functions of the first and second kind. (For simplicity of presentation we keep the notation corresponding

to the constant n and remark that j_ℓ is y_0 and Y_ℓ is the other linearly independent solution of the homogeneous equation with n which is singular at the origin.) Their Wronskian

$$\mathbb{W}(j_\ell, Y_\ell) = j_\ell Y_\ell' - j_\ell' Y_\ell,$$

we know is nonzero, and any particular solution of (18) is given by

$$z_\epsilon = -j_\ell \int \frac{Y_\ell g_\epsilon}{\mathbb{W}} dr + Y_\ell \int \frac{j_\ell g_\epsilon}{\mathbb{W}} dr. \tag{20}$$

By the condition at the origin (19) and the properties of J_l and Y_ℓ there, we must have that

$$z_\epsilon(r) = -j_\ell(r) \int_0^r \frac{Y_\ell g_\epsilon}{\mathbb{W}}(\rho) d\rho + Y_\ell(r) \int_0^r \frac{j_\ell g_\epsilon}{\mathbb{W}}(\rho) d\rho. \tag{21}$$

Hence, we have that

$$y_\epsilon(r) = y_0 + k^2 j_\ell(r) \int_0^r \frac{Y_\ell(n_\epsilon - n)y_\epsilon}{\mathbb{W}}(\rho) d\rho - k^2 Y_\ell(r) \int_0^r \frac{j_\ell(n_\epsilon - n)y_\epsilon}{\mathbb{W}}(\rho) d\rho. \tag{22}$$

From the precompactness of y_ϵ, there exists a strongly convergent L^2 subsequence; call the limit $\hat{y}$. Now we note that at $t \to 0$ we have $\mathbb{W}(t) \sim t^{-2}$, $j_\ell(t) \sim t^\ell$, $Y_\ell(t) \sim t^{-(\ell+1)}$ and $y_\epsilon(t) \sim t^\ell$, so the products with y_ϵ will converge strongly, in particular in $L^1(0,1)$. Since $n_\epsilon - n$ converges weak* L^∞ to zero by assumption, taking the limit in (22) for this subsequence yields $\hat{y} = y_0$, and hence the entire sequence must converge strongly in L^2 to y_0. Once we have this, (22) yields the pointwise convergence of y_ϵ. We can then further calculate

$$z_\epsilon'(r) = -j_\ell'(r) \int_0^r \frac{Y_\ell g_\epsilon}{\mathbb{W}}(\rho) d\rho + Y_\ell'(r) \int_0^r \frac{j_\ell g_\epsilon}{\mathbb{W}}(\rho) d\rho \tag{23}$$

due to the fact that the other terms in the product rule cancel, and hence

$$z_\epsilon'(1) = k^2 j_\ell'(1) \int_0^1 \frac{Y_\ell y_\epsilon}{\mathbb{W}}(\rho) \left(n_\epsilon(\rho) - n(\rho)\right) d\rho - k^2 Y_l'(1) \int_0^1 \frac{j_\ell y_\epsilon}{\mathbb{W}}(\rho) \left(n_\epsilon(\rho) - n(\rho)\right) d\rho. \tag{24}$$

By the weak convergence of n_ϵ and the strong convergence of y_ϵ, the result follows. $\quad\square$

As stated earlier, k being a transmission eigenvalue is equivalent to k being such that the kernel of $T_{n,k} - T_{1,k}$ is nontrivial, i.e., there exists $g \neq 0$ in $L^2(S^2)$ such that $(T_{n,k} - T_{1,k})g \neq 0$, or in other words 0 is an eigenvalue of this operator $T_{n,k} - T_{1,k}$. Here $T_{q,k}$ is the Neumann-to-Dirichlet operator

$$T_{q,k} : g \mapsto u(1, \hat{x})$$

for $q = n$ or $q = 1$, where $u(r, \hat{x})$ solves

$$\Delta u + k^2 q(r)u = 0 \quad \text{for } r < 1 \quad \text{and} \quad \frac{\partial u}{\partial r} = g(\hat{x}) \quad \text{for } r = 1.$$

From now on we assume that k^2 is not a Neumann eigenvalue of the above problem for either $q = 1$ or $q = n(r)$. Let

$$g(\hat{x}) = \sum_{\ell=0}^\infty \sum_{m=-\ell}^\ell g_\ell Y_\ell^m(\hat{x}). \tag{25}$$

Then, the solution of $\Delta v + k^2 v = 0$ is

$$v(r, \hat{x}) = \sum_{\ell=0}^\infty \sum_{m=-\ell}^\ell a_\ell j_\ell(kr) Y_\ell^m(\hat{x}),$$

and with Neumann data *g*, this takes the form

$$v(r, \hat{x}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{g_\ell}{k j'_\ell(k)} j_\ell(kr) Y_\ell^m(\hat{x})$$

where $j'_\ell(t) := \frac{dj_\ell(t)}{dt}$. Hence

$$T_{1,k} g = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{g_\ell j_\ell(k)}{k j'_\ell(k)} Y_\ell^m(\hat{x}).$$

Similarly the solution of $\Delta w + k^2 n(r) w = 0$ is

$$w(r, \hat{x}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} b_\ell y_\ell(r; k, n) Y_\ell^m(\hat{x})$$

and with Neumann data *g*, this takes the form

$$w(r, \hat{x}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{g_\ell}{y'_\ell(1; k, n)} y_\ell(r; k, n) Y_\ell^m(\hat{x}),$$

and therefore

$$T_{n,k} g = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{g_\ell y_\ell(1; k, n)}{y'_\ell(1; k, n)} Y_\ell^m(\hat{x}).$$

Thus, letting $y_\ell(r = 1; k, n) := y(k, n)$ we have that $(T_{n,k} - T_{1,k})g = 0$ yields

$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} g_\ell \left(\frac{y_\ell(k, n)}{y'_\ell(k, n)} - \frac{j_\ell(k)}{k j'_\ell(k)} \right) Y_\ell^m(\hat{x}) = 0.$$

So a transmission eigenvalue *k* corresponds to

$$\frac{y_\ell(k, n)}{y'_\ell(k, n)} - \frac{j_\ell(k)}{k j'_\ell(k)} = 0,$$

that is, *k* are the zeros of determinants

$$D_\ell(k, n) = \det \begin{pmatrix} y_\ell(k, n) & -j_\ell(k) \\ y'_\ell(k, n) & -k j'_\ell(k) \end{pmatrix} = 0, \tag{26}$$

which have been extensively studied (see, e.g., [2]). Note that if $n(r) := n$ is a positive constant, then $y_\ell(k, n) = j_\ell(k\sqrt{n})$ and $y'_\ell(k, n) = \sqrt{n} k j'_\ell(k\sqrt{n})$. In particular if $n(r) \in C^2[0, 1]$ and $\delta := \int_0^1 \sqrt{n(\rho)} d\rho \neq 0$ we already know that there exists real eigenvalues $k > 0$ and the corresponding eigenspace is of finite dimension.

Next we look at the eigenvalues of the self-adjoint compact $\mathbb{T}_{n,k} := T_{n,k} - T_{1,k}$, i.e.,

$$\mathbb{T}_{k,n} g - \Lambda(0, k) g = 0, \quad \text{for } g \neq 0.$$

Hence, for an eigenvector *g* given by (25) we have that

$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} g_\ell \left(\frac{y_\ell(k, n)}{y'_\ell(k, n)} - \frac{j_\ell(k)}{k j'_\ell(k)} - \Lambda(0, k)) \right) Y_\ell^m(\hat{x}) = 0.$$

Thus, eigenvalues are given by

$$\Lambda_\ell(0, k) := \frac{y_\ell(k, n)}{y'_\ell(k, n)} - \frac{j_\ell(k)}{k j'_\ell(k)},$$

with corresponding eigenfunction $g := g_\ell Y_\ell^m(\hat{x})$ and it has at least multiplicity $2\ell + 1$. From the assumption on k^2 we have that all the denominators are different from zero. Note that for $n(r) = n$ constant we have

$$\Lambda_\ell(0, k) := \frac{j_\ell(\sqrt{n}k)}{\sqrt{n}kj'_\ell(\sqrt{n}k)} - \frac{j_\ell(k)}{kj'_\ell(k)}.$$

Again, it is already known that if $n \in C^2[0, 1]$ each $\Lambda_\ell(0, k)$ has infinitely many real zeros which are transmission eigenvalues. Our goal is to show the existence of real transmission eigenvalues for perturbations of a C^2 refractive index $n(r)$.

To this end let $n \in C^2[0, 1]$ and consider a one-parametric perturbation $n_\epsilon \in L^\infty(0, 1)$ of n such that n_ϵ are uniformly bounded in L^∞ and n_ϵ converges weak* L^∞ to n. From the above calculation we have that the eigenvalues $\Lambda(\epsilon, k)$ of $\mathbb{T}_{n_\epsilon, k} := T_{n_\epsilon, k} - T_{1, k}$ are given by

$$\Lambda_\ell(\epsilon, k) := \frac{y_\ell(k, n_\epsilon)}{y'_\ell(k, n_\epsilon)} - \frac{j_\ell(k)}{kj'_\ell(k)}.$$

We fix $\ell \in \mathbb{N}_0$ and consider $k_0 > 0$ such that $\Lambda_\ell(0, k_0) = 0$, i.e., a transmission eigenvalue. Then, we consider the function of two variables $\Lambda_\ell(\epsilon, k)$ as a function defined $\Lambda_\ell : (-\delta, \delta) \times (k_0 - \alpha, k_0 + \alpha)$, with $\alpha > 0$ sufficiently small such that no other transmission eigenvalues of (n, B) are in $(k_0 - \alpha, k_0 + \alpha)$. From Proposition 3.1, we have that $\Lambda_\ell(\epsilon, k)$ satisfies the assumptions 1–3 of the version of the implicit function theorem (Theorem 4.1). Hence, if we in addition require that

$$\frac{d\Lambda_\ell(0, k)}{dk}\Big|_{k=k_0} \neq 0, \tag{27}$$

we can show that there exists $\epsilon < \delta$ such that for every ϵ satisfying $|\epsilon| < \epsilon$ there exists $k := k(\epsilon)$ such that $\Lambda_\ell(\epsilon, k(\epsilon)) = 0$. Thus, there exists at least one real transmission eigenvalue corresponding to n_ϵ in a neighborhood of k_0. Obviously we apply the above reasoning for every transmission eigenvalue separately, corresponding to every $\ell \in \mathbb{N}_0$, as long as the condition (27) is satisfied.

Next, let us investigate a bit further condition (27). We can obviously write

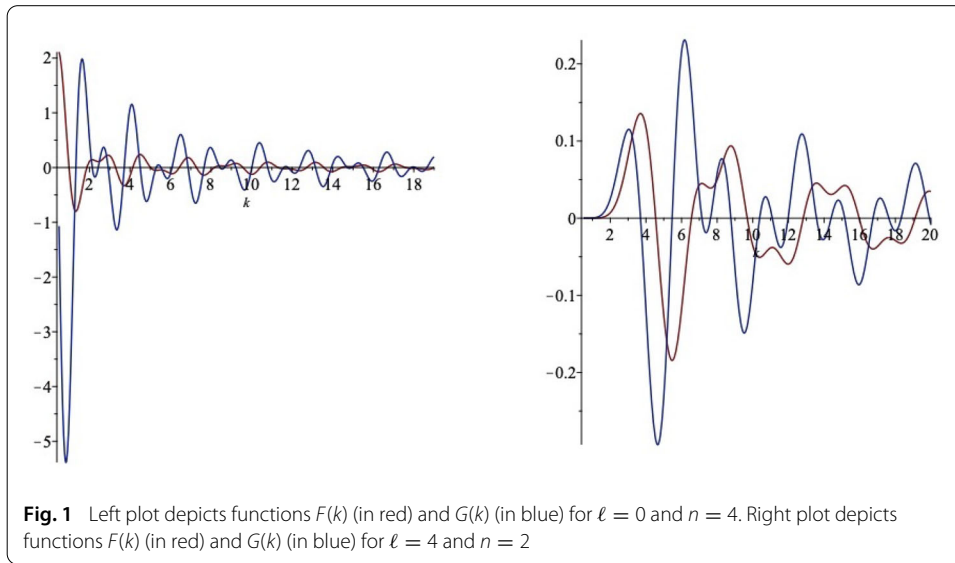$$k\Lambda_\ell(0, k) := \frac{D_\ell(k, n)}{y'_\ell(k, n)j'_\ell(k)}$$

where $D(k, n)$ is the determinant given by (26). If k_0 is a transmission eigenvalue such that $\Lambda_\ell(0, k_0) = 0$, then we have

$$D_\ell(k_0, n) := k_0 j'_\ell(k_0)y_\ell(k_0, n) - j_\ell(k_0)y'_\ell(k_0, n) = 0.$$

Thus, because $D_\ell(k_0, n) = \Lambda_\ell(0, k_0) = 0$ condition (27) is equivalent to

$$\frac{dD_\ell(k, n)}{dk}\Big|_{k=k_0} \neq 0. \tag{28}$$

Remark 3.1 Transmission eigenfunctions corresponding to a transmission eigenvalue k_0 as zero of $D_\ell(k_0, n) = 0$ have $j_\ell(r)$ as the radial part multiplied by $2\ell + 1$ spherical harmonics. The condition (27) means that the algebraic multiplicity of the transmission eigenvalue k_0 is not greater than its geometric multiplicity which is counted by the number of indices ℓ corresponding to the radial part of the eigenfunctions. Note also that it is possible for a transmission eigenvalue k_0 to be a zero of $D_\ell(k, n)$ for more than one index ℓ [12].

Fig. 1 Left plot depicts functions $F(k)$ (in red) and $G(k)$ (in blue) for $\ell = 0$ and $n = 4$. Right plot depicts functions $F(k)$ (in red) and $G(k)$ (in blue) for $\ell = 4$ and $n = 2$

We cannot prove whether the condition (27) is satisfied. Up to date there is no theoretical results regarding when this is the case for spherically symmetric media. Of course, it always holds for simple eigenvalues. However, if we perturb off of constant media, i.e., $n > 0$ is constant, explicit calculations on Maple indicate that this condition is satisfied for a large number of examples that we tried. In particular for constant n the determinant $D_\ell(k, n)$ after dividing by k takes the form

$$D_\ell(k, n) := j'_\ell(k)j_\ell(k\sqrt{n}) - \sqrt{n}j_\ell(k)j'_\ell(k\sqrt{n}).$$

Hence, (27) at a transmission eigenvalue k_0 becomes

$$\frac{dD_\ell(k_0, n)}{dk}\bigg|_{k=k_0} = j''_\ell(k_0)j_\ell(k_0\sqrt{n}) - nj_\ell(k_0)j''_\ell(k_0\sqrt{n}) \neq 0 \qquad (29)$$

For various values of $\ell \geq 0$ and constant $n > 0$, we have used Maple to plot the following function of k

$$F(k) := j'_\ell(k)j_\ell(k\sqrt{n}) - \sqrt{n}j_\ell(k)j'_\ell(k\sqrt{n})$$

whose zeros are transmission eigenvalues and

$$G(k) := j''_\ell(k)j_\ell(k\sqrt{n}) - nj_\ell(k)j''_\ell(k\sqrt{n})$$

and observed that $G(k)$ is not zero at the zeros of $F(k)$. We have shown in Figure 1 two instances of these plots.

Summarizing, we have proven the following theorem.

Theorem 3.1 *Assume that $n \in C^2[0, 1]$ and $n_\epsilon \in L^\infty(0, 1)$ is a one-parametric perturbation of n such that the $L^\infty(0, 1)$-norm of n_ϵ is uniformly bounded on ϵ, and n_ϵ converges to n weak* L^∞. Then, the inhomogeneity (n_ϵ, B) for sufficiently small ϵ has a real transmission eigenvalue in a neighborhood of any transmission eigenvalue k_0 of the unperturbed inhomogeneity (n, B), provided that algebraic multiplicity of k_0 is not greater than its geometric multiplicity as defined in Remark 3.1.*

3.1 Examples of perturbations

The existence of real transmission eigenvalues is known for general media with support D with refractive index $n \in L^\infty(D)$ only under the assumption that either $n(x) > 1$

or $0 < n(x) < 1$ uniformly almost everywhere in D. Otherwise, the existence of real transmission eigenvalues has been shown for spherically symmetric media assuming that $n(r) \in C^2[0, 1]$ such that $\int_0^1 \sqrt{n(\rho)}\,d\rho \neq 1$. Our result of Theorem 3.1 can provide example of existence for spherically symmetric media not covered by the known results. In the following, we provide a few examples.

Example 1 (*L^∞ perturbations; sign changes up to the boundary*) The above theory implies existence of m real transmission eigenvalues for any refractive index of the form

$$n_\epsilon(r) = n(r) + \epsilon \gamma_\epsilon(r)$$

for any $n \in C^2$ and γ_ϵ uniformly bounded in L^∞, for ϵ small enough. In particular, let B be the ball of radius one and define

$$n_\epsilon(r) := n + \epsilon \sin \frac{1}{r-1}$$

for $|\epsilon| \leq \epsilon$, where for example n is a constant and $n \neq 1$. Obviously, $n_\epsilon \in L^\infty(B)$ and is not even in $C[0, 1]$. Theorem 3.1 provides the existence of real transmission eigenvalues for (B, n_ϵ). In fact for sufficiently small $\epsilon_0 > 0$ we can prove the existence of finitely many real transmission eigenvalues and their number depends on how many real transmission eigenvalues corresponding to n satisfy condition (28) and how small ϵ is. More precisely if there are m real transmission eigenvalues that satisfy (28) and ϵ is chosen to be smaller than the minimum of all distances between these eigenvalues, then there exists $m(\epsilon)$ transmission eigenvalues corresponding to $n_\epsilon(r)$.

In particular if $n(1) = 1$, then the above n_ϵ provides an example of the contrast $n_\epsilon - 1$ changing sign in any neighborhood of a boundary point. The spectral analysis for the transmission eigenvalue problem for this case is completely open.

Example 2 (*L^1 perturbations*) We also obtain existence of real transmission eigenvalues for radially symmetric small volume perturbations (thin shells), which could be sign-changing. Let us again let B be the ball of radius 1 and let

$$n_\epsilon(r) = n + \alpha \chi_{[r_0-\epsilon, r_0]} - \beta \chi_{[r_0, r_0+\epsilon]}$$

for constants α, β where χ are radial characteristic functions and r_0 is any radial value between 0 and 1. One could also have small volume but fixed max contrasts which are sign-changing up to the boundary, such as

$$n_\epsilon(r) := n + \chi_{[1-\epsilon, 1]} \sin \frac{1}{r-1}.$$

That is, for any positive integer m, there exists ϵ such that the above scatterers have m real transmission eigenvalues for any $\epsilon < \epsilon$.

Example 3 (*Weak perturbations; periodicity*) Consider a highly oscillating radial refractive index, for example,

$$n_\epsilon(r) = n(\rho) = n\left(\frac{r}{\epsilon}\right)$$

where n is periodic in $\rho \in [0, 1]$. Here n is bounded but possibly discontinuous and sign-changing (perhaps even up to the boundary as in the above examples). It is well known,

see for example [1], that $n(\frac{r}{\epsilon})$ converges weak* in L^∞ to its constant cell average,

$$n = \bar{n} = \int_0^1 n(\rho)d\rho.$$

Hence, by the above result, assuming that $n \neq 1$, for any positive integer m, there exists small enough period size such that the radially periodic scatterer has m real transmission eigenvalues. Note that for such examples, the contrast may be everywhere large. We should point out that it is already known that for non-sign-changing but general (full dimension) periodic refractive index, one has convergence of the transmission eigenvalues to those of the homogenized problem [6] (see also [7] for the scattering problem for periodic media of bounded support).

4 Future directions

It would be of interest to develop perturbation theory for transmission eigenvalues which applies beyond the spherically symmetric case treated in detail in Sect. 3. However, this requires perturbing the zero eigenvalue of a compact operator, which carries with it a number of difficulties. The difficulties were overcome above because of a wealth of spectral information available owing to the spherical symmetry. Nevertheless, in the general case, we offer in this section a future direction for investigation of perturbations of transmission eigenvalues in general. Namely, we give a condition on the unperturbed problem which, if satisfied, guarantees existence of approximate transmission eigenvalues under perturbation. Here, the approximate transmission eigenvalues are transmission eigenvalues for the same problem projected onto finite-dimensional subspaces of any sufficiently large dimension. We now make this precise.

For all k, for all ϵ, the operator $\mathbb{T}_\epsilon^k$ maps the Hilbert space $L^2(\partial D)$ to itself; we denote $X = L^2(\partial D)$, so $\mathbb{T}_\epsilon^k : X \to X$. If k_0 is a transmission eigenvalue of the unperturbed problem, then 0 is an eigenvalue of $\mathbb{T}_0^{k_0}$, with the eigenvector being the Neumann data f. For $j \in \mathbb{N}$, let X_j be a subspace of X, with $X_j \subseteq X_{j+1}$ for all $j \in \mathbb{N}$, and with $X = \bigcup_{j \in \mathbb{N}} X_j$. Assume that the subspaces X_j are such that there exists $J \in \mathbb{N}$ such that for all $j > J$ we have $f \in X_j$. Let the inner product on X_j be the inner product induced by the inner product on X. Let P_j be the projection onto X_j.

We define $\mathbb{T}_{\epsilon,j}^k = P_j \mathbb{T}_\epsilon^k \upharpoonright_{X_j}$, and we note that $\mathbb{T}_{\epsilon,j}^k : X_k \to X_k$. For ease of notation, we will write this simply as $\mathbb{T}_{\epsilon,j}^k = P_j \mathbb{T}_\epsilon^k$, with the understanding that the domain is restricted to be X_j. Then, for $j > J$, we consider $\mathbb{T}_{0,j}^{k_0} f$, and we find

$$\mathbb{T}_{0,j}^{k_0} f = P_j \mathbb{T}_0^{k_0} = P_j 0 = 0.$$

Therefore, 0 is an eigenvalue of $\mathbb{T}_{0,j}^{k_0}$ with eigenvector f. Furthermore, we also have

$$\left(\mathbb{T}_{0,j}^{k_0}\right)^* f = (P_j \mathbb{T}_0^{k_0})^* f = \mathbb{T}_0^{k_0} P_j f = \mathbb{T}_0^{k_0} f = 0.$$

(Here, we have used that projections on Hilbert space are self-adjoint, and that $\mathbb{T}_0^{k_0}$ is also self-adjoint.) Thus, 0 is also an eigenvalue of $\left(\mathbb{T}_{0,j}^{k_0}\right)^*$ with eigenvector f.

Since 0 is an eigenvalue of the finite-dimensional operator $\mathbb{T}_{0,j}^{k_0}$, we can write this eigenvalue as $\Lambda_j(k, \epsilon)$ where $\Lambda_j(k_0, 0) = 0$. We may then use the implicit function theorem to find a function $k_j(\epsilon)$ such that $\Lambda_j(k_j(\epsilon), \epsilon) = 0$ for ϵ in an interval about $\epsilon = 0$. From Proposition 2.1, we have that $\mathbb{T}_0^k$ depends analytically on k in regions of the complex plane $\mathbb{C}$ that

exclude k such that k^2 are eigenvalues of (4) with $q = A$, $p = n$ and $q = I$, $p = 1$. In what follows, $\langle \cdot, \cdot \rangle_{L^2(\partial D)}$ denotes the $L^2(\partial D)$ inner product.

Theorem 4.1 *Assume that*

$$\left\langle \left. \frac{d\mathbb{T}_\epsilon^k}{dk} \right|_{(k,\epsilon)=(k_0,0)} f, f \right\rangle \neq 0.$$

For all $j > J$, there exists $\epsilon_{j,*} > 0$ such that for all $\epsilon \in (-\epsilon_{j,*}, \epsilon_{j,*})$, there exists $k_j(\epsilon)$ with $k_j(0) = k_0$, and such that $\Lambda_j(k_j(0), 0) = 0$.

Proof To apply the implicit function theorem, we need to know that Λ_j is continuous; this is true in the current finite-dimensional setting, since eigenvalues in finite dimensions are the zeros of polynomials, and since the zeros of polynomials depend continuously on the coefficients, which in turn depend continuously on the parameters.

Then, it remains to show that

$$\frac{d\Lambda_j}{dk}(k_0, 0) \neq 0.$$

Since $\mathbb{T}_0^{k_0}$ is analytic with respect to k, we have that the finite-dimensional restriction $\mathbb{T}_{0,j}^{k_0}$ is also analytic with respect to k. By a classical result of Rellich [18] (see also [16]), we then have that the eigenvalues are differentiable with respect to k, with formula

$$\frac{d\Lambda_j}{dk}(k_0, 0) = \frac{\left\langle \left. \frac{d\mathbb{T}_{\epsilon,j}^k}{dk} \right|_{(k,\epsilon)=(k_0,0)} f, f \right\rangle}{\langle f, f \rangle}.$$

This simplifies since $\frac{d\mathbb{T}_{\epsilon,j}^k}{dk} = \frac{d(P_j \mathbb{T}_\epsilon^k)}{dk} = P_j \frac{d\mathbb{T}_\epsilon^k}{dk}$, and also since P_j is self-adjoint and $P_j f = f$. These considerations yield

$$\frac{d\Lambda_j}{d\tau}(k_0, 0) = \frac{\left\langle \left. \frac{d\mathbb{T}_\epsilon^k}{dk} \right|_{(k,\epsilon)=(k_0,0)} f, f \right\rangle}{\langle f, f \rangle}.$$

The numerator of this right-hand side is the quantity which we have assumed to be nonzero. Thus, the implicit function theorem (specifically, Theorem 4.1) applies, and we find the existence of a continuous curve $k_j(\epsilon)$ such that

$$\Lambda_j(k_j(\epsilon), \epsilon) = 0.$$

The curve satisfies $k_j(0) = k_0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Thus, we see that under fairly general conditions on the unperturbed problem, one has approximate transmission eigenvalues for the perturbed problem, at any level, j, of approximation. Of course, to take the limit of these transmission eigenvalues as j goes to infinity would require some uniform control.

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A A specific version of the implicit function theorem

The following theorem is meant to be applied to $f = \Lambda$, with x corresponding to ϵ and y corresponding to k, and $(\bar{x}, \bar{y}) = (0, k_0)$.

Theorem 4.1 (A very specific version of the Implicit Function Theorem from Theorem 2 of [15]) *Let X and Y be open subsets of $\mathbb{R}$. Suppose that $(\bar{x}, \bar{y}) \in X \times Y$ and $f : X \times Y \to \mathbb{R}$. Suppose that*

1. *$f(\bar{x}, \bar{y}) = 0$,*
2. *$f(x, y)$ is continuous in y for every fixed $x \in X$,*
3. *$f(x, y)$ is continuous in x for every fixed $y \neq \bar{y}, y \in Y$,*
4. *$f_y(\bar{x}, \bar{y})$ exists and is not equal to zero,*

then there exist open $X_0 \subset X$ and $Y_0 \subset Y$ with $(\bar{x}, \bar{y}) \in X_0 \times Y_0$ and function $\phi : X_0 \to Y_0$ such that

$$f(x, \phi(x)) = 0 \; \forall x \in X_0$$

and

$$\phi(\bar{x}) = \bar{y}.$$

Proof Consider the function

$$F(x, y) = \frac{f(x, y)}{f_y(\bar{x}, \bar{y})}.$$

For y near $\bar{y}$,

$$f(\bar{x}, y) = f(\bar{x}, \bar{y}) + (y - \bar{y})f_y(\bar{x}, \bar{y}) + o(y - \bar{y})$$

by existence of the partial derivative. Since $f(\bar{x}, \bar{y}) = 0$,

$$F(\bar{x}, y) = y - \bar{y} + \frac{o(y - \bar{y})}{f_y(\bar{x}, \bar{y})}$$

where the denominator is a constant. Hence, for $\delta > 0$ and $(\bar{y} - \delta, \bar{y} + \delta) \subset Y$,

$$F(\bar{x}, y + \delta) = \delta + o(\delta)$$

and

$$F(\bar{x}, y - \delta) = -\delta + o(\delta).$$

So, there exists δ such that

$$F(\bar{x}, y + \delta) > 0$$

and

$$F(\bar{x}, y - \delta) < 0.$$

Keeping δ fixed, by the continuity of F in x at $y = \bar{y} + \delta$ and $y = \bar{y} - \delta$, there exists a ball X_0 around $\bar{x}$ in X such that

$$F(x, y + \delta) > 0$$

and

$$F(x, y - \delta) < 0$$

for every $x \in X_0$. Since F is continuous in y for any $x \in X_0$, the intermediate value theorem applies and hence for each $x \in X_0$ there exists $\phi(x) \in Y_0 := (\bar{y} - \delta, \bar{y} + \delta)$ such that

$$F(x, \phi(x)) = 0.$$

$\square$

We note that the third hypothesis could be relaxed, as long as there were a sequence of points converging to $\bar{y}$ where continuity in x holds.

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