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On vanishing near corners of conductive transmission eigenfunctions

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Abstract

This paper is concerned with the geometric structure of the transmission eigenvalue problem associated with a general conductive transmission condition. We prove that under a mild regularity condition in terms of the Herglotz approximations of one of the pair of the transmission eigenfunctions, the eigenfunctions must be vanishing around a corner on the boundary. The Herglotz approximation is the Fourier extension of the transmission eigenfunction, and the growth rate of the density function can be used to characterize the regularity of the underlying wave function. The geometric structures derived in this paper include the related results in Diao et al. (Commun Partial Differ Equ 46(4):630–679, 2021) and Blåsten and Liu (J Funct Anal 273:3616–3632, 2017) as special cases and verify that the vanishing around corners is a generic local geometric property of the transmission eigenfunctions.

Keywords: Conductive transmission eigenfunctions, Corner singularity, Geometric structures, Vanishing, Herglotz approximation

Mathematics Subject Classification: Primary 35P25, 35R30; Secondary 58J05, 78A05

1 Introduction

1.1 Background

In its general form, the transmission eigenvalue problem is given as follows (cf. [26]):

$$\mathcal{P}_1(\mathbf{x}, D)u = \lambda u, \quad \mathcal{P}_2(x, D)v = \lambda v \text{ in } \Omega; \quad \mathcal{C}(u) = \mathcal{C}(v) \text{ on } \partial\Omega, \tag{1.1}$$

where Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, $n = 2, 3$, the interior of its complement is connected and $P_j(\mathbf{x}, D)$ are two elliptic partial differential operators (PDOs) with D signifying the differentiations with respect to $\mathbf{x} = (x_j)_{j=1}^n \in \mathbb{R}^n$, and $\mathcal{C}$ denotes the Cauchy data set. If there exists a non-trivial pair of solutions (u, v), then $\lambda \in \mathbb{C}$ is called a transmission eigenvalue and (u, v) are the corresponding pair of transmission eigenfunctions.

Though the PDOs $\mathcal{P}_j$, $j = 1, 2$, are generally elliptic, self-adjoint and linear, the transmission eigenvalue problems of the form (1.1) are a type of non-elliptic, non-self-adjoint and nonlinear (in terms of the transmission eigenvalue λ) spectral problems, making the corresponding spectral study highly intriguing and challenging; see [26] for some related discussion. The transmission eigenvalue problems arise in the wave scattering theory and connect to many aspects of the wave scattering theory in a delicate way. Indeed, many of the spectral results established for the transmission eigenvalue problems in the literature have found important applications in the wave scattering theory, including generating novel wave imaging and sensing schemes, producing important implications to invisibility cloaking and proving new uniqueness results for inverse scattering problems. We refer to [11,12,18,26] for historical accounts and surveys on the state-of-the-art developments of the spectral studies for the transmission eigenvalue problems in the literature.

To a great extent, the spectral properties of the (real) transmission eigenvalues resemble those for the classical Dirichlet/Neumann Laplacian: there are infinitely many real transmission eigenvalues which are discrete and accumulate only at infinity. Nevertheless, due to the non-self-adjointness, there are complex transmission eigenvalues; see [11,18] the references cited therein. Recently, several local and global geometric structures of distinct features were discovered for the transmission eigenfunctions [2–9,13–16,21–24,27–29] and all of them have produced interesting applications of practical importance in the scattering theory. In this paper, we are concerned with the vanishing property of the transmission eigenfunctions around a corner on the boundary of the domain, which was first discovered in [5] and further investigated in [23]. Before discussing our major discoveries, we next specify the transmission eigenvalue problem as well as its vanishing properties in our study.

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n = 2, 3$, with a connected complement $\mathbb{R}^n\backslash\overline{\Omega}$, and $V \in L^\infty(\Omega)$ and $\eta \in L^\infty(\partial\Omega)$ be possibly complex-valued functions. Consider the following transmission eigenvalue problem for $v, w \in H^1(\Omega)$ and $\lambda = k^2$, $k \in \mathbb{R}_+$:

$$\begin{cases} \big(\Delta + k^2(1+V)\big)w = 0 & \text{in } \ \Omega, \\ (\Delta + k^2)v = 0 & \text{in } \ \Omega, \\ w = v, \ \ \partial_\nu w = \partial_\nu v + \eta v & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\nu \in \mathbb{S}^{n-1}$ signifies the exterior unit normal to $\partial\Omega$. Two remarks concerning the formulation of the transmission eigenvalue problem (1.2) are in order. First, we introduce k^2 to denote the transmission eigenvalue. On the one hand, k signifies a wavenumber in the physical setup and on the other hand, this notation shall ease the exposition of our subsequent mathematical arguments. Though only $k \in \mathbb{R}_+$ is physically meaningful, some of our subsequent results also hold for the case that k is a complex number, which should be clear from the context. Second, the second transmission condition on $\partial\Omega$ in (1.2) is known as the conductive transmission condition. This type of transmission condition arises in modelling wave interaction with a certain material object and can find important applications in magnetotellurics; see e.g. [14,23] and the references cited therein for more relevant physical backgrounds. On the other hand, if one simply takes $\eta \equiv 0$, (1.2) is reduced to the transmission eigenvalue problem that has been more intensively studied in the literature. In order to signify such a generalization and extension, we refer to the eigenvalue problem (1.2) as the conductive transmission eigenvalue problem, which includes the conventional transmission eigenvalue problem as a special case.

Let $\mathbf{x}_c \in \partial\Omega$ be a corner point, which shall be made more precise in what follows. Let $B_\rho(\mathbf{x}_c)$ denote a ball of radius $\rho \in \mathbb{R}_+$ centred at $\mathbf{x}_c$. The vanishing property of the transmission eigenfunction is described as follows:

$$\lim_{\rho\to+0} \frac{1}{m\left(B\left(\mathbf{x}_c,\rho\right)\cap\Omega\right)} \int_{B(\mathbf{x}_c,\rho)\cap\Omega} |\psi(\mathbf{x})|\,\mathrm{d}\mathbf{x} = 0, \ \ \psi = w \ \text{ or } \ v, \tag{1.3}$$

where m denotes the Lebesgue measure. It is noted that w and v are H^1-functions and the vanishing at a boundary point should be understood in the integral sense. On the other hand, if ψ is a continuous function in a neighbourhood of $\mathbf{x}_c$, (1.3) clearly implies that $\psi(\mathbf{x}_c) = 0$. In fact, the regularity of the transmission eigenfunctions w and v in (1.2) is critical for the establishment of the vanishing property (1.3). Under the regularity condition that both w and v are additionally Hölder continuous, namely C^α continuous with $\alpha \in (0, 1)$, it is shown in [5] and [23] that the vanishing property holds, respectively, in the cases with $\eta \equiv 0$ and $\eta \neq 0$. By the classical result on the quantitative behaviours of the solutions to elliptic PDEs around a corner (cf. [19,20,25]), we have the following decompositions:

$$w = w_{\text{singular}} + w_{\text{regular}}, \quad v = v_{\text{singular}} + v_{\text{regular}}, \tag{1.4}$$

where the regular parts belong to H^2 and hence by the standard Sobolev embedding, they are Hölder continuous. The singular parts may also be Hölder continuous provided the coefficients, namely V, as well as the boundary data of w and v around the corner are sufficiently regular. However, in the transmission eigenvalue problem (1.2), the boundary data, namely $(w|_{\partial\Omega}, \partial_\nu w|_{\partial\Omega})$ and $(v|_{\partial\Omega}, \partial_\nu v|_{\partial\Omega})$, are not specified. Hence, it may happen that the transmission eigenfunctions are H^1 but not Hölder continuous. Clearly, according to our discussion above, the vanishing property may serve as an indicator for such singular behaviours of the transmission eigenfunctions around the corner. Indeed, according to the extensive numerical examples in [4], though the transmission eigenfunctions generically vanish around a corner, there are cases that the transmission eigenfunctions are not vanishing and instead they are localizing around a corner, especially when the corner is concave. Hence, it is mathematically intriguing and physically significant to thoroughly understand such a singularity formation of the transmission eigenfunctions and its connection to the corresponding vanishing behaviour. In [5,23], a regularity criterion of a different mathematical feature, but more physically related, has been investigated in connection to the vanishing property of the transmission eigenfunction. It is given in terms of the Herglotz approximation of the transmission eigenfunction v in (1.2). The Herglotz approximation in a certain sense is the Fourier transform (in terms of the plane waves) of the eigenfunction v who satisfies the homogeneous Helmholtz equation. Hence, the growth rate of the transformed function, i.e. the density function in the Herglotz wave, can naturally be used to characterize the regularity of the underlying wave function. This resembles the classical way of defining the Sobolev space via the Bessel potentials. Indeed, it is related to the Fourier restriction property in harmonic analysis. In this paper, we shall explore along this direction and derive much sharper estimates to show that the vanishing property of the transmission eigenfunctions holds for a much broader class of functions in terms of the Herglotz approximation. The vanishing property of the transmission eigenfunctions derived in this paper include the corresponding results in [5,23] as special cases.

1.2 Statement of the main results and discussions

In order to present a complete and comprehensive study, the statements of our main results are lengthy and technically involved. Nevertheless, in order to give the readers a global picture of our study, we briefly summarize the major findings in the following two theorems. To that end, we first introduce the Herglotz approximation.

For $g_j \in L^2\left(\mathbb{S}^{n-1}\right)$, we set

$$v_j(\mathbf{x}) = \int_{\mathbb{S}^{n-1}} e^{ik\xi\cdot\mathbf{x}} g_j(\xi)\mathrm{d}\sigma(\xi), \quad \xi \in \mathbb{S}^{n-1}, \mathbf{x} \in \mathbb{R}^n, \tag{1.5}$$

v_j is known as a Herglotz wave with kernel g_j. Physically, it is easy to see that v_j is formed by the superposition of plane waves and it is an entire solution to the Helmholtz equation $\Delta v_j + k^2 v_j = 0$. Hence, g_j can be regarded as the Fourier density of the wave function v_j in terms of the plane waves. We have the following denseness property of the Herglotz waves.

In Theorem 2 of [30], letting $k = 0$, we can obtain the following lemma.

Lemma 1.1 [30] *Let $\Omega \Subset \mathbb{R}^n$ be a bounded Lipschitz domain with a connected complement and $\mathscr{H}_k$ be the space of all the Herglotz wave functions of the form (1.5). Define*

$$\mathscr{S}_k(\Omega) = \{u \in C^\infty(\Omega);\ \Delta u + k^2 u = 0\}$$

and

$$\mathscr{H}_k(\Omega) = \{u|_\Omega;\ u \in \mathscr{H}_k\}.$$

Then $\mathscr{H}_k(\Omega)$ is dense in $\mathscr{S}_k(\Omega) \cap L^2(\Omega)$ with respect to the topology induced by the $H^1(\Omega)$-norm.

Theorem 1.2 *Consider the transmission eigenvalue problem (1.2) with $\eta \not\equiv 0$. Let $\mathbf{x}_c \in \partial\Omega$ be a corner point in two and three dimensions and $\mathcal{N}_h$ be a neighbourhood of $\mathbf{x}_c$ within Ω with $h \in \mathbb{R}_+$ sufficiently small. Suppose that $(1+V)w$ and η are both Hölder continuous on $\overline{\mathcal{N}_h}$ and $\partial\mathcal{N}_h \cap \partial\Omega$, respectively, and $\eta(\mathbf{x}_c) \neq 0$. If there exist constants C, ϱ and Υ with $C > 0$, $\Upsilon > 0$ and $\varrho < \Upsilon$ such that the transmission eigenfunction v can be approximated in $H^1(\mathcal{N}_h)$ by the Herglotz functions v_j, $j = 1, 2, \ldots$, with kernels g_j satisfying*

$$\left\|v - v_j\right\|_{H^1(\mathcal{N}_h)} \le j^{-\Upsilon}, \quad \left\|g_j\right\|_{L^2(\mathbb{S}^{n-1})} \le Cj^{\varrho}, \tag{1.6}$$

then w and v vanish near $\mathbf{x}_c$ in the sense of (1.3).

More detailed results are, respectively, given in Theorems 2.3 and 3.1 for the two and three dimensions.

Remark 1.3 As discussed earlier, the vanishing properties were investigated in [23] under a similar setup to Theorem 1.2. Compared to the corresponding results in [23], Theorem 1.2 has two significant improvements in the regularity requirements. First, the Herglotz approximation condition in [23] was required to be

$$\left\|v - v_j\right\|_{H^1(\mathcal{N}_h)} \le j^{-1-\Upsilon}, \quad \left\|g_j\right\|_{L^2(\mathbb{S}^{n-1})} \le Cj^{\varrho}, \tag{1.7}$$

where the constants $C > 0$, $\Upsilon > 0$ and $0 < \varrho < 1$. It is directly verified that the regularity condition (1.7) is included in (1.6) as a special case. Second, it was required in [23] that $w - v$ is H^2-regular away from the corner point $\mathbf{x}_c$, and we relax this regularity requirement in a certain sense in Theorem 1.2.

Theorem 1.4 *Consider the transmission eigenvalue problem (1.2) with $\eta \equiv 0$. Under the same conditions as in Theorem 1.2, one has*

$$\lim_{\rho \to +0} \frac{1}{m\left(B\left(\mathbf{x}_c, \rho\right) \cap \Omega\right)} \int_{B(\mathbf{x}_c,\rho)\cap\Omega} V(\mathbf{x}) w(\mathbf{x}) \mathrm{d}\mathbf{x} = 0. \tag{1.8}$$

The similar result holds in the three-dimensional case with (1.6) *replaced to be* (3.21)*.*

More detailed results are, respectively, given in Corollaries 2.4 *and* 3.2 *for the two and three dimensions.*

Remark 1.5 If $V(\mathbf{x})$ is continuous near the corner $\mathbf{x}_c$ and $V(\mathbf{x}_c) \neq 0$, from the fact that

$$\lim_{\rho\to+0} \frac{1}{m\left(B\left(\mathbf{x}_c,\rho\right)\cap\Omega\right)} \int_{B(\mathbf{x}_c,\rho)\cap\Omega} V(\mathbf{x})w(\mathbf{x})\mathrm{d}\mathbf{x} = V\left(\mathbf{x}_c\right) \lim_{\rho\to+0} \frac{1}{m\left(B\left(\mathbf{x}_c,\rho\right)\cap\Omega\right)} \int_{B(\mathbf{x}_c,\rho)\cap\Omega} w(\mathbf{x})\mathrm{d}\mathbf{x},$$

one can readily see that w vanishes near $\mathbf{x}_c$, which in turn implies the vanishing of v near $\mathbf{x}_c$ by noting that w and v possess the same traces on $\partial\Omega$.

Remark 1.6 The vanishing of the transmission eigenfunctions in the case $\eta \equiv 0$ was also studied in [5,23]. The regularity requirement in [23] is the same as that described in Remark 1.3, whereas in [5], the Herglotz approximation was required to be

$$\left\|v - v_j\right\|_{L^2(\mathcal{N}_h)} \leq e^{-j}, \quad \left\|g_j\right\|_{L^2(\mathbb{S}^{n-1})} \leq C(\ln j)^{\beta}, \tag{1.9}$$

where the constants $C > 0$ and $0 < \beta < 1/(2n+8)$, $(n = 2, 3)$. It is directly verified that the corresponding results in [5,23] are included into Theorems 1.2 and 1.4 as special cases. Nevertheless, it is pointed out that in [5], the technical condition $(1+V)w$ being Hölder continuous is not required and instead it is required that V is Hölder continuous.

The following are three general remarks on the vanishing properties of the transmission eigenfunctions.

Remark 1.7 The vanishing properties established in Theorems 1.2 and 1.4 as well as those in [5,23] are of a completely local feature. That is, all the results hold for the partial-data transmission eigenvalue problem, namely in (1.2) the transmission boundary conditions on $\partial\Omega$ are required to hold only in a small neighbourhood of the corner point. In particular, this means that all the geometric properties also hold for the so-called exterior transmission eigenfunctions; see [11] for discussions on the exterior transmission eigenvalue problems. It is also mentioned that a global rigidity result of the geometric structure of the transmission eigenfunctions was presented in [15].

Remark 1.8 According to our earlier discussion, if the transmission eigenfunctions w and v are Hölder continuous around the corner, then both of them vanish near the corner. Hence, in order to search for the transmission eigenfunctions that are non-vanishing near corners, especially those numerically found in [4] which are actually locally localizing around corners, one should consider transmission eigenfunctions whose regularity lies between H^1 and C^α, $\alpha \in (0, 1)$. By using properties of the Herglotz approximation (cf. [17]), one can show (though not straightforward) that the regularity criterion (1.7) defines a set of functions which includes some functions that are less regular than C^α, but also does not include some functions which are more regular than C^α. Hence, the regularity characterization in terms of the Herglotz approximation is of a different theoretical nature from the standard Sobolev regularity. Nevertheless, Theorems 1.2 and 1.4 indicate that the vanishing near corners is a generic local geometric property of the transmission eigenfunctions.

Remark 1.9 The vanishing properties of the transmission eigenfunctions are closely related to a quantitative result of physical significance in the scattering theory, which asserts that a corner scatters every entire incident field non-trivially. The corner scattering phenomenon has been extensively studied in its own context [3,6,8,10,13] and its connection to the vanishing properties of the transmission eigenfunctions has been extensively remarked in [5,23]. The corner scattering study makes essential use of an assumption that the transmission eigenfunction v can be analytically extended across the corner. In such a case, the regularity of v always fulfils the Hölder-continuity requirement in the vanishing study of the transmission eigenfunctions. Hence, one can easily infer that as long as an incident field is not vanishing at the corner point, it will be scattered non-trivially by the corner. It is interesting to note that there is a special case that the transmission eigenfunction v itself is a Herglotz wave function. It is widely believed that this does not happen unless under very special situations, say Ω is a ball and V is a constant. Hence, it is also believed that a corner always scatterers a Herglotz wave non-trivially. One can easily see that the regularity criteria (1.7) and (1.9) exclude many Herglotz waves, whereas (1.6) includes all the possible Herglotz waves since Υ and ϱ therein can actually be taken to be any real numbers.

In principle, we shall follow the general strategy developed in [23] in establishing the new vanishing results for the transmission eigenfunctions. Nevertheless, we develop technically new ingredients which enable us to sharpen several critical estimates and prove that the vanishing properties hold for a much broader class of transmission eigenfunctions. The method developed shall be useful for the quantitative studies of the geometric properties of transmission eigenfunctions in other contexts. In what follows, Sections 2 and 3 are, respectively, devoted to the vanishing properties of the transmission eigenfunctions in two and three dimensions.

2 Vanishing properties in two dimensions

To facilitate calculation and analysis, we introduce the two-dimensional polar coordinates (r,θ) such that $\mathbf{x}=(x_1,x_2)=(r\cos\theta, r\sin\theta)\in\mathbb{R}^2$. Set $B_h:=B_h(\mathbf{0})$ for $h\in\mathbb{R}_+$. Define the following open sector in $\mathbb{R}^2$,

$$W=\left\{\mathbf{x}\in\mathbb{R}^2 |\ \mathbf{x}\neq\mathbf{0},\ \theta_m<\arg(x_1+\mathbf{i}x_2)<\theta_M\right\}, \tag{2.1}$$

where $-\pi<\theta_m<\theta_M<\pi$, $\mathbf{i}:=\sqrt{-1}$ and let Γ^+ and Γ^-, respectively, be (r,θ_M) and (r,θ_m) with $r>0$. Define that

$$\begin{aligned}&S_h=W\cap B_h, \Gamma_h^{\pm}=\Gamma^{\pm}\cap B_h, \bar{S}_h=\overline{W}\cap B_h,\\ &\Lambda_h=S_h\cap\partial B_h,\ \text{and}\ \Sigma_{\Lambda_h}=S_h\backslash S_{h/2}.\end{aligned} \tag{2.2}$$

In Fig. 1, we present a schematic illustration of the corner introduced above.

We shall make use the following complex geometrical optics (CGO) solution, which was first introduced in [2],

Lemma 2.1 [2, Lemma 2.2] *For* $\mathbf{x}\in\mathbb{R}^2$ *denote* $r=|\mathbf{x}|, \theta=arg(x_1+\mathbf{i}x_2)$*. Define*

$$u_0(\mathbf{x}):=\exp\left(\sqrt{r}\left(\cos\left(\frac{\theta}{2}+\pi\right)+\mathbf{i}\sin\left(\frac{\theta}{2}+\pi\right)\right)\right), \tag{2.3}$$

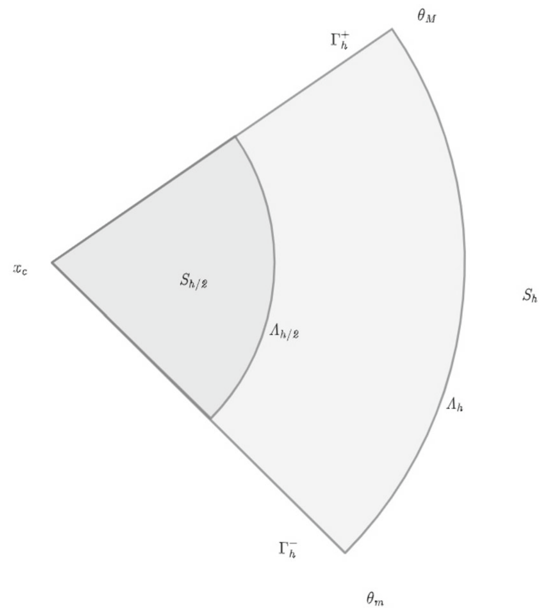

Fig. 1 Schematic illustration of a 2D corner

then $\Delta u_0 = 0$ *in* $\mathbb{R}^2 \setminus \mathbb{R}^2_{0,-}$, *where* $\mathbb{R}^2_{0,-} := \left\{\mathbf{x} \in \mathbb{R}^2 | \, \mathbf{x} = (x_1, x_2);\ x_1 \leq 0, x_2 = 0\right\}$ *and* $s \mapsto u_0(sx)$ *decays exponentially in* $\mathbb{R}_+$. *Choose* $\alpha, s > 0$, *then there holds*

$$\int_W \left|u_0(s\mathbf{x})\|\mathbf{x}\right|^\alpha \mathrm{d}\mathbf{x} \leq \frac{2\,(\theta_M - \theta_m)\,\Gamma(2\alpha + 4)}{\delta_W^{2\alpha+4}} s^{-\alpha-2}. \tag{2.4}$$

where we define $\delta_W =: -\max_{\theta_m < \theta < \theta_M} \cos(\theta/2 + \pi) > 0$. *Moreover, one has*

$$\int_W u_0(s\mathbf{x})\mathrm{d}x = 6\mathbf{i}\left(e^{-2\theta_M \mathbf{i}} - e^{-2\theta_m \mathbf{i}}\right) s^{-2}, \tag{2.5}$$

and for a positive constant $h > 0$, *there holds*

$$\int_{W \setminus B_h} \left|u_0(s\mathbf{x})\right| \mathrm{d}\mathbf{x} \leq \frac{6\,(\theta_M - \theta_m)}{\delta_W^4} s^{-2} e^{-\delta_W \sqrt{hs}/2}. \tag{2.6}$$

By direct calculations, one can obtain the following estimates for the CGO solution $u_0(s\mathbf{x})$:

Corollary 2.1 $u_0 \notin H^2(B_\varepsilon)$ *near the origin and* $|u_0(s\mathbf{x})| \leq 1$ *in* $B_\varepsilon \cap W$ *for sufficiently small* ε.

Furthermore, one has the following estimates where the first and last formulas can also be found in [23, Lemma 2.3].

Corollary 2.2 *The following estimates hold for the* L^2 *norm of* u_0:

$$\left\|u_0(s\mathbf{x})\right\|^2_{L^2(S_h)} \leq \frac{(\theta_M - \theta_m)\, e^{-2\sqrt{s\Theta}\delta_W} h^2}{2}, \quad \Theta \in [0, h], \tag{2.7}$$

$$\left\|u_0(s\mathbf{x})\right\|_{L^2(\Lambda_h)} \leq \sqrt{h} e^{-\delta_W \sqrt{sh}} \sqrt{\theta_M - \theta_m}, \tag{2.8}$$

$$\left\|\partial_\nu u_0(s\mathbf{x})\right\|_{L^2(\Lambda_h)} \leq \frac{1}{2}\sqrt{s} e^{-\delta_W \sqrt{sh}} \sqrt{\theta_M - \theta_m}, \tag{2.9}$$

$$\left\|\partial_\theta u_0(s\mathbf{x})\right\|_{L^2(\Lambda_h)} \leq \frac{\sqrt{s}}{2} h^2 e^{-\delta_W \sqrt{sh}} \sqrt{\theta_M - \theta_m}, \tag{2.10}$$

$$\left\||\mathbf{x}|^\alpha u_0(s\mathbf{x})\right\|^2_{L^2(S_h)} \leq s^{-(2\alpha+2)} \frac{2\left(\theta_M - \theta_m\right)}{\left(4\delta_W^2\right)^{2\alpha+2}} \Gamma(4\alpha+4), \tag{2.11}$$

where δ_W is defined in (2.4) and S_h, Λ_h are defined in (2.2).

Proof First, by making use of integral mean value theorem, one has

$$\begin{aligned}
\left\|u_0(s\mathbf{x})\right\|^2_{L^2(S_h)} &= \int_0^h r\mathrm{d}r \int_{\theta_m}^{\theta_M} e^{2\sqrt{sr}\cos(\theta/2+\pi)} \mathrm{d}\theta \\
&\leq \int_0^h r\mathrm{d}r \int_{\theta_m}^{\theta_M} e^{-2\sqrt{sr}\delta_W} \mathrm{d}\theta \\
&= \frac{\left(\theta_M - \theta_m\right) e^{-2\sqrt{s\Theta}\delta_W} h^2}{2},
\end{aligned}$$

where $\Theta \in [0, h]$. Next, on Λ_h, it can be seen that

$$\begin{aligned}
\left|u_0(s\mathbf{x})\right| &= e^{\sqrt{sh}\cos(\theta/2+\pi)} \leq e^{-\delta_W \sqrt{sh}}, \\
\left|\partial_\nu u_0(s\mathbf{x})\right| &= \left|\frac{\sqrt{s} e^{i\cos(\theta/2+\pi)}}{2\sqrt{h}} e^{\sqrt{sh}\exp(i(\theta/2+\pi))}\right| \leq \frac{1}{2}\sqrt{\frac{s}{h}} e^{-\delta_W \sqrt{sh}}, \\
\left|\partial_\theta u_0(s\mathbf{x})\right| &= \left|\frac{\sqrt{sh}}{2} e^{\sqrt{sh}\exp(i(\theta/2+\pi))}\right| \leq \frac{\sqrt{sh}}{2} e^{-\delta_W \sqrt{sh}},
\end{aligned}$$

which relates to exponentially decay property when $s \to \infty$. By straightforward calculations, one can show that

$$\begin{aligned}
\left\|u_0(s\mathbf{x})\right\|_{L^2(\Lambda_h)} &\leq \sqrt{h} e^{-\delta_W \sqrt{sh}} \sqrt{\theta_M - \theta_m}, \\
\left\|\partial_\nu u_0(s\mathbf{x})\right\|_{L^2(\Lambda_h)} &\leq \frac{1}{2}\sqrt{s} e^{-\delta_W \sqrt{sh}} \sqrt{\theta_M - \theta_m}, \\
\left\|\partial_\theta u_0(s\mathbf{x})\right\|_{L^2(\Lambda_h)} &\leq \frac{\sqrt{s}}{2} h^2 e^{-\delta_W \sqrt{sh}} \sqrt{\theta_M - \theta_m}.
\end{aligned}$$

Finally, by making use of polar coordinates, one can readily deduce that

$$\begin{aligned}
\left\||\mathbf{x}|^\alpha u_0(s\mathbf{x})\right\|^2_{L^2(S_h)} &= \int_0^h r\mathrm{d}r \int_{\theta_m}^{\theta_M} r^{2\alpha} e^{2\sqrt{sr}\cos(\theta/2+\pi)} \mathrm{d}\theta \\
&\leq \int_0^h r\mathrm{d}r \int_{\theta_m}^{\theta_M} r^{2\alpha} e^{-2\sqrt{sr}\delta_w} \mathrm{d}\theta \\
&= \left(\theta_M - \theta_m\right) \int_0^h r^{2\alpha+1} e^{-2\delta_W \sqrt{sr}} \mathrm{d}r \quad \left(t = 2\delta_W \sqrt{sr}\right) \qquad (2.12) \\
&= s^{-(2\alpha+2)} \frac{2\left(\theta_M - \theta_m\right)}{\left(4\delta_W^2\right)^{2\alpha+2}} \int_0^{2\delta_W \sqrt{sh}} t^{4\alpha+3} e^{-t} \mathrm{d}r \\
&\leq s^{-(2\alpha+2)} \frac{2\left(\theta_M - \theta_m\right)}{\left(4\delta_W^2\right)^{2\alpha+2}} \Gamma(4\alpha+4),
\end{aligned}$$

which completes the proof. □

Next, by direct computations and the compact embeddings of Hölder spaces, one can easily obtain the following result:

Lemma 2.2 *For the Herglotz wave function v_j defined in (1.5) in two dimensions,*

$$\begin{aligned} \|v_j\|_{C^1(S_h)} &\leq \sqrt{2\pi}(1+k)\,\|g_j\|_{L^2(\mathbb{S})}, \\ \|v_j\|_{C^\alpha(S_h)} &\leq \operatorname{diam}(S_h)^{1-\alpha}\,\|v_j\|_{C^1(S_h)}. \end{aligned} \tag{2.13}$$

where $0<\alpha<1$ and $\operatorname{diam}(S_h)$ is the diameter of S_h.

Furthermore, using the Jacobi–Anger expansion [17, Page 75] in $\mathbb{R}^2$, we can obtain the following lemma.

Lemma 2.3 *The Herglotz wave function v_j defined in (1.5) admits the following asymptotic expansion:*

$$\begin{aligned} v_j(\mathbf{x}) &= J_0(k|\mathbf{x}|)\int_{\mathbb{S}} g_j(\theta)\mathrm{d}\sigma(\theta) + 2\sum_{p=1}^{\infty}\mathbf{i}^p J_p(k|\mathbf{x}|)\int_{\mathbb{S}} g_j(\theta)\cos(p\varphi)\mathrm{d}\sigma(\theta) \\ &:= v_j(0)J_0(k|\mathbf{x}|) + 2\sum_{p=1}^{\infty}\mathbf{i}^p J_p(k|\mathbf{x}|)\gamma_{pj}, \quad \mathbf{x}\in\mathbb{R}^2, \end{aligned} \tag{2.14}$$

where $J_p(t)$ stands for the pth Bessel function. Indeed,

$$J_p(t) = \frac{t^p}{2^p p!} + \frac{t^p}{2^p}\sum_{\ell=1}^{\infty}\frac{(-1)^\ell t^{2\ell}}{4^\ell(\ell!)^2}, \quad p=0,1,2,\ldots \tag{2.15}$$

The following lemma shows the elementary expansion result for functions with C^α regularity:

Lemma 2.4 *Suppose $f(\mathbf{x})\in C^\alpha$, then the following expansion holds near the origin:*

$$f(\mathbf{x}) = f(\mathbf{0}) + \delta f(\mathbf{x}), \quad |\delta f(\mathbf{x})| \leq \|f\|_{C^\alpha}|\mathbf{x}|^\alpha. \tag{2.16}$$

Lemma 2.5 *Let $v,w\in H^1(\Omega)$ be a pair of conductive transmission eigenfunctions to (1.2), $D_\varepsilon = S_h\setminus B_\varepsilon$ for $0<\varepsilon<h$, $\eta\in C^\alpha\left(\bar{\Gamma}_h^\pm\right)$ for $0<\alpha<1$ and $\Gamma_h^\pm$, S_h be defined in (2.2), then*

$$\begin{aligned} \lim_{\varepsilon\to 0}\int_{D_\varepsilon}\Delta(v-w)u_0(s\mathbf{x})\mathrm{d}\mathbf{x} = &\int_{\Lambda_h}\left(u_0(s\mathbf{x})\partial_\nu(v-w) - (v-w)\partial_\nu u_0(s\mathbf{x})\right)\mathrm{d}\sigma \\ &- \int_{\Gamma_h^\pm}\eta(\mathbf{x})u_0(s\mathbf{x})v(\mathbf{x})\mathrm{d}\sigma. \end{aligned} \tag{2.17}$$

Proof By making use of Green's formula, together with the boundary condition in (1.2), one can deduce that

$$\begin{aligned} &\int_{D_\varepsilon}\Delta(v-w)u_0(s\mathbf{x})\mathrm{d}\mathbf{x} \\ &\quad = \int_{D_\varepsilon}(v-w)\Delta u_0(s\mathbf{x})\mathrm{d}\mathbf{x} + \int_{\partial D_\varepsilon}\left(u_0(s\mathbf{x})\partial_\nu(v-w) - (v-w)\partial_\nu u_0(s\mathbf{x})\right)\mathrm{d}\sigma(\mathbf{x}) \\ &\quad = \int_{\Lambda_h}\left(u_0(s\mathbf{x})\partial_\nu(v-w) - (v-w)\partial_\nu u_0(s\mathbf{x})\right)\mathrm{d}\sigma(\mathbf{x}) \\ &\qquad + \int_{\Lambda_\varepsilon}\left(u_0(s\mathbf{x})\partial_\nu(v-w) - (v-w)\partial_\nu u_0(s\mathbf{x})\right)\mathrm{d}\sigma(\mathbf{x}) \\ &\qquad - \int_{\Gamma_{(\varepsilon,h)}^\pm}\eta(\mathbf{x})u_0(s\mathbf{x})v(\mathbf{x})\mathrm{d}\sigma(\mathbf{x}), \end{aligned} \tag{2.18}$$

where $\Gamma^{\pm}_{(\varepsilon,h)} = \Gamma^{\pm} \cap (B_h \backslash B_\varepsilon)$ and $\Lambda_\varepsilon = S_h \cap \partial B_\varepsilon$. Using the trace theorem, we have $v \in L^2\left(\Gamma^{\pm}_{(0,\varepsilon)}\right)$, where $\Gamma^{\pm}_{(0,\varepsilon)} = \Gamma^{\pm} \cap B_\varepsilon$. By Corollary 2.1, one can see that

$$\lim_{\varepsilon \to 0} \int_{\Lambda_\varepsilon} (u_0 \partial_\nu (v - w) - (v - w) \partial_\nu u_0)\, \mathrm{d}\sigma = 0,$$

and

$$\lim_{\varepsilon \to 0} \int_{\Gamma^{\pm}_{(0,\varepsilon)}} \eta u_0 v \, \mathrm{d}\sigma = 0.$$

Thus, (2.17) is obtained by taking the limit on both sides of Eq. (2.18). □

We next present two auxiliary lemmas which are extended from the related estimates in [23].

Lemma 2.6 *Suppose* $\eta \in C^\alpha\left(\bar{\Gamma}^{\pm}_h\right)$ *for* $0 < \alpha < 1$, θ_M, θ_m *are defined in* (2.1) *and* $\theta_M - \theta_m \neq \pi$. *Define*

$$\omega(\theta) = -\cos(\theta/2 + \pi), \quad \mu(\theta) = -\cos(\theta/2 + \pi) - \mathbf{i} \sin(\theta/2 + \pi), \tag{2.19}$$

and

$$I_2^{\pm} = \int_{\Gamma_h^{\pm}} \eta(\mathbf{x}) u_0(s\mathbf{x}) v_j(\mathbf{x}) \mathrm{d}\sigma. \tag{2.20}$$

Then, it holds that

$$\begin{aligned} I_2^- =& 2\eta(\mathbf{0}) v_j(\mathbf{0}) s^{-1} \left(\mu(\theta_m)^{-2} - \mu(\theta_m)^{-2} e^{-\sqrt{sh}\mu(\theta_m)} - \mu(\theta_m)^{-1} \sqrt{sh} e^{-\sqrt{sh}\mu(\theta_m)} \right) \\ &+ v_j(\mathbf{0}) \eta(\mathbf{0}) I_{21}^- + \eta(\mathbf{0}) I_{22}^- + I_\eta^-, \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} I_2^+ =& 2\eta(\mathbf{0}) v_j(\mathbf{0}) s^{-1} \left(\mu(\theta_M)^{-2} - \mu(\theta_M)^{-2} e^{-\sqrt{sh}\mu(\theta_M)} - \mu(\theta_M)^{-1} \sqrt{sh} e^{-\sqrt{sh}\mu(\theta_M)} \right) \\ &+ v_j(\mathbf{0}) \eta(\mathbf{0}) I_{21}^+ + \eta(\mathbf{0}) I_{22}^+ + I_\eta^+, \end{aligned} \tag{2.22}$$

where

$$\begin{aligned} &I_{21}^- \leq \mathcal{O}(s^{-3}), \quad I_{21}^+ \leq \mathcal{O}(s^{-3}), \\ &I_{22}^- \leq \mathcal{O}(\|g_j\|_{L^2(\mathbb{S}^{n-1})} s^{-2}), \quad I_{22}^+ \leq \mathcal{O}(\|g_j\|_{L^2(\mathbb{S}^{n-1})} s^{-2}), \\ &|I_\eta^-| \leq \|\eta\|_{C^\alpha} \left(v_j(\mathbf{0}) \mathcal{O}(s^{-1-\alpha}) + \mathcal{O}(\|g_j\|_{L^2(\mathbb{S}^{n-1})} s^{-2-\alpha}) \right), \\ &|I_\eta^+| \leq \|\eta\|_{C^\alpha} \left(v_j(\mathbf{0}) \mathcal{O}(s^{-1-\alpha}) + \mathcal{O}(\|g_j\|_{L^2(\mathbb{S}^{n-1})} s^{-2-\alpha}) \right). \end{aligned}$$

Proof Using Lemma 2.4, we have

$$I_2^- = \eta(\mathbf{0}) \int_{\Gamma_h^-} u_0(s\mathbf{x}) v_j(\mathbf{x}) \mathrm{d}\sigma + \int_{\Gamma_h^-} \delta\eta(\mathbf{x}) u_0(s\mathbf{x}) v_j(\mathbf{x}) \mathrm{d}\sigma.$$

Note that $\omega(\theta) > 0$ for $\theta_m \leq \theta \leq \theta_M$. By Lemma 2.3, we can obtain that

$$\begin{aligned} \int_{\Gamma_h^-} u_0(s\mathbf{x}) v_j(\mathbf{x}) \mathrm{d}\sigma &= \int_0^h \left(v_j(\mathbf{0}) J_0(kr) + 2 \sum_{p=1}^{\infty} \gamma_{pj} \mathbf{i}^p J_p(kr) \right) e^{-\sqrt{sr}\mu(\theta_m)} \mathrm{d}r \\ &= v_j(\mathbf{0}) \left[\int_0^h e^{-\sqrt{sr}\mu(\theta_m)} \mathrm{d}r + \sum_{p=1}^{\infty} \frac{(-1)^p k^{2p}}{4^p (p!)^2} \int_0^h r^{2p} e^{-\sqrt{sr}\mu(\theta_m)} \mathrm{d}r \right] \end{aligned}$$

$$
\begin{aligned}
&+\int_0^h 2\sum_{p=1}^{\infty}\gamma_{pj}\mathbf{i}^p J_p(kr)e^{-\sqrt{sr}\mu(\theta_m)}\mathrm{d}r\\
&:= v_j(\mathbf{0})\left[\int_0^h e^{-\sqrt{sr}\mu(\theta_m)}\mathrm{d}r + I_{21}^-\right] + I_{22}^-.
\end{aligned}
\tag{2.23}
$$

Furthermore, we have

$$
\int_0^h e^{-\sqrt{sr}\mu(\theta_m)}\mathrm{d}r = 2s^{-1}\left(\mu(\theta_m)^{-2} - \mu(\theta_m)^{-2}e^{-\sqrt{sh}\mu(\theta_m)} - \mu(\theta_m)^{-1}\sqrt{sh}e^{-\sqrt{sh}\mu(\theta_m)}\right),
\tag{2.24}
$$

and

$$
\left|I_{21}^-\right| \le \sum_{p=1}^{\infty}\frac{k^{2p}h^{2p-2}}{4^p(p!)^2}\int_0^h r^2 e^{-\sqrt{sr}\omega(\theta_m)}\mathrm{d}r = \mathcal{O}\left(s^{-3}\right).
\tag{2.25}
$$

By using Lemma 2.3, one can drive that

$$
\begin{aligned}
\left|I_{22}^-\right| &\le 2\left\|g_j\right\|_{L^2(\mathbb{S})}\sum_{p=1}^{\infty}\left[\frac{k^p}{2^p p!}\int_0^h r^p e^{-\sqrt{sr}\omega(\theta_m)}\mathrm{d}r\right.\\
&\qquad\left. + \frac{(kh)^p}{2^p}\sum_{\ell=1}^{\infty}\frac{k^{2\ell}h^{2(\ell-1)}}{4^\ell(\ell!)^2}\int_0^h r^2 e^{-\sqrt{sr}\omega(\theta_m)}\mathrm{d}r\right]\\
&\le 2\left\|g_j\right\|_{L^2(\mathbb{S})}\sum_{p=1}^{\infty}\left[\frac{k^p h^{p-1}}{2^p p!}\int_0^h r e^{-\sqrt{sr}\omega(\theta_m)}\mathrm{d}r + \mathcal{O}\left(s^{-3}\right)\right]\\
&= \mathcal{O}\left(\left\|g_j\right\|_{L^2(\mathbb{S})}s^{-2}\right),
\end{aligned}
\tag{2.26}
$$

with the assumption that $kh < 1$ for sufficiently small h. Taking

$$
I_\eta^- = \int_{\Gamma_h^-}\delta\eta(\mathbf{x})u_0(s\mathbf{x})v_j(\mathbf{x})\mathrm{d}\sigma.
\tag{2.27}
$$

By Lemmas 2.3 and 2.4, it holds that

$$
\begin{aligned}
\left|I_\eta^-\right| &\le \|\eta\|_{C^\alpha}\int_0^h r^\alpha\left(v_j(0)J_0(kr) + 2\sum_{p=1}^{\infty}\left|\gamma_{pj}\right|J_p(kr)\right)e^{-\sqrt{sr}\omega(\theta_m)}\mathrm{d}r\\
&= \|\eta\|_{C^\alpha}v_j(0)(\int_0^h r^\alpha e^{-\sqrt{sr\omega}(\theta_m)}\mathrm{d}r\\
&\qquad + \sum_{p=1}^{\infty}\frac{(-1)^p k^{2p}}{4^p(p!)^2}\int_0^h r^{\alpha+2p}e^{-\sqrt{sr}\omega(\theta_m)}\mathrm{d}r)\\
&\quad + 2\|\eta\|_{C^\alpha}\sum_{p=1}^{\infty}\int_0^h r^\alpha\gamma_{pj}\mathbf{i}^p J_p(kr)e^{-\sqrt{sr}\mu(\theta_m)}\mathrm{d}r.
\end{aligned}
\tag{2.28}
$$

Since $\omega(\theta_m) > 0$, as $s \to \infty$, we have

$$
\int_0^h r^\alpha e^{-\sqrt{sr\omega}(\theta_m)}\mathrm{d}r = \mathcal{O}\left(s^{-1-\alpha}\right),
\tag{2.29}
$$

and

$$
\begin{aligned}
&\left|\sum_{p=1}^{\infty} \frac{(-1)^p k^{2p}}{4^p (p!)^2} \int_0^h r^{\alpha+2p} e^{-\sqrt{sr}\omega(\theta_m)} \mathrm{d}r)\right| \\
&\quad \leq \sum_{p=1}^{\infty} \frac{h^{2p-2} k^{2p}}{4^p (p!)^2} \int_0^h r^{\alpha+2} e^{-\sqrt{sr}\omega(\theta_m)} \mathrm{d}r = \mathcal{O}\left(s^{-3-\alpha}\right),
\end{aligned}
\tag{2.30}
$$

for $kh < 1$. Furthermore, by direct computations, one can obtain that

$$
\begin{aligned}
&\left|\sum_{p=1}^{\infty} \int_0^h r^{\alpha} \gamma_{pj} \mathbf{i}^p J_p(kr) e^{-\sqrt{sr}\mu(\theta_m)} \mathrm{d}r\right| \\
&\quad \leq \|g_j\|_{L^2(\mathbb{S})} \sum_{p=1}^{\infty} \Bigg[\frac{k^p}{2^p p!} \int_0^h r^{p+\alpha} e^{-\sqrt{sr}\omega(\theta_m)} \mathrm{d}r \\
&\qquad + \frac{k^p}{2^p} \sum_{\ell=1}^{\infty} \frac{k^{2\ell} h^{2(\ell-1)}}{4^{\ell} (\ell!)^2} \left(\int_0^h r^{p+\alpha+2} e^{-\sqrt{sr}\omega(\theta_m)} \mathrm{d}r \right) \Bigg] \\
&\quad \leq \|g_j\|_{L^2(\mathbb{S})} \sum_{p=1}^{\infty} \Bigg[\frac{k^p}{2^p p!} \int_0^h r^{p+\alpha} e^{-\sqrt{sr}\omega(\theta_m)} \mathrm{d}r \\
&\qquad + \frac{(kh)^p}{2^p} \sum_{\ell=1}^{\infty} \frac{k^{2\ell} h^{2(\ell-1)}}{4^{\ell} (\ell!)^2} \left(\int_0^h r^{\alpha+2} e^{-\sqrt{sr}\omega(\theta_m)} \mathrm{d}r \right) \Bigg]\Bigg] \\
&\quad \leq \|g_j\|_{L^2(\mathbb{S})} \sum_{p=1}^{\infty} \left[\frac{k^p h^{p-1}}{2^p p!} \int_0^h r^{\alpha+1} e^{-\sqrt{sr}\omega(\theta_m)} \mathrm{d}r + \mathcal{O}\left(s^{-\alpha-3}\right) \right] \\
&\quad = \mathcal{O}\left(\|g_j\|_{L^2(\mathbb{S})} s^{-\alpha-2} \right).
\end{aligned}
\tag{2.31}
$$

Combining (2.27)-(2.31), one finally obtains

$$
|I_\eta^-| \leq \|\eta\|_{C^\alpha} \left(v_j(\mathbf{0}) \mathcal{O}\left(s^{-1-\alpha}\right) + \mathcal{O}\left(\|g_j\|_{L^2(\mathbb{S}^{n-1})} s^{-\alpha-2} \right) \right).
\tag{2.32}
$$

Similarly, one can prove (2.22), which completes the proof. □

Lemma 2.7 *Suppose that $\eta \in C^\alpha\left(\bar{\Gamma}_h^\pm\right)$ for $0 < \alpha < 1$, $\Gamma_h^\pm$ is defined in (2.2), θ_M, θ_m are defined in (2.1) and $\theta_M - \theta_m \neq \pi$. Define*

$$
\xi_j^\pm(s) = \int_{\Gamma_h^\pm} \eta(\mathbf{x}) u_0(s\mathbf{x}) \left(v(\mathbf{x}) - v_j(\mathbf{x})\right) \mathrm{d}\sigma.
\tag{2.33}
$$

Then, the following estimate holds,

$$
\begin{aligned}
\left|\xi_j^\pm(s)\right| &\leq C \Bigg(|\eta(0)| \frac{\sqrt{\theta_M - \theta_m} e^{-\sqrt{s\Theta\delta_W}h}}{\sqrt{2}} \\
&\quad + \|\eta\|_{C^\alpha} s^{-(\alpha+1)} \frac{\sqrt{2\left(\theta_M - \theta_m\right)\Gamma(4\alpha+4)}}{(2\delta_W)^{2\alpha+2}} \Bigg) \|v - v_j\|_{H^1(S_h)},
\end{aligned}
\tag{2.34}
$$

where δ_w is defined in (2.4).

Proof By Lemma 2.4, the trace theorem and Cauchy–Schwarz inequality, one has

$$
\begin{aligned}
\left|\xi_j^{\pm}(s)\right| &\le |\eta(\mathbf{0})| \int_{\Gamma_h^{\pm}} \left|u_0(s\mathbf{x})\right| \left|v(\mathbf{x}) - v_j(\mathbf{x})\right| \mathrm{d}\sigma \\
&\quad + \|\eta\|_{C^\alpha} \int_{\Gamma_h^{\pm}} |\mathbf{x}|^\alpha \left|u_0(s\mathbf{x})\right| \left|v(\mathbf{x}) - v_j(\mathbf{x})\right| \mathrm{d}\sigma \\
&\le |\eta(\mathbf{0})| \left\|v - v_j\right\|_{H^{1/2}(\Gamma_h^{\pm})} \left\|u_0(s\mathbf{x})\right\|_{H^{-1/2}(\Gamma_h^{\pm})} \\
&\quad + \|\eta\|_{C^\alpha} \left\|v - v_j\right\|_{H^{1/2}(\Gamma_h^{\pm})} \left\||\mathbf{x}|^\alpha u_0(s\mathbf{x})\right\|_{H^{-1/2}(\Gamma_h^{\pm})} \\
&\le |\eta(\mathbf{0})| \left\|v - v_j\right\|_{H^1(S_h)} \left\|u_0(s\mathbf{x})\right\|_{L^2(S_h)} \\
&\quad + \|\eta\|_{C^\alpha} \left\|v - v_j\right\|_{H^1(S_h)} \left\| |\mathbf{x}|^\alpha u_0(s\mathbf{x})\right\|_{L^2(S_h)}.
\end{aligned}
$$

Combining with Corollary 2.2, one readily obtains (2.34).

The proof is complete. □

With the above preliminary results, we present our first main result on the vanishing properties of v in two dimensions.

Theorem 2.3 *Let $v \in H^1(\Omega)$ and $w \in H^1(\Omega)$ be a pair of eigenfunctions to (1.2) for $k \in \mathbb{R}_+$. Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz domain which contains a corner $\Omega \cap W$, with $\mathbf{x}_c$ be the vertex, where W is a sector defined in (2.1). Let S_h be defined in (2.2) with $h > 0$ sufficiently small. Furthermore, suppose that $qw \in C^\alpha\left(\bar{S}_h\right)$ with $q := 1 + V$ and $\eta \in C^\alpha\left(\bar{\Gamma}_h^{\pm}\right)$ for $0 < \alpha < 1$. If the following conditions are fulfilled:*

(a) there exists Herglotz functions $v_j, j = 1, 2, \ldots$, defined in (1.5), with kernels g_j such that

$$\left\|v - v_j\right\|_{H^1(S_h)} \le j^{-\Upsilon}, \quad \left\|g_j\right\|_{L^2(\mathbb{S})} \le C j^{\varrho}. \tag{2.35}$$

hold for some constants $C > 0$, $\Upsilon > 0$ and $\varrho < \Upsilon$.

(b) the function $\eta(\mathbf{x})$ does not vanish at the corner $\mathbf{x}_c$, that is,

$$\eta\left(\mathbf{x}_c\right) \ne 0. \tag{2.36}$$

(c) the corner in the sector W is a real corner, that is

$$-\pi < \theta_m < \theta_M < \pi \ \text{ and } \ \theta_M - \theta_m \ne \pi. \tag{2.37}$$

Then, there holds the following vanishing property:

$$\lim_{\rho \to +0} \frac{1}{m\left(B\left(\mathbf{x}_c, \rho\right) \cap \Omega\right)} \int_{B(\mathbf{x}_c, \rho) \cap \Omega} |v(\mathbf{x})| \mathrm{d}\mathbf{x} = 0. \tag{2.38}$$

Proof of Theorem 2.3 It is well known that $\Delta + k^2$ is invariant under rigid motions; thus, we assume, without loss of generality, that $\mathbf{x}_c = \mathbf{0}$. By (1.2) we define

$$\Delta v = -k^2 v := f_1, \quad \Delta w = -k^2 q w := f_2, \tag{2.39}$$

which in combination with the boundary conditions in (1.2), yields that

$$\Delta(v - w) = f_1 - f_2 \text{ in } S_h, \quad v - w = 0, \quad \partial_\nu (v - w) = -\eta v \text{ on } \Gamma_h^{\pm}.$$

For national simplicity, we define $f_{1j}(\mathbf{x}) = -k^2 v_j$. Consider the following integral:

$$\begin{aligned}\int_{S_h} \Delta(v-w)u_0(s\mathbf{x})\mathrm{d}\mathbf{x} &= \int_{S_h} (k^2 q w - k^2 v)u_0(s\mathbf{x})\mathrm{d}\mathbf{x} \\ &= \int_{S_h} u_0(s\mathbf{x})(f_{1j}-f_2)\mathrm{d}\mathbf{x} - k^2\int_{S_h}(v-v_j)u_0(s\mathbf{x})\mathrm{d}\mathbf{x} \\ &:= I_1 + \Delta_j(s).\end{aligned} \tag{2.40}$$

By Corollary 2.1, one has that $u_0 \notin H^2$ near the origin. Consider the domain $D_\varepsilon = S_h \backslash B_\varepsilon$ for $0 < \varepsilon < h$. Using the fact that

$$\int_{S_h} \Delta(v-w)u_0(s\mathbf{x})\mathrm{d}\mathbf{x} = \lim_{\varepsilon\to 0}\int_{D_\varepsilon} \Delta(v-w)u_0(s\mathbf{x})\mathrm{d}\mathbf{x}, \tag{2.41}$$

and by Lemma 2.5, one can drive that

$$I_1 + \Delta_j(s) = I_3 - I_2^{\pm} - \xi_j^{\pm}(s), \tag{2.42}$$

where I_1 and $\Delta_j(s)$ are defined in (2.40), and

$$\begin{aligned} I_2^{\pm} &= \int_{\Gamma_h^{\pm}} \eta(\mathbf{x})u_0(s\mathbf{x})v_j(\mathbf{x})\mathrm{d}\sigma, \\ I_3 &= \int_{\Lambda_h} \left(u_0(s\mathbf{x})\partial_\nu(v-w) - (v-w)\partial_\nu u_0(s\mathbf{x})\right)\mathrm{d}\sigma, \\ \xi_j^{\pm}(s) &= \int_{\Gamma_h^{\pm}} \eta(\mathbf{x})u_0(s\mathbf{x})\left(v(\mathbf{x})-v_j(\mathbf{x})\right)\mathrm{d}\sigma. \end{aligned} \tag{2.43}$$

Recalling that I_1 is defined in (2.40), by Lemma 2.4 and the compact embedding, one can deduce that

$$I_1 = \left(f_{1j}(0)-f_2(0)\right)\int_{S_h} u_0(s\mathbf{x})\mathrm{d}\mathbf{x} + \int_{S_h}\delta f_{1j}(\mathbf{x})u_0(s\mathbf{x})\mathrm{d}\mathbf{x} - \int_{S_h}\delta f_2(\mathbf{x})u_0(s\mathbf{x})\mathrm{d}\mathbf{x}, \tag{2.44}$$

where $\delta f_{1j}(\mathbf{x})$ and $\delta f_2(\mathbf{x})$ are deduced by Lemma 2.4. Using the fact that

$$\int_{S_h} u_0(s\mathbf{x})\mathrm{d}\mathbf{x} = \int_W u_0(s\mathbf{x})\mathrm{d}\mathbf{x} - \int_{W\backslash S_h} u_0(s\mathbf{x})\mathrm{d}\mathbf{x},$$

and combining (2.5) and (2.6) in Lemma 2.1, it can be derived that

$$\left|\int_{S_h} u_0\left(s\mathbf{x}\right)\mathrm{d}\mathbf{x}\right| \le 6\left|e^{-2\theta_M \mathbf{i}} - e^{-2\theta_m \mathbf{i}}\right| s^{-2} + \frac{6\left(\theta_M-\theta_m\right)}{\delta_W^4} s^{-2} e^{-\frac{\delta_W\sqrt{hs}}{2}}. \tag{2.45}$$

From Lemma 2.1, Corollary 2.2, Lemmas 2.4 and 2.2, one can derive that

$$\begin{aligned}\left|\int_{S_h}\delta f_{1j}(\mathbf{x})u_0(s\mathbf{x})\mathrm{d}\mathbf{x}\right| &\le \int_{S_h}\left|\delta f_{1j}(\mathbf{x})\right|\left|u_0(s\mathbf{x})\right|\mathrm{d}\mathbf{x} \\ &\le \left\|f_{1j}\right\|_{C^\alpha}\int_W \left|u_0(s\mathbf{x})\right|\left|\mathbf{x}\right|^\alpha \mathrm{d}\mathbf{x} \\ &\le \frac{2\left(\theta_M-\theta_m\right)\Gamma(2\alpha+4)}{\delta_W^{2\alpha+4}}\left\|f_{1j}\right\|_{C^\alpha} s^{-\alpha-2} \\ &\le \frac{2\sqrt{2\pi}\left(\theta_M-\theta_m\right)\Gamma(2\alpha+4)}{\delta_W^{2\alpha+4}} k^2 \operatorname{diam}\left(S_h\right)^{1-\alpha} \\ &\quad \times (1+k)Cj^{\varrho}s^{-\alpha-2},\end{aligned} \tag{2.46}$$

where $\delta f_{1j}(\mathbf{x})$ is deduced by Lemma 2.4. Using the assumption (2.35), we further obtain that

$$\left|\int_{S_h}\delta f_{1j}(\mathbf{x})u_0(s\mathbf{x})\mathrm{d}\mathbf{x}\right| \le C' j^{\varrho} s^{-\alpha-2}, \tag{2.47}$$

where $C' = (1+k)C\frac{2\sqrt{2\pi}(\theta_M-\theta_m)\Gamma(2\alpha+4)}{\delta_W^{2\alpha+4}}k^2\,\mathrm{diam}\,(S_h)^{1-\alpha}$ and C is defined in (2.35). Similarly, we have

$$\left|\int_{S_h}\delta f_2(\mathbf{x})u_0(s\mathbf{x})\mathrm{d}\mathbf{x}\right| \le \frac{2\left(\theta_M-\theta_m\right)\Gamma(2\alpha+4)}{\delta_W^{2\alpha+4}}\left\|f_2\right\|_{C^\alpha}s^{-\alpha-2}. \tag{2.48}$$

As for the $\Delta_j(s)$ defined in (2.40), using the Cauchy–Schwarz inequality, Corollary 2.2 and the assumption (2.35), we can drive that

$$\begin{aligned}\left|\Delta_j(s)\right| &\le k^2\left\|v-v_j\right\|_{L^2(S_h)}\left\|u_0(s\mathbf{x})\right\|_{L^2(S_h)}\\ &\le \frac{\sqrt{\theta_M-\theta_m}k^2e^{-\sqrt{s\Theta}\delta_w}h}{\sqrt{2}}j^{-\Upsilon}.\end{aligned} \tag{2.49}$$

By using the Hölder inequality, Corollary 2.2, and the trace theorem, one can prove that

$$\begin{aligned}|I_3| &\le \left\|u_0(s\mathbf{x})\right\|_{H^{1/2}(\Lambda_h)}\left\|\partial_\nu(v-w)\right\|_{H^{-1/2}(\Lambda_h)}+\left\|\partial_\nu u_0(s\mathbf{x})\right\|_{L^2(\Lambda_h)}\left\|v-w\right\|_{L^2(\Lambda_h)}\\ &\le \left(\left\|u_0(s\mathbf{x})\right\|_{H^1(\Lambda_h)}+\left\|\partial_\nu u_0(s\mathbf{x})\right\|_{L^2(\Lambda_h)}\right)\left\|v-w\right\|_{H^1\left(\Sigma_{\Lambda_h}\right)} \le Ce^{-c'\sqrt{s}},\end{aligned} \tag{2.50}$$

where $c' > 0$ as $s \to \infty$.

By multiplying s on both sides of (2.42), using Lemma 2.6 and (2.44), one can then get that

$$\begin{aligned}&2v_j(\mathbf{0})\eta(\mathbf{0})\Big[\left(\mu\left(\theta_M\right)^{-2}-\mu\left(\theta_M\right)^{-2}e^{-\sqrt{sh}(\theta_M)}-\mu\left(\theta_M\right)^{-1}\sqrt{sh}e^{-\sqrt{sh}\mu(\theta_M)}\right)\\ &\quad+\left(\mu\left(\theta_m\right)^{-2}-\mu\left(\theta_m\right)^{-2}e^{-\sqrt{sh}\mu(\theta_m)}-\mu\left(\theta_m\right)^{-1}\sqrt{sh}e^{-\sqrt{sh}\mu(\theta_m)}\right)\Big]\\ &=s\Big[I_3-\left(f_{1j}(\mathbf{0})-f_2(\mathbf{0})\right)\int_{S_h}u_0(s\mathbf{x})\mathrm{d}\mathbf{x}-\Delta_j(s)\\ &\quad-v_j(\mathbf{0})\eta(\mathbf{0})\left(I_{21}^+ + I_{21}^-\right)-\eta(\mathbf{0})\left(I_{22}^+ + I_{22}^-\right)\\ &\quad-I_\eta^+ - I_\eta^- - \int_{S_h}\delta f_{1j}(\mathbf{x})u_0(s\mathbf{x})\mathrm{d}\mathbf{x}+\int_{S_h}\delta f_2(\mathbf{x})u_0(s\mathbf{x})\mathrm{d}\mathbf{x}-\xi_j^{\pm}(s)\Big].\end{aligned} \tag{2.51}$$

Taking $s = j^\beta$, where $\max\{\varrho, 0\} < \beta < \Upsilon$, and letting $j \to \infty$, by (2.45), (2.47), (2.48), (2.50), Lemmas 2.6 and 2.7, we can deduce that

$$\eta(\mathbf{0})\left(\mu\left(\theta_m\right)^{-2}+\mu\left(\theta_M\right)^{-2}\right)\lim_{j\to\infty}v_j(\mathbf{0})=0. \tag{2.52}$$

Using the assumption of (2.37), one has that

$$\mu\left(\theta_m\right)^{-2}+\mu\left(\theta_M\right)^{-2}=\frac{\left(\cos\theta_m+\cos\theta_M\right)+\mathbf{i}\left(\sin\theta_m+\sin\theta_M\right)}{\left(\cos\theta_m+\mathbf{i}\sin\theta_m\right)\left(\cos\theta_M+\mathbf{i}\sin\theta_M\right)}\ne 0. \tag{2.53}$$

By assumption (2.36), it holds that $\eta(0) \ne 0$. Combining (2.52) with (2.53), we know that

$$\lim_{j\to\infty}v_j(\mathbf{0})=0.$$

Thus, it is an easy consequence that

$$\lim_{j\to\infty}|v_j(\mathbf{0})|=\lim_{\rho\to+0}\frac{1}{m\left(B\left(\mathbf{x}_c,\rho\right)\cap\Omega\right)}\int_{B(\mathbf{x}_c,\rho)\cap\Omega}\left|v_j(\mathbf{x})\right|\mathrm{d}\mathbf{x}.$$

Combining with the triangular inequality

$$\begin{aligned}&\lim_{\rho\to+0}\frac{1}{m\left(B\left(\mathbf{x}_c,\rho\right)\cap\Omega\right)}\int_{B(\mathbf{x}_c,\rho)\cap\Omega}|v(\mathbf{x})|\mathrm{d}\mathbf{x}\\ &\le\lim_{j\to\infty}\left(\lim_{\rho\to+0}\frac{1}{m\left(B\left(\mathbf{x}_c,\rho\right)\cap\Omega\right)}\int_{B(\mathbf{x}_c,\rho)\cap\Omega}\left|v(\mathbf{x})-v_j(\mathbf{x})\right|\mathrm{d}\mathbf{x}\right.\\ &\quad\left.+\lim_{\rho\to+0}\frac{1}{m\left(B\left(\mathbf{x}_c,\rho\right)\cap\Omega\right)}\int_{B(\mathbf{x}_c,\rho)\cap\Omega}\left|v_j(\mathbf{x})\right|\mathrm{d}\mathbf{x}\right),\end{aligned}$$

one finally arrives at

$$\lim_{\rho\to+0}\frac{1}{m\left(B\left(\mathbf{x}_c,\rho\right)\cap\Omega\right)}\int_{B(\mathbf{x}_c,\rho)\cap\Omega}|v(\mathbf{x})|\mathrm{d}\mathbf{x}=0.$$

The proof is complete. □

If $\eta\equiv 0$, then by slightly modifying the proof of Theorem 2.3, one can show that the following result holds:

Corollary 2.4 *Let $v\in H^1(\Omega)$ and $w\in H^1(\Omega)$ be a pair of eigenfunctions to (1.2) with $\eta\equiv 0$ and $k\in\mathbb{R}_+$. Let the corner $\mathbf{x}_c$ be the same as that in Theorem 2.3. Suppose $qw\in C^\alpha\left(\bar{S}_h\right)$ for $0<\alpha<1$ and there exists Herglotz functions $v_j, j=1,2,\ldots$, defined in (1.5), with kernels g_j such that*

$$\left\|v-v_j\right\|_{H^1(S_h)}\leq j^{-\Upsilon},\quad \left\|g_j\right\|_{L^2(\mathbb{S})}\leq Cj^{\varrho}, \tag{2.54}$$

for some constants $C>0$, $\Upsilon>0$ and $\varrho<\alpha\Upsilon/2$, then one has

$$\lim_{\rho\to+0}\frac{1}{m\left(B\left(\mathbf{x}_c,\rho\right)\cap\Omega\right)}\int_{B(\mathbf{x}_c,\rho)\cap\Omega}V(\mathbf{x})w(\mathbf{x})\mathrm{d}\mathbf{x}=0. \tag{2.55}$$

Proof The proof is similar to that of Theorem 2.3, we only outline some necessary modifications in the following. Again, we assume that $\mathbf{x}_c=0$. Since $\eta(\mathbf{x})\equiv 0$, from (2.42), (2.43) and (2.44), we have the following integral identity

$$\begin{aligned}&\left(f_{1j}(0)-f_2(0)\right)\int_{S_h}u_0(s\mathbf{x})\mathrm{d}\mathbf{x}+\Delta_j(s)\\&\quad=I_3-\int_{S_h}\delta f_{1j}(\mathbf{x})u_0(s\mathbf{x})\mathrm{d}\mathbf{x}+\int_{S_h}\delta f_2(\mathbf{x})u_0(s\mathbf{x})\mathrm{d}\mathbf{x},\end{aligned} \tag{2.56}$$

where $\Delta_j(s)$ and I_3 are defined in (2.40) and (2.43), $\delta f_{1j}(\mathbf{x})$ and $\delta f_2(\mathbf{x})$ are deduced by Lemma 2.4 with $f_2(\mathbf{x})$ defined in (2.39) and $f_{1j}(\mathbf{x})=-k^2v_j$. Multiplying s^2 on both sides of (2.56), taking $s=J^\beta$ where $\max\{\varrho/\alpha,0\}<\beta<\Upsilon/2$, together with the assumptions (2.54) and (2.37), from (2.45), (2.46), (2.48), (2.49) and (2.50), one can derive that

$$\lim_{j\to\infty}v_j(\mathbf{0})=\frac{f_2(\mathbf{0})}{-k^2}. \tag{2.57}$$

Since

$$\begin{aligned}\lim_{j\to\infty}v_j(\mathbf{0})&=\lim_{j\to\infty}\lim_{\rho\to+0}\frac{1}{m(B(\mathbf{0},\rho)\cap\Omega)}\int_{B(\mathbf{0},\rho)\cap\Omega}v_j(\mathbf{x})\mathrm{d}\mathbf{x}\\&=\lim_{\rho\to+0}\frac{1}{m(B(\mathbf{0},\rho)\cap\Omega)}\int_{B(\mathbf{0},\rho)\cap\Omega}v(\mathbf{x})\mathrm{d}\mathbf{x},\end{aligned}$$

and

$$\frac{f_2(\mathbf{0})}{-k^2}=\lim_{\rho\to+0}\frac{1}{m(B(\mathbf{0},\rho)\cap\Omega)}\int_{B(\mathbf{0},\rho)\cap\Omega}qw(\mathbf{x})\mathrm{d}\mathbf{x},$$

which together with the fact that

$$\lim_{\rho\to+0}\frac{1}{m(B(\mathbf{0},\rho)\cap\Omega)}\int_{B(\mathbf{0},\rho)\cap\Omega}v(\mathbf{x})\mathrm{d}\mathbf{x}=\lim_{\rho\to+0}\frac{1}{m(B(\mathbf{0},\rho)\cap\Omega)}\int_{B(\mathbf{0},\rho)\cap\Omega}w(\mathbf{x})\mathrm{d}\mathbf{x},$$

readily implies (2.55).

The proof is complete. □

3 Vanishing properties in three dimensions

In this section, we shall consider the vanishing properties of the transmission eigenfunctions in the three-dimensional case. First, we introduce the notion of edge corner geometry in the three-dimensional setting, we mention that this kind of notation is also given in [23]. Let W is a sector defined in (2.1) and $M \in \mathbb{R}_+$. It can be readily seen that $W \times (-M, M)$ actually describes an edge singularity and we call it a 3D corner. In what follows, suppose a Lipschitz domain $\Omega \subset \mathbb{R}^3$ possesses a 3D corner. Let $\mathbf{x}_c \in \mathbb{R}^2$ be the vertex of W and $x_3^c \in (-M, M)$. Then, $\left(\mathbf{x}_c, x_3^c\right)$ is defined as an edge point of $W \times (-M, M)$. We shall use $\mathbf{x} = (\mathbf{x}', x_3) \in W \times (-M, M)$ to signify a point in the edge corner. Now, we consider the following three dimensional conductive transmission eigenvalue problem:

$$\begin{cases} \big(\Delta + k^2(1+V)\big)w = 0 & \text{in } W \times (-M, M), \\ (\Delta + k^2)v = 0 & \text{in } W \times (-M, M), \\ w = v, \ \ \partial_\nu w = \partial_\nu v + \eta v & \text{on } \Gamma^{\pm} \times (-M, M), \end{cases} \tag{3.1}$$

where $\Gamma^{\pm}$ are the two boundary pieces of W.

For the subsequent use, we introduce the following dimension reduction operator.

Definition 3.1 [23, Definition 3.1] Let $W \subset \mathbb{R}^2$ be defined in (2.1), $M > 0$. For a given function g with the domain $W \times (-M, M)$. Pick up any point $x_3^c \in (-M, M)$. Suppose $\psi \in C_0^\infty\left(\left(x_3^c - L, x_3^c + L\right)\right)$ is a non-negative function and $\psi \not\equiv 0$, where L is sufficiently small such that $\left(x_3^c - L, x_3^c + L\right) \subset (-M, M)$, and write $\mathbf{x} = \left(\mathbf{x}', x_3\right) \in \mathbb{R}^3, \mathbf{x}' \in \mathbb{R}^2$. The dimension reduction operator $\mathcal{R}$ is defined by

$$\mathcal{R}(g)\left(\mathbf{x}'\right) = \int_{x_3^c - L}^{x_3^c + L} \psi\left(x_3\right) g\left(\mathbf{x}', x_3\right) \, \mathrm{d}x_3. \tag{3.2}$$

where $\mathbf{x}' \in W$.

For later usage, we need the following regularity result of the dimension reduction operator.

Lemma 3.1 *Let* $g \in H^1(W \times (-M, M)) \cap C^\alpha(\bar{W} \times [-M, M])$, *where* $0 < \alpha < 1$. *Then,*

$$\mathcal{R}(g)\left(\mathbf{x}'\right) \in H^1(W) \cap C^\alpha(\bar{W}).$$

Proof We first show that $\mathcal{R} : H^1(W \times (-M, M)) \to H^1(W)$ is a bounded operator. Let $g \in H^1(W \times (-M, M))$, by the dominated convergence theorem, we know that $\partial^\alpha_{\mathbf{x}'} \mathcal{R}(g)(\mathbf{x}') = \mathcal{R}(\partial^\alpha_{\mathbf{x}'} g)(\mathbf{x}')$ for $\alpha = (i, j)$ with $i, j = 0, 1$ and $i + j \le 1$, so

$$|\partial^\alpha_{\mathbf{x}'} \mathcal{R}(g)(\mathbf{x}')| \le \int_{x_3^c - L}^{x_3^c + L} \|\psi\|_\infty \left|\partial^\alpha_{\mathbf{x}'} g(\mathbf{x}', x_3)\right| \, \mathrm{d}x_3.$$

Furthermore, by Minkowski integral, we have

$$\|\mathcal{R}(g)(\mathbf{x}')\|_{H^1(W)} \le \|\psi\|_\infty \|g(\mathbf{x}', x_3)\|_{H^1(W \times (-M, M))}.$$

When $g \in C^\alpha(\bar{W} \times [-M, M])$, it can be easily derived that

$$|\mathcal{R}(g)(\mathbf{x}') - \mathcal{R}(g)(\mathbf{y}')| \le 2\|\psi\|_\infty \|g\|_{C^\alpha} |\mathbf{x}' - \mathbf{y}'|^\alpha.$$

which means that $\mathcal{R}(g)(\mathbf{x}') \in C^\alpha(\bar{W})$. □

Similar to Lemma 2.2, one can prove similarly the following result:

Lemma 3.2 *For the Herglotz wave function v_j defined in (1.5) in three dimensions, it holds that*

$$\begin{aligned} &\|\mathcal{R}(v_j)\|_{C^1} \leq 4L\sqrt{\pi}\|\psi\|_{C^\infty}(1+k)\|g_j\|_{L^2(\mathbb{S}^2)}, \\ &\|\mathcal{R}(v_j)\|_{C^\alpha} \leq \operatorname{diam}(S_h)^{1-\alpha}\|\mathcal{R}(v_j)\|_{C^1}, \end{aligned} \tag{3.3}$$

where $0<\alpha<1$ and $\operatorname{diam}(S_h)$ is the diameter of S_h.

Using the Jacobi–Anger expansion [17, Page 33] in $\mathbb{R}^3$, one can derive the following result:

Lemma 3.3 *The Herglotz function v_j has the expansion in three dimensions as follows:*

$$v_j(\mathbf{x}) = v_j(\mathbf{0})j_0(k|\mathbf{x}|) + \sum_{\ell=1}^{\infty}\gamma_{\ell j}\mathbf{i}^\ell(2\ell+1)j_\ell(k|\mathbf{x}|), \quad \mathbf{x}\in\mathbb{R}^3, \tag{3.4}$$

where

$$v_j(\mathbf{0}) = \int_{\mathbb{S}^2} g_j(\mathbf{d})\mathrm{d}\sigma(\mathbf{d}), \quad \gamma_{\ell j} := \int_{\mathbb{S}^2} g_j(\mathbf{d})P_\ell(\cos(\varphi))\mathrm{d}\sigma(\mathbf{d}), \quad \mathbf{d}\in\mathbb{S}^2. \tag{3.5}$$

Here, P_ℓ denotes the Legendre polynomial and φ is the angle between $\mathbf{x}$ and $\mathbf{d}$. $j_\ell(t)$ stands for the spherical Bessel function of order ℓ (see e.g. [1]), and more explicitly, one has

$$j_\ell(t) = \frac{t^\ell}{(2\ell+1)!!}\left(1-\sum_{l=1}^{\infty}\frac{(-1)^l t^{2l}}{2^l l! N_{\ell,l}}\right), \ \ell = 0,1,2,\ldots, \tag{3.6}$$

where $N_{\ell,l} = (2\ell+3)\cdots(2\ell+2l+1)$.

Applying the dimension reduction operator to the above spherical Bessel function, we can obtain the following lemma.

Lemma 3.4 [23, Lemma 3.5] *$\mathcal{R}(j_0)(\mathbf{x}')$ and $\mathcal{R}(j_\ell)(\mathbf{x}')$ have the deformation as following:*

$$\begin{aligned} \mathcal{R}(j_0)(\mathbf{x}') &= C(\psi)\left[1-\sum_{l=1}^{\infty}\frac{(-1)^l k^{2l}}{2^l l!(2l+1)!!}\left(\left|\mathbf{x}'\right|^2 + a_{0,l}^2\right)^l\right], \\ \mathcal{R}(j_\ell)(\mathbf{x}') &= \frac{k^\ell\left(\left|\mathbf{x}'\right|^2 + a_\ell^2\right)^{(\ell-1)/2}}{(2\ell+1)!!}\left[1-\sum_{l=1}^{\infty}\frac{(-1)^l k^{2l}\left(\left|\mathbf{x}'\right|^2 + a_{\ell,l}^2\right)^l}{2^l l! N_{\ell,l}}\right] C_1(\psi)\left|\mathbf{x}'\right|^2, \end{aligned} \tag{3.7}$$

where

$$N_{\ell,l} = (2\ell+3)\cdots(2\ell+2l+1), \quad a_{0,l}, a_\ell, a_{\ell,l} \in [-L, L],$$

and

$$C(\psi) = \int_{-L}^{L}\psi(x_3)\,\mathrm{d}x_3, \quad C_1(\psi) = \int_{-\arctan L/|\mathbf{x}'|}^{\arctan L/|\mathbf{x}'|}\psi\left(\left|\mathbf{x}'\right|\tan\varpi\right)\sec^3\varpi\,\mathrm{d}\varpi,$$

with

$$\varpi\in\left[-\arctan L/\left|\mathbf{x}'\right|, \arctan L/\left|\mathbf{x}'\right|\right].$$

We further derive several critical auxiliary lemmas.

Lemma 3.5 *Let $v, w \in H^1(W \times (-M, M))$ be a pair of conductive transmission eigenfunctions to (3.1) and $D_\varepsilon = S_h \setminus B_\varepsilon$ for $0 < \varepsilon < h$, $\eta \in C^\alpha(\bar{\Gamma}_h^\pm \times [-M, M])$ for $0 < \alpha < 1$ and $\eta = \eta(\mathbf{x}')$ is independent of x_3. Then, it holds that*

$$\begin{aligned}\lim_{\varepsilon\to\infty} \int_{D_\varepsilon} & u_0(s\mathbf{x}')\,(\Delta_{\mathbf{x}'}\mathcal{R}(v) - \Delta_{\mathbf{x}'}\mathcal{R}(w))\,\mathrm{d}\mathbf{x}' \\ &= \int_{\Lambda_h} \left(u_0\left(s\mathbf{x}'\right)\partial_\nu \mathcal{R}(v-w)\left(\mathbf{x}'\right) - \mathcal{R}(v-w)\left(\mathbf{x}'\right)\partial_\nu u_0\left(s\mathbf{x}'\right)\right)\mathrm{d}\sigma \\ &\quad - \int_{\Gamma_h^\pm} \eta\left(\mathbf{x}'\right)\mathcal{R}(v)\left(\mathbf{x}'\right)u_0\left(s\mathbf{x}'\right)\mathrm{d}\sigma,\end{aligned} \tag{3.8}$$

where $\Lambda_h = S_h \cap \partial B_h$ and $\Gamma_h^\pm = \Gamma^\pm \cap B_h$.

Proof Since $w\left(\mathbf{x}', x_3\right) = v\left(\mathbf{x}', x_3\right)$ when $\mathbf{x}' \in \Gamma$ and $-L < x_3 < L$, we have

$$\mathcal{R}(w)\left(\mathbf{x}'\right) = \mathcal{R}(v)\left(\mathbf{x}'\right) \text{ on } \Gamma. \tag{3.9}$$

Similarly, using the fact that η is independent of x_3, we can obtain that

$$\partial_\nu \mathcal{R}(v)\left(\mathbf{x}'\right) + \eta\left(\mathbf{x}'\right)\mathcal{R}(v)\left(\mathbf{x}'\right) = \partial_\nu \mathcal{R}(w)\left(\mathbf{x}'\right) \text{ on } \Gamma. \tag{3.10}$$

Therefore, by Green's formula, we have

$$\begin{aligned}\int_{D_\varepsilon} & \Delta_{\mathbf{x}'}\left(\mathcal{R}(v)\left(\mathbf{x}'\right) - \mathcal{R}(w)\left(\mathbf{x}'\right)\right)u_0\left(s\mathbf{x}'\right)\mathrm{d}\mathbf{x}' \\ &= \int_{\partial D_\varepsilon} \left(u_0\left(s\mathbf{x}'\right)\partial_\nu \mathcal{R}(v-w)\left(\mathbf{x}'\right) - \mathcal{R}(v-w)\left(\mathbf{x}'\right)\partial_\nu u_0\left(s\mathbf{x}'\right)\right)\mathrm{d}\sigma \\ &= \int_{\Lambda_h} \left(u_0\left(s\mathbf{x}'\right)\partial_\nu \mathcal{R}(v-w)\left(\mathbf{x}'\right) - \mathcal{R}(v-w)\left(\mathbf{x}'\right)\partial_\nu u_0\left(s\mathbf{x}'\right)\right)\mathrm{d}\sigma \\ &\quad + \int_{\Lambda_\varepsilon} \left(u_0\left(s\mathbf{x}'\right)\partial_\nu \mathcal{R}(v-w)\left(\mathbf{x}'\right) - \mathcal{R}(v-w)\left(\mathbf{x}'\right)\partial_\nu u_0\left(s\mathbf{x}'\right)\right)\mathrm{d}\sigma \\ &\quad - \int_{\Gamma_{(\varepsilon,h)}^\pm} \eta\left(\mathbf{x}'\right)\mathcal{R}(v)\left(\mathbf{x}'\right)u_0\left(s\mathbf{x}'\right)\mathrm{d}\sigma,\end{aligned}$$

where $\Lambda_h = S_h \cap \partial B_h$, $\Lambda_\varepsilon = S_h \cap \partial B_\varepsilon$ and $\Gamma_{(\varepsilon,h)}^\pm = \Gamma^\pm \cap (B_h \backslash B_\varepsilon)$.

Since $v, w \in H^1\left(S_h \times (-L, L)\right)$, from Lemma 3.1 we know that $\mathcal{R}(v-w) \in H^1(S_h)$, and it can be derived that

$$\lim_{\varepsilon\to 0} \int_{\Lambda_c} \left(u_0\left(s\mathbf{x}'\right)\partial_\nu \mathcal{R}(v-w)\left(\mathbf{x}'\right) - \mathcal{R}(v-w)\left(\mathbf{x}'\right)\partial_\nu u_0\left(s\mathbf{x}'\right)\right)\mathrm{d}\sigma = 0.$$

One the other hand, since $v \in H^1\left((S_h \cap B_\varepsilon) \times (-L, L)\right)$, then by Lemma 3.1, one also has $\mathcal{R}(v)\left(\mathbf{x}'\right) \in H^1\left(S_h \cap B_\varepsilon\right)$. The trace theorem then implies that $\mathcal{R}(v)\left(\mathbf{x}'\right) \in L^2\left(\Gamma_{(0,e)}^\pm\right)$ where $\Gamma_{(0,c)}^\pm = \Gamma^\pm \cap B_\varepsilon$. For sufficiently small ε, we further see that $\left|u_0\left(s\mathbf{x}'\right)\right| \leq 1$ and $\eta \in C^\alpha\left(\bar{\Gamma}_h^\pm \times [-M, M]\right)$. Hence, we have

$$\lim_{\varepsilon\to 0} \int_{\Gamma_{(0,c)}^\pm} \eta\left(\mathbf{x}'\right)\mathcal{R}(v)\left(\mathbf{x}'\right)u_0\left(s\mathbf{x}'\right)\mathrm{d}\sigma = 0.$$

The proof is complete. □

Lemma 3.6 *Suppose that $\eta \in C^\alpha\left(\bar{\Gamma}_h^\pm \times [-M, M]\right)$ for $0 < \alpha < 1$ and $\eta = \eta(\mathbf{x}')$ is independent of x_3, and θ_M, θ_m are defined in (2.1) and $\theta_M - \theta_m \neq \pi$. Define*

$$I_2^\pm = \int_{\Gamma_h^\pm} \eta\left(\mathbf{x}'\right)u_0\left(s\mathbf{x}'\right)\mathcal{R}\left(v_j\right)\left(\mathbf{x}'\right)\mathrm{d}\sigma. \tag{3.11}$$

Then, it holds that

$$I_2^- = 2\eta(\mathbf{0})v_j(\mathbf{0})s^{-1}\left(\mu(\theta_m)^{-2} - \mu(\theta_m)^{-2} e^{-\sqrt{sh}\mu(\theta_m)} - \mu(\theta_m)^{-1}\sqrt{sh}e^{-\sqrt{sh}\mu(\theta_m)}\right)C_2^- + v_j(\mathbf{0})\eta(\mathbf{0})I_{21}^- + \eta(\mathbf{0})I_{22}^- + I_\eta^-, \tag{3.12}$$

and

$$I_2^+ = 2\eta(\mathbf{0})v_j(\mathbf{0})s^{-1}\left(\mu(\theta_M)^{-2} - \mu(\theta_M)^{-2} e^{-\sqrt{sh}\mu(\theta_M)} - \mu(\theta_M)^{-1}\sqrt{sh}e^{-\sqrt{sh}\mu(\theta_M)}\right)C_2^+ + v_j(\mathbf{0})\eta(\mathbf{0})I_{21}^+ + \eta(\mathbf{0})I_{22}^+ + I_\eta^+. \tag{3.13}$$

where $C_2^\pm$ are positive constants, and

$$\begin{aligned}
I_{21}^- &\le \mathcal{O}\left(s^{-3}\right), \quad I_{22}^- \le \mathcal{O}\left(\|g_j\|_{L^2(\mathbb{S}^2)} s^{-3}\right),\\
\left|I_\eta^-\right| &\le \|\eta\|_{C^\alpha}\left(v_j(\mathbf{0})\mathcal{O}\left(s^{-1-\alpha}\right) + \mathcal{O}\left(\|g_j\|_{L^2(\mathbb{S}^2)} s^{-3-\alpha}\right)\right),\\
I_{21}^+ &\le \mathcal{O}\left(s^{-3}\right), \quad I_{22}^+ \le \mathcal{O}\left(\|g_j\|_{L^2(\mathbb{S}^2)} s^{-3}\right),\\
\left|I_\eta^+\right| &\le \|\eta\|_{C^\alpha}\left(v_j(\mathbf{0})\mathcal{O}\left(s^{-1-\alpha}\right) + \mathcal{O}\left(\|g_j\|_{L^2(\mathbb{S}^2)} s^{-3-\alpha}\right)\right).
\end{aligned}$$

Proof This is basically Lemma 3.6 in [23]. Its proof shall be needed in our subsequent analysis. For self-containedness and convenient reference of the readers, we present its proof in what follows.

Using Lemma 2.4, we have

$$I_2^- = \eta(\mathbf{0})\int_{\Gamma_h^-} u_0(s\mathbf{x}')\mathcal{R}(v_j)(\mathbf{x}')\,\mathrm{d}\sigma + \int_{\Gamma_h^-}\delta\eta(\mathbf{x}')u_0(s\mathbf{x}')\mathcal{R}(v_j)(\mathbf{x}')\,\mathrm{d}\sigma.$$

Recall the definition in (2.19). Combining Lemmas 3.3 and 3.4, we can obtain that

$$\begin{aligned}
&\int_{\Gamma_h^-} u_0(s\mathbf{x}')\mathcal{R}(v_j)(\mathbf{x}')\,\mathrm{d}\sigma\\
&= v_j(\mathbf{0})\int_{\Gamma_h^-} u_0(s\mathbf{x}')\mathcal{R}(j_0)(\mathbf{x}')\,\mathrm{d}\sigma\\
&\quad + \sum_{\ell=1}^{\infty}\gamma_{\ell j}\mathbf{i}^\ell(2\ell+1)\int_{\Gamma_h^-} u_0(s\mathbf{x}')\mathcal{R}(j_\ell)(\mathbf{x}')\,\mathrm{d}\sigma.
\end{aligned}$$

Using Newton's binomial expansion (see also Lemma 3.4), we have

$$\begin{aligned}
&\int_{\Gamma_h^-} u_0(s\mathbf{x}')\mathcal{R}(j_0)(\mathbf{x}')\,\mathrm{d}\sigma\\
&= C(\psi)\int_0^h\left[1 - \sum_{l=1}^{\infty}\frac{(-1)^l k^{2l}}{(2l+1)!!}\left(r^2 + a_{0,l}^2\right)^l\right]e^{-\sqrt{sr}\mu(\theta_m)}\mathrm{d}r\\
&= C(\psi)\left[1 - \sum_{l=1}^{\infty}\frac{(-1)^l k^{2l}}{(2l+1)!!}a_{0,l}^{2l}\right]\int_0^h e^{-\sqrt{sr}\mu(\theta_m)}\mathrm{d}r \qquad (3.14)\\
&\quad - C(\psi)\sum_{l=1}^{\infty}\frac{(-1)^l k^{2l}}{(2l+1)!!}\left(\sum_{i_1=1}^{l} C(l,i_1)a_{0,l}^{2(l-i_1)}\int_0^h r^{2i_1}e^{-\sqrt{sr}\mu(\theta_m)}\mathrm{d}r\right)\\
&:= 2s^{-1}\left(\mu(\theta_m)^{-2} - \mu(\theta_m)^{-2}e^{-\sqrt{sh}\mu(\theta_m)} - \mu(\theta_m)^{-1}\sqrt{sh}e^{-\sqrt{sh}\mu(\theta_m)}\right)C_2^- + I_{21}^-,
\end{aligned}$$

where $C(\psi) = \int_{-L}^{L}\psi(x_3)\,\mathrm{d}x_3 > 0$, $C(l, i_1) = \frac{l!}{i_1!(l-i_1)!}$ is the combinatorial number of order l and $C_2^- = C(\psi)\left[1 - \sum_{l=1}^{\infty}\frac{(-1)^l k^{2l}}{(2l+1)!!}a_{0,l}^{2l}\right]$. By choosing L such that $kL < 1$, we have that

$$\left|\sum_{l=1}^{\infty}\frac{(-1)^l k^{2l}}{(2l+1)!!}a_{0,l}^{2l}\right| \le \sum_{l=1}^{\infty}(kL)^{2l} = \frac{(kL)^2}{1-(kL)^2}.$$

Therefore, we can deduce that

$$0 < \frac{C(\psi)\left(1-2(kL)^2\right)}{1-(kL)^2} \le C_2^- \le \frac{C(\psi)}{1-(kL)^2}. \tag{3.15}$$

For I_{21}^-, choosing h and L such that $k^2(h^2+L^2) < 1$, we can deduce that

$$\begin{aligned}
|I_{21}^-| &\le |C(\psi)| \sum_{l=1}^{\infty}\frac{k^{2l}}{(2l+1)!!}\sum_{i_1=1}^{l} C\left(l,i_1\right) h^{2(i_1-1)}L^{2(l-i_1)}\int_0^h r^2 e^{-\sqrt{sr}\omega(\theta_m)}\mathrm{d}r\\
&= |C(\psi)| \sum_{l=1}^{\infty}\frac{k^{2l}}{(2l+1)!!h^2}\sum_{i_1=1}^{l} C\left(l,i_1\right) h^{2i_1}L^{2(l-i_1)}\int_0^h r^2 e^{-\sqrt{sr}\omega(\theta_m)}\mathrm{d}r\\
&= |C(\psi)| \sum_{l=1}^{\infty}\frac{k^{2l}}{(2l+1)!!h^2}\left(\left(h^2+L^2\right)^l - L^{2l}\right)\int_0^h r^2 e^{-\sqrt{sr}\omega(\theta_m)}\mathrm{d}r\\
&\le 2L\|\psi\|_{\infty} \sum_{l=1}^{\infty}\frac{k^{2l}}{(2l+1)!!h^2}\left(\left(h^2+L^2\right)^l - L^{2l}\right)\mathcal{O}\left(s^{-3}\right)\\
&= \mathcal{O}\left(s^{-3}\right).
\end{aligned} \tag{3.16}$$

Taking

$$I_{22}^- = \sum_{\ell=1}^{\infty}\gamma_{\ell j}\mathbf{i}^{\ell}(2\ell+1)\int_{\Gamma_h^-} u_0\left(s\mathbf{x}'\right)\mathcal{R}\left(j_\ell\right)\left(\mathbf{x}'\right)\mathrm{d}\sigma,$$

we then have

$$\begin{aligned}
\left|I_{22}^-\right| &\le C_1(\psi)\left\|g_j\right\|_{L^2(\mathbb{S}^2)}\\
&\quad\cdot\sum_{\ell=1}^{\infty}\int_0^h r^2 e^{-\sqrt{sr}\omega(\theta_m)}\frac{k^{\ell}\left(|r|^2+a_\ell^2\right)^{(\ell-1)/2}}{(2\ell-1)!!}\left|1-\sum_{l=1}^{\infty}\frac{(-1)^l k^{2l}\left(|r|^2+a_{\ell,l}^2\right)^l}{2^l l! N_{\ell,l}}\right|\mathrm{d}r\\
&= C_1(\psi)\left\|g_j\right\|_{L^2(\mathbb{S}^2)}\sum_{\ell=1}^{\infty}\frac{k^{\ell}\left(|\beta_\ell|^2+a_\ell^2\right)^{(\ell-1)/2}}{(2\ell-1)!!}\left|1-\sum_{l=1}^{\infty}\frac{(-1)^l k^{2l}\left(\left|\beta_{\ell,l}\right|^2+a_{\ell,l}^2\right)^l}{2^l l! N_{\ell,l}}\right|\\
&\quad\cdot\int_0^h r^2 e^{-\sqrt{sr}\omega(\theta_m)}\mathrm{d}r\\
&= \mathcal{O}\left(\left\|g_j\right\|_{L^2(\mathbb{S}^2)} s^{-3}\right),
\end{aligned} \tag{3.17}$$

where $C_1(\psi)$ is defined in Lemma 3.4. Taking

$$I_\eta^- = \int_{\Gamma_h^-}\delta\eta\left(\mathbf{x}'\right)u_0\left(s\mathbf{x}'\right)\mathcal{R}\left(v_j\right)\left(\mathbf{x}'\right)\mathrm{d}\sigma,$$

and by Lemmas 3.3 and 3.4, we can obtain that

$$\begin{aligned}
I_\eta^- &= v_j(0)\int_{\Gamma_h^-}\delta\eta\left(\mathbf{x}'\right)u_0\left(s\mathbf{x}'\right)\mathcal{R}\left(j_0\right)\left(\mathbf{x}'\right)\mathrm{d}\sigma\\
&\quad+\sum_{\ell=1}^{\infty}\gamma_{\ell j}\mathbf{i}^{\ell}(2\ell+1)\int_{\Gamma_h^-}\delta\eta\left(\mathbf{x}'\right)u_0\left(s\mathbf{x}'\right)\mathcal{R}\left(j_\ell\right)\left(\mathbf{x}'\right)\mathrm{d}\sigma.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
&\left|\int_{\Gamma_h^-} \delta\eta\left(\mathbf{x}'\right) u_0\left(s\mathbf{x}'\right) \mathcal{R}\left(j_0\right)\left(\mathbf{x}'\right) \mathrm{d}\sigma\right| \\
&\leq |C(\psi)|\|\eta\|_{C^\alpha} \int_0^h r^\alpha \left|1-\sum_{l=1}^{\infty} \frac{(-1)^l k^{2l}}{(2l+1)!!}\left(r^2+a_{0,l}^2\right)^l\right| e^{-\sqrt{sr}\omega(\theta_m)} \mathrm{d}r \\
&= 2L\|\psi\|_\infty \|\eta\|_{C^\alpha}\left|1-\sum_{l=1}^{\infty} \frac{(-1)^l k^{2l}}{(2l+1)!!}\left(\beta_{0,l}^2+a_{0,l}^2\right)^l\right| \int_0^h r^\alpha e^{-\sqrt{sr}\omega(\theta_m)} \mathrm{d}r \\
&\leq \mathcal{O}\left(s^{-\alpha-1}\right),
\end{aligned} \tag{3.18}$$

where $\beta_{0,l} \in [0, h]$ such that $k^2\left(\beta_{0,l}^2+a_{0,l}^2\right) \leq k^2\left(h^2+L^2\right) < 1$ for sufficiently small h and L. Next, we can deduce that

$$\begin{aligned}
&\left|\sum_{\ell=1}^{\infty} \gamma_{\ell j} \mathbf{i}^\ell (2\ell+1) \int_{\Gamma_h^-} \delta\eta\left(\mathbf{x}'\right) u_0\left(s\mathbf{x}'\right) \mathcal{R}\left(j_\ell\right)\left(\mathbf{x}'\right) \mathrm{d}\sigma\right| \\
&\leq C_1(\psi)\|\eta\|_{C^\alpha} \\
&\quad \cdot \sum_{\ell=1}^{\infty} |\gamma_{\ell j}| \int_0^h r^\alpha \frac{k^\ell\left(r^2+a_\ell^2\right)^{(\ell-1)/2}}{(2\ell-1)!!}\left|1-\sum_{l=1}^{\infty} \frac{k^{2l}\left(r^2+a_{\ell,l}^2\right)^l}{2^l l! N_{\ell,l}}\right| r^2 e^{-\sqrt{sr}\omega(\theta_m)} \mathrm{d}r \\
&\leq 2C_1(\psi)\|\eta\|_{C^\alpha}\left\|g_j\right\|_{L^2(\mathbb{S}^2)} \int_0^h r^{2+\alpha} e^{-\sqrt{sr}\omega(\theta_m)} \mathrm{d}r \\
&\quad \cdot \sum_{\ell=1}^{\infty} \frac{k^\ell\left(\beta_\ell^2+a_\ell^2\right)^{(\ell-1)/2}}{(2\ell-1)!!}\left|1-\sum_{l=1}^{\infty} \frac{k^{2l}\left(\beta_{\ell,l}^2+a_{\ell,l}^2\right)^l}{2^l l! N_{\ell,l}}\right| \\
&= \mathcal{O}\left(\left\|g_j\right\|_{L^2(\mathbb{S}^2)} s^{-\alpha-3}\right),
\end{aligned} \tag{3.19}$$

where we have used the estimates $k^2\left(\beta_\ell^2+a_\ell^2\right) \leq k^2\left(h^2+L^2\right) < 1$, $k^2\left(\beta_{\ell,l}^2+a_{\ell,l}^2\right) \leq k^2\left(h^2+L^2\right) < 1$ for sufficiently small h and L, as well as the estimate

$$\left|\gamma_{\ell j}\right| = \left|\int_{\mathbb{S}^2} g_j(\mathbf{d}) P_\ell(\hat{\mathbf{x}}|_{\Gamma_h^-} \cdot \mathbf{d}) \mathrm{d}\sigma(\mathbf{d})\right| \leq \sqrt{2\pi}\sqrt{\frac{2}{2l+1}}\left\|g_j\right\|_{L^2(\mathbb{S}^2)} \leq 2\sqrt{\pi}\left\|g_j\right\|_{L^2(\mathbb{S}^2)}.$$

Finally, by combining (3.14), (3.16), (3.17), (3.18) and (3.19), we can prove (3.12). Using a similar argument, one can prove (3.13).

The proof is complete. □

Lemma 3.7 *Let $\eta \in C^\alpha\left(\bar{\Gamma}_h^\pm \times [-M, M]\right)$ for $0 < \alpha < 1$ and $\eta = \eta(\mathbf{x}')$ be independent of x_3, θ_M, θ_m be defined in (2.1) and $\theta_M - \theta_m \neq \pi$. Set*

$$\xi_j^\pm(s) = \int_{\Gamma_h^\pm} \eta\left(\mathbf{x}'\right) u_0\left(s\mathbf{x}'\right) \mathcal{R}\left(v\left(\mathbf{x}', x_3\right) - v_j\left(\mathbf{x}', x_3\right)\right) \mathrm{d}\sigma. \tag{3.20}$$

Then, it holds that

$$\begin{aligned}
\left|\xi_j^\pm(s)\right| \leq C\|\psi\|_\infty &\left(|\eta(\mathbf{0})|\left\|u_0\left(s\mathbf{x}'\right)\right\|_{L^2(S_h)} + \|\eta\|_{C^\alpha}\left\||\mathbf{x}'|^\alpha u_0\left(s\mathbf{x}'\right)\right\|_{L^2(S_h)}\right) \\
&\left\|v - v_j\right\|_{H^1(S_h \times (-L,L))},
\end{aligned}$$

where C is a positive constant.

Proof Using Cauchy–Schwarz inequality and the trace theorem, we can deduce as follows:

$$\begin{aligned}
\left|\xi_j^{\pm}(s)\right| \leq & |\eta(\mathbf{0})| \int_{\Gamma_h^{\pm}} \left|u_0\left(s\mathbf{x}'\right)\right|\left|\mathcal{R}\left(v\left(\mathbf{x}', x_3\right)-v_j\left(\mathbf{x}', x_3\right)\right)\right| \mathrm{d}\sigma \\
& + \|\eta\|_{C^\alpha} \int_{\Gamma_h^{\pm}} \left|\mathbf{x}'\right|^\alpha \left|u_0\left(s\mathbf{x}'\right)\right|\left|\mathcal{R}\left(v\left(\mathbf{x}', x_3\right)-v_j\left(\mathbf{x}', x_3\right)\right)\right| \mathrm{d}\sigma \\
\leq & |\eta(\mathbf{0})| \left\|\mathcal{R}\left(v-v_j\right)\right\|_{H^{1/2}(\Gamma_h^{\pm})} \left\|u_0\left(s\mathbf{x}'\right)\right\|_{H^{-1/2}(\Gamma_h^{\pm})} \\
& + \|\eta\|_{C^\alpha} \left\|\mathcal{R}\left(v-v_j\right)\right\|_{H^{1/2}(\Gamma_h^{\pm})} \left\|\left|\mathbf{x}'\right|^\alpha u_0\left(s\mathbf{x}'\right)\right\|_{H^{-1/2}(\Gamma_h^{\pm})} \\
\leq & |\eta(\mathbf{0}) \left\|\mathcal{R}\left(v-v_j\right)\right\|_{H^1(S_h)} \left\|u_0\left(s\mathbf{x}'\right)\right\|_{L^2(S_h)} \\
& + \|\eta\|_{C^\alpha} \left\|\mathcal{R}\left(v-v_j\right)\right\|_{H^1(S_h)} \left\|\left|\mathbf{x}'\right|^\alpha u_0\left(s\mathbf{x}'\right)\right\|_{L^2(S_h)} \\
\leq & C\|\psi\|_\infty \left\|v-v_j\right\|_{H^1(S_h\times(-L,L))} \\
& \left(|\eta(\mathbf{0})| \left\|u_0\left(s\mathbf{x}'\right)\right\|_{L^2(S_h)} + \|\eta\|_{C^\alpha} \left\|\left|\mathbf{x}'\right|^\alpha u_0\left(s\mathbf{x}'\right)\right\|_{L^2(S_h)}\right),
\end{aligned}$$

which readily completes the proof. □

We are now in a position of showing the vanishing properties of v in the three-dimensional case.

Theorem 3.1 *Let $v, w \in H^1(W \times (-M, M))$ be a pair of eigenfunctions to (3.1) associated with $k \in \mathbb{R}_+$, where $W \subset \mathbb{R}^2$ is defined in (2.1) and $M > 0$. For any fixed $x_3^c \in (-M, M)$, suppose $L > 0$, defined in Definition 3.1, is sufficiently small such that $\left(x_3^c - L, x_3^c + L\right) \subset (-M, M)$. Moreover, there exists a sufficiently small neighbourhood S_h of $\mathbf{x}_c \in \mathbb{R}^2$ such that $qw \in C^\alpha\left(\overline{S}_h \times [-M, M]\right)$, $v - w \in C^\alpha(\overline{S_h} \times [-M, M])$ and $\eta \in C^\alpha\left(\bar{\Gamma}_h^{\pm} \times [-M, M]\right)$ for $0 < \alpha < 1$, where $q := 1 + V$. If the following conditions are fulfilled:*

(a) there exists Herglotz functions $v_j, j = 1, 2, \ldots,$ defined in (1.5), with kernels g_j such that

$$\left\|v - v_j\right\|_{H^1(S_h\times(-M,M))} \leq j^{-\Upsilon}, \quad \left\|g_j\right\|_{L^2(\mathbb{S}^2)} \leq Cj^{\varrho}, \tag{3.21}$$

for some constants $C > 0$, $\Upsilon > 0$ and $\varrho < (1 + \alpha)\Upsilon$;

(b) the function $\eta = \eta\left(\mathbf{x}'\right)$ is independent of x_3 and does not vanish at $\mathbf{x}_c$, that is,

$$\eta\left(\mathbf{x}_c\right) \neq 0; \tag{3.22}$$

(c) the corner in the sector W is a real corner, that is

$$-\pi < \theta_m < \theta_M < \pi \text{ and } \theta_M - \theta_m \neq \pi; \tag{3.23}$$

then for every edge point $\left(\mathbf{x}_c, x_3^c\right) \in \mathbb{R}^3$ of $W \times (-M, M)$ where $x_3^c \in (-M, M)$, one has

$$\lim_{\rho\to+0} \frac{1}{m\left(B\left(\left(\mathbf{x}_c, x_3^c\right), \rho\right) \cap (W \times (-M, M))\right)} \int_{B((\mathbf{x}_c, x_3), \rho)\cap(W\times(-M,M))} |v(\mathbf{x})| \mathrm{d}\mathbf{x} = 0.$$

Proof For an edge point $(\mathbf{x}_c, x_3^c) \in W \times (-M, M)$, we assume, without loss of generality, that the vertex $(\mathbf{x}_c, x_3^c) = \mathbf{0}$. By direct calculations, we have

$$
\begin{aligned}
\Delta_{\mathbf{x}'}\mathcal{R}(v)\left(\mathbf{x}'\right) &= \Delta_{\mathbf{x}'}\int_{-L}^{L}\psi(x_3)v(\mathbf{x}', x_n)\mathrm{d}x_3 \\
&= \int_{-L}^{L}\psi(x_3)\left(-k^2 v(\mathbf{x}', x_n) - \partial_{x_3}^2 v(\mathbf{x}', x_n)\right)\mathrm{d}x_3 \\
&= -\int_{-L}^{L}\psi(x_3)\partial_{x_3}^2 v(\mathbf{x}', x_n)\mathrm{d}x_3 - k^2\mathcal{R}(v)(\mathbf{x}') \\
&= \int_{-L}^{L}\psi''(x_3)\, v\left(\mathbf{x}', x_3\right) dx_3 - k^2\mathcal{R}(v)\left(\mathbf{x}'\right).
\end{aligned}
\tag{3.24}
$$

Similarly, we can obtain that

$$
\Delta_{\mathbf{x}'}\mathcal{R}(w)\left(\mathbf{x}'\right) = \int_{-L}^{L}\psi''(x_3)\, w\left(\mathbf{x}', x_3\right) dx_3 - k^2\mathcal{R}(qw)\left(\mathbf{x}'\right).
\tag{3.25}
$$

Therefore, we have

$$
\begin{aligned}
&\Delta_{\mathbf{x}'}\mathcal{R}(v)\left(\mathbf{x}'\right) - \Delta_{\mathbf{x}'}\mathcal{R}(w)\left(\mathbf{x}'\right) \\
&\quad = \int_{-L}^{L}\psi''(x_3)\left(v\left(\mathbf{x}', x_3\right) - w\left(\mathbf{x}', x_3\right)\right)\mathrm{d}x_3 + k^2\mathcal{R}(qw)\left(\mathbf{x}'\right) - k^2\mathcal{R}(v)\left(\mathbf{x}'\right) \\
&\quad := F_1\left(\mathbf{x}'\right) + F_2\left(\mathbf{x}'\right) + F_3\left(\mathbf{x}'\right).
\end{aligned}
\tag{3.26}
$$

Next, we set

$$
F_{3j}\left(\mathbf{x}'\right) = -k^2\mathcal{R}\left(v_j\right)\left(\mathbf{x}'\right),
\tag{3.27}
$$

and consider the following integral

$$
\begin{aligned}
&\int_{S_h}\left(\Delta_{\mathbf{x}'}\mathcal{R}(v)\left(\mathbf{x}'\right) - \Delta_{\mathbf{x}'}\mathcal{R}(w)\left(\mathbf{x}'\right)\right)u_0\left(s\mathbf{x}'\right)\mathrm{d}\mathbf{x}' \\
&\quad = \int_{S_h}\left(F_1\left(\mathbf{x}'\right) + F_2\left(\mathbf{x}'\right) + F_{3j}\left(\mathbf{x}'\right)\right)u_0\left(s\mathbf{x}'\right)\mathrm{d}\mathbf{x}' \\
&\qquad + \int_{S_h}\left(F_3\left(\mathbf{x}'\right) - F_{3j}\left(\mathbf{x}'\right)\right)\mathrm{d}\mathbf{x}' \\
&\quad := I_1 + \Delta_j(s).
\end{aligned}
\tag{3.28}
$$

Using the fact that

$$
\begin{aligned}
&\int_{S_h}\left(\Delta_{\mathbf{x}'}\mathcal{R}(v)\left(\mathbf{x}'\right) - \Delta_{\mathbf{x}'}\mathcal{R}(w)\left(\mathbf{x}'\right)\right)u_0\left(s\mathbf{x}'\right)\mathrm{d}\mathbf{x}' \\
&\quad = \lim_{\varepsilon\to 0}\int_{D_\varepsilon}\left(\Delta_{\mathbf{x}'}\mathcal{R}(v)\left(\mathbf{x}'\right) - \Delta_{\mathbf{x}'}\mathcal{R}(w)\left(\mathbf{x}'\right)\right)u_0\left(s\mathbf{x}'\right)\mathrm{d}\mathbf{x}',
\end{aligned}
\tag{3.29}
$$

where $D_\varepsilon = S_h \backslash B_\varepsilon$ for $0 < \varepsilon < h$, and by Lemma 3.5, we can deduce that

$$
I_1 + \Delta_j(s) = I_3 - I_2^{\pm} - \xi_j^{\pm}(s),
\tag{3.30}
$$

where $\xi_j^{\pm}(s)$ is defined in Lemma 3.7, $I_2^{\pm}$ is defined in Lemma 3.6, and

$$
I_3 = \int_{\Lambda_h}\left(u_0\left(s\mathbf{x}'\right)\partial_\nu\mathcal{R}(v-w) - \mathcal{R}(v-w)\partial_\nu u_0\left(s\mathbf{x}'\right)\right)\mathrm{d}\sigma,
\tag{3.31}
$$

with $\Lambda_h = S_h \cap \partial B_h$. Since $v - w \in H^1(S_h \times (-L, L))$ and $qw \in C^\alpha(\bar{S}_h \times [-L, L])$, $\alpha \in (0, 1)$, from Lemma 3.1 we know that $F_1\left(\mathbf{x}'\right) \in C^\alpha(\overline{S}_h)$ and $F_2\left(\mathbf{x}'\right) \in C^\alpha(\overline{S}_h)$. In addition, we have $\mathcal{R}(v_j)(\mathbf{x}') \in C^\alpha(\overline{S}_h)$. Therefore, by Lemma 2.4, we have

$$\begin{aligned} I_1 =& \left(F_1(\mathbf{0}) + F_2(\mathbf{0}) + F_{3j}(\mathbf{0})\right) \int_{S_h} u_0\left(s\mathbf{x}'\right) \mathrm{d}\mathbf{x}' + \int_{S_h} \delta F_1\left(\mathbf{x}'\right) u_0\left(s\mathbf{x}'\right) \mathrm{d}\mathbf{x}' \\ &+ \int_{S_h} \delta F_2\left(\mathbf{x}'\right) u_0\left(s\mathbf{x}'\right) \mathrm{d}\mathbf{x}' + \int_{S_h} \delta F_{3j}\left(\mathbf{x}'\right) u_0\left(s\mathbf{x}'\right) \mathrm{d}\mathbf{x}'. \end{aligned} \tag{3.32}$$

where $\delta F_1(\mathbf{x}')$, $\delta F_2(\mathbf{x}')$ and $\delta F_{3j}(\mathbf{x}')$ are deduced by Lemma 2.4 with $F_1(\mathbf{x}')$, $F_2(\mathbf{x}')$ and $F_{3j}(\mathbf{x}')$ defined in (3.26) and (3.27). By Lemmas 3.2 and 2.1, we can deduce that

$$\begin{aligned} \left|\int_{S_h} \delta F_1\left(\mathbf{x}'\right) u_0\left(s\mathbf{x}'\right) \mathrm{d}\mathbf{x}'\right| \le& \frac{2\left\|F_1\right\|_{C^\alpha}\left(\theta_M - \theta_m\right)\Gamma(2\alpha+4)}{\delta_W^{2\alpha+4}} s^{-\alpha-2}, \\ \left|\int_{S_h} \delta F_2\left(\mathbf{x}'\right) u_0\left(s\mathbf{x}'\right) \mathrm{d}\mathbf{x}'\right| \le& \frac{2\left\|F_2\right\|_{C^\alpha}\left(\theta_M - \theta_m\right)\Gamma(2\alpha+4)}{\delta_W^{2\alpha+4}} s^{-\alpha-2}, \\ \left|\int_{S_h} \delta F_{3j}\left(\mathbf{x}'\right) u_0\left(s\mathbf{x}'\right) \mathrm{d}\mathbf{x}'\right| \le& \frac{8L\sqrt{\pi}\|\psi\|_{C^\infty}\left(\theta_M - \theta_m\right)\Gamma(2\alpha+4)}{\delta_W^{2\alpha+4}} k^2 \operatorname{diam}\left(S_h\right)^{1-\alpha} \\ &\times (1+k)\left\|g_j\right\|_{L^2\left(\mathbb{S}^2\right)} s^{-\alpha-2}. \end{aligned} \tag{3.33}$$

For $\Delta_j(s)$, using Cauchy–Schwarz inequality, Corollary 2.2 and the assumption (3.21), we can drive that

$$\begin{aligned} \left|\Delta_j(s)\right| \le& k^2\left\|\mathcal{R}(v) - \mathcal{R}\left(v_j\right)\right\|_{L^2\left(S_h\right)}\left\|u_0\left(s\mathbf{x}'\right)\right\|_{L^2\left(S_h\right)} \\ \le& \frac{k^2\|\psi\|_\infty\sqrt{C(L,h)\left(\theta_M - \theta_m\right)} e^{-\sqrt{s\Theta}\delta_W} h}{\sqrt{2}} j^{-\Upsilon}, \end{aligned} \tag{3.34}$$

where $C(L, h)$ is a positive constant depending on L and h and $\Theta \in [0, h]$.

By Lemma 3.1, and the same arguments in (2.50), we have

$$|I_3| \le Ce^{-c'\sqrt{s}}, \tag{3.35}$$

where $c' > 0$ as $s \to \infty$.

By Lemma 3.6 and (3.32), multiplying s on the both sides of (3.30), we can deduce that

$$\begin{aligned} &2v_j(\mathbf{0})\eta(\mathbf{0})\Big[\left(\mu\left(\theta_M\right)^{-2} - \mu\left(\theta_M\right)^{-2} e^{-\sqrt{sh}\mu\left(\theta_M\right)} - \mu\left(\theta_M\right)^{-1}\sqrt{sh}e^{-\sqrt{sh}\mu\left(\theta_M\right)}\right) C_2^+ \\ &\quad + \left(\mu\left(\theta_m\right)^{-2} - \mu\left(\theta_m\right)^{-2} e^{-\sqrt{sh}\mu\left(\theta_m\right)} - \mu\left(\theta_m\right)^{-1}\sqrt{sh}e^{-\sqrt{sh}\mu\left(\theta_m\right)}\right) C_2^-\Big] \\ &= s\Big[I_3 - \left(F_1(\mathbf{0}) + F_2(\mathbf{0}) + F_{3j}(\mathbf{0})\right)\int_{S_h} u_0\left(s\mathbf{x}'\right)\mathrm{d}\mathbf{x}' - \Delta_j(s) \\ &\quad - \eta(\mathbf{0})\left(I_{22}^+ + I_{22}^-\right) - I_\eta^+ - I_\eta^- - \int_{S_h} \delta F_1\left(\mathbf{x}'\right) u_0\left(s\mathbf{x}'\right)\mathrm{d}\mathbf{x}' \\ &\quad - \int_{S_h} \delta F_2\left(\mathbf{x}'\right) u_0\left(s\mathbf{x}'\right)\mathrm{d}\mathbf{x}' \\ &\quad - \int_{S_h} \delta F_{3j}\left(\mathbf{x}'\right) u_0\left(s\mathbf{x}'\right)\mathrm{d}\mathbf{x}' - v_j(\mathbf{0})\eta(\mathbf{0})\left(I_{21}^- + I_{21}^+\right) - \xi_j^\pm(s)\Big]. \end{aligned}$$

Taking $s = j^\beta$, with $\max\left\{0, \frac{\varrho}{1+\alpha}\right\} < \beta < \Upsilon$ and letting $j \to \infty$, together with the use of (2.45), (3.33),(3.34), (3.35), Lemmas 3.6 and 3.7, we can deduce that

$$\lim_{j\to\infty} \eta(\mathbf{0})\left(C_2^-\mu\left(\theta_m\right)^{-2} + C_2^+\mu\left(\theta_M\right)^{-2}\right) v_j(\mathbf{0}) = 0. \tag{3.36}$$

Moreover, by straightforward calculations, we know that

$$\begin{aligned} &C_2^-\mu\left(\theta_m\right)^{-2} + C_2^+\mu\left(\theta_M\right)^{-2} \\ &= \frac{\left(C_2^+\cos\theta_m + C_2^-\cos\theta_M\right) + \mathbf{i}\left(C_2^+\sin\theta_m + C_2^-\sin\theta_M\right)}{\left(\cos\theta_m + \mathbf{i}\sin\theta_m\right)\left(\cos\theta_M + \mathbf{i}\sin\theta_M\right)} \end{aligned}$$

Because of the assumption (3.23), one can directly verify that the following two quantities

$$\left(\cos\theta_m+\cos\theta_M\right) \text{ and } \left(\sin\theta_m+\sin\theta_M\right)$$

cannot be vanishing simultaneously. Suppose that $\sin\theta_m+\sin\theta_M\neq 0$, while the same result can be obtained if the other one is not vanishing. The arguments are divided into the following two cases:

(1) $\sin\theta_m+\sin\theta_M>0$, and

(2) $\sin\theta_m+\sin\theta_M<0$.

For the first case, if $\sin\theta_m$ and $\sin\theta_M$ have the same sign, then from (3.15), we know that $C_2^+\sin\theta_m+C_2^-\sin\theta_M\neq 0$ which means that

$$C_2^-\mu\left(\theta_m\right)^{-2}+C_2^+\mu\left(\theta_M\right)^{-2}\neq 0. \tag{3.37}$$

If $\sin\theta_m$ and $\sin\theta_M$ have different signs, under the assumption (3.23) one has $\sin\theta_m<0$ and $\sin\theta_M>0$. From (3.15), we have

$$\begin{aligned}\frac{C(\psi)}{1-(kL)^2}\left(\sin\theta_m+\left(1-2(kL)^2\right)\sin\theta_M\right)&\leq C_2^+\sin\theta_m+C_2^-\sin\theta_M\\&\leq\frac{C(\psi)}{1-(kL)^2}\left(\left(1-2(kL)^2\right)\sin\theta_m+\sin\theta_M\right).\end{aligned}$$

For fixed $0<\varepsilon<1$, one can choose L sufficiently small such that $0<kL<\sqrt{\varepsilon/2}$, and from which one can derive the bounds as follows:

$$\begin{aligned}\frac{C(\psi)}{1-(kL)^2}\left(\sin\theta_m+(1-\varepsilon)\sin\theta_M\right)&\leq C_2^+\sin\theta_m+C_2^-\sin\theta_M\\&\leq\frac{C(\psi)}{1-(kL)^2}\left((1-\varepsilon)\sin\theta_m+\sin\theta_M\right).\end{aligned} \tag{3.38}$$

Since $\sin\theta_m+\sin\theta_M>0$, we can establish the lower bound in (3.38) as follows. Denote $\varepsilon_0=\min\left\{\frac{\sin\theta_m+\sin\theta_M}{2\sin\theta_M},1\right\}$ and choose $\varepsilon\in(0,\varepsilon_0)$. It can be verified that

$$C_2^+\sin\theta_m+C_2^-\sin\theta_M\geq\frac{C(\psi)}{1-(kL)^2}\left(\sin\theta_m+(1-\varepsilon)\sin\theta_M\right)>0,$$

which indicates that (3.37) holds as well.

For the second case, if $\sin\theta_m=0$ or $\sin\theta_M=0$ is satisfied, from (3.38) we know that

$$C_2^+\sin\theta_m+C_2^-\sin\theta_M<0. \tag{3.39}$$

Otherwise, if $|\sin\theta_m|\leq|\sin\theta_M|$, by using the fact that $(1-\varepsilon)|\sin\theta_m|\leq|\sin\theta_M|$ and the condition that $\sin\theta_m+\sin\theta_M<0$, (3.39) still holds by using the upper bound in (3.38). If $|\sin\theta_m|>|\sin\theta_M|$ then one can choose ε with $\varepsilon>1-|\sin\theta_M|/|\sin\theta_m|>0$ such that (3.39) still holds by using the upper bound of (3.38). Therefore, for the second case, (3.37) is always fulfilled. Thus, by (3.36) and (3.22) we know that

$$\lim_{j\to\infty}v_j(\mathbf{0})=0.$$

Next, in order to simplify the notations, we define

$$\kappa:=B((\mathbf{x}_c,x_n^c),\rho)\cap(W\times(-M,M)).$$

Then, by using the fact that

$$\begin{aligned}&\lim_{\rho\to+0}\frac{1}{m(\kappa)}\int_\kappa|v(\mathbf{x})|\mathrm{d}\mathbf{x}\\&\quad\leq\lim_{j\to\infty}\left(\lim_{\rho\to+0}\frac{1}{m(\kappa)}\int_\kappa\left|v(\mathbf{x})-v_j(\mathbf{x})\right|\mathrm{d}\mathbf{x}+\lim_{\rho\to+0}\frac{1}{m(\kappa)}\int_\kappa\left|v_j(\mathbf{x})\right|\mathrm{d}\mathbf{x}\right),\end{aligned}$$

we finally have

$$\lim_{\rho\to+0}\frac{1}{m\left(\kappa\right)}\int_{\kappa}|v(\mathbf{x})|\mathrm{d}\mathbf{x}=0.$$

The proof is complete. □

Similar to Corollary 2.4, we consider the vanishing property of the transmission eigenfunctions in the case $\eta\equiv 0$ in three dimensions.

Corollary 3.2 *Let $v,w\in H^1(W\times(-M,M))$ be a pair of eigenfunctions to (3.1) associated with $\eta\equiv 0, k\in\mathbb{R}_+$ and $W\subset\mathbb{R}^2$ being defined in (2.1), and $M>0$. For any fixed $x_3^c\in(-M,M)$, suppose $L>0$, defined in Definition 3.1, is sufficiently small such that $\left(x_3^c-L,x_3^c+L\right)\subset(-M,M)$. Moreover, there exists a sufficiently small neighbourhood S_h of $\mathbf{x}_c\in\mathbb{R}^2$ such that $qw\in C^\alpha\left(\overline{S}_h\times[-M,M]\right)$ and $v-w\in C^\alpha(S_h\times(-M,M))$ for $0<\alpha<1$. If the following conditions are fulfilled:*

(a) there exists Herglotz functions $v_j, j=1,2,\dots$, defined in (1.5), with kernels g_j such that

$$\left\|v-v_j\right\|_{H^1(S_h\times(-M,M))}\le j^{-\Upsilon},\quad \left\|g_j\right\|_{L^2(\mathbb{S}^2)}\le Cj^{\varrho},\tag{3.40}$$

for some constants $C>0, \Upsilon>0$ and $\varrho<\alpha\Upsilon/2$;

(b) the corner in the sector W is a real corner, that is

$$-\pi<\theta_m<\theta_M<\pi\quad \text{and}\quad \theta_M-\theta_m\neq\pi;\tag{3.41}$$

then it holds that

$$\lim_{\rho\to+0}\frac{1}{m\left(B\left(\mathbf{x}_c,\rho\right)\cap W\right)}\int_{B(\mathbf{x}_c,\rho)\cap W}\mathcal{R}(Vw)\left(\mathbf{x}'\right)\mathrm{d}\mathbf{x}'=0,$$

where $q\left(\mathbf{x}',x_3\right)=1+V\left(\mathbf{x}',x_3\right)$.

Proof We assume that $\mathbf{x}_c=\mathbf{0}$. Since $\eta\equiv 0$, from (3.30), (3.32), we can obtain that

$$\begin{aligned}&\left(F_1(\mathbf{0})+F_2(\mathbf{0})+F_{3j}(\mathbf{0})\right)\int_{S_h}u_0\left(s\mathbf{x}'\right)\mathrm{d}\mathbf{x}'+\Delta_j(s)\\ &\quad=I_3-\int_{S_h}\delta F_1\left(\mathbf{x}'\right)u_0\left(s\mathbf{x}'\right)\mathrm{d}\mathbf{x}'\\ &\qquad-\int_{S_h}\delta F_2\left(\mathbf{x}'\right)u_0\left(s\mathbf{x}'\right)\mathrm{d}\mathbf{x}'-\int_{S_h}\delta F_{3j}\left(\mathbf{x}'\right)u_0\left(s\mathbf{x}'\right)\mathrm{d}\mathbf{x}',\end{aligned}\tag{3.42}$$

where $\Delta_j(s)$ and I_3 are defined in (3.28) and (3.31), and $\delta F_1(\mathbf{x}')$, $\delta F_2(\mathbf{x}')$ and $\delta F_{3j}(\mathbf{x}')$ are deduced by Lemma 2.4 with $F_1(\mathbf{x}')$, $F_2(\mathbf{x}')$ and $F_{3j}(\mathbf{x}')$ defined in (3.26) and (3.27). Since $v=w$ on $\Gamma^\pm\times(-M,M)$, it is easy to see that

$$F_1(\mathbf{0})=\int_{-L}^{L}\psi''\left(x_3\right)\left(v\left(\mathbf{0},x_3\right)-w\left(\mathbf{0},x_3\right)\right)\mathrm{d}x_3=0.$$

Multiplying s^2 on both sides of (3.42), taking $s=j^\beta$ with $\max\{\varrho/\alpha,0\}<\beta<\Upsilon/2$, using the assumptions (3.40) and (3.41), and by letting $j\to\infty$, from (3.43), (2.45), (3.33), (3.34) and (3.35), we can prove that

$$\lim_{j\to\infty}F_{3j}(\mathbf{0})=-F_2(\mathbf{0}),$$

which in turn implies that

$$\lim_{j\to\infty} \mathcal{R}\left(v_j\right)(\mathbf{0}) = \mathcal{R}(qw)(\mathbf{0}).$$

Using the boundary condition in (3.1) and Definition 3.1, we have that $\mathcal{R}(w)\left(\mathbf{x}'\right) = \mathcal{R}(v)\left(\mathbf{x}'\right)$ on Γ. Hence, we have

$$\begin{aligned}&\lim_{\rho\to+0} \frac{1}{m(B(\mathbf{0},\rho)\cap W)} \int_{B(\mathbf{0},\rho)\cap W} \mathcal{R}(v)\left(\mathbf{x}'\right) \mathrm{d}\mathbf{x}' \\ &\quad = \lim_{\rho\to+0} \frac{1}{m(B(\mathbf{0},\rho)\cap W)} \int_{B(\mathbf{0},\rho)\cap W} \mathcal{R}(w)\left(\mathbf{x}'\right) \mathrm{d}\mathbf{x}'.\end{aligned}$$

which together with the facts that

$$\begin{aligned}\lim_{j\to\infty} \mathcal{R}\left(v_j\right)(\mathbf{0}) &= \lim_{j\to\infty}\lim_{\rho\to+0} \frac{1}{m(B(\mathbf{0},\rho)\cap W)} \int_{B(\mathbf{0},\rho)\cap W} \mathcal{R}\left(v_j\right)\left(\mathbf{x}'\right) \mathrm{d}\mathbf{x}' \\ &= \lim_{\rho\to+0} \frac{1}{m(B(\mathbf{0},\rho)\cap W)} \int_{B(\mathbf{0},\rho)\cap W} \mathcal{R}(v)\left(\mathbf{x}'\right) \mathrm{d}\mathbf{x}', \\ \mathcal{R}(qw)(\mathbf{0}) &= \lim_{\rho\to+0} \frac{1}{m(B(\mathbf{0},\rho)\cap W)} \int_{B(\mathbf{0},\rho)\cap W} \mathcal{R}(qw)\left(\mathbf{x}'\right) \mathrm{d}\mathbf{x}'.\end{aligned}$$

readily completes the proof of the corollary. □

Remark 3.8 If $V(\mathbf{x}', x_n)$ is continuous near the edge point $(\mathbf{x}_c, x_3^c)$ and $V(\mathbf{x}_c, x_3^c) \neq 0$, by the dominant convergent theorem and Definition 3.1, we can prove that

$$\lim_{\rho\to+0} \frac{1}{m\left(B\left(\mathbf{x}_c,\rho\right)\cap W\right)} \int_{B(\mathbf{x}_c,\rho)\cap W} \int_{x_3^c-L}^{x_3^c+L} \psi\left(x_3\right) w\left(\mathbf{x}', x_3\right) \mathrm{d}\mathbf{x}'\mathrm{d}x_3 = 0.$$

Furthermore, if $\psi(x_3^c) \neq 0$, we can show that

$$\lim_{\rho\to+0} \frac{1}{m\left(B\left(\mathbf{x}_c,\rho\right)\cap W\right)} \int_{B(\mathbf{x}_c,\rho)\cap W} \int_{x_3^c-L}^{x_3^c+L} w\left(\mathbf{x}', x_3\right) \mathrm{d}\mathbf{x}'\mathrm{d}x_3 = 0.$$

which describes the vanishing property of the transmission eigenfunctions near the edge corner in three dimensions.

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