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On the modified transmission eigenvalue problem with an artificial metamaterial background



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Abstract

The modified transmission eigenvalue problem arises in inverse scattering theory for inhomogeneous media, by embedding the relevant medium into an other, artificially introduced inhomogeneous medium. In the present work, we examine the case where the artificial medium is characterised as a metamaterial, i.e. having a negative valued refractive index. Our aim is to construct an appropriate spectral Galerkin method to compute the modified transmission eigenvalues, with potential use to the inverse spectral problem as well. We show that the modified transmission eigenvalue problem corresponds to a compact and self-adjoint operator for which the eigenfunction system is not complete in the solution space. By introducing an auxiliary Dirichlet–Neumann eigenvalue problem, we construct an eigenfunction system which has the desired completeness property. We use this complete system to define the Galerkin scheme and by applying some results for compact and positive operator eigenvalue problems, we prove the convergence of our method. We present some numerical examples which validate the eigenvalues approximation. Finally, we pose the corresponding inverse spectral problem and show that the largest eigenvalue can determine an unknown constant refractive index.

Keywords: Transmission eigenvalues, Inhomogeneous medium, Metamaterial refractive index, Modified far field equations, Galerkin approximation

1 Introduction

Transmission eigenvalues play an important role in the inverse scattering of acoustic, electromagnetic and elastic waves for inhomogeneous media and have been studied thoroughly in the last years. We refer the reader to [3,6,7] and the references therein for a detailed review. However, the use of transmission eigenvalues in non-destructive testing of materials suffers from some drawbacks. Firstly, measured scattering data may be used for the determination of only real transmission eigenvalues and thus cannot be used for absorbing media. Moreover, since transmission eigenvalues are determined from the physical properties of the scattering object, we do not a priori know the range of the interrogating wave frequencies which in turn, are required for successful measurements.

As a result, we need to collect multifrequency data in order to accurately measure the eigenvalues. In practical terms, this is translated as high engineering costs.

To overcome the disadvantages mentioned above, a new approach for the scattering problem has been developed recently [4]. The main idea is to modify the far field operator so that the wavenumber can be fixed and define a new parameter which is now the “signature” eigenvalue of the scattering problem. Injectivity of the modified far field operator is associated with the new eigenvalues in a similar way as the classic far field operator is related to transmission eigenvalues [2, 18]. Moreover, this modification of the interior transmission problem has led into two different spectral problems. One is a Steklov-type eigenvalue problem for which the corresponding eigenvalues can be measured from scattering data and may be used to detect flaws in the material properties of an inhomogeneous medium [1, 4, 8]. The latter corresponds to a problem, similar in structure with the Steklov eigenvalue problem, for which the corresponding eigenvalues can be determined from scattering data as well [1, 10, 11].

For the second case, it is assumed that the inhomogeneous medium is embedded inside another, artificially introduced inhomogeneous medium. More specifically, let $D_b \subset \mathbb{R}^m$ be a bounded region with smooth Lipschitz boundary ∂D_b . The *modified transmission eigenvalue problem* has the following formulation:

$$\Delta w + k^2 \eta(x)w = 0, \quad x \in D_b \tag{1}$$

$$-a \Delta v + \lambda k^2 \eta_0(x)v = 0, \quad x \in D_b \tag{2}$$

$$w = v, \quad x \in \partial D_b \tag{3}$$

$$\frac{\partial w}{\partial \nu} = -a \frac{\partial v}{\partial \nu}, \quad x \in \partial D_b \tag{4}$$

where $k > 0$ is the fixed wavenumber, $\eta \in L^\infty(D_b)$ is the refractive index, $\eta_0 \in L^\infty(D_b)$ is the artificial refractive index, a is a positive constant and ν is the outward unit normal to ∂D_b . We furthermore suppose that $\Re(\eta), \Re(\eta_0) > 0$ and $\Im(\eta), \Im(\eta_0) = 0$ so that the metamaterial refractive index $-\eta_0(x)/a$ is always negative valued. We denote as $\eta_{\max} := \sup_{D_b}(\eta)$, $\eta_{0,\max} := \sup_{D_b}(\eta_0)$ and assume that $\eta_{\min} := \inf_{D_b}(\eta)$, $\eta_{0,\min} := \inf_{D_b}(\eta_0)$ are strictly positive. The complex values of λ for which (1)–(4) has a non-trivial solution (w, v) are the *modified transmission eigenvalues*.

The modified transmission eigenvalue problem was initially introduced in [10], for $-a > 0$ and $\eta_0 = 1$. The authors of [1], considered the possibility of a constant metamaterial background, instead. For the purposes of this paper, we examine the spectral properties of (1)–(4) as a generalization of the problem introduced in [1], by allowing the artificial metamaterial background to be a variable function. Firstly, we show that this problem is equivalent with a self-adjoint and compact operator eigenvalue problem which secures existence and discreteness of the spectrum. We also show, that under a sufficient condition on the wavenumber, there exists an equivalent generalized eigenvalue problem for a coercive operator which provides an upper positive bound for all eigenvalues. Using this result we infer that a constant refractive index is uniquely determined by the knowledge of the largest eigenvalue. Next, we show that surprisingly, the eigenfunction system is not complete in the solution space. We construct an orthonormal basis on the solution space by introducing an auxiliary Dirichlet–Neumann eigenvalue problem and we use its eigenelements as an orthonormal basis. From this complete system, we develop

a Galerkin scheme for the numerical approximation of the eigenvalues. We prove that the approximate eigenvalues converge to the corresponding original by using some results for convergence of compact and positive operators. Furthermore, we verify our approximation method with some numerical examples of discs in \mathbb{R}^2 , for which we can also analytically compute the eigenvalues by applying the separation of variables technique. Finally, we address the corresponding inverse spectral problem by minimizing the error of the largest eigenvalue.

2 Formulation of the problem

Consider the system of equations (1)–(4). We define as an eigenpair of the problem, all combinations of values $\lambda \in \mathbb{C}$ and non-trivial solutions (w, v) of (1)–(4). These λ are the eigenvalues and (w, v) are the corresponding eigenfunction pairs of the problem. In order to study the spectral properties of (1)–(4), following the ideas of [1], we proceed with a variational formulation.

We consider the Hilbert space $\mathcal{H}(D_b) := \{(w, v) \in H^1(D_b) \times H^1(D_b) : w = v \text{ on } \partial D_b\}$. Then, a weak solution of (1)–(4) is defined to be a function pair (w, v) that solves the following equation:

$$\begin{aligned} \Phi_\lambda((w, v), (w', v')) &= \int_{D_b} \nabla w \cdot \nabla \overline{w'} \, dx + a \int_{D_b} \nabla v \cdot \nabla \overline{v'} \, dx \\ &\quad - k^2 \int_{D_b} \eta(x) w \cdot \overline{w'} \, dx + k^2 \lambda \int_{D_b} \eta_0(x) v \cdot \overline{v'} \, dx = 0 \end{aligned} \tag{5}$$

for all $(w', v') \in \mathcal{H}(D_b)$, where Φ_λ can be seen as a sesquilinear form defined on $\mathcal{H}(D_b) \times \mathcal{H}(D_b)$.

By substituting as a test function one of the eigenfunctions (w, v) , we notice that all eigenvalues λ of (5), if they exist, have to be real. In fact, we later demonstrate that the eigenvalues λ correspond to a self-adjoint operator. It is also easy to show that system of equations (1)–(4) is strongly elliptic [19].

Further study of the spectral properties of the above eigenvalue problem requires to proceed by defining an equivalent operator eigenvalue problem by means of Riesz Representation Theorem:

Define $A_\lambda : \mathcal{H}(D_b) \rightarrow \mathcal{H}(D_b)$, $\lambda \in \mathbb{C}$ such that:

$$(A_\lambda(w, v), (w', v'))_{\mathcal{H}(D_b) \times \mathcal{H}(D_b)} = \Phi_\lambda((w, v), (w', v'))$$

Thus, problem (5) is equivalent to the following eigenvalue problem:

$$(A_\lambda(w, v), (w', v'))_{\mathcal{H}(D_b) \times \mathcal{H}(D_b)} = 0, \quad \forall (w', v') \in \mathcal{H}(D_b) \tag{6}$$

We can also define $K_\lambda : \mathcal{H}(D_b) \rightarrow (\mathcal{H}(D_b))^*$, $\lambda \in \mathbb{C}$ such that:

$$(K_\lambda(w, v), (w', v'))_{\mathcal{H}^*(D_b) \times \mathcal{H}(D_b)} = \Phi_\lambda((w, v), (w', v')),$$

where the above notation corresponds to the duality pairing of $(\mathcal{H}(D_b))^* \times \mathcal{H}(D_b)$.

Thus, problem (5) is also equivalent to the following eigenvalue problem:

$$(K_\lambda(w, v), (w', v'))_{\mathcal{H}^*(D_b) \times \mathcal{H}(D_b)} = 0, \quad \forall (w', v') \in \mathcal{H}(D_b) \tag{7}$$

We note that $K_\lambda(w, v)$ is an antilinear functional defined on $\mathcal{H}(D_b)$ and $(\mathcal{H}(D_b))^*$ is the space of all antilinear functionals defined on $\mathcal{H}(D_b)$. These operators, defined by means of the Riesz representation Theorem, are linear and bounded. They are linked through the following relation:

$$A_\lambda = \Gamma \cdot K_\lambda, \quad \forall \lambda \in \mathbb{C}, \tag{8}$$

Operator $\Gamma : (\mathcal{H}(D_b))^* \rightarrow \mathcal{H}(D_b)$ is the topological isomorphism, defined by the Riesz Representation Theorem, such that for every $f \in (\mathcal{H}(D_b))^*$ we have $\Gamma(f) = x$, where $f(v) = (x, v)_{\mathcal{H}(D_b) \times \mathcal{H}(D_b)}$.

It can be easily verified that the family of operators A_λ are of Fredholm type for all $\lambda \in \mathbb{C}$, since they can be written as sum of a coercive and a compact operator. They also depend analytically on the spectral parameter λ . Since the Fredholm property remains invariant under linear isomorphisms, namely Γ^{-1} , we infer that $\{K_\lambda\}_{\lambda \in \mathbb{C}}$ is also a family of analytical Fredholm operators.

Moreover, if we choose a $\tau \in \mathbb{R}$ and set $t = i\tau$, we can see that $N(A_t) = 0$, i.e. $A_t(w, v) = 0$ possesses only the trivial solution. Since A_t is of Fredholm type, it must then be a bijective mapping. Hence, $(A_z)^{-1}$ exists as a linear and bounded operator, for a complex number $z = t$. Now, we make use of a Theorem from Analytic Fredholm Theory [12, Theorem 8.26]

Theorem 1 *Suppose $\{Q_z\}_{z \in \mathbb{C}}$ is a family of operators that depend analytically on the parameter z . Then, exactly one of the two alternatives is correct:*

- (a) $(Q_z)^{-1}$ does not exist for any $z \in \mathbb{C}$ or,
- (b) $(Q_z)^{-1}$ exists at every z , excluding at most a discrete subset of the complex plane.

Thus, for the family of operators $\{A_\lambda\}_{\lambda \in \mathbb{C}}$ alternative (b) is valid and this proves the discreteness for the spectrum of our eigenvalue problem. Also, as a consequence of alternative (b), we can always find a $\beta \in \mathbb{R}$, such that A_β^{-1} exists as a linear, bounded operator. From this point onward, we always assume that $\beta \in \mathbb{R}$ has this property. We are now able to proceed in proving the existence of an infinite set of eigenvalues.

We define the compact, linear operator $B : \mathcal{H}(D_b) \rightarrow \mathcal{H}(D_b)$

$$(B(w, v), (w', v'))_{\mathcal{H}(D_b) \times \mathcal{H}(D_b)} = \int_{D_b} \eta_0 v \bar{v}' dx$$

Then, eigenvalue problem (6) is equivalent with:

$$A_\beta(w, v) = k^2(\beta - \lambda)B(w, v)$$

Next, we formulate an equivalent eigenvalue problem to (6) and (7) that corresponds to a compact and self-adjoint operator. For each $f \in L^2(D_b)$, we also have that $\eta_0 f \in L^2(D_b)$. Define an operator, depending on the fixed parameter β

$$T : L^2(D_b) \rightarrow L^2(D_b) \text{ such that } Tf = v_f \in H^1(D_b)$$

where v_f is the unique element in $H^1(D_b)$ such that $K_\beta(w_f, v_f) = (0, \eta_0 f) \in (\mathcal{H}(D_b))^*$. Since $\beta \in \mathbb{R}$ was selected in such a way that A_β^{-1} exists as a linear and bounded operator, from (8), K_β^{-1} also exists as a linear and bounded operator. In what follows, we demonstrate the properties of T and its corresponding eigenvalue problem. To begin, we define the weighted inner product $(f, g)_{L^2(D_b, \eta_0 dx)} := \int_{D_b} \eta_0(x) f \bar{g} dx$, where $f, g \in L^2(D_b)$. As a consequence of $\eta_{0, \min} \leq \eta_0 \leq \eta_{0, \max}$, the weighted inner product is equivalent to the usual inner product of $L^2(D_b)$. We are now able to state the following:

Proposition 1 *$T : L^2(D_b, \eta_0 dx) \rightarrow L^2(D_b, \eta_0 dx)$ is a compact and self-adjoint operator. Furthermore, the eigenvalue problem that arises for T :*

$$Tv = \frac{1}{k^2(\beta - \lambda)} v, \tag{9}$$

where v is identified as the second eigenfunction from an eigenpair (w, v) of (5), is an equivalent eigenvalue problem to (5).

Proof T is compact because the mapping from $L^2(D_b) \rightarrow H^1(D_b)$ is bounded, the embedding $I : H^1(D_b) \rightarrow L^2(D_b)$ is compact and the equivalence of the usual and weighted inner products. Moreover, since $A_\beta, \beta \in \mathbb{R}$ is a family of self-adjoint operators, T is self-adjoint. We proceed by proving the equivalence of eigenvalue problems (5) and (9).

Let $\{\lambda, (w, v)\}$ be a non-trivial solution of (5). Then,

$$\langle K_\lambda(w, v), (w', v') \rangle = 0, \text{ for all } (w', v') \in \mathcal{H}(D_b)$$

or,

$$\langle K_\beta(w, v), (w', v') \rangle = k^2(\beta - \lambda) \int_{D_b} \eta_0 v \bar{v}' dx, \text{ for all } (w', v') \in \mathcal{H}(D_b).$$

Also, let $f \in L^2(D_b)$. Then, $(0, \eta_0 f) \in L^2(D_b) \times L^2(D_b)$. Since $L^2(D_b) \times L^2(D_b) \subset H^{-1}(D_b) \times H^{-1}(D_b) \subset (\mathcal{H}(D_b))^*$, we have that:

$$\langle (0, \eta_0 f), (w', v') \rangle = \int_{D_b} \eta_0 f \bar{v}' dx = \langle K_\beta(w_f, v_f), (w', v') \rangle$$

Hence,

$$\begin{aligned} &\langle K_\beta(k^2(\beta - \lambda)w_f - w, k^2(\beta - \lambda)v_f - v), (w', v') \rangle \\ &= k^2(\beta - \lambda) \langle K_\beta(w_f, v_f), (w', v') \rangle - \langle K_\beta(w, v), (w', v') \rangle \\ &= k^2(\beta - \lambda) \langle (0, \eta_0 f), (w', v') \rangle - k^2(\beta - \lambda) \langle (0, \eta_0 v), (w', v') \rangle \end{aligned}$$

If we set $f = v$, where v is the second eigenfunction in any eigenpair (w, v) , the expression above is equal to zero for all $(w', v') \in \mathcal{H}(D_b)$. Since K_β is invertible, its nullspace is trivial, that is $N(K_\beta) = \{0\}$. Consequently,

$$k^2(\beta - \lambda)w_f = w \text{ and } k^2(\beta - \lambda)v_f = v, \text{ thus } Tv = \frac{1}{k^2(\beta - \lambda)}v$$

Conversely, if (t, u) is an arbitrary eigenelement of T , with $t \neq 0$, then we can show that it satisfies eigenvalue problem (5).

We have that $\langle K_\beta(w_u, tu), (w', v') \rangle = \langle (0, \eta_0 u), (w', v') \rangle$, thus

$$\begin{aligned} &\int_{D_b} \nabla w_u \cdot \nabla \bar{w}' dx + a \int_{D_b} \nabla tu \cdot \nabla \bar{v}' dx - k^2 \int_{D_b} \eta(x) w_u \cdot \bar{w}' dx \\ &+ k^2 \beta \int_{D_b} \eta_0(x) tu \cdot \bar{v}' dx = \int_{D_b} \eta_0(x) u \cdot \bar{v}' dx \end{aligned}$$

or equivalently,

$$\begin{aligned} &\int_{D_b} \nabla \frac{w_u}{t} \cdot \nabla \bar{w}' dx + a \int_{D_b} \nabla u \cdot \nabla \bar{v}' dx - k^2 \int_{D_b} \eta(x) \frac{w_u}{t} \cdot \bar{w}' dx \\ &+ k^2 \beta \int_{D_b} \eta_0(x) u \cdot \bar{v}' dx = \frac{1}{t} \int_{D_b} \eta_0(x) u \cdot \bar{v}' dx \end{aligned}$$

and we conclude that

$$\int_{D_b} \nabla \frac{w_u}{t} \cdot \nabla \bar{w}' dx + a \int_{D_b} \nabla u \cdot \nabla \bar{v}' dx - k^2 \int_{D_b} \eta(x) \frac{w_u}{t} \cdot \bar{w}' dx$$

$$+ k^2(\beta - \frac{1}{tk^2}) \int_{D_b} \eta_0(x)u \cdot \bar{v}' dx = 0.$$

Hence, $\{\beta - \frac{1}{tk^2}, (\frac{1}{t}w_w, u)\}$ is an eigenpair for eigenvalue problem (5). □

Proposition 2 *Operator T is injective:*

Proof Suppose $f \in N(T)$. Then, it satisfies the following equation for any choice of test functions (w', v') :

$$\int_{D_b} \nabla w_f \cdot \nabla \bar{w}' dx - k^2 \int_{D_b} \eta w_f \cdot \bar{w}' dx = \int_{D_b} \eta_0 f \bar{v}' dx$$

We fix a $w' \in H_0^1(D_b)$, $w' \neq 0$ and we pick $v'_1 = 0, v'_2 \in H_0^1(D_b)$. Then,

$$\langle K_\beta(w_f, 0), (w', v'_1) \rangle = \langle K_\beta(w_f, 0), (w', v'_2) \rangle = \int_{D_b} \eta_0 f \bar{v}'_2 dx = 0$$

Since $\eta_0 f \in (H_0^1(D_b))^*$ and $v'_2 \in H_0^1(D_b)$ is an arbitrary element, $\eta_0 f$ has to be the zero functional of $(H_0^1(D_b))^*$. Thus, $f = 0$. □

Remark 1 Since T is injective, from the following Hilbert space decomposition:

$$L^2(D_b, \eta_0 dx) = N(T) \oplus \overline{R(T)} = \overline{R(T)}$$

and the Hilbert-Schmidt theorem, the eigenelements of T form an orthonormal basis for $L^2(D_b, \eta_0 dx)$ and this also proves the existence of an infinite, discrete set of eigenvalues $\{\lambda_n\}_{n=1}^\infty$.

Until this point, we have selected a $\beta \in \mathbb{R}$, such that A_β^{-1} exists as a linear and bounded operator. In what follows, we attempt to find sufficient conditions, under which there exists a $\Lambda > 0$, such that A_Λ is a coercive operator. Note that, all coercive operators are invertible, due to the Lax-Milgram theorem.

Lemma 1 *Suppose that $(w, v) \in \mathcal{H}(D_b)$. Then, the following inequality holds:*

$$\frac{\epsilon - 1}{\epsilon} \|w\|^2 \leq \frac{2}{\lambda_0(D_b)} [\|\nabla w\|^2 + \|\nabla v\|^2] + (\epsilon - 1)\|v\|^2, \tag{10}$$

where $\epsilon > 1$ and $\lambda_0(D_b)$ is the first Dirichlet eigenvalue on D_b .

Proof As a consequence of Poincare's inequality on $H_0^1(D_b)$, we have:

$$\|w - v\|^2 \leq \frac{1}{\lambda_0(D_b)} \|\nabla w - \nabla v\|^2$$

and so,

$$(\|w\| - \|v\|)^2 \leq \frac{2}{\lambda_0(D_b)} [\|\nabla w\|^2 + \|\nabla v\|^2]$$

Using Young's inequality for an $\epsilon > 1$, we derive that:

$$\left(1 - \frac{1}{\epsilon}\right) \|w\|^2 + (1 - \epsilon)\|v\|^2 \leq \frac{2}{\lambda_0(D_b)} [\|\nabla w\|^2 + \|\nabla v\|^2]$$

□

Lemma 2 *Let $\sigma \in (0, 1)$, $\epsilon > 1$ and $c_1 = \min\{1, a\}$. Then, for the operator A_Λ to be coercive, it suffices:*

$$k^2 < \frac{(1 - \sigma)c_1\lambda_0(D_b)(\epsilon - 1)}{2\epsilon\eta_{\max}}, \quad \Lambda > \frac{(1 - \sigma)c_1\lambda_0(D_b)(\epsilon - 1)}{2k^2\eta_{0,\min}} \tag{11}$$

Proof Inequality (10) can be used, in combination with a sufficient condition on the wavenumber k^2 , to show that there exists a positive number $\Lambda > 0$, such that A_Λ is a coercive operator. Choose a $\Lambda > 0$. Then,

$$\begin{aligned} (A_\Lambda(w, v), (w, v)) &= \int_{D_b} \nabla w \cdot \nabla \bar{w} dx + a \int_{D_b} \nabla v \cdot \nabla \bar{v} dx \\ &\quad - k^2 \int_{D_b} \eta(x)w \cdot \bar{w} dx + k^2 \Lambda \int_{D_b} \eta_0(x)v \cdot \bar{v} dx \\ &\geq c_1 [\|\nabla w\|^2 + \|\nabla v\|^2] - k^2 \eta_{\max} \|w\|^2 + k^2 \Lambda \eta_{0,\min} \|v\|^2 \\ &= \sigma c_1 [\|\nabla w\|^2 + \|\nabla v\|^2] + (1 - \sigma)c_1 [\|\nabla w\|^2 + \|\nabla v\|^2] \\ &\quad - k^2 \eta_{\max} \|w\|^2 + k^2 \Lambda \eta_{0,\min} \|v\|^2 \\ &\geq \sigma c_1 [\|\nabla w\|^2 + \|\nabla v\|^2] + (1 - \sigma)c_1 \left[\frac{\lambda_0(D_b)(\epsilon - 1)}{2\epsilon} \|w\|^2 \right. \\ &\quad \left. - \frac{\lambda_0(D_b)(\epsilon - 1)}{2} \|v\|^2 \right] - k^2 \eta_{\max} \|w\|^2 + k^2 \Lambda \eta_{0,\min} \|v\|^2 \end{aligned}$$

□

Choosing values for σ, ϵ provides a lower bound for Λ that also serves as an upper bound for all eigenvalues $\{\lambda_n\}_{n=1}^\infty$. Moreover, it is possible to derive a coercivity constraint on the wavenumber k^2 which explicitly determines σ, ϵ and Λ , such that (11) holds.

Corollary 1 *Suppose that the wavenumber k^2 satisfies the inequality*

$$k^2 < \frac{c_1\lambda_0(D_b)}{2\eta_{\max}} \tag{12}$$

Then, there exists a $\Lambda > 0$, such that A_Λ is coercive.

Proof Suppose that (12) is satisfied. Then, it is easy to see that there are suitable choices for σ, ϵ and a $\Lambda > 0$ large enough, such that (11) holds. As a result, the operator A_Λ is coercive. □

Theorem 2 *Suppose (12) is satisfied. Then, there exists an infinite sequence of eigenvalues $\{\lambda_n\}_{n=1}^\infty$ of (5), with $-\infty$ as their only accumulation point. As a consequence, there exist at most finite positive eigenvalues.*

Proof Suppose (12) is satisfied. Then, there exists a $\Lambda > 0$, such that A_Λ is coercive. Hence, (5) is equivalent to the generalized eigenvalue problem:

$$\begin{aligned} \int_{D_b} \nabla w \cdot \nabla \bar{w}' dx + a \int_{D_b} \nabla v \cdot \nabla \bar{v}' dx - k^2 \int_{D_b} \eta(x)w \cdot \bar{w}' dx \\ + k^2 \Lambda \int_{D_b} \eta_0(x)v \cdot \bar{v}' dx = k^2(\Lambda - \lambda) \int_{D_b} \eta_0(x)v \bar{v}' dx \end{aligned}$$

or,

$$(A_\Lambda(w, v), (w', v')) = k^2(\Lambda - \lambda)(B(w, v), (w', v'))$$

A_Λ is a linear, coercive, self-adjoint operator. Moreover, B is also linear, self-adjoint, compact and positive. Operator eigenvalue problem:

$$A_\Lambda(w, v) = k^2(\Lambda - \lambda)B(w, v) \tag{13}$$

which is equivalent to (5), has been shown to have an increasing and positive sequence of eigenvalues $\{k^2(\Lambda - \lambda_j)\}$ that have $+\infty$ as their only possible accumulation point (if they are indeed infinite) [12, Theorem 10.23]. From Remark 1, λ_j is an infinite sequence. Thus, $k^2(\Lambda - \lambda_j) \rightarrow +\infty$ and so, $\lambda_j \rightarrow -\infty$. Furthermore, since $\Lambda - \lambda_j > 0$, we conclude that:

$$\lambda_j < \Lambda \quad \forall j \in \mathbb{N} \tag{14}$$

As we can see, (14) is the aforementioned upper bound for all eigenvalues λ_j .

For the second part of the theorem, let's assume to the contrary that there exists an infinite number of positive eigenvalues, i.e. $\{\lambda_j\}_{j \in J}$, is a subsequence of positive eigenvalues. Since (14) is an upper bound for all eigenvalues, the subset of positive eigenvalues $\{\lambda_j\}_{j \in J}$ is contained in $(0, \Lambda)$. From Bolzano-Weierstrass theorem, there exists a convergent subsequence of positive eigenvalues $\{\lambda_{j_k}\}_{k \in I}$. That is a contradiction, since their limit would be a second accumulation point for the eigenvalues $\{\lambda_n\}_{n=1}^\infty$ of (5). \square

The sequence of generalized eigenvalues $k^2(\Lambda - \lambda_j)$ follows the Courant-Fischer type min-max principles, with the first (smallest) eigenvalue $k^2(\Lambda - \lambda_1)$ being an infimum [12, Theorem 10.24]:

$$\begin{aligned} k^2(\Lambda - \lambda_1) &= \inf_{(w,v) \in \mathcal{H}(D_b), v \neq 0} \frac{(A_\Lambda(w, v), (w, v))_{\mathcal{H}(D_b) \times \mathcal{H}(D_b)}}{(B(w, v), (w, v))_{\mathcal{H}(D_b) \times \mathcal{H}(D_b)}} \\ &= \inf_{(w,v) \in \mathcal{H}(D_b), v \neq 0} \frac{\|\nabla w\|^2 + a\|\nabla v\|^2 - k^2 \int_{D_b} \eta |w|^2 dx + k^2 \Lambda \int_{D_b} \eta_0 |v|^2 dx}{\int_{D_b} \eta_0 |v|^2 dx} \end{aligned} \tag{15}$$

From inf – sup properties and (15), the following min-max principle for λ_1 holds:

$$\lambda_1 = \frac{1}{k^2} \sup_{(w,v) \in \mathcal{H}(D_b), v \neq 0} \frac{-\|\nabla w\|^2 - a\|\nabla v\|^2 + k^2 \int_{D_b} \eta |w|^2 dx}{\int_{D_b} \eta_0 |v|^2 dx} \tag{16}$$

Corollary 2 *Suppose that the coercivity constraint (12) is met. Then, there exists a positive, lower bound for the largest eigenvalue λ_1 :*

$$\lambda_1 \geq \frac{\eta_{\min}}{\eta_{0,\max}} > 0$$

That is to say, there always exists a positive eigenvalue for (5).

Proof By substituting $(1, 1) \in \mathcal{H}(D_b)$ in the right-hand side of (16), we have that

$$\lambda_1 \geq \frac{\int_{D_b} \eta dx}{\int_{D_b} \eta_0 dx} \geq \frac{\int_{D_b} \eta_{\min} dx}{\int_{D_b} \eta_{0,\max} dx} = \frac{\eta_{\min}}{\eta_{0,\max}}$$

\square

Furthermore, (16) implies that the largest eigenvalue λ_1 is monotonically increasing with respect to the refractive index η . As a result, if we consider (1)–(4) for a constant refractive index, we have the following uniqueness result for the inverse spectral problem:

Theorem 3 *Under the coercivity constraint (12), a constant refractive index is uniquely determined from the knowledge of the largest eigenvalue $\lambda_1 = \lambda_1(\eta)$.*

Proof Suppose that $\eta_1(x) = \eta_1 \neq \eta_2 = \eta_2(x)$ are two distinct, constant indices of refraction and that $\lambda_1(\eta_1) = \lambda_1(\eta_2)$. We furthermore assume that k , a and $\eta_0 = \eta_0(x)$ are identical for both problems. Without loss of generality, we suppose that $\eta_1 < \eta_2$. From (16), we have that:

$$\lambda_1(\eta_1) = \frac{1 - \|\nabla w_1\|^2 - a\|\nabla v_1\|^2 + k^2\eta_1\|w_1\|^2}{\int_{D_b} \eta_0|v_1|^2 dx}$$

where (w_1, v_1) is a maximizer of (16). Then,

$$m = \frac{1 - \|\nabla w_1\|^2 - a\|\nabla v_1\|^2 + k^2\eta_2\|w_1\|^2}{\int_{D_b} \eta_0|v_1|^2 dx}$$

satisfies the inequality $\lambda_1(\eta_1) < m \leq \lambda_1(\eta_2)$, which is a contradiction. Similarly, we arrive at a contradiction, if we assume that $\eta_2 < \eta_1$. □

Remark 2 In contrast to [5], uniqueness for the inverse spectral problem for a constant index of refraction is not associated with any constraints on the sign of $1 - \eta$.

3 Non-completeness for the eigenfunction system

One of the main goals of our study is to implement a Galerkin Scheme for the numerical approximation of the modified transmission eigenvalues. To this end, an orthonormal basis for the solution space $\mathcal{H}(D_b)$ is required. At first glance, an attempt to define an auxiliary spectral problem, similar to (1)–(4), would be reasonable. A natural choice for the latter occurs as a special case of (1)–(4), by choosing η and η_0 to be constants.

Obviously, this potential eigenvalue problem would then have properties similar to the starting eigenvalue problem and thus, in order to determine whether it produces a complete set of eigenfunctions, one could firstly study the completeness of (1)–(4).

According to the previous analysis, if the coercivity condition for our wavenumber is satisfied, then the following eigenvalue problems are equivalent:

$$(A_\lambda(w, v), (w', v')) = 0$$

or,

$$(A_\beta(w, v), (w', v')) = k^2(\beta - \lambda)(B(w, v), (w', v'))$$

or,

$$(A_\Lambda(w, v), (w', v')) = k^2(\Lambda - \lambda)(B(w, v), (w', v')) \tag{17}$$

where A_Λ in particular can be chosen to be a coercive, self-adjoint operator. We will find out, later in our study, that the coercivity property is of great significance, both to the theoretical and the numerical aspects of the spectral problem.

Define $\tilde{T} : \mathcal{H}(D_b) \rightarrow \mathcal{H}(D_b)$, such that

$$\tilde{T} := A_\Lambda^{-\frac{1}{2}} B A_\Lambda^{-\frac{1}{2}}$$

$A_\Lambda^{-\frac{1}{2}}, A_\Lambda^{+\frac{1}{2}}$ are well defined and the property $A_\Lambda^{-\frac{1}{2}} \cdot A_\Lambda^{+\frac{1}{2}} = I = A_\Lambda^{+\frac{1}{2}} \cdot A_\Lambda^{-\frac{1}{2}}$ holds, since A_Λ is a coercive, self-adjoint operator (as a consequence of coercive operators being positive operators). Thus, (17) is equivalent with the operator eigenvalue problem for \tilde{T} :

$$\left[\frac{1}{k^2(\Lambda - \lambda)} I - \tilde{T} \right] u = 0, \quad u = (u_1, u_2) \in \mathcal{H}(D_b) \tag{18}$$

where $(u_1, u_2) = A_\Lambda^{+\frac{1}{2}}(w, v)$, excluding the zero eigenvalue for \tilde{T} .

We denote by:

$$\mu = \frac{1}{k^2(\Lambda - \lambda)}$$

the eigenvalues of (18).

We note that the operator \tilde{T} is compact, self-adjoint and positive. Thus, the following decomposition holds: $\mathcal{H}(D_b) = N(\tilde{T}) \oplus \overline{R(\tilde{T})}$.

Proposition 3 *There exists an orthonormal basis $\{v'_i\}_{i \in J}$ consisting of eigenelements of \tilde{T} that correspond to the zero eigenvalue of T , that spans $N(\tilde{T})$.*

Proof $N(\tilde{T})$ is a closed subspace of $\mathcal{H}(D_b)$, thus is a Hilbert space itself. It is also separable, since it is a subspace of $\mathcal{H}(D_b)$, which is a separable space. Hence, there exists an orthonormal basis for $N(\tilde{T})$. □

Proposition 4 *$\overline{R(\tilde{T})}$ is spanned by an orthonormal system of eigenelements $\{u_i\}_{i \in I}$ of \tilde{T} that correspond to all nonzero eigenvalues of \tilde{T} .*

Proof Corollary of Hilbert-Schmidt theorem. □

Corollary 3 *If we combine the orthonormal basis $\{v'_i\}_{i \in J}$ and $\{u_i\}_{i \in I}$ for $N(\tilde{T})$ and $\overline{R(\tilde{T})}$ respectively, we get an orthonormal basis for $\mathcal{H}(D_b)$. That is, $\{v'_i\}_{i \in J} \cup \{u_i\}_{i \in I}$ is an orthonormal basis for $\mathcal{H}(D_b)$ since $(N(\tilde{T}))^\perp = \overline{R(\tilde{T})}$.*

We examine the existence of a system of eigenfunctions (w_n, v_n) that correspond to the eigenvalues $\{\lambda_n\}_{n=1}^\infty$, which constitutes a basis for $\mathcal{H}(D_b)$. Operator $A_\Lambda^{-\frac{1}{2}}$ is a topological isomorphism, defined on $\mathcal{H}(D_b)$. If $\{\tilde{u}_n\}_{n=1}^\infty$ is any orthonormal basis, then the system $\{A_\Lambda^{-\frac{1}{2}}\tilde{u}_n\}_{n=1}^\infty$ is a Riesz basis for $\mathcal{H}(D_b)$. Furthermore, we notice that if (w, v) is an eigenfunction pair for (5), then from equivalence of the two problems (5) and (18), $u := A_\Lambda^{+\frac{1}{2}}(w, v)$ is an eigenelement of \tilde{T} .

Proposition 5 *Suppose that $\{u_n\}_{n=1}^\infty$ is an orthonormal basis of $\overline{R(\tilde{T})}$, consisting of eigenelements of \tilde{T} and define the eigenfunction system $(w_n, v_n) := A_\Lambda^{-\frac{1}{2}}u_n$ of (5). Then,*

$$\overline{\text{span}\{(w_n, v_n)\}} = R\left(A_\Lambda^{-\frac{1}{2}} \Big|_{\overline{R(\tilde{T})}}\right)$$

Proof Let $\omega \in \overline{R(\tilde{T})}$. Then it can be written uniquely as $\omega = \sum_{n=1}^\infty a_n u_n$, for some complex constants a_n . Since $A_\Lambda^{-\frac{1}{2}}$ is bounded, we have that

$$\begin{aligned} A_\Lambda^{+\frac{1}{2}}\omega &= A_\Lambda^{+\frac{1}{2}} \sum_{n=1}^\infty a_n u_n \\ &= A_\Lambda^{+\frac{1}{2}} \sum_{n=1}^\infty A_\Lambda^{-\frac{1}{2}} a_n (w_n, v_n) \\ &= A_\Lambda^{+\frac{1}{2}} A_\Lambda^{-\frac{1}{2}} \sum_{n=1}^\infty a_n (w_n, v_n) = \sum_{n=1}^\infty a_n (w_n, v_n) \end{aligned}$$

This concludes the proof. □

In the following lemma, we present a dimensional analysis for the nullspace $N(\tilde{T})$, which is necessary in order to study the completeness of eigenfunction pairs $\{(w_n, v_n)\}_{n=1}^\infty$.

Lemma 3 *Suppose the coercivity condition (12) is satisfied. Then,*

$$\dim N(\tilde{T}) = \infty$$

Proof Since $A_\Lambda^{-\frac{1}{2}}$ is a topological isomorphism, $N(\tilde{T}) = N(A_\Lambda^{-\frac{1}{2}}BA_\Lambda^{-\frac{1}{2}}) = N(BA_\Lambda^{-\frac{1}{2}})$. Let $u \in N(\tilde{T})$. Then, $\tilde{T}u = 0$ is equivalent with $BA_\Lambda^{-\frac{1}{2}}u = 0$ and $A_\Lambda^{-\frac{1}{2}}u \in N(B)$.

Furthermore, the nullspace of B can be described as follows:

$$N(B) = \{(w, 0) : w \in H_0^1(D_b)\},$$

as a result of the orthogonality relation

$$(B(w, v), (w', v')) = 0 \text{ for all } (w', v') \in \mathcal{H}(D_b)$$

which is equivalent to $v = 0$ and $w \in H_0^1(D_b)$.

Hence, we have that $\dim N(B) = \dim H_0^1(D_b) = \infty$. Moreover, $A_\Lambda^{-\frac{1}{2}}u \in N(B)$, that is, $u \in A_\Lambda^{+\frac{1}{2}}N(B)$ and since $A_\Lambda^{+\frac{1}{2}}$ is a topological isomorphism, it will also conserve dimensions. Thus we have the desired result, $\dim N(\tilde{T}) = \dim(A_\Lambda^{+\frac{1}{2}}N(B)) = \dim N(B) = \infty$. \square

Theorem 4 *Suppose the coercivity condition (12) is satisfied. Then, eigenfunction pairs $\{(w_n, v_n)\}_{n=1}^\infty$ of (5) are not complete in $\mathcal{H}(D_b)$, in the sense that they are not a Riesz Basis.*

Proof Since $A_\Lambda^{-\frac{1}{2}}$ is a topological isomorphism and $\overline{R(\tilde{T})}$ is a proper subset of $\mathcal{H}(D_b)$ as a consequence of Lemma 3, we conclude that $R\left(A_\Lambda^{-\frac{1}{2}}|_{\overline{R(\tilde{T})}}\right) \subsetneq \mathcal{H}(D_b)$. Invoking Proposition 5 yields that $\overline{\text{span}\{(w_n, v_n)\}} \subsetneq \mathcal{H}(D_b)$, that is the system of eigenfunctions $\{(w_n, v_n)\}_{n=1}^\infty$ of our eigenvalue problem isn't complete in the Riesz-Basis sense, in $\mathcal{H}(D_b)$. \square

Remark 3 Note that the previous procedure can be applied to any self-adjoint, generalized eigenvalue problem. More specifically, if it is possible to define an equivalent eigenvalue problem, corresponding to an operator with the form $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, where A is self-adjoint and coercive and B is a compact operator, then spectral completeness in the Riesz Basis sense is dependent on whether compact operator B is injective or not.

4 An auxiliary spectral problem

In what follows, we define an auxiliary eigenvalue problem, in order to construct an orthonormal basis for the solution space $\mathcal{H}(D_b)$. We consider the system:

$$\Delta\phi - \sigma\phi = 0, \quad x \in D_b \tag{19}$$

$$\Delta\psi - \sigma\psi = 0, \quad x \in D_b \tag{20}$$

$$\phi = \psi, \quad x \in \partial D_b \tag{21}$$

$$\frac{\partial\phi}{\partial\nu} = -\frac{\partial\psi}{\partial\nu}, \quad x \in \partial D_b \tag{22}$$

which is similar in structure with eigenvalue problem (1)–(4), but as we will discuss later on, it also has the spectral completeness property that (1)–(4) lacks. It is also a degenerate

system, in the sense that the differential operator $\Delta - \sigma$ repeats itself on both equations (19) and (20). We will examine the analytical implications of the above definition in the following chapter.

We proceed by illustrating the spectral properties of the auxiliary problem defined above. Consider the variational formulation of (19)–(22):

Find $(\phi, \psi) \in \mathcal{H}(D_b)$ and a $\sigma \in \mathbb{C}$, such that

$$\int_{D_b} \nabla \phi \cdot \nabla \bar{\phi}' dx + \int_{D_b} \nabla \psi \cdot \nabla \bar{\psi}' dx + \sigma \int_{D_b} \phi \bar{\phi}' dx + \sigma \int_{D_b} \psi \bar{\psi}' dx = 0, \tag{23}$$

$$\forall (\phi', \psi') \in \mathcal{H}(D_b)$$

By means of Riesz representation theorem, we define an operator eigenvalue problem, equivalent to (23):

$$(\mathbb{A}_\sigma(\phi, \psi), (\phi', \psi')) := \int_{D_b} \nabla \phi \cdot \nabla \bar{\phi}' dx + \int_{D_b} \nabla \psi \cdot \nabla \bar{\psi}' dx + \sigma \int_{D_b} \phi \bar{\phi}' dx + \sigma \int_{D_b} \psi \bar{\psi}' dx = 0$$

for all $(\phi', \psi') \in \mathcal{H}(D_b)$, where $\mathbb{A}_\sigma : \mathcal{H}(D_b) \rightarrow \mathcal{H}(D_b)$ is a linear bounded operator that depends analytically on the spectral parameter $\sigma \in \mathbb{C}$.

Remark 4 \mathbb{A}_σ is self-adjoint if and only if $\sigma \in \mathbb{R}$. It is easy to verify that if σ is not a real number, then $N(\mathbb{A}_\sigma) = \{0\}$.

Remark 5 The family of operators $\{\mathbb{A}_\sigma\}_{\sigma \in \mathbb{C}}$ is of Fredholm type and depends analytically on σ . Thus, if we set: $\tilde{\sigma} := \sigma_0 + i\sigma_1$, $\sigma_1 \neq 0$, then from Remark 4, $R(\mathbb{A}_{\tilde{\sigma}}) = \mathcal{H}(D_b)$, and as a consequence, $\mathbb{A}_{\tilde{\sigma}}^{-1}$ exists as a bounded, linear operator. From the theory for analytic Fredholm operators, we conclude that $\{\mathbb{A}_\sigma\}_{\sigma \in \mathbb{C}}$ is invertible for all $\sigma \in \mathbb{C}$, except mostly on a discrete subset of the complex plane. Thus, we can choose one $\beta \in \mathbb{R}$, such that \mathbb{A}_β^{-1} exists as a bounded operator.

We can also define, by means of the Riesz Representation theorem, a bounded linear operator $\mathbb{K}_\beta : \mathcal{H}(D_b) \rightarrow (\mathcal{H}(D_b))^*$ in the following way:

$$\langle \mathbb{K}_\beta(\phi, \psi), (\phi', \psi') \rangle_{(\mathcal{H}(D_b))^* \times \mathcal{H}(D_b)} = (\mathbb{A}_\beta(\phi, \psi), (\phi', \psi'))_{\mathcal{H}(D_b) \times \mathcal{H}(D_b)}$$

Hence, \mathbb{K}_β is also of Fredholm type and $\mathbb{K}_\beta^{-1} : (\mathcal{H}(D_b))^* \rightarrow \mathcal{H}(D_b)$ exists as a bounded operator.

This operator can be used to define an equivalent eigenvalue problem with (23). Define $\mathbb{T} : L^2(D_b) \times L^2(D_b) \rightarrow L^2(D_b) \times L^2(D_b)$ which is depending on the fixed parameter β , such that $\mathbb{T}(f, g) := (\phi_f, \psi_g) \in \mathcal{H}(D_b)$, where $\mathbb{K}_\beta^{-1}(f, g) = (\phi_f, \psi_g)$. Since \mathbb{K}_β is invertible, \mathbb{T} is well defined.

Proposition 6 *Operator $\mathbb{T} : L^2(D_b) \times L^2(D_b) \rightarrow L^2(D_b) \times L^2(D_b)$ is compact and self-adjoint. Furthermore, the eigenvalue problem (23) is equivalent with the following eigenvalue problem for \mathbb{T} , excluding the zero eigenvalue of \mathbb{T} (if zero is an eigenvalue).*

$$\mathbb{T}(\phi, \psi) = \frac{1}{\beta - \sigma}(\phi, \psi) \tag{24}$$

Proof \mathbb{T} is compact, since it is a bounded mapping from $L^2(D_b) \times L^2(D_b)$ to $\mathcal{H}(D_b)$ and the compact embedding of $\mathcal{H}(D_b)$ into $L^2(D_b) \times L^2(D_b)$. Also, \mathbb{T} is self-adjoint as a consequence

of \mathbb{A}_β , $\beta \in \mathbb{R}$ being self-adjoint. We proceed by proving the equivalence of eigenvalue problems (23) and (24):

Suppose (ϕ, ψ) is a non-trivial solution of (23) for some σ . Then (ϕ, ψ) belongs to the nullspace of K_σ , i.e.

$$\langle \mathbb{K}_\sigma(\phi, \psi), (\phi', \psi') \rangle = 0 \quad \forall (\phi', \psi') \in \mathcal{H}(D_b)$$

Alternatively, the relation above can be written as:

$$\langle \mathbb{K}_\beta(\phi, \psi), (\phi', \psi') \rangle - (\beta - \sigma)\langle (\phi, \psi), (\phi', \psi') \rangle = 0$$

for all $(\phi', \psi') \in \mathcal{H}(D_b)$, since $\{\sigma, (\phi, \psi)\}$ is an eigenpair for (23). As a consequence,

$$\mathbb{K}_\beta(\phi, \psi) = (\beta - \sigma)(\phi, \psi) \text{ and thus, } \mathbb{T}(\phi, \psi) = \mathbb{K}_\beta^{-1}(\phi, \psi) = \frac{1}{\beta - \sigma}(\phi, \psi).$$

Conversely, suppose $\{t, (u_1, u_2)\}$, $t \neq 0$ is an eigenpair of \mathbb{T} . Then, it is also a non-trivial solution of (23). Indeed, we have that $T(u_1, u_2) = t(u_1, u_2)$ is equivalent to $K_\beta^{-1}(u_1, u_2) = t(u_1, u_2)$. This implies that $K_\beta K_\beta^{-1}(u_1, u_2) = tK_\beta(u_1, u_2)$ and so $(u_1, u_2) = tK_\beta(u_1, u_2)$. As a result, $\langle \mathbb{K}_\beta(u_1, u_2), (\phi', \psi') \rangle = \langle \frac{1}{t}(u_1, u_2), (\phi', \psi') \rangle$ which can be written as: $\langle \mathbb{K}_{\beta - \frac{1}{t}}(u_1, u_2), (\phi', \psi') \rangle = 0$ for all $(\phi', \psi') \in \mathcal{H}(D_b)$. That is, $\beta - \frac{1}{t}$ is an eigenvalue and (u_1, u_2) the corresponding eigenfunction pair for (23). \square

As a result, the eigenvalues σ of (23) form a discrete set with no accumulation points in \mathbb{R} .

Lemma 4 *Operator \mathbb{T} is injective, i.e. $N(\mathbb{T}) = \{0\}$*

Proof Let $(f_0, g_0) \in N(\mathbb{T})$. Then, $\mathbb{T}(f_0, g_0) = (0, 0) = \mathbb{K}_\beta^{-1}(f_0, g_0) \in L^2(D_b) \times L^2(D_b)$ and since \mathbb{K}_β^{-1} is an isomorphism, we conclude that $(f_0, g_0) = (0, 0) \in \mathcal{H}(D_b)$ \square

From the decomposition $L^2(D_b) \times L^2(D_b) = N(\mathbb{T}) \oplus \overline{R(\mathbb{T})}$, injectivity of \mathbb{T} implies that $L^2(D_b) \times L^2(D_b) = \overline{R(\mathbb{T})}$ and thus, the eigenelements $\{(\phi_n, \psi_n)\}_{n=1}^\infty$ of \mathbb{T} can form an orthonormal basis for the space $L^2(D_b) \times L^2(D_b)$. Moreover, injectivity of \mathbb{T} also implies that eigenvalues $\{\sigma_n\}_{n=1}^\infty$ are infinite.

In what follows, we illustrate that eigenfunction pairs of (23) are orthogonal with respect to the inner products of both $L^2(D_b) \times L^2(D_b)$ and $\mathcal{H}(D_b)$.

Proposition 7 *Let $\sigma_1 \neq \sigma_2$ are two distinct eigenvalues of (23) and $(\phi_1, \psi_1), (\phi_2, \psi_2)$ are corresponding eigenfunction pairs. Then, they are orthogonal with respect to the $L^2(D_b) \times L^2(D_b)$ -inner product. That is,*

$$\langle (\phi_1, \psi_1), (\phi_2, \psi_2) \rangle_{L^2(D_b) \times L^2(D_b)} = 0.$$

Proof If $t_1 \neq t_2$ are two distinct eigenvalues of \mathbb{T} and $(f_1, g_1), (f_2, g_2)$ are the corresponding eigenelements, they are orthogonal with respect to the $L^2(D_b) \times L^2(D_b)$ inner product, due to \mathbb{T} being self-adjoint. Equivalence of eigenvalue problems (23) and (24) implies that all eigenfunction pairs of (23) that correspond to distinct eigenvalues σ are also orthogonal with respect to the $L^2(D_b) \times L^2(D_b)$ inner product. \square

Lemma 5 *Suppose $\{\sigma_1, (\phi_1, \psi_1)\}$ and $\{\sigma_2, (\phi_2, \psi_2)\}$ are two eigenpairs of (23), with $\sigma_1 \neq \sigma_2$. Then, they are orthogonal with respect to the $\mathcal{H}(D_b)$ -inner product, i.e.*

$$\langle (\phi_1, \psi_1), (\phi_2, \psi_2) \rangle_{\mathcal{H}(D_b) \times \mathcal{H}(D_b)} = 0.$$

Proof Let $\{\sigma_1, (\phi_1, \psi_1)\}$ and $\{\sigma_2, (\phi_2, \psi_2)\}$ be two eigenpairs of (23), with $\sigma_1 \neq \sigma_2$. Since (ϕ_1, ψ_1) is an eigenfunction pair, it is a solution of (23) for any choice of test functions. Choosing (ϕ_2, ψ_2) as test functions yields

$$\int_{D_b} \nabla \phi_1 \cdot \nabla \overline{\phi_2} dx + \int_{D_b} \nabla \psi_1 \cdot \nabla \overline{\psi_2} dx + \sigma_1 \left[\int_{D_b} \phi_1 \overline{\phi_2} dx + \int_{D_b} \psi_1 \overline{\psi_2} dx \right] = 0 \tag{25}$$

From Proposition 7,

$$\int_{D_b} \phi_1 \overline{\phi_2} dx + \int_{D_b} \psi_1 \overline{\psi_2} dx = 0 \tag{26}$$

Equations (25) and (26) imply that

$$\int_{D_b} \nabla \phi_1 \cdot \nabla \overline{\phi_2} dx + \int_{D_b} \nabla \psi_1 \cdot \nabla \overline{\psi_2} dx = 0$$

As a result,

$$\begin{aligned} ((\phi_1, \psi_1), (\phi_2, \psi_2))_{\mathcal{H}(D_b) \times \mathcal{H}(D_b)} &= ((\phi_1, \psi_1), (\phi_2, \psi_2))_{L^2(D_b) \times L^2(D_b)} \\ &\quad + ((\nabla \phi_1, \nabla \psi_1), (\nabla \phi_2, \nabla \psi_2))_{L^2(D_b) \times L^2(D_b)} \\ &= 0 \end{aligned}$$

□

In what follows, an equivalent generalized eigenvalue problem to (23) is formulated, which grants a min-max characterization to the eigenvalues $\{\sigma_n\}_{n=1}^\infty$ of (23):

Let $\Lambda > 0$ be an arbitrary positive number. Then, eigenvalue problem (23) is equivalent with the following:

$$\begin{aligned} \int_{D_b} \nabla \phi \cdot \nabla \overline{\phi'} dx + \int_{D_b} \nabla \psi \cdot \nabla \overline{\psi'} dx + \Lambda \left[\int_{D_b} \phi \overline{\phi'} dx + \int_{D_b} \psi \overline{\psi'} dx \right] \\ = (\Lambda - \sigma) \left[\int_{D_b} \phi \overline{\phi'} dx + \int_{D_b} \psi \overline{\psi'} dx \right] \end{aligned} \tag{27}$$

We define a bounded, linear operator $\mathbb{B} : \mathcal{H}(D_b) \rightarrow \mathcal{H}(D_b)$ by means of Riesz Representation theorem:

$$(\mathbb{B}(\phi, \psi), (\phi', \psi'))_{\mathcal{H}(D_b) \times \mathcal{H}(D_b)} := \int_{D_b} \phi \overline{\phi'} dx + \int_{D_b} \psi \overline{\psi'} dx$$

Then, (27) can be written as an generalized, operator eigenvalue problem:

$$\mathbb{A}_\Lambda(\phi, \psi) = (\Lambda - \sigma)\mathbb{B}(\phi, \psi),$$

where \mathbb{A}_Λ is a linear, self-adjoint, coercive operator and \mathbb{B} is a compact, linear operator. Note that, in contrast to our original eigenvalue problem (5), there are no coercivity constraints.

The eigenvalues $\Lambda - \sigma_n$ of this operator eigenvalue problem are positive, increasing, with only possible accumulation point at $+\infty$, if they are infinite [12, Theorem 10.23]. They also have a min-max characterization (Courant-Fischer), with the smallest eigenvalue, $\Lambda - \sigma_1$, being an infimum:

$$\Lambda - \sigma_1 = \inf_{(\phi, \psi) \in \mathcal{H}(D_b), (\phi, \psi) \neq (0,0)} \frac{(\mathbb{A}_\Lambda(\phi, \psi), (\phi, \psi))_{\mathcal{H}(D_b) \times \mathcal{H}(D_b)}}{(\mathbb{B}(\phi, \psi), (\phi, \psi))_{\mathcal{H}(D_b) \times \mathcal{H}(D_b)}}$$

$$\begin{aligned}
 &= \inf_{(\phi, \psi) \in \mathcal{H}(D_b), (\phi, \psi) \neq (0,0)} \frac{\|\nabla\phi\|^2 + \|\nabla\psi\|^2 + \Lambda \int_{D_b} |\phi|^2 dx + \Lambda \int_{D_b} |\psi|^2 dx}{\int_{D_b} |\phi|^2 dx + \int_{D_b} |\psi|^2 dx} \\
 &= \inf_{(\phi, \psi) \in \mathcal{H}(D_b), (\phi, \psi) \neq (0,0)} \frac{\|\nabla\phi\|^2 + \|\nabla\psi\|^2}{\int_{D_b} |\phi|^2 dx + \int_{D_b} |\psi|^2 dx} + \Lambda \\
 &= \Lambda, \quad (\text{by choosing } (\phi_0, \psi_0) = (1, 1))
 \end{aligned}$$

These properties are then passed onto the sequence $\{\sigma_n\}_{n=1}^\infty$. Hence, $\{\sigma_n\}_{n=1}^\infty$ is a decreasing sequence, with only possible accumulation point at $-\infty$. Since Λ can be chosen to be an arbitrarily small positive number, $\sigma_n \leq 0 \forall n \in \mathbb{N}$. Also,

$$\sigma_1 = \Lambda - \Lambda = 0$$

We will later find out that σ_1 corresponds to the first Neumann eigenvalue. In fact, the sequence $\{-\sigma_n\}_{n \in \mathbb{N}}$ consists of either Dirichlet or Neumann eigenvalues for the Laplacian in D_b .

In order to construct the orthonormal basis for our solution space $\mathcal{H}(D_b)$, we define an operator $\tilde{\mathbb{T}} : \mathcal{H}(D_b) \rightarrow \mathcal{H}(D_b)$ as follows:

$$\tilde{\mathbb{T}} := \mathbb{A}_\Lambda^{-\frac{1}{2}} \mathbb{B} \mathbb{A}_\Lambda^{-\frac{1}{2}}$$

Hence, $\tilde{\mathbb{T}}$ is a self-adjoint, compact and positive linear operator. From definition of \mathbb{B} , $N(\mathbb{B}) = \{0\}$ which implies that $N(\tilde{\mathbb{T}}) = \{0\}$. Injectivity of \mathbb{B} is crucial for spectral completeness, as from the the Hilbert space decomposition $\mathcal{H}(D_b) = N(\tilde{\mathbb{T}}) \oplus R(\tilde{\mathbb{T}}) = R(\tilde{\mathbb{T}})$ and the application of the Hilbert-Schmidt theorem to $\tilde{\mathbb{T}}$, it is possible to select an orthonormal basis of $\mathcal{H}(D_b)$ consisting of eigenlements of $\tilde{\mathbb{T}}$.

The eigenvalue problem (23) is equivalent with the following:

$$\left[\frac{1}{\Lambda - \sigma} I - \tilde{\mathbb{T}} \right] u = 0, \quad u = (u_1, u_2) \in \mathcal{H}(D_b) \tag{28}$$

where $(u_1, u_2) = \mathbb{A}_\Lambda^{+\frac{1}{2}}(\phi, \psi)$, (ϕ, ψ) is an eigenfunction pair of (23) and both operators $\mathbb{A}_\Lambda^{+\frac{1}{2}}, \mathbb{A}_\Lambda^{-\frac{1}{2}}$ are topological isomorphisms. In contrast to (18), this eigenvalue problem corresponds to an injective operator, thus zero isn't one of its eigenvalues.

Suppose that $\{\tilde{u}_j\}_{j=1}^\infty := \{(u_1^{(j)}, u_2^{(j)})\}_{j=1}^\infty$ is an orthonormal basis for $\mathcal{H}(D_b)$, consisting of eigenlements of $\tilde{\mathbb{T}}$. Then, $\{\mathbb{A}_\Lambda^{-\frac{1}{2}} \tilde{u}_j\}_{j=1}^\infty$ is from definition, a Riesz basis for $\mathcal{H}(D_b)$. It consists of eigenfunction pairs (ϕ, ψ) of (23), since (23) and (28) are equivalent eigenvalue problems.

Using the results mentioned above, we present a convenient method to construct the desired orthonormal basis. For each eigenvalue σ_n , $n \in \mathbb{N}$, we select an orthonormal basis for its corresponding eigenspace: $\{(\phi_1^{(n)}, \psi_1^{(n)}), (\phi_2^{(n)}, \psi_2^{(n)}),$

$\dots, (\phi_{k(n)}^{(n)}, \psi_{k(n)}^{(n)})\}$, where $k(n) < +\infty$ is the geometric multiplicity of σ_n . Then, from the equivalence of (23) and (28), we have that:

$$\mathbb{A}_\Lambda^{-\frac{1}{2}} \tilde{u}_n \in \text{span}\{(\phi_1^{(n)}, \psi_1^{(n)}), (\phi_2^{(n)}, \psi_2^{(n)}), \dots, (\phi_{k(n)}^{(n)}, \psi_{k(n)}^{(n)})\}, \quad \forall n \in \mathbb{N} \tag{29}$$

Theorem 5 *The system:*

$$\mathcal{F} := \bigcup_{n=1}^\infty \{(\phi_1^{(n)}, \psi_1^{(n)}), (\phi_2^{(n)}, \psi_2^{(n)}), \dots, (\phi_{k(n)}^{(n)}, \psi_{k(n)}^{(n)})\}$$

is an orthonormal basis for $\mathcal{H}(D_b)$.

Proof From Lemma 5, \mathcal{F} is orthonormal. Let $\omega \in \mathcal{H}(D_b)$. Since $\{\mathbb{A}_\Lambda^{-\frac{1}{2}}\tilde{u}_n\}_{n=1}^\infty$ is a Riesz basis for $\mathcal{H}(D_b)$, ω can be described from a series expansion

$$\omega = \sum_{n=1}^\infty c_n \mathbb{A}_\Lambda^{-\frac{1}{2}} \tilde{u}_n, \tag{30}$$

for some choice of constants c_n . Relation (29) implies that

$$\mathbb{A}_\Lambda^{-\frac{1}{2}} \tilde{u}_n = \sum_{i=1}^{k(n)} a_i^{(n)} (\phi_i^{(n)}, \psi_i^{(n)}), \quad n \in \mathbb{N} \tag{31}$$

As a consequence of (30) and (31),

$$\omega = \sum_{n=1}^\infty c_n \sum_{i=1}^{k(n)} a_i^{(n)} (\phi_i^{(n)}, \psi_i^{(n)})$$

Hence, $\omega \in \overline{\text{span}\mathcal{F}}$, implying that $\mathcal{H}(D_b) \subset \overline{\text{span}\mathcal{F}}$. On the other hand, $\overline{\text{span}\mathcal{F}} \subset \mathcal{H}(D_b)$, so we have that $\overline{\text{span}\mathcal{F}} = \mathcal{H}(D_b)$ and the proof is concluded. \square

5 Analytic construction of the orthonormal basis

In the previous chapter, we showed that (19)–(22) provides a complete eigenfunction system in $\mathcal{H}(D_b)$. Firstly, we restrict ourselves to the simple case of D_b being the disc $D(0, R) \subset \mathbb{R}^2$. Hence, we can apply separation of variables and obtain analytical formulas for the eigenfunction system. However, as we will mention later on, this procedure can be generalized for all feasible geometries D_b .

For simplicity, in what follows we substitute $-\sigma$ with σ and the auxiliary eigenvalue problem is re-written in the form:

$$\Delta\phi + \sigma\phi = 0, \quad x \in D_b \tag{32}$$

$$\Delta\psi + \sigma\psi = 0, \quad x \in D_b \tag{33}$$

$$\phi = \psi, \quad x \in \partial D_b \tag{34}$$

$$\frac{\partial\phi}{\partial\nu} = -\frac{\partial\psi}{\partial\nu}, \quad x \in \partial D_b \tag{35}$$

Thus, any solution pair of problem (32)–(35) for $x = (r, \theta)$ in polar coordinates, must have the form:

$$\phi_m(r, \theta) = a_m J_m(\sqrt{\sigma}r) e^{im\theta}$$

$$\psi_m(r, \theta) = b_m J_m(\sqrt{\sigma}r) e^{im\theta},$$

where $m = 0, 1, 2, \dots$ and J_m are the cylindrical Bessel functions. Since any solution pair must also satisfy the boundary conditions (34)–(35), i.e. $\phi(R, \theta) = \psi(R, \theta)$ and $\partial\phi/\partial\nu(R, \theta) = -\partial\psi/\partial\nu(R, \theta)$, the following homogeneous, linear system of equations emerges for constants a_m and b_m :

$$a_m J_m(\sqrt{\sigma}R) - b_m J_m(\sqrt{\sigma}R) = 0 \tag{36}$$

$$\sqrt{\sigma} a_m J'_m(\sqrt{\sigma}R) + \sqrt{\sigma} b_m J'_m(\sqrt{\sigma}R) = 0 \tag{37}$$

Non-trivial eigenfunctions $w(r, \theta)$ and $v(r, \theta)$ stem from values of σ that are roots of the determinant:

$$\det \begin{pmatrix} J_m(\sqrt{\sigma}R) & -J_m(\sqrt{\sigma}R) \\ \sqrt{\sigma} J'_m(\sqrt{\sigma}R) & \sqrt{\sigma} J'_m(\sqrt{\sigma}R) \end{pmatrix} = 0$$

Equivalently, $2\sqrt{\sigma}J'_m(\sqrt{\sigma}R) \cdot J_m(\sqrt{\sigma}R) = 0$, that is,

$$J_m(\sqrt{\sigma}R) = 0 \tag{38}$$

or,

$$\sqrt{\sigma}J'_m(\sqrt{\sigma}R) = 0 \tag{39}$$

To summarize, if σ is an eigenvalue of the auxiliary spectral problem, then it has to be either a Dirichlet eigenvalue (38), or a Neumann eigenvalue (39) for the Laplacian, in the disk $D_b = D(0, R)$. We notice that (38) and (39) are mutually exclusive (for fixed $m \in \mathbb{N}$), since a Dirichlet eigenfunction can not be simultaneously a Neumann eigenfunction.

In the case where (38) holds, the eigenfunction pairs take the following form, as a consequence of (36) and (37):

$$c_1 \left(J_m \left(\frac{j_{m,l}}{R} r \right) e^{im\theta}, -J_m \left(\frac{j_{m,l}}{R} r \right) e^{im\theta} \right), \quad c_1 \in \mathbb{C}, \tag{40}$$

where $j_{m,l}$ is the l -th zero of J_m , $m \in \mathbb{N}_0$

In the case where (39) holds and $\sigma \neq 0$, the eigenfunction pairs take the following form, as a consequence of (36) and (37):

$$c_2 \left(J_m \left(\frac{j'_{m,l}}{R} r \right) e^{im\theta}, J_m \left(\frac{j'_{m,l}}{R} r \right) e^{im\theta} \right), \quad c_2 \in \mathbb{C}, \tag{41}$$

where $j'_{m,l}$ is the l -th zero of J'_m , $m \in \mathbb{N}_0$

The case when $\sigma = 0$ is special, because the corresponding Neumann eigenfunctions for $-\Delta$ is the family of constant functions.

Lemma 6 *For $\sigma = 0$, eigenfunction pairs take the following form:*

$$c_3(1, 1), \quad c_3 \in \mathbb{C}$$

Proof Let (ϕ, ψ) be an eigenfunction pair for the auxiliary eigenvalue problem, corresponding to $\sigma = 0$. Define $u := \phi - \psi$. Then, u satisfies the differential equation:

$$\begin{aligned} \Delta u &= \Delta\phi - \Delta\psi = 0, \quad \text{in } D_b \\ u|_{\partial D_b} &= (\phi - \psi)|_{\partial D_b} = 0 \end{aligned}$$

This differential equation is well posed and only possesses the trivial solution $u = 0$. Thus, our eigenfunction pair (ϕ, ψ) satisfies the relation $\phi = \psi$, in $\overline{D_b}$. Hence, by substituting in the auxiliary eigenvalue problem, ϕ must be a solution of the differential equation:

$$\begin{aligned} \Delta\phi &= 0, \quad \text{in } D_b \\ \frac{\partial\phi}{\partial\nu} &= 0, \quad \text{on } \partial D_b \end{aligned}$$

That is, both ϕ and ψ are Neumann eigenfunctions for $\sigma = 0$, i.e. $\phi = \psi = c$, $c \in \mathbb{C}$. \square

To summarize the results above, we state the following:

Corollary 4 *Let $\mathcal{D}(D(0, R))$ and $\mathcal{N}(D(0, R))$ denote the set of all Dirichlet and Neumann eigenvalues respectively, for $-\Delta$ in the disk $D(0, R)$. Then, the eigenvalues σ of the auxiliary*

eigenvalue problem consist of the set $\mathcal{D}(D(0, R)) \cup \mathcal{N}(D(0, R))$, where the corresponding eigenpairs are of the form:

$$\begin{aligned} c_1(J_m(\sqrt{\sigma}r)e^{im\theta}, -J_m(\sqrt{\sigma}r)e^{im\theta}), & \text{ if } \sigma \in \mathcal{D}(D(0, R)) \\ c_2(J_m(\sqrt{\sigma}r)e^{im\theta}, J_m(\sqrt{\sigma}r)e^{im\theta}), & \text{ if } 0 \neq \sigma \in \mathcal{N}(D(0, R)) \\ c_3(1, 1), & \text{ if } \sigma = 0 \end{aligned}$$

The above result, as mentioned earlier, can be generalized to all feasible geometries D_b . That is, the problem of constructing an orthonormal basis for $\mathcal{H}(D_b)$, where D_b is an arbitrary geometry, reduces to solving the corresponding Dirichlet and Neumann eigenvalue problems on D_b . This can be proven by using the fundamental properties of Dirichlet and Neumann eigenfunctions, for $-\Delta$ on a general geometry D_b . Therefore, the following result can be inferred:

Theorem 6 *The spectrum of the auxiliary eigenvalue problem (32)–(35) consists of Dirichlet and Neumann eigenvalues for $-\Delta$ on D_b . The corresponding eigenpairs are of the form:*

$$\begin{aligned} c_1(u, -u), & \text{ if } \sigma \in \mathcal{D}(D_b) \\ c_2(\tilde{u}, \tilde{u}), & \text{ if } 0 \neq \sigma \in \mathcal{N}(D_b) \\ c_3(1, 1), & \text{ if } \sigma = 0 \end{aligned}$$

where u, \tilde{u} are Dirichlet and Neumann eigenfunctions respectively.

The previous method of constructing an orthonormal basis for $\mathcal{H}(D_b)$ focuses in the special case where D_b is chosen to be a disk. However, it is possible to obtain a basis in more general geometries. Theorem 6 implies that the only practical difficulty appears when solving the Dirichlet and Neumann eigenvalue problem on D_b . If the geometry allows separation of variables, it is possible to derive analytical expressions for Dirichlet and Neumann eigenfunctions, after solving the characteristic equations.

6 A spectral Galerkin approximation method for the eigenvalues

In what follows, a spectral Galerkin method is implemented to calculate the eigenvalues of (5). The computation of modified transmission eigenvalues is also considered in [9, 10]. Furthermore, for the numerical approximation of Steklov eigenvalues associated with a modified far field operator, we refer the reader to [4, 16, 17]. Here, we adopt a similar technique with [14, 21], where a spectral Galerkin method was introduced for the computation of transmission eigenvalues. In that case, the bilaplacian eigenfunctions were used as the appropriate basis in the corresponding solution space. According to the previous analysis, we assume that the coercivity condition (12) for the wavenumber k^2 is satisfied and that $\{(\phi_n, \psi_n)\}_{n=1}^\infty$ is an orthonormal basis for $\mathcal{H}(D_b)$. We have already demonstrated one method of constructing such basis, through means of the auxiliary, degenerate spectral problem (32)–(35).

Let $\{\lambda, (w, v)\}$ be an eigenpair for (5). Since $(w, v) \in \mathcal{H}(D_b)$, we can consider its Fourier expansion with respect to $\{(\phi_n, \psi_n)\}_{n=1}^\infty$,

$$(w, v) = \sum_{n=1}^\infty c_n (\phi_n, \psi_n)$$

Hence, an approximation for the eigenpair (w, v) can be defined through the partial sum:

$$(w, v)^{(N)} := \sum_{n=1}^N c_n (\phi_n, \psi_n)$$

i.e.

$$w^{(N)} := \sum_{n=1}^N c_n \phi_n \quad \text{and} \quad v^{(N)} := \sum_{n=1}^N c_n \psi_n$$

By substituting $w^{(N)}$ and $v^{(N)}$ in variational formulation (5), and choosing as test functions the orthonormal system $\{(\phi_n, \psi_n)\}_{n=1}^N$, an approximate generalized, matrix eigenvalue problem arises:

$$M_1^{(N)} \mathbf{c} = -\lambda^{(N)} M_2^{(N)} \mathbf{c} \tag{42}$$

where M_1, M_2 are $N \times N$ matrices, defined as follows:

$$M_1^{(N)} := \int_{D_b} \nabla \phi_i \cdot \nabla \bar{\phi}_j dx + a \int_{D_b} \nabla \psi_i \cdot \nabla \bar{\psi}_j dx - k^2 \int_{D_b} \eta(x) \phi_i \bar{\phi}_j dx$$

$$M_2^{(N)} := k^2 \int_{D_b} \eta_0(x) \psi_i \bar{\psi}_j dx$$

and $\mathbf{c} = (c_1, c_2, \dots, c_N)^T \in \mathbb{R}^N, \quad i, j = 1, 2, \dots, N$

Equation (42) is a generalized matrix eigenvalue problem and is a discrete analogue of the eigenvalue problem (5). Since the coercivity condition is met, (5) and (13), are also equivalent eigenvalue problems. Thus, approximation results for eigenvalues of matrix equation (42) can be obtained, through an application of spectral approximation theory for the generalized eigenvalue problem (13).

It is useful to introduce an abstract framework for convergence of projection methods [15]. Then, a Galerkin discretization scheme can be defined, since it falls into such category.

Let X be a Hilbert space and $\{X_N\}_{N=1}^\infty$ be a sequence of finite-dimensional subspaces of X , where $\dim(X_N) = N, \forall N \in \mathbb{N}$. Also, let $P_N : X \rightarrow X$ be bounded orthogonal projectors on the finite-dimensional subspaces $X_N, \forall N \in \mathbb{N}$, such that $P_N x \rightarrow x, \forall x \in X$. Given an invertible operator $A \in \mathcal{L}(H)$, the projection method for A seeks to approximate a solution of the equation $Ax = y$, by a sequence of solutions $\{x_N\}_{N=1}^\infty$ to the equations:

$$P_N A P_N x = P_N y, \quad N = 1, 2, \dots \tag{43}$$

Definition 1 The projection method for A is said to converge, if there exists an integer n_0 such that for every $y \in X$ and for every $N \geq n_0$, there exists a unique solution x_N to equation (43) and, additionally, it is required that $x_N \rightarrow A^{-1}y$.

We denote by $\prod(P_N)$ the set of all linear, invertible operators for which the projection method converges. The following result is useful for our analysis [15, Theorem 17.1]:

Theorem 7 Let $A \in \mathcal{L}(X)$. Then, $A \in \prod(P_N)$ iff there exists an integer n_0 , such that for all $N \geq n_0, P_N A|_{X_N} = P_N A P_N|_{X_N} : X_N \rightarrow X_N$ has a bounded inverse and also, $\sup_{N \geq n_0} \|(P_N A P_N|_{X_N})^{-1}\| < \infty$.

We also make use of the following result [15, Corollary 17.5]:

Corollary 5 Let X be a Hilbert space and $A : X \rightarrow X$ be a linear, coercive operator. If $P_N : X \rightarrow X$ is a sequence of orthogonal projectors, such that $P_N x \rightarrow x$, then $A \in \prod(P_N)$.

Combining the above considerations, we derive the following:

Corollary 6 *Let $A \in \prod(P_N)$ and set $A^{(N)} := P_N A|_{X_N} : X_N \rightarrow X_N$. Then, as a consequence of (43) and Theorem 7,*

$$\left(A^{(N)}\right)^{-1} P_N y \rightarrow A^{-1} y$$

Furthermore, the asymptotic behaviour of eigenvalues corresponding to matrix equation (42) can be revealed, by using a spectral approximation result for positive and compact operators [13, pg. 134]. We note that the assumptions for spectral convergence are stronger than in [14, 21], where the corresponding eigenvalue problem was not self-adjoint.

Theorem 8 *Let H be a Hilbert space and $K_N : H \rightarrow H$ be a sequence of positive and compact operators, that converge uniformly to a positive and compact operator $K : H \rightarrow H$ (norm-convergence). If we denote by $\{t_{N,k}\}$ the sequence of eigenvalues of K_N and by $\{t_k\}$ the sequence of eigenvalues of K , then we have that:*

$$\lim_{N \rightarrow \infty} t_{N,k} = t_k$$

where the convergence is uniform with respect to k .

Using the aforementioned framework for projection methods and spectral approximation, allows us to define a Galerkin discretization scheme which intends to approximate the eigenvalues $\{\lambda_n\}_{n=1}^\infty$ of (5) and examine its convergence properties.

Let $X := \mathcal{H}(D_b)$ and the N -dimensional subspaces $X_N \subset X, \quad N \in \mathbb{N}$:

$$X_N := span\{(\phi_1, \psi_1), (\phi_2, \psi_2), \dots, (\phi_N, \psi_N)\}$$

Moreover, we consider the orthogonal projection operators $P_N : \mathcal{H}(D_b) \rightarrow \mathcal{H}(D_b)$, with range in X_N . Then, we define the following linear, bounded operators:

$$\begin{aligned} A_\Lambda^{(N)} &:= P_N A_\Lambda|_{X_N} : X_N \rightarrow X_N \\ B^{(N)} &:= P_N B|_{X_N} : X_N \rightarrow X_N \end{aligned}$$

Since A_Λ is coercive, from Corollary 5, $A_\Lambda \in \prod(P_n)$. It is also easy to see that $A_\Lambda^{(N)}$ inherits the coercivity property from A_Λ . Hence, the operators $\left(A_\Lambda^{(N)}\right)^{-\frac{1}{2}}, \left(A_\Lambda^{(N)}\right)^{+\frac{1}{2}} : X_N \rightarrow X_N$ can be defined, since both $A_\Lambda^{(N)}$ and $\left(A_\Lambda^{(N)}\right)^{-1}$ are positive operators. Furthermore, we can define the discrete analogue of operator eigenvalue equation (13) as follows:

$$\begin{aligned} \left(A_\Lambda^{(N)}(w^{(N)}, v^{(N)}), (w', v')\right) &= k^2(\Lambda - \lambda^{(N)}) \left(B^{(N)}(w^{(N)}, v^{(N)}), (w', v')\right), \\ \forall (w', v') &\in X_N \end{aligned} \tag{44}$$

Define the linear, bounded and self-adjoint operator $\tilde{T}^{(N)} : X_N \rightarrow X_N$ such that:

$$\tilde{T}^{(N)} := \left(A_\Lambda^{(N)}\right)^{-\frac{1}{2}} B^{(N)} \left(A_\Lambda^{(N)}\right)^{-\frac{1}{2}}$$

Then,

$$\left[\frac{1}{k^2(\Lambda - \lambda^{(N)})} I - \tilde{T}^{(N)} \right] u^{(N)} = 0, \quad u^{(N)} = (u_1^{(N)}, u_2^{(N)}) \in X_N$$

where $(u_1^{(N)}, u_2^{(N)}) = \left(A_\Lambda^{(N)}\right)^{+\frac{1}{2}} (w^{(N)}, v^{(N)})$, is an equivalent eigenvalue problem with (44), excluding the eigenvalue $\mu^{(N)} = 0$ for $\tilde{T}^{(N)}$.

The finite-dimensional operator eigenvalue problem (44) is equivalent with the matrix eigenvalue problem (42). Set:

$$\mu^{(N)} := \frac{1}{k^2(\Lambda - \lambda^{(N)})} \quad \text{and} \quad \mu = \frac{1}{k^2(\Lambda - \lambda)}$$

Then, to obtain the convergence result $\lambda^{(N)} \rightarrow \lambda, \quad N \rightarrow \infty$, it suffices to show that $\mu^{(N)} \rightarrow \mu, \quad N \rightarrow \infty$. To achieve this, we also make use of the following results:

Lemma 7 *Let H be a Hilbert space and $F, F_N : H \rightarrow H$ positive operators, $N \in \mathbb{N}$. If $F_N \rightarrow F$ strongly, then:*

$$F_N^{\frac{1}{2}} \rightarrow F^{\frac{1}{2}} \text{ strongly}$$

The above Lemma can be proven by using a Taylor series representation for the square root operators and Tannery’s Lemma, a Corollary of the Dominated Convergence Theorem. It is also easy to show that:

Proposition 8 *Let H be a Hilbert space and $A : H \rightarrow H$ be a coercive, linear operator. Let $P_N : H \rightarrow H$ be orthogonal projections with range in a finite dimensional subspace X_N . Then, $A^{(N)} := P_N A|_{X_N} : X_N \rightarrow X_N$ is a sequence of coercive, linear operators that satisfy $\left\| \left(A^{(N)} \right)^{-\frac{1}{2}} \right\| \leq C$ for some $C > 0$. Moreover, $\left\{ \left(A^{(N)} \right)^{-1} P_N \right\}_{N=1}^\infty$ is a sequence of positive operators.*

By combining these, we can prove the following:

Theorem 9 *The eigenvalues of matrix equation (42) converge to the eigenvalues $\{\lambda_n\}_{n=1}^\infty$ of (5).*

Proof Consider the operator $\tilde{T}^{(N)} P_N : \mathcal{H}(D_b) \rightarrow \mathcal{H}(D_b), \quad N \in \mathbb{N}$. This is a linear extension of $\tilde{T}^{(N)}$ onto $\mathcal{H}(D_b), \forall N \in \mathbb{N}$. Since the spectrum of $\tilde{T}^{(N)} P_N$ and $\tilde{T}^{(N)}$ are identical, apart from (possibly) the zero eigenvalue, we will attempt to use the spectral approximation setting for the sequence of operators $\{\tilde{T}^{(N)} P_N\}_{N=1}^\infty$.

Set $K := \tilde{T}$ and $K^{(N)} := \tilde{T}^{(N)} P_N$. Since both K and $K^{(N)}$ are linear, compact and positive operators, it suffices to show that $\|K - K^{(N)}\| \rightarrow 0$. Consider the estimation

$$\begin{aligned} \|K - K^{(N)}\| &= \left\| A_\Lambda^{-\frac{1}{2}} B A_\Lambda^{-\frac{1}{2}} - \left(A_\Lambda^{(N)} \right)^{-\frac{1}{2}} B^{(N)} \left(A_\Lambda^{(N)} \right)^{-\frac{1}{2}} \right\| \\ &= \left\| A_\Lambda^{-\frac{1}{2}} B A_\Lambda^{-\frac{1}{2}} - \left(A_\Lambda^{(N)} \right)^{-\frac{1}{2}} P_N B A_\Lambda^{-\frac{1}{2}} \right. \\ &\quad \left. + \left(A_\Lambda^{(N)} \right)^{-\frac{1}{2}} P_N B A_\Lambda^{-\frac{1}{2}} - \left(A_\Lambda^{(N)} \right)^{-\frac{1}{2}} B^{(N)} \left(A_\Lambda^{(N)} \right)^{-\frac{1}{2}} \right\| \\ &\leq \left\| \left(A_\Lambda^{-\frac{1}{2}} - \left(A_\Lambda^{(N)} \right)^{-\frac{1}{2}} P_N \right) B A_\Lambda^{-\frac{1}{2}} \right\| \\ &\quad + \left\| \left(A_\Lambda^{(N)} \right)^{-\frac{1}{2}} \left(P_N B A_\Lambda^{-\frac{1}{2}} - B^{(N)} \left(A_\Lambda^{(N)} \right)^{-\frac{1}{2}} P_N \right) \right\| \end{aligned}$$

By setting

$$a_1(N) := \left\| \left(A_\Lambda^{-\frac{1}{2}} - \left(A_\Lambda^{(N)} \right)^{-\frac{1}{2}} P_N \right) B A_\Lambda^{-\frac{1}{2}} \right\|$$

and

$$a_2(N) := \left\| \left(A_\Lambda^{(N)} \right)^{-\frac{1}{2}} \left(P_N B A_\Lambda^{-\frac{1}{2}} - B^{(N)} \left(A_\Lambda^{(N)} \right)^{-\frac{1}{2}} P_N \right) \right\|$$

it suffices to show that $\lim_{N \rightarrow \infty} a_1(N) = \lim_{N \rightarrow \infty} a_2(N) = 0$.

The second term satisfies the broader estimate:

$$\begin{aligned} a_2(N) &= \left\| \left(A_\Lambda^{(N)} \right)^{-\frac{1}{2}} \left(P_N B A_\Lambda^{-\frac{1}{2}} - B^{(N)} \left(A_\Lambda^{(N)} \right)^{-\frac{1}{2}} P_N \right) \right\| \\ &= \left\| \left(A_\Lambda^{(N)} \right)^{-\frac{1}{2}} \left(P_N B A_\Lambda^{-\frac{1}{2}} - P_N B \left(A_\Lambda^{(N)} \right)^{-\frac{1}{2}} P_N \right) \right\| \\ &\leq \left\| \left(A_\Lambda^{(N)} \right)^{-\frac{1}{2}} \right\| \cdot \|P_N\| \cdot \left\| B \left(A_\Lambda^{-\frac{1}{2}} - \left(A_\Lambda^{(N)} \right)^{-\frac{1}{2}} P_N \right) \right\| \end{aligned}$$

Operator A_Λ is coercive. From Corollaries 5 and 6, $\left(A_\Lambda^{(N)} \right)^{-1} P_N \rightarrow A_\Lambda^{-1}$ strongly. Applying Lemma 7 to the sequence of positive operators $\left(A_\Lambda^{(N)} \right)^{-1} P_N$, we have that $\left(\left(A_\Lambda^{(N)} \right)^{-1} P_N \right)^{\frac{1}{2}} = \left(A_\Lambda^{(N)} \right)^{-\frac{1}{2}} P_N \rightarrow A_\Lambda^{-\frac{1}{2}}$. We also notice that $B \left(A_\Lambda^{-\frac{1}{2}} - \left(A_\Lambda^{(N)} \right)^{-\frac{1}{2}} P_N \right) : \mathcal{H}(D_b) \rightarrow \mathcal{H}(D_b)$ is a sequence of compact operators, since $B : \mathcal{H}(D_b) \rightarrow \mathcal{H}(D_b)$ is compact. Furthermore, $A_\Lambda^{-\frac{1}{2}} - \left(A_\Lambda^{(N)} \right)^{-\frac{1}{2}} P_N : \mathcal{H}(D_b) \rightarrow \mathcal{H}(D_b)$ is self-adjoint and B is self-adjoint. Hence,

$$\left\| B \left(A_\Lambda^{-\frac{1}{2}} - \left(A_\Lambda^{(N)} \right)^{-\frac{1}{2}} P_N \right) \right\| = \left\| \left(A_\Lambda^{-\frac{1}{2}} - \left(A_\Lambda^{(N)} \right)^{-\frac{1}{2}} P_N \right) B \right\|$$

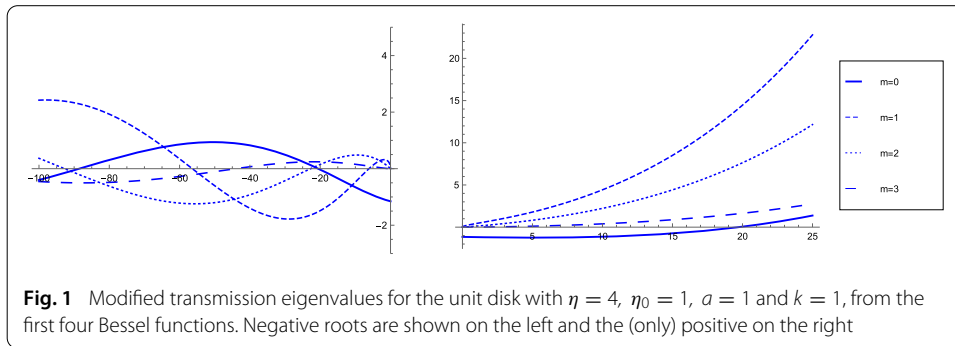
We also apply the following [15, Lemma 17.8]: Multiplying a strongly convergent operator sequence with a compact operator on the right hand-side, it becomes norm convergent. Thus, $\left\| B \left(A_\Lambda^{-\frac{1}{2}} - \left(A_\Lambda^{(N)} \right)^{-\frac{1}{2}} P_N \right) \right\| \rightarrow 0$. Also, $\left\| \left(A_\Lambda^{(N)} \right)^{-\frac{1}{2}} \right\| \leq C$ for some $C > 0$ (uniformly bounded), as a consequence of the coercivity property of $A_\Lambda^{(N)}$. The orthogonal projection operators are also uniformly bounded, since $\|P_N\| \leq 1$.

As a consequence, $\lim_{N \rightarrow \infty} a_1(N) = \lim_{N \rightarrow \infty} a_2(N) = 0$ and the proof is concluded. \square

Remark 6 Since (44) is a generalized operator eigenvalue problem for the linear, coercive operator $A_\Lambda^{(N)} : X_N \rightarrow X_N$ and the linear, compact and positive operator $B^{(N)} : X_N \rightarrow X_N$, its corresponding generalized eigenvalues satisfy the property $k^2(\Lambda - \lambda^{(N)}) > 0$ for all $N \in \mathbb{N}$. As a result, $\Lambda > 0$ serves as an upper bound for all approximate eigenvalues: $\Lambda > \lambda^{(N)}$ for all $N \in \mathbb{N}$. We note that this is a property inherited by the original eigenvalue problem (13).

7 Numerical results for the modified transmission eigenvalue problem

We now present some numerical results, to validate the Galerkin method, as described above. We are interested in the direct and in the inverse spectral problem as well. For the direct spectral problem we aim to calculate the modified eigenvalues, given known material properties (i.e. shape of the scatterer and refractive index). For the inverse spectral problem, we attempt to determine the unknown physical parameters, with an a priori knowledge of a subset of the spectrum. We note that uniqueness for the inverse spectral problem is completely open.



We restrict our analysis to circular domains with constant refractive index where in this case, modified transmission eigenvalues can also be computed analytically. Let D_b be the unit disk $B(0, 1) \subset \mathbb{R}^2$. We assume that the wave number $k > 0$ is fixed, the metamaterial parameters a and η_0 are both positive constants and the refractive index is constant as well, $\eta(x) := \eta > 0$. We can compute the modified transmission eigenvalues using separation of variables for (1)–(4). The eigenfunctions can be expressed in terms of Bessel functions:

$$v_m(r, \theta) = a_m J_m \left(k \sqrt{\frac{\lambda \eta_0}{-a}} r \right) e^{im\theta}$$

$$w_m(r, \theta) = b_m J_m(k \sqrt{\eta} r) e^{im\theta}$$

where $m \in \mathbb{N}_0$. From the boundary conditions, we conclude that λ is a modified transmission eigenvalue, if and only if:

$$\det \begin{pmatrix} J_m(k \sqrt{\eta}) & -J_m \left(k \sqrt{\frac{\lambda \eta_0}{-a}} \right) \\ \frac{d}{dr} J_m(k \sqrt{\eta} r) \Big|_{r=1} & a \frac{d}{dr} J_m \left(k \sqrt{\frac{\lambda \eta_0}{-a}} r \right) \Big|_{r=1} \end{pmatrix} = 0, \quad m = 0, 1, \dots \tag{45}$$

Since $\lambda \in \mathbb{R}$, equation (45) may have both positive and negative roots. When $\lambda > 0$, we use the property that $J_m(ix) = i^m I_m(x)$ in order to compute the positive eigenvalues, if any, where I_m is the modified Bessel function of the first kind. An example is shown in Fig. 1.

On the other hand, we construct an orthogonal system $\{(\phi_n, \psi_n)\}_{n=1}^N$ and apply the Galerkin method to approximate the modified transmission eigenvalues $\lambda^{(N)}$. We use the method described in Sect. 5, and from Corollary 4 we represent the eigenfunctions as Dirichlet and Neumann pairs:

$$(J_m(\sqrt{\sigma} r) e^{im\theta}, -J_m(\sqrt{\sigma} r) e^{im\theta}), \quad \text{if } \sigma \in \mathcal{D}(B(0, 1))$$

$$(J_m(\sqrt{\sigma} r) e^{im\theta}, J_m(\sqrt{\sigma} r) e^{im\theta}), \quad \text{if } 0 \neq \sigma \in \mathcal{N}(B(0, 1))$$

$$(1, 1), \quad \text{if } \sigma = 0$$

The Dirichlet and Neumann eigenvalues $\sqrt{\sigma}$, correspond to Bessel function roots given in (40)–(41) and can be easily computed [23]. We form a basis with 40 eigenfunction pairs and compute the 40×40 matrices of the generalized eigenvalue problem (42) with 2-D numerical integration.

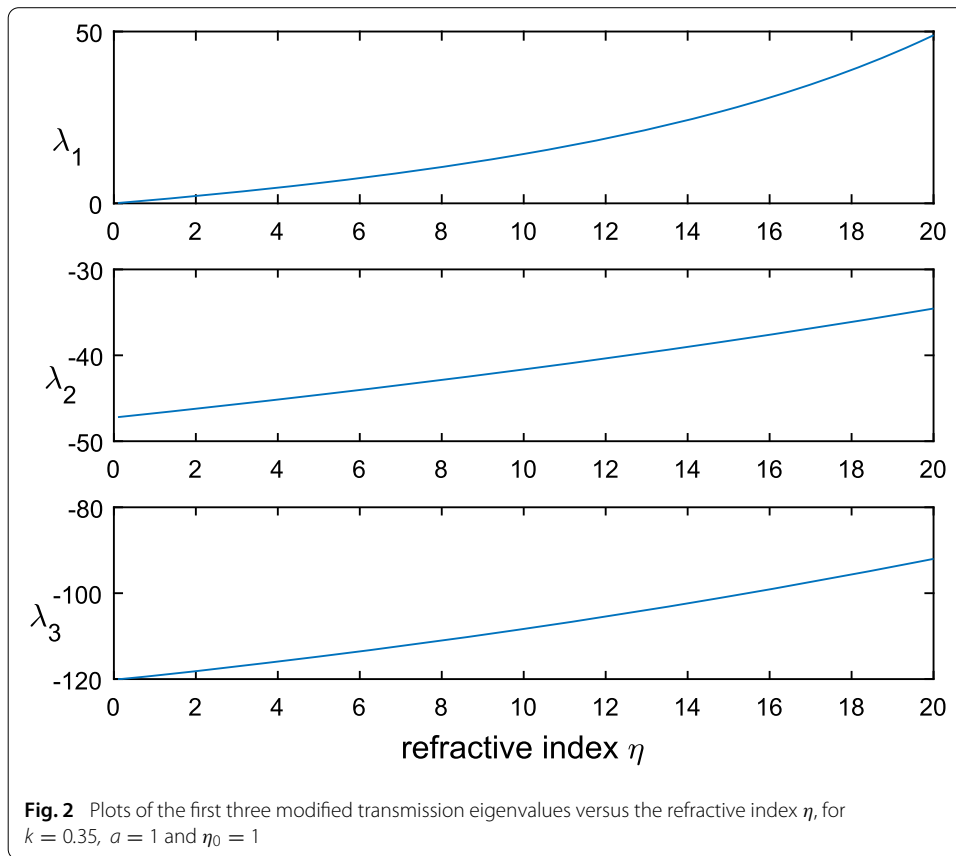


Table 1 Approximation of modified transmission eigenvalues for disks with different material properties and fixed wavenumber $k = 0.35$

(η, η_0, a)	λ_1	$ \lambda_1 - \lambda_1^{(N)} $	λ_2	$ \lambda_2 - \lambda_2^{(N)} $	λ_3	$ \lambda_3 - \lambda_3^{(N)} $
(0.1, 0.5, 1)	0.2006	1.9×10^{-6}	-94.3193	0.0713	-239.5053	0.4822
(0.5, 1, 1)	0.5077	2.4×10^{-5}	-46.9584	0.0349	-119.3490	0.3753
(4, 1, 2)	4.4093	0.0109	-74.1301	0.1276	-200.3112	0.8096
(15.8, 6, 4)	3.8892	0.1463	-20.2322	0.0189	-58.2437	0.2978

For the direct problem, we assume that all physical parameters are known and using the MATLAB function `eig`, we compute the approximate modified transmission eigenvalues $\lambda^{(N)}$. We note that we use a sufficiently small wavenumber k , to satisfy the coercivity constraint (12). For the unit disc of \mathbb{R}^2 and for refractive indices in the range $\eta \in (0, 20)$ with $a \geq 1$, we must take $k < 0.38$.

In Table 1, we report the first three eigenvalues for disks with different material properties. We also calculate the corresponding errors between analytically known and approximated eigenvalues. In all cases, we take $k = 0.35$, to satisfy the coercivity constraint.

Furthermore, in Table 2, we verify the convergence of our Galerkin approximation method. Relative error is reduced as we increase the basis dimension, as we expect from Theorem 9.

We noticed that all material parameters affect the distribution of eigenvalues. Of particular interest is the monotonic relationship between eigenvalues and refractive index,

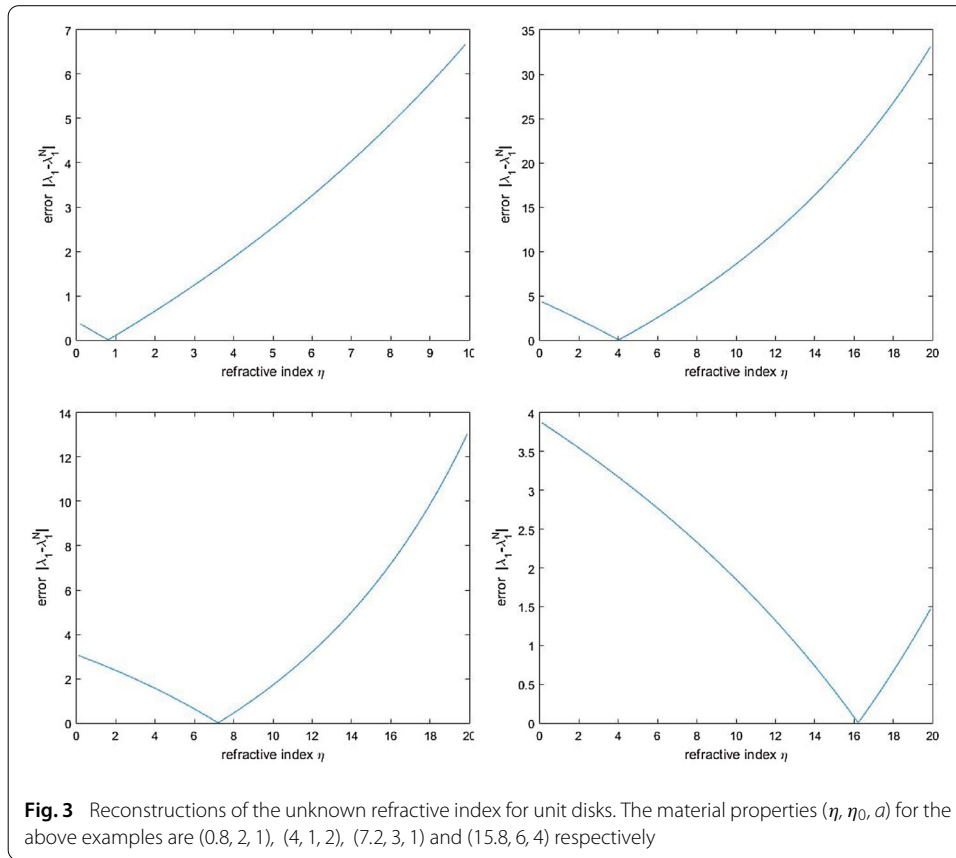


Table 2 Convergence for the first three modified transmission eigenvalues, for a disk with $\eta = 4$, $\eta_0 = 1$, $a = 2$ and $k = 0.35$

Error	$N = 10$	$N = 20$	$N = 30$	$N = 40$
$ \lambda_1 - \lambda_1^{(N)} $	0.0320	0.0155	0.0151	0.0109
$ \lambda_2 - \lambda_2^{(N)} $	0.4018	0.2681	0.1704	0.1276
$ \lambda_3 - \lambda_3^{(N)} $	2.3993	1.2239	0.9311	0.8096

which is demonstrated in Fig. 2. This result is theoretically addressed in (16), for the largest eigenvalue.

Next, as a preliminary approach to the inverse spectral problem, we assume that the largest eigenvalue is known and we estimate η . We also fix parameters k , a and η_0 . In Theorem 3 we have shown that the largest positive eigenvalue can uniquely determine the constant refractive index. We calculate the 40×40 matrices of the generalized eigenvalue problem (42) for $\eta \in (0, 20)$ and step 0.1. We construct a database with the eigenvalues $\lambda_1^{(N)}$, and reconstruct η by minimizing the error $|\lambda_1 - \lambda_1^{(N)}|$ where we consider $\lambda_1^{(N)} = \lambda_1^{(N)}(\eta)$. Some plots of the error versus η are shown in Fig. 3. We see that the error is minimized for estimated η very close to the original one, which corresponds to λ_1 .

Remark 7 We note that for the above numerical examples, we do not a priori assume that $\eta > 1$ or $\eta < 1$, which is the case for the classical transmission eigenvalue problem [14, 21].

Remark 8 Galerkin schemes could also be utilized in geometries that allow separation of variables, since it is then possible to obtain analytical expressions for the orthonormal basis and derive the Galerkin discretization scheme (42). For more general domains, one could potentially examine whether other numerical methods, such as finite elements [20, 22], are applicable for the modified transmission eigenvalue problem.

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