B. Heim, M. Neuhauser *Res Math Sci* (2020) 7:3
https://doi.org/10.1007/s40687-019-0201-5

Research in
the Mathematical Sciences

RESEARCH

The Dedekind eta function and D'Arcais-type polynomials

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Abstract

D'Arcais-type polynomials encode growth and non-vanishing properties of the coefficients of powers of the Dedekind eta function. They also include associated Laguerre polynomials. We prove growth conditions and apply them to the representation theory of complex simple Lie algebras and to the theory of partitions, in the direction of the Nekrasov–Okounkov hook length formula. We generalize and extend results of Kostant and Han.

Keywords: Dedekind eta function, Nekrasov–Okounkov, Partitions, Polynomials, Recursions

Mathematics Subject Classification: Primary 11F20, 26C10; Secondary 05A17, 11F11

1 Introduction

Since the times of Euler, Gauss, and Jacobi properties and formulas of the coefficients $a_n(r)$ of integral powers of Euler products, now known as powers of the Dedekind eta function η^r, have been studied [1,2,18,23]. These involve pentagonal numbers, partition numbers, and the Ramanujan tau-function [19] as the most prominent examples. Non-vanishing properties are of particular interest, e. g. the Lehmer conjecture [17] addressing $r = 24$. A still outstanding result was given by Serre [23] in 1985 for r positive and even. The sequence $a_n(r)$ is lacunary if and only if $r \in \{2, 4, 6, 8, 10, 14, 26\}$.

In this paper, we significantly improve results of Kostant [16] and Han [9] on the non-vanishing of the coefficients. Kostant proved that $a_n\left(m^2 - 1\right) \neq 0$ for $m \geq \max\{4, n\}$ by using Macdonald's fundamental theory on affine root systems [18] and the identification of $\left|a_n\left(m^2 - 1\right)\right|$ with the dimension of some special Lie algebras. In 2010, Han extended Kostant's result to $r \in \mathbb{R}$ applying the Nekrasov–Okounkov [20] hook length formula.

The starting point is the Dedekind eta function. Its powers η^r $(r \in \mathbb{Z})$ are one of the most well-known and most studied functions in mathematics [2,3,6,14,18,22].

$$\eta(\tau)^r := q^{\frac{r}{24}} \prod_{m=1}^{\infty} (1 - q^m)^r = q^{\frac{r}{24}} \sum_{n=0}^{\infty} a_n(r)\, q^n, \tag{1}$$

where $q := e^{2\pi i \tau}$, $\mathrm{Im}(\tau) > 0$. The coefficients are special values of the D'Arcais polynomials $P_n(x)$ [4,5,13,21,23,24]. Let $z \in \mathbb{C}$. Let

$$\sum_{n=0}^{\infty} P_n(z)\, q^n := q^{\frac{z}{24}}\, \eta(\tau)^{-z}. \tag{2}$$

These polynomials are special cases of recursively defined polynomials $P_n^g(x)$ associated with a normalized, i. e. $g(1) = 1$, arithmetic function $g : \mathbb{N} \longrightarrow \mathbb{N}$. In this paper, we are mainly interested in the cases $g(n) = \sigma(n)$ and $g(n) = n^l$, where $l \in \mathbb{N}_0$. Let $P_0^g(x) := 1$ and

$$P_n^g(x) := \frac{x}{n} \sum_{k=1}^{n} g(k)\, P_{n-k}^g(x) \tag{3}$$

for $n \geq 1$. Let $g(n) = \sigma(n) := \sum_{d|n} d$, then $P_n(x) = P_n^{\sigma}(x)$ and $a_n(r) = P_n^{\sigma}(-r)$. Let $g(n) = n$, then it can be shown that $P_n^g(x)$ is proportional to an associated Laguerre polynomial, see [10]. As a special case of Theorem 6 and Corollary 3, we obtain our main results with respect to the Dedekind eta function. We prove in this paper, see Sect. 4:

Theorem *Let $\kappa = 15$. For all $z \in \mathbb{C}$ and $n \in \mathbb{N}$ with $|z| > \kappa(n-1)$, we obtain the growth condition*

$$|P_n(z)| > \frac{|z|}{2n}\ |P_{n-1}(z)|. \tag{4}$$

Corollary *Let $|z| > \kappa(n-1)$, then $P_n(z) \neq 0$.*

Let for example $z = -(10^6 - 1)$. Then, the result of Kostant [16, Theorem 4.28] implies that $P_n(z) \neq 0$ for all $n \leq 10^3$. Our Theorem implies that this is already true for $n \leq 6 \cdot 10^4$.

The work of Kostant and Nekrasov–Okounkov led to new non-vanishing results towards coefficients of the powers of the Dedekind eta function. In this work, we utilize properties of the D'Arcais polynomials to obtain new results beyond their results. Actually, we even get an unexpected new type of result in the context of Kostant's study of complex simple Lie algebras $\mathfrak{g}$. We denote by $\lfloor\ \rfloor$ the Gauss bracket.

Application *Let $\mathfrak{g}$ be a complex simple Lie algebra (to simplify we exclude the types A_1, A_2, G_2). Let $h^{\vee}$ be the dual Coxeter number and*

$$n_0 := \min\left\{h^{\vee}, \left\lfloor \frac{\dim\ \mathfrak{g}}{\kappa} + 1 \right\rfloor\right\}.$$

Let $C_n \subset \wedge^n \mathfrak{g}$ be the span of all 1-dimensional subspaces of the form $\wedge^n \mathfrak{a}$, where $\mathfrak{a} \subset \mathfrak{g}$ is any n-dimensional abelian subalgebra of $\mathfrak{g}$ (see also Sect. 2). Then, $\dim\ C_n \neq 0$ *if and only if $1 \leq n \leq h^{\vee}$. Further* $\dim C_n = a_n(\dim\ \mathfrak{g})$ *(result of Kostant [16]). Our Theorem implies:*

$$\frac{\dim\ C_n}{\dim\ C_{n-1}} > \frac{\dim\ \mathfrak{g}}{2n} \quad (n \leq n_0). \tag{5}$$

2 Kostant's formula

We recall a result of Kostant [16] involving alternating sums of dim V_λ, where V_λ is the irreducible module with the highest weight λ of a complex simple Lie algebra. The highest weight uniquely determines the representation π_λ up to equivalence and also the Casimir operator. We obtain growth results and significantly improve the non-vanishing results obtained by Kostant.

Let $\mathfrak{g}$ be a complex simple Lie algebra. We choose a simply connected compact group K, such that $\mathfrak{k} = \text{Lie}\ K$ is a compact form of $\mathfrak{g}$. Let $T \subset K$ be a maximal torus and $\mathfrak{h} := i\ \text{Lie}\ T$. We identify $\mathfrak{h}$ with its dual with respect to the Killing form such that $\Delta \subset \mathfrak{h}$ for the set of roots for the pair $(\mathfrak{h}_{\mathbb{C}}, \mathfrak{g})$. Here, Δ^+ denotes a set of positive roots and $\mathfrak{h}^+$ the corresponding Weyl chamber.

Let $D \subset \mathfrak{h}^+$ be the set of dominant integral forms of $\mathfrak{h}$. Then, every $\lambda \in D$ corresponds to an irreducible representation

$$\pi_\lambda : K \longrightarrow \text{Aut } V_\lambda$$

with the highest weight λ. For the following (including notation), we refer to Theorem 3.1 in [15] and Theorem 0.1 in [16]. Let ρ be the Weyl element and $a_P := \exp(2\pi\, i\, 2\, \rho)$.

Theorem 1 (Kostant [15]) *For any $\lambda \in D$, the value of the character χ_λ of π_λ evaluated at a_P is an element of $\{-1, 0, 1\}$. Let $\text{Cas}(\lambda)$ be the scalar value taken by the Casimir element of V_λ. Then,*

$$\prod_{n=1}^{\infty} (1 - X^n)^{\dim K} = \sum_{\lambda \in D} \chi_\lambda(a_P) \ \dim \ V_\lambda \ X^{\text{Cas}(\lambda)}. \tag{6}$$

We are interested in the vanishing properties of the coefficients $a_n = a_n(\dim \ K)$ defined by

$$\sum_{n=0}^{\infty} a_n(\dim K) \ X^n = \prod_{n=1}^{\infty} (1 - X^n)^{\dim \ K}. \tag{7}$$

Let W_f denote the affine Weyl group acting on $\mathfrak{h}$. Let $\psi \in \Delta^+$ be the highest root. Then,

$$A_1 := \{x \in \mathfrak{h}^+ \mid \psi(x) \leq 1\} \tag{8}$$

is a fundamental domain. Let $\sigma \in W_f$, then $A_\sigma := \sigma(A_1)$ is called an alcove. An alcove is dominant if $A_\sigma \subset \mathfrak{h}^+$. We put

$$W_f^+ := \{\sigma \in W_f \mid A_\sigma \subset \mathfrak{h}^+\}. \tag{9}$$

Let ρ be the Weyl element and $\sigma \in W_f^+$ then

$$\lambda^\sigma := \frac{\sigma(2\rho)}{2} - \rho. \tag{10}$$

Theorem 2 (Kostant [16]) *Let $\lambda \in D$. Then, $\chi_\lambda(a_P) \in \{-1, 1\}$ if and only if*

$$\lambda \in D_{\mathsf{alcove}} = \{\lambda^\sigma \mid \sigma \in W_f^+\}.$$

In this case, $\chi_\lambda(a_P) = (-1)^{l(\sigma)}$, where $l(\sigma)$ is the length of σ. The coefficients $a_n = a_n(\dim K)$ are given by

$$a_n = \sum_{\substack{\sigma \in W_f^+, \\ \text{Cas}(\lambda^\sigma)=n}} (-1)^{l(\sigma)} \dim \ V_{\lambda^\sigma}. \tag{11}$$

As Kostant already indicated [16, Sect. 4.6]: one major difficulty in using formula (11) to determine a_n is the cancellation in the sums due to the alternation of signs. He discovered that if $n \leq h^\vee$ (dual Coxeter number), then this alternation does not occur, and $(-1)^n a_n$ can be identified with the dimension of certain algebras, see Kostant [16], Theorem 4.23. Here, we recall one of these identifications and an application towards non-vanishing of the coefficients a_n.

Let $n \in \mathbb{N}_0$. Then, we denote by $C_n \subset \wedge^n \mathfrak{g}$ the span of all 1-dimensional subspaces of the form $\wedge^n \mathfrak{a}$, where $\mathfrak{a} \subset \mathfrak{g}$ is any n-dimensional abelian subalgebra of $\mathfrak{g}$. Then, $C_n \neq 0 \Leftrightarrow n \leq M$, where M is the maximal dimension of a commutative subalgebra of $\mathfrak{g}$. Malcev computed M for each simple Lie algebra $\mathfrak{g}$. We give a complete list. The cases A_{m-1} and

G_2 had been worked out in [16]. Note that $M < h^\vee \Leftrightarrow \mathfrak{g}$ is of type A_1, A_2 and G_2 and $M = h^\vee$ otherwise.

$\mathfrak{g}$ type	h	$h^\vee$	dim $\mathfrak{k}$
A_{m-1}	m	m	m^2-1
$B_{(m+1)/2}$	$m+1$	m	$(m+1)(m+2)/2$
C_{m-1}	$2m-2$	m	m^2-1
$D_{(m+2)/2}$	m	m	$(m+1)(m+2)/2$
E_6	12	12	78
E_7	18	18	133
E_8	30	30	248
F_4	12	9	52
G_2	6	4	14

Theorem 3 (Kostant [16]) *Let* $\mathfrak{g}$ *be a complex simple Lie algebra. Let* $n \leq h^\vee$*. Then,*

$$(-1)^n a_n(\dim K) = \dim C_n. \tag{12}$$

These coefficients are zero if and only if $M \leq n \leq h^\vee$*. Hence,* $a_n = 0$ *if and only if the* $\mathfrak{g}$ *type is* A_1 *and* $n = 2$*, or* A_2 *and* $n = 3$*, or* G_2 *and* $n = 4$*.*

The direct application of our theorem given in the introduction (see also Theorem 6) leads to new insights and improvement of the results of Kostant (we also refer to [16] Theorem 4.28 and [9] Theorem 1.6).

Theorem 4 *Let* $\mathfrak{g}$ *be a complex simple Lie algebra. Let* $\lambda^\sigma = \frac{\sigma(2\rho)}{2} - \rho$*, where* ρ *is the Weyl element and* $\sigma \in W_f^+$*. Let* $\pi_{\lambda^\sigma} : K \longrightarrow \mathrm{Aut}(V_{\lambda^\sigma})$ *be the corresponding irreducible representation. We denote by* $l(\sigma)$ *the length of the Weyl group element. Let* $\dim K > \kappa\,(n-1)$*, where* $\kappa = 15$*. Then,*

$$(-1)^n \sum_{\substack{\sigma \in W_f^+,\\ \mathrm{Cas}(\lambda^\sigma)=n}} (-1)^{l(\sigma)} \dim V_{\lambda^\sigma} > \frac{(-1)^{n-1} \dim K}{2n} \sum_{\substack{\sigma \in W_f^+,\\ \mathrm{Cas}(\lambda^\sigma)=n-1}} (-1)^{l(\sigma)} \dim V_{\lambda^\sigma} \tag{13}$$

Corollary 1 *Let* $\dim K > \kappa\,(n-1)$*. Then,*

$$(-1)^n \sum_{\substack{\sigma \in W_f^+,\\ \mathrm{Cas}(\lambda^\sigma)=n}} (-1)^{l(\sigma)} \dim V_{\lambda^\sigma} > 0. \tag{14}$$

Example Let $\mathfrak{g}$ be of type A_{m-1}. Let $m = 10^3$ then $\dim K = 10^6 - 1$. Then, Kostant's result implies that (14) is true for $n \leq 10^3$. Theorem 4 implies that (14) is already true for

$$n \leq (10^6 - 1)/15 \approx 6.7 \cdot 10^4.$$

3 The Nekrasov–Okounkov hook length formula

Almost at the same time as Kostant published his paper, Nekrasov and Okounkov [9,20,24] discovered a new type of hook length formula.

We follow the introduction given in [12]. Random partitions and the Seiberg–Witten theory lead to an identity between a sum over products of partition hook lengths [7,8] and the coefficients of complex powers of Euler products [11,21,23], which is essentially a power of the Dedekind eta function.

Let λ be a partition of n denoted by $\lambda \vdash n$ with weight $|\lambda| = n$. We denote by $\mathcal{H}(\lambda)$ the multiset of hook lengths associated with λ and by $\mathcal{P}$ the set of all partitions. The Nekrasov–Okounkov hook length formula [9, Theorem 1.2] is given by

$$\sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{z}{h^2}\right) = \prod_{m=1}^{\infty} (1 - q^m)^{z-1}. \tag{15}$$

The identity (15) is valid for all $z \in \mathbb{C}$. Our result in this context is the following.

Theorem 5 *Let $n \in \mathbb{N}$ and $\kappa = 15$. Let $z \in \mathbb{C}$ and $|z| > \kappa\,(n-1)$. Then,*

$$\left| \sum_{\lambda \vdash n} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{1-z}{h^2}\right) \right| > \frac{|z|}{2n} \left| \sum_{\lambda \vdash n-1} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{1-z}{h^2}\right) \right|. \tag{16}$$

This is a new type of growth condition related to the Nekrasov–Okounkov hook length formula.

Remark a) Let z be a positive real number. Then, (15) implies:

$$\sum_{\lambda \vdash n} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{1-z}{h^2}\right) > 0. \tag{17}$$

b) Let z be a negative real number. Han observed [9, Theorem 1.6]: let $z < 1 - n^2$, then

$$(-1)^n \sum_{\lambda \vdash n} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{1-z}{h^2}\right) > 0. \tag{18}$$

If $-z \geq 4$, then (18) is already true for $z \leq 1 - n^2$.

Theorem 5 implies the following non-vanishing result.

Corollary 2 *Let $n \in \mathbb{N}$ and $\kappa = 15$. Let $z \in \mathbb{C}$ and $|z| > \kappa\,(n-1)$. Then,*

$$\left| \sum_{\lambda \vdash n} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{1-z}{h^2}\right) \right| > 0. \tag{19}$$

Let z be a negative real number. Then, $z < \kappa\,(1-n)$ implies

$$(-1)^n \sum_{\lambda \vdash n} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{1-z}{h^2}\right) > 0. \tag{20}$$

This is true, since the left hand side of (19) is a polynomial in z of degree n, which is non-vanishing for real z smaller than $\kappa\,(1-n)$ and thus behaves like z^n.

4 Growth conditions on D'Arcais-type polynomials $P_n^g(x)$

Recall the setting from the introduction. Let $g : \mathbb{N} \to \mathbb{N}$ with $g(1) = 1$ be a normalized arithmetic function. We further associate with $g(n)$ a family of polynomials $P_n^g(x)$ and a (shifted) generating function $G(T)$ with positive radius R of convergence. We put $P_0^g(x) = 1$ and

$$P_n^g(x) := \frac{x}{n} \sum_{k=1}^{n} g(k)\, P_{n-k}^g(x) \tag{21}$$

for $n \geq 1$. Let further

$$G\,(T) := \sum_{k=1}^{\infty} g(k+1)\, T^k. \tag{22}$$

Theorem 6 *Let a normalized arithmetic function g be given. Let $P_n^g(x)$ be the associated polynomials and let $G(T)$ be the generating function with positive radius of convergence. Then, for any constant $\kappa > 0$ with $G(\frac{2}{\kappa}) \leq \frac{1}{2}$ we have the following estimation.*

$$\left|P_n^g(x)\right| > \frac{|x|}{2n}\left|P_{n-1}^g(x)\right| \tag{23}$$

if $|x| > \kappa(n-1)$, for all $n \in \mathbb{N}$.

Proof The proof will be given by induction on n. We start with $n = 1$. Let $|x| > 0$. Then,

$$\left|P_1^g(x)\right| = |x| > \frac{|x|}{2}.$$

Let us assume the theorem is true for $1 \leq j \leq n-1$. The induction step is based on the following inverse triangle inequality, employing (21)

$$\left|P_n^g(x)\right| \geq \frac{|x|}{n}\left(\left|P_{n-1}^g(x)\right| - \left|\sum_{k=2}^{n} g(k) P_{n-k}^g(x)\right|\right). \tag{24}$$

We are allowed to assume for $1 \leq j \leq n-1$:

$$\left|P_{j-1}^g(x)\right| < \frac{2j}{|x|}\left|P_j^g(x)\right| \qquad \text{for } |x| > \kappa(j-1).$$

Iterating this inequality leads to

$$\left|P_{n-k}^g(x)\right| < \left|P_{n-1}^g(x)\right|\left(\frac{2n-2}{|x|}\right)^{k-1} \qquad \text{for } |x| > \kappa(n-1)$$

for all $k = 2, \ldots, n$. Using this, we can now estimate the sum in (24), which is involved in the lower bound of $|P_n^g(x)|$:

$$\begin{aligned}\left|\sum_{k=2}^{n} g(k) P_{n-k}^g(x)\right| &\leq \sum_{k=2}^{n} g(k)\left|P_{n-k}^g(x)\right| \\ &< \left|P_{n-1}^g(x)\right| \sum_{k=2}^{n} g(k)\left(\frac{2n-2}{|x|}\right)^{k-1}.\end{aligned}$$

This leads to the crucial inequality

$$\left|P_n^g(x)\right| > \frac{\left|x P_{n-1}^g(x)\right|}{n}\left(1 - \sum_{k=2}^{n} g(k)\left(\frac{2n-2}{|x|}\right)^{k-1}\right).$$

Estimating the sum and using the assumption from the theorem, we obtain

$$\sum_{k=2}^{n} g(k)\left(\frac{2n-2}{|x|}\right)^{k-1} \leq G\left(\frac{2n-2}{|x|}\right) \leq G\left(\frac{2}{\kappa}\right) \leq \frac{1}{2}.$$

Note that $\frac{2n-2}{|x|} < \frac{2}{\kappa}$. Since G is increasing on $[0, R)$ as $g(k) > 0$ for all $k \in \mathbb{N}$, the theorem is proven. □

In particular, for the sum of divisors function we obtain:

Corollary 3 *Let $g = \sigma$ and $|x| > 15(n-1)$ for $n \geq 1$. Then,*

$$|P_n(x)| > \frac{|x|}{2n}|P_{n-1}(x)|.$$

Proof We have to find an upper bound on $G(q) = \sum_{k=1}^{\infty} \sigma(k+1) q^k$. Let $h(k) = \sigma(k)$ for $1 \leq k \leq 4$ and $h(k) = (k+1)k$ for $k \geq 4$. Then, obviously $\sigma(k) \leq h(k)$ for all $k \in \mathbb{N}$. This implies $G(q) \leq \sum_{k=1}^{\infty} h(k+1) q^n = F(q)$ for $0 \leq q \leq 1 \leq R$. The series F is now almost (except for the first 4 terms) the second derivative of the geometric series of q:

$$G(q) \leq \sum_{k=1}^{\infty} h(k+1) q^k = \sum_{k=0}^{\infty} (k+2)(k+1) q^k - 2 - 3q - 8q^2 - 13q^3$$
$$= \frac{2}{(1-q)^3} - 2 - 3q - 8q^2 - 13q^3.$$

For $q = \frac{2}{15}$, we obtain

$$G\left(\frac{2}{15}\right) \leq \frac{3701502}{7414875} < \frac{1}{2}.$$

The claim now follows from the previous theorem. □

Taking more values of h equal to σ does not seem to yield a significant improvement any more. For example, taking $h(k) = \sigma(k)$ for $k \leq 9$ would only allow us to take $\kappa = 14.76$.

The previous estimate on the growth of the polynomials $P_n(x)$ has important implications.

Corollary 4 *$P_n(x) \neq 0$ for $|x| > 15(n-1)$, $n \geq 1$.*

Similarly, the theorem can be exploited to find uniform constants $\kappa = \kappa_m$ only depending on a function $h : \mathbb{N} \to \mathbb{N}$ for all functions $g : \mathbb{N} \to \mathbb{N}$ that satisfy $g(k) \leq h(k)$. One example is the following:

Corollary 5 *Let $m \in \mathbb{N} \cup \{0\}$ be fixed. Suppose $g : \mathbb{N} \to \mathbb{N}$ satisfies*

$$g(k) \leq h_m(k) = \binom{k+m-1}{m}$$

for all $n \in \mathbb{N}$. Then, for all such g the constant c in Theorem 6 can be chosen as

$$\kappa_m = \frac{2}{1 - \sqrt[m+1]{2/3}}.$$

Proof By our assumption, the power series $G(q) = \sum_{k=1}^{\infty} g(k) q^k$ satisfies for $0 \leq q < 1 \leq R$:

$$G(q) \leq \sum_{k=1}^{\infty} h_m(k+1) q^k = \frac{1}{(1-q)^{m+1}} - 1$$

since the series is essentially the mth derivative of the geometric series in q. For $\kappa_m = \frac{2}{1 - \sqrt[m+1]{2/3}}$, we obtain $\frac{1}{2}$ for $q = \frac{2}{\kappa_m}$ in the series. □

In the following, we list the values of κ_m for $m = 0, 1, 2$ and integer bounds on them. They are related to the interesting cases considered in [13]. Here, $m = 0$ leads to polynomials which have Stirling numbers of the first kind as their coefficients. The case $m = 1$ leads to associated Laguerre polynomials. And $m = 2$ leads to polynomials which can be considered as an upper bound of the D'Arcais polynomials $P_n^{\sigma}(x)$.

Corollary 6 *For* $m = 0, 1, 2$, *we obtain* $\kappa_0 = \frac{2}{1-\frac{2}{3}} = 6$, $\kappa_1 = \frac{2}{1-\sqrt{\frac{2}{3}}} < 11$, *and* $\kappa_2 = \frac{2}{1-\sqrt[3]{\frac{2}{3}}} < 16$.

Acknowledgements
We thank both reviewers for carefully reading the manuscript and useful comments.

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Received: 11 July 2019 Accepted: 26 December 2019 Published online: 16 January 2020

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