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# On the stability of static domain wall profiles in ferromagnetic thin film

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## Abstract

In this article, we consider a two-dimensional model of ferromagnetic material. Our prime goal is to analyze the stability of static domain wall configuration calculated by Walker. The dynamics of magnetization inside the material is governed by the Landau–Lifschitz equation which is nonlinear and parabolic in nature. We prove the stability of the static waves solutions for the Landau–Lifschitz equation with a simplified expression of the stray field which is not unique in general, because of the non-convexity constraint  $|u| = 1$ .

**Keywords:** Landau–Lifschitz equation, Micromagnetics, Domain walls, Stability

**Mathematics Subject Classification:** Primary 35B35, 34D05, 35Q60; Secondary 35K55

## 1 Introduction

The study of ferromagnetic systems is of great importance because of its huge application in the development of modern technological devices. Ferromagnetic materials are used in numerous technological devices, such as hard disks, recording heads, cellular phones, magnetic sensors. In particular, ferromagnetic thin films are of great interest because of their sensitive response to applied magnetic fields which makes them useful for the design of many devices such as giant magnetoresistive sensors (GMR) and thin-film memories. The detailed applications of thin film can be found in excellent book by Hubert and Schäfer [20]. For the characteristics and a general description of ferromagnetic materials, we refer the reader to [3, 9, 20, 24].

The general setting in 3D model is the following. We consider an infinite homogeneous ferromagnetic medium. The ferromagnetic materials are characterized by spontaneous magnetization. We denote the magnetization vector field by  $u$  and is given by:

$$u : \mathbb{R}^+ \times \mathbb{R}^3 \rightarrow \mathbb{R}^3.$$

The magnetic moment  $u$  links the magnetic induction  $B$  and the magnetic field  $H$  by the relation  $B = u + H$ . In addition, we assume that the material is saturated so that the magnitude of  $u$  is constant. After renormalization, we assume that

$$|u(t, X)| = 1 \text{ for } (t, X) \in \mathbb{R}^+ \times \mathbb{R}^3 \text{ a.e.} \quad (1.1)$$

The dynamics of  $u$  is described by the Landau–Lifschitz equation:

$$\frac{\partial u}{\partial t} = -u \times H_{\text{eff}} - u \times (u \times H_{\text{eff}}).$$

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The effective field  $H_{\text{eff}} = -\nabla \mathcal{E}$  is derived from the micromagnetism energy  $\mathcal{E}$  given by:

$$\begin{cases} \mathcal{E} = \mathcal{E}_{\text{exch}} + \mathcal{E}_{\text{anis}} + \mathcal{E}_{\text{dem}}, \\ H_{\text{eff}} = h_e(u) + h_a(u) + h_d(u). \end{cases}$$

Next, we describe the various energy contributions and corresponding fields in the following way:

- Exchange Energy ( $\mathcal{E}_{\text{exch}}$ ): The ferromagnetic behavior is essentially due to a quantistic force which tends to align the magnetic dipole moment parallel to each other. The most important contribution is due to the exchange energy:

$$\mathcal{E}_{\text{exch}} = \frac{A}{2} \int_{\mathbb{R}^3} |\nabla u|^2,$$

with

$$h_e(u) = A\Delta u,$$

where the exchange constant  $A$  depends on the material. For simplicity, we choose  $A$  to be 1.

- Anisotropy Energy ( $\mathcal{E}_{\text{anis}}$ ): The anisotropy energy reflects the existence of a preferential axis (easy axis) of magnetization:

$$\mathcal{E}_{\text{anis}} = -\frac{1}{2} \tilde{\gamma} \int_{\mathbb{R}^3} (u \cdot e)^2,$$

where the unit vector  $e$  gives the direction of the easy axis ( $\tilde{\gamma} > 0$ ) or the orientation of the easy plane ( $\tilde{\gamma} < 0$ ). we have

$$h_a(u) = \tilde{\gamma}(u \cdot e)e.$$

- Demagnetizing (Stray Field) Energy ( $\mathcal{E}_{\text{dem}}$ ): This energy is connected with the magnetic field generated by the medium itself:

$$\mathcal{E}_{\text{dem}} = \frac{1}{2} \int_{\mathbb{R}^3} |h_d(u)|^2.$$

The demagnetizing field  $h_d(u)$  is characterized by the Maxwell's equations:

$$\begin{cases} \text{curl } h_d(u) = 0 \text{ in } \mathcal{D}'(\mathbb{R}^3), \\ \text{div}(h_d(u) + u) = 0 \text{ in } \mathcal{D}'(\mathbb{R}^3), \end{cases} \tag{1.2}$$

where  $\mathcal{D}'(\mathbb{R}^3)$  denotes the space of distributions on  $\mathbb{R}^3$ . Therefore, we obtain that

$$H_{\text{eff}} = A\Delta u + \tilde{\gamma}(u \cdot e)e + h_d(u).$$

This effective field offers a simple interpretation of the micromagnetic equations. In the form  $u \times H_{\text{eff}} = 0$ , it means that the effective field must at every point be directed along the magnetization vector: The torque exerted on any magnetization vector must vanish in static equilibrium.

Existence results for the Landau–Lifschitz equation can be found in [6, 12, 13, 19, 28] and [18] for the weak solutions, and the strong solutions are considered in [14, 15] and known to exist locally in time. Numerical simulations of ferromagnetic materials can be

found in [7,8,23] and the references therein. Stability and controllability results related to one-dimensional model of ferromagnetic materials are studied by many authors under different conditions [2,5,10,16,25]. Higher-dimensional models of these objects are very complex, and there are very few mathematical description available [4,11,17].

The motivation of our work comes from the results presented by Gilles [11,16]. In [16] he discussed the stability of static wave solutions for one-dimensional model and extended this result in [10] by proving the controllability of traveling waves solution in ferromagnetic nanowire, while in [11], he considered three-dimensional model and established nice result of its stability. Stability properties of static wave solutions are very helpful when we study the behavior of traveling wave solutions. Here, we are interested in the study of static walls stability for two-dimensional model of Landau–Lifschitz equation which is equally important and will help to fill the existing gap. 2D models mainly arise in case of thin film. It is well known that a variety of patterns (uniformly magnetized regions) of magnetization vectors appears on thin ferromagnetic films [3,20] which are separated by a thin boundary layers called the domain wall. In the absence of external magnetic field, a pattern arises on the ferromagnetic film called the Bloch wall (static wall or stationary wave solutions). The study of stability properties of these static walls is very relevant for practical purposes. For example, in many thin-film devices, switching or reversing the magnetization of relevant configurations is of particular interest (for details see 6.5 in [20]), and in order to switch the magnetization, we need to have stability property for such relevant configurations. It is worth to mention that obtained results are mainly based on the energy method and variational estimate technique.

**1.1 Considered model**

We consider a two-dimensional model of ferromagnetic material. We deal with the static wall configuration calculated by Walker [29]. Walker executed the exact integration of the equations of motion for a planar wall [27]. From the theory of ferromagnetism, it is well known that static walls of infinite nanowires are Bloch walls, whose main characteristic is to produce two almost linear regimes separated by a wall. We investigate the stability features of these exact solutions for the Landau–Lifschitz equation with a simplified expression for the stray field. In case of flat domain wall, magnetic moment depends only on the  $x$  variable which in turn gives the expression for demagnetizing field as  $h_d(u) = -u_1e_1$  (where  $u(t, x, y) = (u_1, u_2, u_3)$  and  $(e_1, e_2, e_3)$  represents the canonical basis of  $\mathbb{R}^3$ ). With this expression of the stray field, the static wave solution to Landau–Lifschitz equation is given by:

$$\mathcal{U}_0(x, y) = \mathcal{U}_0(x) = \begin{pmatrix} 1/\text{ch } x \\ 0 \\ \text{th } x \end{pmatrix}. \tag{1.3}$$

In our study, we simplify the model by taking  $h_d(u)$  to  $-u_1e_1$  even for the perturbations of  $\mathcal{U}_0$  and choose  $\tilde{\gamma} = 1$ . Also, we consider the energetically preferred direction of magnetization (easy axis) along the  $e_2$ -direction, *i.e.*,  $e = e_2$ . Hence, we investigate the following system:

$$\begin{aligned} \frac{\partial u}{\partial t} &= -u \times H_{\text{eff}} - u \times (u \times H_{\text{eff}}), \\ H_{\text{eff}} &= \Delta u + u_2e_2 - u_1e_1. \end{aligned} \tag{1.4}$$

Next, we establish the stability result of the static wave solution  $\mathcal{U}_0$  for the system (1.4).

## 1.2 Main result

Statement of our main result is as follows:

**Theorem 1.1** *Suppose  $u$  is the solution of the Landau–Lifschitz equation (1.4) with initial condition  $u(0, x, y) = u_0(x, y)$ , where  $u_0$  satisfies the saturation condition (1.1). Let  $\varepsilon > 0$ ; then there exist  $\delta > 0$  such that if  $u_0$  holds  $\|u_0 - \mathcal{U}_0\|_{H^2(\mathbb{R}^2)} \leq \delta$ , for all  $u_0 \in H^2(\mathbb{R}^2; \mathbb{R}^3)$ , then  $u$  satisfies,*

$$\forall t \geq 0, \quad \|u(t, \cdot) - \mathcal{U}_0\|_{H^2(\mathbb{R}^2)} \leq \varepsilon.$$

This article is organized as follows:

In Sect. 2, we describe the perturbations  $u$  of the static wall profile  $\mathcal{U}_0$  satisfying the saturation constraint  $|u| = 1$  in the mobile frame  $(\mathcal{U}_0(x), \mathcal{U}_1(x), \mathcal{U}_2)$ , where

$$\mathcal{U}_1(x) = \begin{pmatrix} -\operatorname{th} x \\ 0 \\ 1/\operatorname{ch} x \end{pmatrix} \quad \text{and} \quad \mathcal{U}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

writing

$$u(t, x, y) = \xi_1(t, x, y)\mathcal{U}_1(x) + \xi_2(t, x, y)\mathcal{U}_2 \\ + \left(1 - (\xi_1(t, x, y))^2 - (\xi_2(t, x, y))^2\right)^{1/2} \mathcal{U}_0(x).$$

We transform the Landau–Lifschitz equation (1.4) in the new unknown  $\xi$ , where  $\xi = (\xi_1, \xi_2)$  takes its values in  $\mathbb{R}^2$ . We obtain that Eq. (1.4) is equivalent to a nonlinear equation on  $\xi$  and the stability feature of  $\mathcal{U}_0$  is equivalent to the stability of 0 for the transformed equation. In Sect. 3, we describe the properties of the linear operator of the nonlinear equation on  $\xi$ . Since Landau–Lifschitz equation (1.4) is translation invariance in the  $x$ -variable, the linear part of the perturbed equation admits 0 as a simple eigenvalue which prevents establishing the stability result directly. In order to overcome this situation, we introduced the new coordinate system in Sect. 4. We adapt the new coordinate system in such a way that the linear parts of the transformed equations in the new system behave independently, and we can apply variational estimates technique. Kapitula in [21, 22] developed the techniques concerning the stability of traveling waves to semilinear parabolic equations in which the linearization about the wave contains 0 as an eigenvalue. We decompose the perturbations into a spatial translation component and a normal component. The spatial component satisfies a quasilinear parabolic equation, and the normal component exhibits a very dissipative quasilinear parabolic equation. The main difficulty here is that the equations are quasilinear and methods discussed in [21] cannot be applied directly.

In Sect. 5, we established some preliminary estimates to derive the main result. In Sect. 6, we prove the stability result using the variational estimates. In the last section, we conclude this article with some remarks and suggest further work in the same research lines.

*Remark 1.1* The static wall profile  $\mathcal{U}_0$  for the Landau–Lifschitz equation (1.4) is not unique, because of the non-convexity of the constraint  $|u| = 1$ . We have the following static solution  $\tilde{\mathcal{U}}_0$  which also satisfies the Landau–Lifschitz equation (1.4) and is given by

$$\tilde{\mathcal{U}}_0(x) = \begin{pmatrix} 0 \\ \text{th } x \\ 1/\text{ch } x \end{pmatrix}.$$

In this paper, we use the following notations. Throughout the article,  $\text{ch}$ ,  $\text{sh}$ , and  $\text{th}$  represent the hyperbolic cosine, sine, and tangent functions, respectively. The canonical basis of  $\mathbb{R}^3$  is  $(e_1, e_2, e_3)$  and the letter  $C$  denotes generic constant.

**2 Transformation to new coordinates**

We transform Landau–Lifschitz equation (1.4) in new coordinates system  $(\mathcal{U}_0(x), \mathcal{U}_1(x), \mathcal{U}_2)$  with

$$\forall x \in \mathbb{R}, \quad \mathcal{U}_0(x) = \begin{pmatrix} 1/\text{ch } x \\ 0 \\ \text{th } x \end{pmatrix}, \quad \mathcal{U}_1(x) = \begin{pmatrix} -\text{th } x \\ 0 \\ 1/\text{ch } x \end{pmatrix} \text{ and } \mathcal{U}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

We consider  $u$  as a small perturbation of  $\mathcal{U}_0$  and write it as:

$$u(t, x, y) = \mathcal{U}_0(x) + \xi_1(t, x, y)\mathcal{U}_1(x) + \xi_2(t, x, y)\mathcal{U}_2 + \lambda(\xi(t, x, y))\mathcal{U}_0(x), \tag{2.1}$$

so that it satisfies the saturation constraint  $|u| = 1$ . Here  $\lambda : B(0, 1) \rightarrow \mathbb{R}$  is a smooth map defined as

$$\lambda(\zeta) = \sqrt{1 - (\zeta_1^2) - (\zeta_2^2)} - 1,$$

where  $\zeta = (\zeta_1, \zeta_2)$  and  $B(0, 1) = \{(\zeta_1, \zeta_2), (\zeta_1)^2 + (\zeta_2)^2 < 1\}$  is the unit ball of  $\mathbb{R}^2$ .

We notice that  $\xi_1(t, x, y) = u(t, x, y) \cdot \mathcal{U}_1(x)$  and  $\xi_2(t, x, y) = u(t, x, y) \cdot \mathcal{U}_2$ , with the unknown  $\xi(t, x, y) = \begin{pmatrix} \xi_1(t, x, y) \\ \xi_2(t, x, y) \end{pmatrix}$  using the perturbations (2.1) of  $\mathcal{U}_0$ . Furthermore,

$$H_{\text{eff}}(u) = \mu_0\mathcal{U}_0 + \mu_1\mathcal{U}_1 + \mu_2\mathcal{U}_2,$$

with

$$\begin{aligned} \mu_0 &= \lambda'(\xi)\Delta\xi + \lambda''(\xi) \{(\partial_x\xi)^2 + (\partial_y\xi)^2\} - 2\frac{(1+\lambda)}{\text{ch}^2x} - 2\frac{\partial_x\xi_1}{\text{ch } x} + 2\xi_1\frac{\text{sh } x}{\text{ch}^2x}, \\ \mu_1 &= \Delta\xi_1 + 2\lambda'(\xi)\frac{\partial_x\xi}{\text{ch } x} - \xi_1, \\ \mu_2 &= \Delta\xi_2 + \xi_2. \end{aligned}$$

We transform Eq. (1.4) by using (2.1) and obtain that

$$\begin{aligned} \frac{\partial\lambda}{\partial t}\mathcal{U}_0 + \frac{\partial\xi_1}{\partial t}\mathcal{U}_1 + \frac{\partial\xi_2}{\partial t}\mathcal{U}_2 &= -(\xi_1\mu_2 - \xi_2\mu_1)\mathcal{U}_0 - (\xi_2\mu_0 - (1+\lambda)\mu_2)\mathcal{U}_1 \\ &\quad - ((1+\lambda)\mu_1 - \xi_1\mu_0)\mathcal{U}_2 \\ &\quad - (1+\lambda)(\xi_2\mu_0 - (1+\lambda)\mu_2)\mathcal{U}_2 \\ &\quad + (1+\lambda)((1+\lambda)\mu_1 - \xi_1\mu_0)\mathcal{U}_1 \\ &\quad + \xi_1(\xi_1\mu_2 - \xi_2\mu_1)\mathcal{U}_2 \\ &\quad - \xi_1((1+\lambda)\mu_1 - \xi_1\mu_0)\mathcal{U}_0 \\ &\quad - \xi_2(\xi_1\mu_2 - \xi_2\mu_1)\mathcal{U}_1 + \xi_2(\xi_2\mu_0 - (1+\lambda)\mu_2)\mathcal{U}_0. \end{aligned} \tag{2.2}$$

Taking the projection of Eq. (2.2) in the directions of  $\mathcal{U}_1$  and  $\mathcal{U}_2$ , we obtain that if  $u$  is the solution of (1.4), then

$$\begin{aligned} \frac{\partial\xi_1}{\partial t} &= -\xi_2\mu_0 + (1+\lambda)\mu_2 + (1+\lambda)((1+\lambda)\mu_1 - \xi_1\mu_0) - \xi_2(\xi_1\mu_2 - \xi_2\mu_1), \\ \frac{\partial\xi_2}{\partial t} &= -((1+\lambda)\mu_1 - \xi_1\mu_0) - (1+\lambda)(\xi_2\mu_0 - (1+\lambda)\mu_2) + \xi_1(\xi_1\mu_2 - \xi_2\mu_1). \end{aligned} \tag{2.3}$$

Replacing the  $\mu'_i$ s by their values in Eq. (2.3), We obtain that Landau–Lifschitz equation is equivalent for the small perturbations of  $\mathcal{U}_0$  to the following system:

$$\begin{aligned} \frac{\partial \xi_1}{\partial t} &= -\xi_2 \lambda'(\xi) \Delta \xi - \xi_2 \lambda''(\xi) \{(\partial_x \xi)^2 + (\partial_y \xi)^2\} \\ &\quad + 2\xi_2(1 + \lambda(\xi)) \frac{1}{\text{ch}^2 x} + 2\xi_2(\partial_x \xi_1) \frac{1}{\text{ch} x} \\ &\quad - 2\xi_1 \xi_2 \frac{\text{sh} x}{\text{ch}^2 x} + (1 + \lambda(\xi)) \Delta \xi_2 + (1 + \lambda(\xi)) \xi_2 + (1 + \lambda(\xi))^2 \Delta \xi_1 \\ &\quad + 2(1 + \lambda(\xi))^2 \lambda'(\xi) (\partial_x \xi) \frac{1}{\text{ch} x} - (1 + \lambda(\xi))^2 \xi_1 - \xi_1(1 + \lambda(\xi)) \lambda'(\xi) \Delta \xi \\ &\quad - \xi_1(1 + \lambda(\xi)) \lambda''(\xi) \{(\partial_x \xi)^2 + (\partial_y \xi)^2\} + 2\xi_1(1 + \lambda(\xi))^2 \frac{1}{\text{ch}^2 x} \\ &\quad + 2\xi_1(1 + \lambda(\xi)) (\partial_x \xi_1) \frac{1}{\text{ch} x} - 2\xi_1^2(1 + \lambda(\xi)) \frac{\text{sh} x}{\text{ch}^2 x} \\ &\quad - 2\xi_1 \xi_2^2 - \xi_1 \xi_2 \Delta \xi_2 + \xi_2^2 \Delta \xi_1 + 2\lambda'(\xi) \xi_2^2 (\partial_x \xi) \frac{1}{\text{ch} x}, \\ \frac{\partial \xi_2}{\partial t} &= -(1 + \lambda(\xi)) \Delta \xi_1 - 2(1 + \lambda(\xi)) \lambda'(\xi) (\partial_x \xi) \frac{1}{\text{ch} x} + (1 + \lambda(\xi)) \xi_1 + \xi_1 \lambda'(\xi) \Delta \xi \\ &\quad + \xi_1 \lambda''(\xi) \{(\partial_x \xi)^2 + (\partial_y \xi)^2\} \\ &\quad - 2\xi_1(1 + \lambda(\xi)) \frac{1}{\text{ch}^2 x} - 2\xi_1(\partial_x \xi_1) \frac{1}{\text{ch} x} + 2\xi_1^2 \frac{\text{sh} x}{\text{ch}^2 x} \\ &\quad - \xi_2(1 + \lambda(\xi)) \lambda'(\xi) \Delta \xi - \xi_2(1 + \lambda(\xi)) \lambda''(\xi) \{(\partial_x \xi)^2 + (\partial_y \xi)^2\} \\ &\quad + 2\xi_2(1 + \lambda(\xi))^2 \frac{1}{\text{ch}^2 x} + 2\xi_2(1 + \lambda(\xi)) (\partial_x \xi_1) \frac{1}{\text{ch} x} \\ &\quad - 2\xi_1 \xi_2(1 + \lambda(\xi)) \frac{\text{sh} x}{\text{ch}^2 x} + (1 + \lambda(\xi))^2 \Delta \xi^2 \\ &\quad + (1 + \lambda(\xi))^2 \xi_2 + \xi_1^2 \Delta \xi_2 + 2\xi_1^2 \xi_2 - \xi_1 \xi_2 \Delta \xi_1 - 2\xi_1 \xi_2 \lambda'(\xi) (\partial_x \xi) \frac{1}{\text{ch} x}. \end{aligned} \tag{2.4}$$

Writing Eq. (2.4) in an operator form, we obtain that if  $u$  satisfies (1.4) then  $\xi$  verifies:

$$\partial_t \xi = \mathcal{T} \xi + \mathcal{L}(x, \xi, \nabla \xi, \Delta \xi), \tag{2.5}$$

where

$$\mathcal{T} \xi = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} T \xi_1 \\ T \xi_2 - 2\xi_2 \end{pmatrix},$$

with  $T = -\Delta + \eta$ ,  $\eta(x) = 2\text{th}^2 x - 1$ .

The nonlinear term  $\mathcal{L} : \mathbb{R} \times B(0, 1) \times \mathbb{R}^4 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is defined by

$$\mathcal{L}(x, \xi, \nabla \xi, \Delta \xi) = \mathcal{P}(\xi) \Delta \xi + \sum_{i=1}^2 \mathcal{Q}(\xi) (\partial_i \xi, \partial_i \xi) + \mathcal{B}(x, \xi) (\partial_x \xi) + \mathcal{S}(x, \xi),$$

with the following notations:

- $\partial_1(\xi) = \partial_x \xi = \frac{\partial \xi}{\partial x}$ ,  $\partial_2(\xi) = \partial_y \xi = \frac{\partial \xi}{\partial y}$ .
- $\mathcal{P} \in C^\infty(B(0, 1); \mathcal{M}_2(\mathbb{R}))$  ( $\mathcal{M}_2(\mathbb{R})$  is the set of the real  $2 \times 2$  matrices):

$$\mathcal{P}(\xi) = \begin{pmatrix} -\xi_1^2 & \lambda(\xi) - \xi_1 \xi_2 \\ \lambda(\xi) - \xi_1 \xi_2 & -\xi_2^2 \end{pmatrix} + \begin{pmatrix} -\xi_2 - \xi_1(1 + \lambda(\xi)) \\ \xi_1 - \xi_2(1 + \lambda(\xi)) \end{pmatrix} \lambda'(\xi).$$

- $\mathcal{Q} \in C^\infty(B(0, 1); \mathcal{L}_2(\mathbb{R}^2))$  ( $\mathcal{L}_2(\mathbb{R}^2; \mathbb{R}^2)$ ) is the set of the bilinear functions defined on  $\mathbb{R}^2 \times \mathbb{R}^2$  with values in  $\mathbb{R}^2$ :

$$\mathcal{Q}(\xi)(\zeta, \zeta) = \begin{pmatrix} -\xi_2 - \xi_1 - \xi_1 \lambda(\xi) \\ \xi_1 - \xi_2 - \xi_2 \lambda(\xi) \end{pmatrix} \lambda''(\zeta, \zeta).$$

- $\mathcal{R} \in C^\infty(\mathbb{R} \times B(0, 1); \mathcal{M}_2(\mathbb{R}))$ :

$$\begin{aligned} \mathcal{R}(x, \xi)(\zeta) &= \frac{2}{\text{ch } x} \begin{pmatrix} \xi_2 + \xi_1(1 + \lambda(\xi)) \\ -\xi_1 + \xi_2(1 + \lambda(\xi)) \end{pmatrix} \zeta_1 \\ &\quad + \frac{2}{\text{ch } x} \begin{pmatrix} 1 - \xi_1^2 \\ -1 - \lambda(\xi) - \xi_1 \xi_2 \end{pmatrix} \lambda'(\xi)(\zeta). \end{aligned}$$

- $\mathcal{S} \in C^\infty(\mathbb{R} \times B(0, 1); \mathbb{R}^2)$   $\mathcal{S}(x, \xi) = \begin{pmatrix} \mathcal{S}_1 \\ \mathcal{S}_2 \end{pmatrix}$ , with

$$\begin{aligned} \mathcal{S}_1 &= -(-2\xi_2 + \xi_2 \eta + 2\xi_1 \eta + 2\xi_1 \eta \lambda(\xi)) \lambda(\xi) - 2\xi_1 \xi_2^2 \\ &\quad - 2 \frac{\text{sh } x}{\text{ch}^2 x} \xi_1 (\xi_2 + \xi_1 + \xi_1 \lambda(\xi)), \\ \mathcal{S}_2 &= (\xi_1 \eta - 2\xi_2 \eta - \xi_2 \eta \lambda(\xi) + 4\xi_2 + 2\xi_2 \lambda(\xi)) \lambda(\xi) + 2\xi_1^2 \xi_2 \\ &\quad + 2 \frac{\text{sh } x}{\text{ch}^2 x} \xi_1 (\xi_1 - \xi_2 - \xi_2 \lambda(\xi)). \end{aligned}$$

In the form of following proposition, we prove that Landau–Lifschitz equation (1.4) and the perturbed equation (2.5) are equivalent.

**Proposition 2.1** *Assume that  $u \in C^1(0, T; H^2(\mathbb{R}^2; \mathbb{R}^3))$  with saturation constraint  $|u| = 1$  and satisfying:*

$$\forall t \in [0, T], \quad \forall (x, y) \in \mathbb{R}^2, \quad |u(t, x, y) - \mathcal{U}_0(x)| < \sqrt{2}$$

We introduce  $\xi = (\xi_1, \xi_2) \in C^1(0, T; H^2(\mathbb{R}^2; \mathbb{R}^2))$  defined by

$$u(t, x, y) = \mathcal{U}_0(x) + \xi_1(t, x, y) \mathcal{U}_1(x) + \xi_2(t, x, y) \mathcal{U}_2 + \lambda(\xi(t, x, y)) \mathcal{U}_0(x).$$

Then  $u$  is solution to the Landau–Lifschitz equation (1.4) if and only if  $\xi$  is solution to (2.5) and  $\mathcal{U}_0$  is stable for (1.4) if and only if 0 is stable for (2.5).

*Proof* We derive the result using the similar arguments which have been used in [11, 16]. By projection on both  $\mathcal{U}_1$  and  $\mathcal{U}_2$ , it is clear that if  $u$  satisfies (1.4), then  $\xi$  satisfies (2.5). Conversely, we write (1.4) on the form

$$\frac{\partial u}{\partial t} = \mathcal{J}(u),$$

since  $u$  satisfies the physical constraint  $|u| = 1$ , renders  $u \cdot \frac{\partial u}{\partial t} = 0$ . Moreover,  $u \cdot \mathcal{J}(u) = 0$ . We remark that  $u$  satisfies:

$$\left( \frac{\partial u}{\partial t} - \mathcal{J}(u) \right) \cdot \mathcal{U}_k = 0, \quad \forall k \in \{1, 2\}$$

yields

$$\left( \frac{\partial u}{\partial t} - \mathcal{J}(u) \right) \cdot (1 + \lambda) \mathcal{U}_0 = 0$$

since  $\lambda \neq -1$ , implies that  $u$  satisfies (1.4). This completes the proof of Proposition 2.1.  $\square$

### 3 Properties of the linear operator

The linear operator  $T$  acting on  $H^2(\mathbb{R}^2)$  is defined by:

$$T = -\Delta + \eta$$

with  $\eta(x) = 2\text{th}^2x - 1$  and  $\Delta = \partial_{xx} + \partial_{yy}$ .

We denote by  $T_1$  the reduced operator acting on  $H^2(\mathbb{R})$  given by

$$T_1 = -\partial_{xx} + \eta.$$

We remark that  $T_1$  is a self-adjoint operator on  $H^2(\mathbb{R})$ . Furthermore,  $T_1$  is positive operator since we can write  $T_1 = \tau^* \circ \tau$  where  $\tau = \partial_x + \text{th } x$ , and 0 is the simple eigenvalue associated with the eigenvector  $\frac{1}{\text{ch } x}$ . Hence  $\text{Ker } T_1$  is the one-dimensional space spanned by  $\frac{1}{\text{ch } x}$ . The self-adjoint operator  $T_1$  is a compact perturbations of  $-\partial_{xx} + 1$ , and thus its essential spectrum is  $[1, +\infty)$  (using Weyl Theorem, see in [1,26]). The spectrum of  $T_1$  is  $\{0\} \cup [1, +\infty)$ , where 0 is the unique eigenvalue. We denote  $\mathcal{G}_1 = (\text{Ker } T_1)^\perp$ . The restriction of  $T_1$  on  $\mathcal{G}_1$  is a symmetric definite positive operator. We define  $\mathcal{G}_1$  by:

$$\mathcal{G}_1 = (\text{Ker } T_1)^\perp = \left\{ \alpha \in H^2(\mathbb{R}), \int_{\mathbb{R}} \alpha(x) \frac{1}{\text{ch } x} dx = 0 \right\}.$$

Then, for all  $u \in \mathcal{G}_1$ , the  $H^2$ -norm is equivalent to  $\|T_1 u\|_{L^2(\mathbb{R})}$  and the  $H^3$ -norm is equivalent to  $\|T_1^{3/2} u\|_{L^2(\mathbb{R})}$ . (For details, see [16].)

**Proposition 3.1** *The operator  $T = -\Delta + \eta$  is a positive self-adjoint operator defined on  $H^2(\mathbb{R}^2)$ . We introduce  $\mathcal{G}$  and defined it as*

$$\mathcal{G} = \left\{ \alpha \in H^2(\mathbb{R}^2), \forall y \in \mathbb{R}, \int_{x \in \mathbb{R}} \alpha(x, y) \frac{1}{\text{ch } x} dx = 0 \right\}.$$

There exists a constant  $C$  such that

$$\begin{aligned} \forall \alpha \in \mathcal{G}, \quad \|\alpha\|_{H^2(\mathbb{R}^2)} &\leq C \|T\alpha\|_{L^2(\mathbb{R}^2)}, \\ \forall \alpha \in H^3(\mathbb{R}^2) \cap \mathcal{G}, \quad \|\alpha\|_{H^3(\mathbb{R}^2)} &\leq C \|T^{3/2}\alpha\|_{L^2(\mathbb{R}^2)}. \end{aligned}$$

*Proof* We have that, for all  $u \in \mathcal{G}_1$ , the  $H^2$ -norm is equivalent to  $\|T_1 u\|_{L^2(\mathbb{R})}$  which implies

$$\|u\|_{L^2(\mathbb{R})}^2 + \|\partial_{xx}u\|_{L^2(\mathbb{R})}^2 \leq C \|T_1 u\|_{L^2(\mathbb{R})}^2.$$

Now for  $\alpha \in \mathcal{G}$ , we have for almost every  $y \in \mathbb{R}$ :

$$\int_{x \in \mathbb{R}} (|\alpha(x, y)|^2 + |\partial_{xx}\alpha(x, y)|^2) dx \leq C \int_{x \in \mathbb{R}} |T_1\alpha(x, y)|^2 dx$$

On integrating  $y \in \mathbb{R}$ , we obtain

$$\|\alpha\|_{L^2(\mathbb{R}^2)}^2 + \|\partial_{xx}\alpha\|_{L^2(\mathbb{R}^2)}^2 \leq C \|T_1\alpha\|_{L^2(\mathbb{R}^2)}^2.$$

Moreover,

$$\|T\alpha\|_{L^2(\mathbb{R}^2)}^2 = \int_{\mathbb{R}^2} |T\alpha|^2 = \int_{\mathbb{R}^2} |T_1\alpha|^2 + \int_{\mathbb{R}^2} |\partial_{yy}\alpha|^2 - 2 \int_{\mathbb{R}^2} T_1\alpha \partial_{yy}\alpha.$$

The last term is positive:

$$\therefore -2 \int_{\mathbb{R}^2} T_1\alpha \partial_{yy}\alpha = -2 \int_{\mathbb{R}^2} \tau^* \circ \tau \alpha \cdot \partial_{yy}\alpha = 2 \int_{\mathbb{R}^2} |\partial_y \tau \alpha|^2.$$



So,

$$\int_{\mathbb{R}^2} |T\alpha|^2 \geq \int_{\mathbb{R}^2} |T_1\alpha|^2 + \int_{\mathbb{R}^2} |\partial_{yy}\alpha|^2,$$

which implies

$$\|\alpha\|_{L^2(\mathbb{R}^2)}^2 + \|\Delta\alpha\|_{L^2(\mathbb{R}^2)}^2 \leq C\|T\alpha\|_{L^2(\mathbb{R}^2)}^2.$$

We establish an estimate on  $H^3$ -norm in a similar lines using the equivalence of  $H^3$ -norm to  $\|T_1^{3/2}u\|_{L^2(\mathbb{R})}$ . □

#### 4 Change of coordinates

A one-parameter family of static solutions to Landau–Lifschitz equation (1.4) is being constructed using the translational invariance for solutions depending only on the  $x$ -variable. Furthermore, for  $\sigma \in \mathbb{R}$ ,  $x \mapsto \mathcal{U}_0(x - \sigma)$  satisfies (1.4). We introduce the one-parameter family  $(\beta(\sigma))_{\sigma \in \mathbb{R}}$  of static wave solutions to (2.5) obtained from  $\mathcal{U}_0(x - \sigma)$  in the mobile frame  $(\mathcal{U}_1(x), \mathcal{U}_2(x))$ :

$$\beta(\sigma)(x) = \begin{pmatrix} \mathcal{U}_0(x - \sigma) \cdot \mathcal{U}_1(x) \\ \mathcal{U}_0(x - \sigma) \cdot \mathcal{U}_2(x) \end{pmatrix} = \begin{pmatrix} \rho(\sigma)(x) \\ 0 \end{pmatrix},$$

where  $\rho(\sigma)(x) = \frac{-\text{th } x}{\text{ch}(x-\sigma)} + \frac{\text{th}(x-\sigma)}{\text{ch } x}$ .

Using the techniques from [21], we write small perturbations of  $\xi$  in a neighborhood of 0 in the new coordinate system given by  $(\phi, \psi, \mathcal{N})$  as

$$\xi(t, x, y) = \beta(\phi(t, y))(x) + \begin{pmatrix} 0 \\ \frac{1}{\text{ch } x} \end{pmatrix} \psi(t, y) + \mathcal{N}(t, x, y), \tag{4.1}$$

where both coordinates of  $\mathcal{N}$  take their values in  $\mathcal{G}$ . In order to prove that this system is relevant to our analysis, we start with the following notations.

We denote by  $\Pi$  the following space:

$$\Pi := H^2(\mathbb{R}) \times H^2(\mathbb{R}) \times \mathcal{G} \times \mathcal{G}. \tag{4.2}$$

We define the norm on  $\Pi$  as

$$\|(\phi, \psi, \mathcal{N})\|_{\mathcal{H}^2} := \|\phi\|_{H^2(\mathbb{R})} + \|\psi\|_{H^2(\mathbb{R})} + \|T\mathcal{N}_1\|_{L^2(\mathbb{R}^2)} + \|T\mathcal{N}_2\|_{L^2(\mathbb{R}^2)}.$$

Using Proposition 3.1, we have the following equivalence of norms on  $\Pi$ :

$$\|(\phi, \psi, \mathcal{N})\|_{\mathcal{H}^2} \sim \|\phi\|_{H^2(\mathbb{R})} + \|\psi\|_{H^2(\mathbb{R})} + \|\mathcal{N}_1\|_{H^2(\mathbb{R}^2)} + \|\mathcal{N}_2\|_{H^2(\mathbb{R}^2)}.$$

Similarly, on  $\Pi \cap H^3$ , we define

$$\|(\phi, \psi, \mathcal{N})\|_{\mathcal{H}^3} := \|\phi\|_{H^3(\mathbb{R})} + \|\psi\|_{H^3(\mathbb{R})} + \|T^{3/2}\mathcal{N}_1\|_{L^2(\mathbb{R}^2)} + \|T^{3/2}\mathcal{N}_2\|_{L^2(\mathbb{R}^2)},$$

and this norm is equivalent to the following norm on  $\Pi \cap H^3$ :

$$\|(\phi, \psi, \mathcal{N})\|_{\mathcal{H}^3} \sim \|\phi\|_{H^3(\mathbb{R})} + \|\psi\|_{H^3(\mathbb{R})} + \|\mathcal{N}_1\|_{H^3(\mathbb{R}^2)} + \|\mathcal{N}_2\|_{H^3(\mathbb{R}^2)}.$$

In the following proposition, we prove the relevance of such perturbations of  $\xi$  in a neighborhood of 0.

**Proposition 4.1** *There exists  $\delta > 0$ , such that if  $\xi \in H^2(\mathbb{R}^2; \mathbb{R}^2)$  verifies  $\|\xi\|_{H^2(\mathbb{R}^2)} \leq \delta$ , there exists*

$(\phi, \psi, \mathcal{N}) \in \Pi$  such that

$$\xi(x, y) = \beta(\phi(y))(x) + \begin{pmatrix} 0 \\ \frac{1}{\text{ch } x} \end{pmatrix} \psi(y) + \mathcal{N}(x, y).$$

Moreover, there exists  $C$  such that for  $\xi \in H^2(\mathbb{R}^2; \mathbb{R}^2)$  in a neighborhood of zero

$$\frac{1}{C} \|(\phi, \psi, \mathcal{N})\|_{\mathcal{H}^2} \leq \|\xi\|_{H^2(\mathbb{R}^2)} \leq C \|(\phi, \psi, \mathcal{N})\|_{\mathcal{H}^2},$$

and for  $H^3(\mathbb{R}^2; \mathbb{R}^2)$  in a neighborhood of zero,

$$\frac{1}{C} \|(\phi, \psi, \mathcal{N})\|_{\mathcal{H}^3} \leq \|\xi\|_{H^3(\mathbb{R}^2)} \leq C \|(\phi, \psi, \mathcal{N})\|_{\mathcal{H}^3}. \tag{4.3}$$

*Proof* We introduce the linear mappings  $f$  and  $g$  defined for  $\xi = (\xi_1, \xi_2) \in H^2(\mathbb{R}^2; \mathbb{R}^2)$  by

$$f(\xi)(y) = \frac{1}{2} \int_{x \in \mathbb{R}} \xi_1(x, y) \frac{1}{\text{ch } x} dx,$$

$$g(\xi)(y) = \frac{1}{2} \int_{x \in \mathbb{R}} \xi_2(x, y) \frac{1}{\text{ch } x} dx.$$

The operators  $f$  and  $g$  are bounded linear transformations from  $H^2(\mathbb{R}^2; \mathbb{R}^2)$  (resp.  $H^3(\mathbb{R}^2; \mathbb{R}^2)$ ) into  $H^2(\mathbb{R})$  (resp.  $H^3(\mathbb{R})$ ). For a fixed  $\xi$  in a neighborhood of 0,  $(\phi, \psi, \mathcal{N})$  can be obtained in the following way:

We operate  $g$  on (4.1) and obtain

$$g(\xi)(y) = \psi(y),$$

By operating  $f$  on (4.1), we get

$$f(\xi)(y) = \frac{1}{2} \int_{x \in \mathbb{R}} \rho(\phi(y))(x) \frac{1}{\text{ch } x} dx.$$

We define  $\Psi \in C^\infty(\mathbb{R}; \mathbb{R})$  given by

$$\Psi(\sigma) = \frac{1}{2} \int_{x \in \mathbb{R}} \rho(\sigma)(x) \frac{1}{\text{ch } x} dx.$$

Since  $\Psi(0) = 0$  and  $\Psi'(0) = 1$ , there exists  $\delta > 0$  such that  $\Psi$  is a  $C^\infty$ -diffeomorphism from  $(-\delta, \delta)$  to neighborhood of zero. We get

$$f(\xi)(y) = \Psi(\phi(y)),$$

Also  $\phi$  is given by

$$\phi(y) = \Psi^{-1}(f(\xi)(y)).$$

We set  $\mathcal{N}$  as

$$\mathcal{N}(x, y) = \xi(x, y) - \beta(\phi(y))(x) - \begin{pmatrix} 0 \\ \frac{1}{\text{ch } x} \end{pmatrix} \psi(y), \tag{4.4}$$

and by construction  $f(\mathcal{N}) = g(\mathcal{N}) = 0$ , i.e.,  $\mathcal{N} \in \mathcal{G}^2$ .

As we remark that  $\mathcal{G}^2 = \{\mathcal{N} \in H^2(\mathbb{R}^2; \mathbb{R}^2), f(\mathcal{N}) = g(\mathcal{N}) = 0\}$ .

Using  $\rho(0)(x) = 0$ ,  $\partial_\sigma \rho(0)(x) = -\frac{1}{\text{ch } x}$  and  $|\beta(\sigma)(x)| \leq C \frac{|\sigma|}{\text{ch } x}$ , we obtain that for  $\phi \in H^2(\mathbb{R})$  sufficiently small,

$$\|\beta(\phi(y))(x)\|_{H^2(\mathbb{R}^2)} \leq C \|\phi\|_{H^2(\mathbb{R})}, \tag{4.5}$$

so  $\xi \in H^2(\mathbb{R}^2; \mathbb{R}^2)$  we have

$$\|\xi\|_{H^2(\mathbb{R}^2)} \leq C (\|\phi\|_{H^2(\mathbb{R})} + \|\psi\|_{H^2(\mathbb{R})} + \|\mathcal{N}\|_{H^2(\mathbb{R}^2)}) \leq C\|\phi, \psi, \mathcal{N}\|_{\mathcal{H}^2}.$$

Now using the boundedness of the linear operators  $f$  and  $g$  for the  $H^2$ -norm, since  $\Psi^{-1}$  is smooth in a neighborhood of 0 and satisfies  $\Psi^{-1}(\sigma) = \sigma + O(\sigma^2)$ , we get the following relation:

$$\|\phi\|_{H^2(\mathbb{R})} + \|\psi\|_{H^2(\mathbb{R})} \leq C\|\xi\|_{H^2(\mathbb{R}^2)}.$$

Using (4.4) and (4.5), we get

$$\|\mathcal{N}\|_{H^2(\mathbb{R}^2)} \leq C\|\xi\|_{H^2(\mathbb{R}^2)},$$

which yields

$$\|\phi, \psi, \mathcal{N}\|_{\mathcal{H}^2} \leq C\|\xi\|_{H^2(\mathbb{R}^2)}.$$

We prove (4.3) in a similar fashion. This concludes the proof of Proposition 4.1. □

In a neighborhood of zero, we define  $\xi$  in the new coordinates system  $(\phi, \psi, \mathcal{N})$  given by (4.1). We transform (2.5) in these coordinates. We choose  $\delta$  to be sufficiently small so that  $\|\xi\|_{L^\infty} < 1$ . We notice that in the one-dimensional case, for a fixed  $\sigma$ , the map  $x \mapsto \beta(\sigma)(x)$  is a static wave solution to (2.5). We denote  $\mathcal{T}_1$  the reduced operator given by:

$$\mathcal{T}_1\gamma = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} T_1\gamma_1 \\ T_1\gamma_2 - 2\gamma_2 \end{pmatrix},$$

with

$$\begin{aligned} \mathcal{T}_1\beta(\phi) + \mathcal{P}(\beta(\phi))\partial_{xx}\beta(\phi) + \mathcal{Q}(\beta(\phi))(\partial_x\beta(\phi), \partial_x\beta(\phi)) \\ + \mathcal{R}(x, \beta(\phi))(\partial_x\beta(\phi)) + \mathcal{S}(x, \beta(\phi)) = 0. \end{aligned} \tag{4.6}$$

Furthermore,

$$\partial_t(\beta(\phi(t, y))(x)) = \partial_\sigma R(\phi(t, y))\partial_t\phi(t, y),$$

and

$$\Delta(R(\phi(t, y))(x)) = \partial_{xx}R(\phi(t, y)) + \partial_\sigma R(\phi(t, y))\partial_{yy}\phi + \partial_{\sigma\sigma}R(\phi(t, y))|\partial_y\phi|^2.$$

We also have

$$\mathcal{T}\beta(\phi) = \mathcal{T}_1\beta(\phi) + \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} (-\partial_\sigma\beta(\phi)\partial_{yy}\phi - \partial_{\sigma\sigma}\beta(\phi)|\partial_y\phi|^2).$$

Substituting (4.1) in (2.5) and using (4.6), we obtain

$$\begin{aligned} \partial_\sigma\beta(\phi)\partial_t\phi + \begin{pmatrix} 0 \\ \frac{1}{\text{ch } x} \end{pmatrix} \partial_t\psi + \partial_t\mathcal{N} = (\partial_\sigma\rho(\phi)\partial_{yy}\phi + \partial_{\sigma\sigma}\rho(\phi)|\partial_y\phi|^2) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ + \frac{1}{\text{ch } x} (\partial_{yy}\psi + 2\psi) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \mathcal{T}\mathcal{N} + \Lambda. \end{aligned} \tag{4.7}$$

The nonlinear term  $\Lambda$  is defined as

$$\Lambda = \sum_{i=1}^5 \Lambda_i \tag{4.8}$$

where

$$\begin{aligned} \Lambda_1 &= \mathcal{P}(\beta(\phi))\partial_{yy}\beta(\phi) + \tilde{\mathcal{P}}(\beta(\phi), \gamma)(\gamma)(\partial_{xx}\beta(\phi)) \\ &\quad + \tilde{\mathcal{P}}(\beta(\phi), \gamma)(\gamma)(\partial_{yy}\beta(\phi)) + \mathcal{P}(\xi)\Delta\gamma. \\ \Lambda_2 &= 2\mathcal{Q}(\beta(\phi))(\partial_x\beta(\phi), \partial_x\gamma) + \mathcal{Q}(\beta(\phi))(\partial_x\gamma, \partial_x\gamma) + \tilde{\mathcal{Q}}(\beta(\phi), \gamma)(\gamma)(\partial_x\xi, \partial_x\xi). \\ \Lambda_3 &= \mathcal{Q}(\xi)(\partial_y\beta(\phi), \partial_y\beta(\phi)) + 2\mathcal{Q}(\xi)(\partial_y\beta(\phi), \partial_y\gamma) + \mathcal{Q}(\xi)(\partial_y\gamma, \partial_y\gamma). \\ \Lambda_4 &= \tilde{\mathcal{R}}(x, \beta(\phi))(\partial_x\gamma) + \tilde{\mathcal{R}}(x, \beta(\phi), \gamma)(\gamma)(\partial_x\xi). \\ \Lambda_5 &= \tilde{\mathcal{S}}(x, \beta(\phi), \gamma)(\gamma). \end{aligned}$$

with the following notations:

- $\gamma(x, y) = \psi(y) \begin{pmatrix} 0 \\ \frac{1}{\text{ch } x} \end{pmatrix} + \mathcal{N}(x, y)$  and  $\xi(x, y) = \beta(\phi(y))(x) + \gamma(x, y)$ ,
- $\tilde{\mathcal{P}} \in C^\infty(B(0, 1/2) \times (B(0, 1/2); \mathcal{L}(\mathbb{R}^2; \mathcal{M}_2(\mathbb{R}))) (\mathcal{L}(\mathbb{R}^2; \mathcal{M}_2(\mathbb{R}))$  is the set of linear transformations defined on  $\mathbb{R}^2$  with values in  $\mathcal{M}_2(\mathbb{R})$ ):

$$\tilde{\mathcal{P}}(u, v) = \int_0^1 \mathcal{P}'(u + sv) ds.$$

- $\tilde{\mathcal{Q}} \in C^\infty(B(0, 1/2) \times (B(0, 1/2); \mathcal{L}(\mathbb{R}^2; \mathcal{L}_2(\mathbb{R}^2; \mathbb{R}^2))))$ :

$$\tilde{\mathcal{Q}}(u, v) = \int_0^1 \mathcal{Q}'(u + sv) ds.$$

- $\tilde{\mathcal{R}} \in C^\infty(B(0, 1/2) \times (B(0, 1/2); \mathcal{L}(\mathbb{R}^2; \mathcal{M}_2(\mathbb{R}))))$ :

$$\tilde{\mathcal{R}}(x, u, v) = \int_0^1 \partial_\zeta \mathcal{R}(x, u + sv) ds.$$

- $\tilde{\mathcal{S}} \in C^\infty(B(0, 1/2) \times (B(0, 1/2); \mathcal{L}(\mathbb{R}^2; \mathbb{R}^2)))$ :

$$\tilde{\mathcal{S}}(x, u, v) = \int_0^1 \partial_\zeta \mathcal{S}(x, u + sv) ds.$$

(the tilde terms come from the Taylor expansion for  $\mathcal{P}$ ,  $\mathcal{Q}$ ,  $\mathcal{R}$  and  $\mathcal{S}$  applied between  $\beta(\phi)$  and  $\beta(\phi) + \gamma$ ).

We obtain the transformed system in new coordinates  $(\phi, \psi, \mathcal{N})$  with the help of operators  $f$  and  $g$ .

We multiply (4.7) by  $\begin{pmatrix} \frac{1}{2\text{ch } x} \\ 0 \end{pmatrix}$ , and we integrate in the  $x$  variable. We get

$$\mathcal{A}(\phi)\partial_t(\phi) = \mathcal{A}(\phi)\partial_{yy}\phi + \mathcal{B}(\phi)|\partial_y\phi|^2 + (\partial_{yy}\psi + 2\psi) + f(\Lambda),$$

where

$$\begin{aligned} \mathcal{A}(\sigma) &= \frac{1}{2} \int_{\mathbb{R}} \partial_\sigma \rho(\sigma)(x) \frac{1}{\text{ch } x} dx \\ &= -\frac{1}{2} \int_{\mathbb{R}} \left[ \frac{\text{sh}(x - \sigma) \text{th } x}{\text{ch}^2(x - \sigma)} + \frac{1}{\text{ch}^2(x - \sigma) \text{ch } x} \right] \frac{1}{\text{ch } x} dx, \end{aligned}$$

and

$$\begin{aligned} \mathcal{B}(\sigma) &= \frac{1}{2} \int_{\mathbb{R}} \partial_{\sigma\sigma} \rho(\sigma)(x) \frac{1}{\operatorname{ch} x} dx \\ &= \int_{\mathbb{R}} \left[ \frac{\operatorname{th} x}{\operatorname{ch}(x - \sigma)} - 2 \frac{\operatorname{sh}^2(x - \sigma) \operatorname{th} x}{\operatorname{ch}^3(x - \sigma)} - 2 \frac{\operatorname{sh}(x - \sigma)}{\operatorname{ch}^3(x - \sigma) \operatorname{ch} x} \right] \frac{1}{\operatorname{ch} x} dx. \end{aligned}$$

We notice that  $\mathcal{A}$  and  $\mathcal{B}$  are in  $C^\infty(\mathbb{R}, \mathbb{R})$  and that  $\mathcal{A}(0) = 1$  and  $\mathcal{B}(0) = 0$ .

In a neighborhood of zero, we write  $\frac{1}{\mathcal{A}(\sigma)} = 1 + \nu(\sigma)$  where  $\nu(\sigma) = O(|\sigma|)$  and  $\mathcal{B}(\sigma) = O(\sigma)$ . We obtain that

$$\partial_t \phi = \partial_{yy} \phi + (\partial_{yy} \psi + 2\psi) + \mathcal{F}_1(\phi, \psi, \mathcal{N}), \tag{4.9}$$

where the nonlinear term  $\mathcal{F}_1$  is given by

$$\mathcal{F}_1(\phi, \psi, \mathcal{N}) = \nu(\phi) (\partial_{yy} \psi + 2\psi) + \frac{\mathcal{B}(\phi)}{\mathcal{A}(\phi)} |\partial_y \phi|^2 + \frac{1}{\mathcal{A}(\phi)} f(\Lambda). \tag{4.10}$$

We multiply (4.7) by  $\begin{pmatrix} 0 \\ \frac{1}{2 \operatorname{ch} x} \end{pmatrix}$  and integrate in the  $x$  variable. We are left with:

$$\partial_t \psi = -\partial_{yy} \phi + (\partial_{yy} \psi + 2\psi) + \mathcal{F}_2(\phi, \psi, \mathcal{N}) \tag{4.11}$$

with

$$\mathcal{F}_2(\phi, \psi, \mathcal{N}) = (1 - \mathcal{A}(\phi)) \partial_{yy} \phi - \mathcal{B}(\phi) |\partial_y \phi|^2 + g(\Lambda). \tag{4.12}$$

In order to get the equation for  $\mathcal{N}$ , we multiply (4.8) by  $\partial_\sigma \beta(\phi)$ , (4.11) by  $\begin{pmatrix} 0 \\ \frac{1}{\operatorname{ch} x} \end{pmatrix}$  and subtracting from (4.7) which gives

$$\partial_t \mathcal{N} = \mathcal{F} \mathcal{N} + \mathcal{F}_3(x, \phi, \psi, \mathcal{N}), \tag{4.13}$$

where

$$\begin{aligned} \mathcal{F}_3(x, \phi, \psi, \mathcal{N}) &= \Lambda \\ &+ \left( |\partial_y \phi|^2 \partial_{\sigma\sigma} \rho(\phi) + (\partial_{yy} \psi + 2\psi) \left( \frac{1}{\operatorname{ch} x} - \partial_\sigma \rho(\phi) \right) - \partial_\sigma \rho(\phi) \mathcal{F}_1(\phi, \psi, \mathcal{N}) \right. \\ &\quad \left. - |\partial_y \phi|^2 \partial_{\sigma\sigma} \rho(\phi) + \partial_{yy} \phi \left( \frac{1}{\operatorname{ch} x} - \partial_\sigma \rho(\phi) \right) - \frac{1}{\operatorname{ch} x} \mathcal{F}_2(\phi, \psi, \mathcal{N}) \right), \end{aligned} \tag{4.14}$$

and with this, we complete the details of the following proposition:

**Proposition 4.2** *Assume that  $(\phi, \psi, \mathcal{N}) \in C^1(0, T; \Pi)$  is given by proposition (4.1). Consider  $\xi \in C^1(0, T; H^2(\mathbb{R}^2, \mathbb{R}^2))$  such that for all  $t \geq 0$ ,  $\|\xi(t, \cdot)\|_{H^2(\mathbb{R}^2)} \leq \delta$ . Then  $\xi$  satisfies (2.5) if and only if  $(\phi, \psi, \mathcal{N})$  satisfies the system (4.9)–(4.11)–(4.13), and 0 is stable for (2.5) if and only if  $(0, 0, 0)$  is stable for (4.9)–(4.11)–(4.13).*

### 5 Preliminary estimates

We recall that from Proposition 4.1, for  $\xi \in H^2(\mathbb{R}^2; \mathbb{R}^2)$  in a neighborhood of 0, we have the following representation,

$$\xi(x, y) = \beta(\phi(y))(x) + \begin{pmatrix} 0 \\ \frac{1}{\operatorname{ch} x} \end{pmatrix} \psi(y) + \mathcal{N}(x, y),$$

with  $(\phi, \psi, \mathcal{N}) \in \Pi$ , and there exists  $C$  such that for  $k = 2$  or  $3$ ,

$$\frac{1}{C} \|(\phi, \psi, \mathcal{N})\|_{\mathcal{H}^k} \leq \|\xi\|_{H^k(\mathbb{R}^2)} \leq C \|(\phi, \psi, \mathcal{N})\|_{\mathcal{H}^k}.$$

We introduce  $\kappa > 0$  such that if  $\|(\phi, \psi, \mathcal{N})\|_{\mathcal{H}^2} \leq \kappa$ , then  $\|\xi\|_{L^\infty} \leq \delta$ , so that we are in the framework of Proposition 4.2. We state the following proposition:

**Proposition 5.1** *There exists  $C$  such that for all  $(\phi, \psi, \mathcal{N}) \in \Pi$ , if  $\|(\phi, \psi, \mathcal{N})\|_{\mathcal{H}^2} \leq \kappa$ , then*

$$\begin{aligned} & \|\mathcal{F}_1\|_{H^1(\mathbb{R})} + \|\mathcal{F}_2\|_{H^1(\mathbb{R})} + \|\mathcal{F}_3\|_{H^1(\mathbb{R}^2)} \\ & \leq C \|(\phi, \psi, \mathcal{N})\|_{\mathcal{H}^2} (\|\partial_y \phi\|_{H^2(\mathbb{R})} + \|\psi\|_{H^3(\mathbb{R})} + \|\mathcal{N}\|_{H^3(\mathbb{R}^2)}). \end{aligned} \tag{5.1}$$

Moreover, we can write  $\mathcal{F}_1 - \mathcal{F}_2$  on the form:  $\mathcal{F}_1 - \mathcal{F}_2 = \tilde{\mathcal{F}}_a + \tilde{\mathcal{F}}_b$ , where  $\tilde{\mathcal{F}}_a$  and  $\tilde{\mathcal{F}}_b$  satisfy the following estimates: There exists  $C$  such that for all  $(\phi, \psi, \mathcal{N}) \in \Pi$ , if  $\|(\phi, \psi, \mathcal{N})\|_{\mathcal{H}^2} \leq \kappa$ , then

$$\|\tilde{\mathcal{F}}_a\|_{L^1(\mathbb{R})} \leq C(\|\partial_y \phi\|_{H^2(\mathbb{R})} + \|\psi\|_{H^3(\mathbb{R})} + \|\mathcal{N}\|_{H^3(\mathbb{R}^2)})^2, \tag{5.2}$$

$$\|\tilde{\mathcal{F}}_b\|_{L^{\frac{4}{3}}(\mathbb{R})} \leq C(\|\partial_y \phi\|_{H^2(\mathbb{R})} + \|\psi\|_{H^3(\mathbb{R})} + \|\mathcal{N}\|_{H^3(\mathbb{R}^2)})\|\phi\|_{L^4(\mathbb{R})}. \tag{5.3}$$

First we establish some preliminary estimates before going to the proof of this proposition. We establish a Sobolev-type inequality in the following lemma:

**Lemma 5.1** *There exists a constant  $C$  such that for all  $u \in H^2(\mathbb{R})$  and for  $k = 1, 2, 4$ ,*

$$\|\partial_y u\|_{L^{2k}(\mathbb{R})}^2 \leq C \|u\|_{L^\infty(\mathbb{R})} \|\partial_{yy} u\|_{L^k(\mathbb{R})}.$$

*Proof* For  $k = 1, 2, 4$ , we obtain

$$\begin{aligned} \|\partial_y u\|_{L^{2k}(\mathbb{R})}^{2k} &= \int_{\mathbb{R}} (\partial_y u)^{2k} = \int_{\mathbb{R}} \partial_y u (\partial_y u)^{2k-1} = -(2k-1) \int_{\mathbb{R}} u \partial_{yy} u (\partial_y u)^{2k-2} \\ &\leq C \|u\|_{L^\infty(\mathbb{R})} \|\partial_{yy} u\|_{L^k(\mathbb{R})} \|\partial_y u\|_{L^{2k}(\mathbb{R})}^{2k-2}, \end{aligned}$$

which completes the proof. □

To get a estimate for the nonlinear term  $\Lambda$  defined in (4.8), we establish preliminary estimates.

**Lemma 5.2** *There exists  $C$  such that for all  $(\phi, \psi, \mathcal{N}) \in \Pi$ , if  $\|(\phi, \psi, \mathcal{N})\|_{\mathcal{H}^2} \leq \kappa$ , then*

$$\|\beta(\phi)\|_{L^\infty(\mathbb{R}^2)} + \|\nabla \beta(\phi)\|_{L^4(\mathbb{R}^2)} + \|\nabla \partial_x \beta(\phi)\|_{L^4(\mathbb{R}^2)} \leq C \|(\phi, \psi, \mathcal{N})\|_{\mathcal{H}^2}$$

and

$$\begin{aligned} & \|\partial_{yy} \beta(\phi)\|_{L^2(\mathbb{R}^2)} + \|\partial_{yy} \beta(\phi)\|_{L^4(\mathbb{R}^2)} + \|\nabla \partial_{yy} \beta(\phi)\|_{L^2(\mathbb{R}^2)} \\ & \leq C(\|\partial_y \phi\|_{H^2(\mathbb{R})} + \|\psi\|_{H^3(\mathbb{R})} + \|\mathcal{N}\|_{H^3(\mathbb{R}^2)}). \end{aligned}$$

*Proof* We recall that there exists  $C$  such that for  $\sigma \in B(0,1/2)$ , we obtain

$$\begin{cases} |\beta(\sigma)(x)| + |\partial_x \beta(\sigma)(x)| + |\partial_{xx} \beta(\sigma)(x)| \leq C \frac{|\sigma|}{\text{ch } x}, \\ |\partial_\sigma \beta(\sigma)(x)| + |\partial_x \partial_\sigma \beta(\sigma)(x)| \leq \frac{C}{\text{ch } x}, \\ |\partial_{\sigma\sigma} \beta(\sigma)(x)| + |\partial_x \partial_{\sigma\sigma} \beta(\sigma)(x)| \leq \frac{C}{\text{ch } x}, \\ |\partial_{\sigma\sigma\sigma} \beta(\sigma)(x)| \leq \frac{C}{\text{ch } x}. \end{cases}$$

From Sobolev embedding,  $H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$  and using (4.5), we obtain

$$\|\beta(\phi)\|_{L^\infty(\mathbb{R}^2)} \leq C \|\phi\|_{H^2(\mathbb{R})}$$

using previous remarks, and we have

$$|\nabla \beta(\phi)| \leq \frac{C}{\text{ch } x} (|\phi| + |\partial_y \phi|),$$

From Sobolev embedding,  $H^2(\mathbb{R}^2) \hookrightarrow W^{1,4}(\mathbb{R}^2)$ , we get

$$\|\nabla\beta(\phi)\|_{L^4(\mathbb{R}^2)} \leq C\|\phi\|_{H^2(\mathbb{R})}$$

and

$$\|\nabla\partial_x\beta(\phi)\|_{L^4(\mathbb{R}^2)} \leq C\|\phi\|_{H^2(\mathbb{R})}.$$

This concludes the claimed estimate of the first part.

For the second part, we have

$$\begin{aligned} \partial_{yy}(\beta(\phi)) &= \partial_\sigma\beta(\phi)\partial_{yy}\phi + \partial_{\sigma\sigma}\beta(\phi)|\partial_y\phi|^2, \\ |\partial_{yy}(\beta(\phi))| &\leq \frac{C}{\text{ch } x} (|\partial_{yy}\phi| + |\partial_y\phi|^2). \end{aligned}$$

Using Lemma 5.1,

$$\|\partial_{yy}(\beta(\phi))\|_{L^2(\mathbb{R}^2)} \leq C\|\partial_{yy}\phi\|_{L^2(\mathbb{R})},$$

we have

$$\begin{aligned} \|\partial_{yy}(\beta(\phi))\|_{L^4(\mathbb{R}^2)} &\leq C(\|\partial_{yy}\phi\|_{L^4(\mathbb{R})} + \|\partial_y\phi\|_{L^8(\mathbb{R})}^2) \\ &\leq C\|\partial_{yy}\phi\|_{L^4(\mathbb{R})} \quad \text{by Lemma 5.1,} \\ &\leq C\|\partial_y\phi\|_{H^2(\mathbb{R})} \quad \text{by Sobolev embedding.} \end{aligned}$$

To conclude, we have

$$\partial_x(\partial_{yy}\beta(\phi)) = \partial_x\partial_\sigma\beta(\phi)\partial_{yy}\phi + \partial_x\partial_{\sigma\sigma}\beta(\phi)|\partial_y\phi|^2,$$

yields

$$\|\partial_x(\partial_{yy}\beta(\phi))\|_{L^2(\mathbb{R}^2)} \leq C\|\partial_{yy}\phi\|_{L^2(\mathbb{R})}$$

also,

$$\begin{aligned} \partial_y(\partial_{yy}\beta(\phi)) &= \partial_{\sigma\sigma}\beta(\phi)(\partial_y\phi)(\partial_{yy}\phi) + \partial_\sigma\beta(\phi)\partial_{yyy}\phi + \partial_{\sigma\sigma\sigma}\beta(\phi)(\partial_y\phi)^3 \\ &\quad + 2\partial_{\sigma\sigma}\beta(\phi)(\partial_y\phi)(\partial_{yy}\phi), \end{aligned}$$

which implies

$$\begin{aligned} \|\partial_y(\partial_{yy}\beta(\phi))\|_{L^2(\mathbb{R}^2)} &\leq C\|\partial_y\phi\|_{L^4(\mathbb{R})}\|\partial_{yy}\phi\|_{L^4(\mathbb{R})} + C\|\partial_{yyy}\phi\|_{L^2(\mathbb{R})} \\ &\quad + C\|\partial_y\phi\|_{L^6(\mathbb{R})}^3 + C\|\partial_y\phi\|_{L^4(\mathbb{R})}\|\partial_{yy}\phi\|_{L^4(\mathbb{R})} \\ &\leq C(\|\partial_{yy}\phi\|_{L^2(\mathbb{R})}^{1/2}\|\partial_{yyy}\phi\|_{L^2(\mathbb{R})}^{1/2} + \|\partial_{yyy}\phi\|_{L^2(\mathbb{R})}) \\ &\quad + \|\partial_{yy}\phi\|_{L^2(\mathbb{R})}^{1/2}\|\partial_y\phi\|_{L^8(\mathbb{R})}^2 \\ &\leq C\|\partial_y\phi\|_{H^2(\mathbb{R})}. \quad \text{by Lemma 5.1} \end{aligned}$$

This concludes the proof of Lemma 5.2. □

In the following lemma, we derive estimates for the term  $\gamma$  defined as

$$\gamma(t, x, y) = \psi(t, y) \begin{pmatrix} 0 \\ \frac{1}{\text{ch } x} \end{pmatrix} + \mathcal{N}(t, x, y).$$

**Lemma 5.3** *There exists a constant  $C$  such that*

$$\|\gamma\|_{L^\infty(\mathbb{R}^2)} + \|\gamma\|_{H^2(\mathbb{R}^2)} + \|\gamma\|_{L^4(\mathbb{R}^2)} + \|\nabla\gamma\|_{L^4(\mathbb{R}^2)} \leq C\|(\phi, \psi, \mathcal{N})\|_{\mathcal{H}^2},$$

and

$$\begin{aligned} & \|\gamma\|_{H^2(\mathbb{R}^2)} + \|\Delta\gamma\|_{L^4(\mathbb{R}^2)} + \|\nabla\Delta\gamma\|_{L^2(\mathbb{R}^2)} \\ & \leq C(\|\partial_y\phi\|_{H^2(\mathbb{R})} + \|\psi\|_{H^3(\mathbb{R})} + \|\mathcal{N}\|_{H^3(\mathbb{R}^2)}). \end{aligned}$$

*Proof* We have

$$|\gamma| \leq \frac{1}{\text{ch } x} |\psi| + |\mathcal{N}|$$

yields

$$\begin{aligned} \|\gamma\|_{L^2(\mathbb{R}^2)} & \leq C(\|\psi\|_{L^2(\mathbb{R})} + \|\mathcal{N}\|_{L^2(\mathbb{R}^2)}), \\ \|\nabla\gamma\|_{L^2(\mathbb{R}^2)} & \leq C(\|\psi\|_{L^2(\mathbb{R})} + \|\partial_y\psi\|_{L^2(\mathbb{R})} + \|\nabla\mathcal{N}\|_{L^2(\mathbb{R}^2)}), \\ \|\Delta\gamma\|_{L^2(\mathbb{R}^2)} & \leq C(\|\psi\|_{L^2(\mathbb{R})} + \|\partial_{yy}\psi\|_{L^2(\mathbb{R})} + \|\Delta\mathcal{N}\|_{L^2(\mathbb{R}^2)}). \end{aligned}$$

which implies

$$\|\gamma\|_{H^2(\mathbb{R}^2)} \leq C\|(\phi, \psi, \mathcal{N})\|_{\mathcal{H}^2}.$$

From the Sobolev embeddings of  $H^2(\mathbb{R}^2)$  into  $L^\infty(\mathbb{R}^2)$  and  $W^{1,4}(\mathbb{R}^2)$ , we obtain the claimed estimate of the first part. Also, we have

$$\begin{aligned} \|\Delta\gamma\|_{L^4(\mathbb{R}^2)} & \leq C\|\nabla\gamma\|_{H^2(\mathbb{R}^2)}, \\ \|\nabla\Delta\gamma\|_{L^2(\mathbb{R}^2)} & \leq C(\|\psi\|_{H^3(\mathbb{R})} + \|\nabla\Delta\mathcal{N}\|_{L^2(\mathbb{R}^2)}). \end{aligned}$$

This completes the proof of Lemma 5.3. □

*Proof of Proposition 5.1* In order to obtain the claimed estimate (5.1), we need to establish an estimate for the nonlinear functions appearing in Eq. (2.5) and the nonlinear term  $\Lambda$  defined in (4.8). We provide these estimates in the following Propositions. □

**Proposition 5.2** *There exists a constant  $C$  such that for  $\xi \in B(0, 1)$  and for  $x \in \mathbb{R}$ ,*

$$\begin{cases} |\mathcal{P}(\xi)| \leq C|\xi| \text{ and } |\mathcal{P}'(\xi)| \leq C, \\ |\mathcal{Q}(\xi)| \leq C|\xi| \text{ and } |\mathcal{Q}'(\xi)| \leq C, \\ |\mathcal{R}(x, \xi)| \leq \frac{C}{\text{ch } x} |\xi| \text{ and } |\partial_\xi \mathcal{R}(x, \xi)| \leq \frac{C}{\text{ch } x}, \\ |\mathcal{S}(x, \xi)| \leq C|\xi|^3 + \frac{C}{\text{ch } x} |\xi|^2 \text{ and } |\partial_\xi \mathcal{S}(x, \xi)| \leq C|\xi|^2 + \frac{C}{\text{ch } x} |\xi|. \end{cases}$$

**Proposition 5.3** *There exists  $C$  such that for all  $(\phi, \psi, \mathcal{N}) \in \Pi$ , if  $\|(\phi, \psi, \mathcal{N})\|_{\mathcal{H}^2} \leq \kappa$ , then*

$$\|\Lambda\|_{H^1(\mathbb{R}^2)} \leq C\|(\phi, \psi, \mathcal{N})\|_{\mathcal{H}^2} (\|\partial_y\phi\|_{H^2(\mathbb{R})} + \|\psi\|_{H^3(\mathbb{R})} + \|\mathcal{N}\|_{H^3(\mathbb{R}^2)}).$$

*Proof* We get the desired estimate for the term  $\Lambda$  in the following way.

- The term  $\Lambda_1$  is given by

$$\begin{aligned} \Lambda_1 & = \mathcal{P}(\beta(\phi))\partial_{yy}\beta(\phi) + \tilde{\mathcal{P}}(\beta(\phi), \gamma)(\gamma)(\partial_{xx}\beta(\phi)) \\ & \quad + \tilde{\mathcal{P}}(\beta(\phi), \gamma)(\gamma)(\partial_{yy}\beta(\phi)) + \mathcal{P}(\beta(\phi) + \gamma)\Delta\gamma \end{aligned}$$

and from Proposition (5.2), there exists  $C$  such that for  $|\zeta| \leq \frac{1}{2}$ ,



$$|\mathcal{P}(\zeta)| \leq C|\zeta|, \quad |\mathcal{P}'(\zeta)| \leq C,$$

$$|\tilde{\mathcal{P}}(u, v)| \leq C(|u| + |v|) \quad \text{and} \quad |\partial_u \tilde{\mathcal{P}}(u, v)| + |\partial_v \tilde{\mathcal{P}}(u, v)| \leq C,$$

this implies

$$\begin{aligned} |\Lambda_1| &\leq C|\beta(\phi)| |\partial_{yy}\beta(\phi)| \\ &\quad + C|\gamma| |\partial_{xx}\beta(\phi)| + C|\gamma| |\partial_{yy}\beta(\phi)| + C(|\beta(\phi)| + |\gamma|) |\Delta\gamma| \end{aligned}$$

and

$$\begin{aligned} \|\Lambda_1\|_{L^2(\mathbb{R}^2)} &\leq C\|\beta(\phi)\|_{L^\infty(\mathbb{R}^2)} \|\partial_{yy}\beta(\phi)\|_{L^2(\mathbb{R}^2)} + C\|\partial_{xx}\beta(\phi)\|_{L^\infty(\mathbb{R}^2)} \|\gamma\|_{L^2(\mathbb{R}^2)} \\ &\quad + C\|\gamma\|_{L^\infty(\mathbb{R}^2)} \|\partial_{yy}\beta(\phi)\|_{L^2(\mathbb{R}^2)} + C\|\beta(\phi)\|_{L^\infty(\mathbb{R}^2)} \|\Delta\gamma\|_{L^2(\mathbb{R}^2)} \\ &\quad + C\|\gamma\|_{L^\infty(\mathbb{R}^2)} \|\Delta\gamma\|_{L^2(\mathbb{R}^2)} \\ &\leq C\|(\phi, \psi, \mathcal{N})\|_{\mathcal{H}^2} (\|\partial_y\phi\|_{H^2(\mathbb{R})} + \|\psi\|_{H^3(\mathbb{R})} + \|\mathcal{N}\|_{H^3(\mathbb{R}^2)}) \end{aligned}$$

from Lemmas 5.2 and 5.3.

Concerning the gradient term

$$\begin{aligned} |\nabla\Lambda_1| &\leq C|\nabla\beta(\phi)| |\partial_{yy}\beta(\phi)| + C|\beta(\phi)| |\nabla\partial_{yy}\beta(\phi)| + C(|\nabla\beta(\phi)| + |\nabla\gamma|) |\gamma| |\partial_{xx}\beta(\phi)| \\ &\quad + C|\nabla\gamma| |\partial_{xx}\beta(\phi)| + C|\gamma| |\nabla\partial_{xx}\beta(\phi)| + C(|\nabla\beta(\phi)| + |\nabla\gamma|) |\gamma| |\partial_{yy}\beta(\phi)| \\ &\quad + C|\nabla\gamma| |\partial_{yy}\beta(\phi)| + C|\gamma| |\nabla\partial_{yy}\beta(\phi)| + C(|\nabla\beta(\phi)| + |\nabla\gamma|) |\Delta\gamma| \\ &\quad + C(|\beta(\phi)| + |\gamma|) |\nabla\Delta\gamma| \end{aligned}$$

yields

$$\begin{aligned} \|\nabla\Lambda_1\|_{L^2(\mathbb{R}^2)} &\leq C\|\nabla\beta(\phi)\|_{L^4(\mathbb{R}^2)} \|\partial_{yy}\beta(\phi)\|_{L^4(\mathbb{R}^2)} + C\|\beta(\phi)\|_{L^\infty(\mathbb{R}^2)} \|\nabla\partial_{yy}\beta(\phi)\|_{L^2(\mathbb{R}^2)} \\ &\quad + C\|\partial_{xx}\beta(\phi)\|_{L^\infty(\mathbb{R}^2)} (\|\nabla\beta(\phi)\|_{L^4(\mathbb{R}^2)} + \|\nabla\gamma\|_{L^4(\mathbb{R}^2)}) \|\gamma\|_{L^4(\mathbb{R}^2)} \\ &\quad + C\|\partial_{xx}\beta(\phi)\|_{L^\infty(\mathbb{R}^2)} \|\nabla\gamma\|_{L^2(\mathbb{R}^2)} + C\|\gamma\|_{L^\infty(\mathbb{R}^2)} \|\nabla\partial_{xx}\beta(\phi)\|_{L^2(\mathbb{R}^2)} \\ &\quad + C\|\gamma\|_{L^\infty(\mathbb{R}^2)} (\|\nabla\beta(\phi)\|_{L^4(\mathbb{R}^2)} + \|\nabla\gamma\|_{L^4(\mathbb{R}^2)}) \|\partial_{yy}\beta(\phi)\|_{L^4(\mathbb{R}^2)} \\ &\quad + C\|\nabla\gamma\|_{L^4(\mathbb{R}^2)} \|\partial_{yy}\beta(\phi)\|_{L^4(\mathbb{R}^2)} + C(\|\nabla\beta(\phi)\|_{L^4(\mathbb{R}^2)} \\ &\quad + \|\nabla\gamma\|_{L^4(\mathbb{R}^2)}) \|\Delta\gamma\|_{L^4(\mathbb{R}^2)} \\ &\quad + C\|\gamma\|_{L^\infty(\mathbb{R}^2)} \|\nabla\partial_{yy}\beta(\phi)\|_{L^2(\mathbb{R}^2)} + C(\|\beta(\phi)\|_{L^\infty(\mathbb{R}^2)} \\ &\quad + \|\gamma\|_{L^\infty(\mathbb{R}^2)}) \|\nabla\Delta\gamma\|_{L^2(\mathbb{R}^2)} \\ &\leq C\|(\phi, \psi, \mathcal{N})\|_{\mathcal{H}^2} (\|\partial_y\phi\|_{H^2(\mathbb{R})} + \|\psi\|_{H^3(\mathbb{R})} + \|\mathcal{N}\|_{H^3(\mathbb{R}^2)}) \end{aligned}$$

by Lemmas 5.2 and 5.3.

- We recall the term  $\Lambda_2$  is given by

$$\begin{aligned} \Lambda_2 &= 2\mathcal{Q}(\beta(\phi))(\partial_x\beta(\phi), \partial_x\gamma) + \mathcal{Q}(\beta(\phi))(\partial_x\gamma, \partial_x\gamma) \\ &\quad + \tilde{\mathcal{Q}}(\beta(\phi), \gamma)(\gamma)(\partial_x\beta(\phi), \partial_x\beta(\phi)) \\ &\quad + 2\tilde{\mathcal{Q}}(\beta(\phi), \gamma)(\gamma)(\partial_x\beta(\phi), \partial_x\gamma) + \tilde{\mathcal{Q}}(\beta(\phi), \gamma)(\gamma)(\partial_x\gamma, \partial_x\gamma). \end{aligned}$$

In addition, from Proposition (5.2), there exists  $C$  such that for  $|\zeta| \leq \frac{1}{2}$ ,

$$|\mathcal{Q}(\zeta)| \leq C|\zeta|, \quad |\mathcal{Q}'(\zeta)| \leq C,$$

and for  $|u| \leq 1/2$  and  $|v| \leq 1/2$ ,

$$|\mathcal{Q}(u, v)| + |\partial_u \mathcal{Q}(u, v)| + |\partial_v \mathcal{Q}(u, v)| \leq C.$$

We have

$$|\Lambda_2| \leq C|\beta(\phi)| |\partial_x \beta(\phi)| |\partial_x \gamma| + C|\beta(\phi)| |\partial_x \gamma|^2 + C|\gamma| |\partial_x \beta(\phi)|^2 + C|\gamma| |\partial_x \beta(\phi)| |\partial_x \gamma| + C|\gamma| |\partial_x \gamma|^2,$$

using Lemmas 5.2 and 5.3, we obtain

$$\begin{aligned} \|\Lambda_2\|_{L^2(\mathbb{R}^2)} &\leq C\|\partial_x \beta(\phi)\|_{L^4(\mathbb{R}^2)} \|\partial_x \gamma\|_{L^4(\mathbb{R}^2)} + C\|\beta(\phi)\|_{L^\infty(\mathbb{R}^2)} \|\partial_x \gamma\|_{L^4(\mathbb{R}^2)}^2 \\ &\quad + C\|\gamma\|_{L^\infty(\mathbb{R}^2)} \|\partial_x \beta(\phi)\|_{L^4(\mathbb{R}^2)}^2 \\ &\quad + C\|\gamma\|_{L^\infty(\mathbb{R}^2)} \|\partial_x \gamma\|_{L^4(\mathbb{R}^2)} \|\partial_x \beta(\phi)\|_{L^4(\mathbb{R}^2)} \\ &\quad + C\|\gamma\|_{L^\infty(\mathbb{R}^2)} \|\partial_x \gamma\|_{L^4(\mathbb{R}^2)}^2 \\ &\leq C\|(\phi, \psi, \mathcal{N})\|_{\mathcal{H}^2} (\|\partial_y \phi\|_{H^2(\mathbb{R})} + \|\psi\|_{H^3(\mathbb{R})} + \|\mathcal{N}\|_{H^3(\mathbb{R}^2)}). \end{aligned}$$

also,

$$\begin{aligned} |\nabla \Lambda_2| &\leq C|\nabla \beta(\phi)| |\partial_x \gamma| |\partial_x \beta(\phi)| + C|\beta(\phi)| (|\nabla \partial_x \beta(\phi)| |\partial_x \gamma| + |\partial_x \beta(\phi)| |\nabla \partial_x \gamma|) \\ &\quad + C|\nabla \beta(\phi)| |\partial_x \gamma|^2 + C|\beta(\phi)| |\partial_x \gamma| |\nabla \partial_x \gamma| \\ &\quad + C(|\nabla \beta(\phi)| + |\nabla \gamma|) |\gamma| |\partial_x \beta(\phi)|^2 \\ &\quad + C|\nabla \gamma| |\partial_x \beta(\phi)|^2 + C|\gamma| |\partial_x \beta(\phi)| |\nabla \partial_x \beta(\phi)| \\ &\quad + C(|\nabla \beta(\phi)| + |\nabla \gamma|) |\gamma| |\partial_x \beta(\phi)| |\partial_x \gamma| \\ &\quad + C|\nabla \gamma| |\partial_x \beta(\phi)| |\partial_x \gamma| + C|\gamma| (|\nabla \partial_x \beta(\phi)| |\partial_x \gamma| + |\partial_x \beta(\phi)| |\nabla \partial_x \gamma|) \\ &\quad + C(|\nabla \beta(\phi)| + |\nabla \gamma|) |\gamma| |\partial_x \gamma|^2 + C|\nabla \gamma| |\partial_x \gamma|^2 + C|\gamma| |\partial_x \gamma| |\nabla \partial_x \gamma| \end{aligned}$$

gives

$$\begin{aligned} \|\nabla \Lambda_2\|_{L^2(\mathbb{R}^2)} &\leq C\|\partial_{xx} \beta(\phi)\|_{L^\infty(\mathbb{R}^2)} \|\nabla \beta(\phi)\|_{L^4(\mathbb{R}^2)} \|\partial_x \gamma\|_{L^4(\mathbb{R}^2)} \\ &\quad + C\|\beta(\phi)\|_{L^\infty(\mathbb{R}^2)} \|\partial_x \gamma\|_{L^4(\mathbb{R}^2)} \|\nabla \partial_x \gamma\|_{L^4(\mathbb{R}^2)} \\ &\quad + C\|\beta(\phi)\|_{L^\infty(\mathbb{R}^2)} (\|\nabla \partial_x \beta(\phi)\|_{L^4(\mathbb{R}^2)} \|\partial_x \gamma\|_{L^4(\mathbb{R}^2)} \\ &\quad + \|\partial_x \beta(\phi)\|_{L^4(\mathbb{R}^2)} \|\nabla \partial_x \gamma\|_{L^4(\mathbb{R}^2)}) \\ &\quad + C\|\nabla \beta(\phi)\|_{L^4(\mathbb{R}^2)} \|\partial_x \gamma\|_{L^8(\mathbb{R}^2)}^2 + C\|\nabla \gamma\|_{L^4(\mathbb{R}^2)} \|\partial_x \beta(\phi)\|_{L^8(\mathbb{R}^2)}^2 \\ &\quad + C\|\partial_x \beta(\phi)\|_{L^\infty(\mathbb{R}^2)} \|\gamma\|_{L^4(\mathbb{R}^2)} (\|\nabla \beta(\phi)\|_{L^4(\mathbb{R}^2)} + \|\nabla \gamma\|_{L^4(\mathbb{R}^2)}) \\ &\quad + C\|\gamma\|_{L^\infty(\mathbb{R}^2)} \|\partial_x \beta(\phi)\|_{L^4(\mathbb{R}^2)} \|\nabla \partial_x \beta(\phi)\|_{L^4(\mathbb{R}^2)} \\ &\quad + C\|\gamma\|_{L^\infty(\mathbb{R}^2)} \|\partial_x \gamma\|_{L^4(\mathbb{R}^2)} \|\nabla \partial_x \gamma\|_{L^4(\mathbb{R}^2)} \\ &\quad + C\|\nabla \gamma\|_{L^4(\mathbb{R}^2)} \|\partial_x \gamma\|_{L^8(\mathbb{R}^2)}^2 \\ &\quad + C\|\gamma\|_{L^\infty(\mathbb{R}^2)} (\|\nabla \partial_x \beta(\phi)\|_{L^4(\mathbb{R}^2)} \|\partial_x \gamma\|_{L^4(\mathbb{R}^2)} \\ &\quad + \|\partial_x \beta(\phi)\|_{L^4(\mathbb{R}^2)} \|\nabla \partial_x \gamma\|_{L^4(\mathbb{R}^2)}) \\ &\quad + C\|\gamma\|_{L^\infty(\mathbb{R}^2)} \|\partial_x \gamma\|_{L^8(\mathbb{R}^2)}^2 (\|\nabla \beta(\phi)\|_{L^4(\mathbb{R}^2)} + \|\nabla \gamma\|_{L^4(\mathbb{R}^2)}) \\ &\quad + C\|\partial_x \beta(\phi)\|_{L^\infty(\mathbb{R}^2)} \|\nabla \gamma\|_{L^4(\mathbb{R}^2)} \|\partial_x \gamma\|_{L^4(\mathbb{R}^2)} \\ &\quad + C\|\gamma\|_{L^\infty(\mathbb{R}^2)} \|\partial_{xx} \beta(\phi)\|_{L^\infty(\mathbb{R}^2)} \|\partial_x \gamma\|_{L^4(\mathbb{R}^2)} (\|\nabla \beta(\phi)\|_{L^4(\mathbb{R}^2)} \end{aligned}$$

$$\begin{aligned}
 &+ \|\nabla \gamma\|_{L^4(\mathbb{R}^2)} \\
 &\leq C\|(\phi, \psi, \mathcal{N})\|_{\mathcal{H}^2}(\|\partial_y \phi\|_{H^2(\mathbb{R})} + \|\psi\|_{H^3(\mathbb{R})} + \|\mathcal{N}\|_{H^3(\mathbb{R}^2)}).
 \end{aligned}$$

We get the claimed estimate using Lemmas 5.1, 5.2, and 5.3.

- To estimate  $\Lambda_3$ , we recall that:

$$\Lambda_3 = \mathcal{Q}(\xi)(\partial_\sigma \beta(\phi), \partial_\sigma \beta(\phi))|\partial_y \phi|^2 + 2\mathcal{Q}(\xi)(\partial_\sigma \beta(\phi), \partial_y \gamma)(\partial_y \phi) + \mathcal{Q}(\xi)(\partial_y \gamma, \partial_y \gamma)$$

we have for  $|\zeta| \leq 1/2$ ,

$$|\mathcal{Q}(\zeta)| \leq C|\zeta| \quad \text{and} \quad |\mathcal{Q}'(\zeta)| \leq C.$$

Therefore,

$$|\Lambda_3| \leq C(|\beta(\phi)| + |\gamma|)(|\partial_\sigma \beta(\phi)|^2 |\partial_y \phi|^2 + |\partial_\sigma \beta(\phi)| |\partial_y \gamma| |\partial_y \phi| + |\partial_y \gamma|^2)$$

so that

$$\begin{aligned}
 \|\Lambda_3\|_{L^2(\mathbb{R}^2)} &\leq C(\|\beta(\phi)\|_{L^\infty(\mathbb{R}^2)} + \|\gamma\|_{L^\infty(\mathbb{R}^2)})(\|\partial_y \phi\|_{L^4(\mathbb{R})}^2 + \|\partial_y \phi\|_{L^4(\mathbb{R})} \|\partial_y \gamma\|_{L^4(\mathbb{R}^2)} \\
 &\quad + \|\partial_y \gamma\|_{L^4(\mathbb{R}^2)}^2) \\
 &\leq C\|(\phi, \psi, \mathcal{N})\|_{\mathcal{H}^2}(\|\partial_y \phi\|_{H^2(\mathbb{R})} + \|\psi\|_{H^3(\mathbb{R})} + \|\mathcal{N}\|_{H^3(\mathbb{R}^2)}).
 \end{aligned}$$

and

$$\begin{aligned}
 |\nabla \Lambda_3| &\leq C(|\nabla \beta(\phi)| + |\nabla \gamma|)|\partial_\sigma \beta(\phi)|^2 |\partial_y \phi|^2 + C|\partial_\sigma \beta(\phi)| |\nabla \partial_\sigma \beta(\phi)| |\partial_y \phi|^2 \\
 &\quad + C|\partial_\sigma \beta(\phi)|^2 |\partial_y \phi| |\partial_{yy} \phi| + C(|\nabla \beta(\phi)| + |\nabla \gamma|)|\partial_\sigma \beta(\phi)| |\partial_y \gamma| |\partial_y \phi| \\
 &\quad + C(|\nabla \partial_\sigma \beta(\phi)| |\partial_y \gamma| + |\partial_\sigma \beta(\phi)| |\nabla \partial_y \gamma|) |\partial_y \gamma| + C|\partial_y \gamma| |\nabla \partial_y \gamma| \\
 &\quad + C|\partial_\sigma \beta(\phi)| |\partial_y \gamma| |\partial_{yy} \phi| + C(|\nabla \beta(\phi)| + |\nabla \gamma|) |\partial_y \gamma|^2
 \end{aligned}$$

yields

$$\begin{aligned}
 \|\nabla \Lambda_3\|_{L^2(\mathbb{R}^2)} &\leq C(\|\nabla \beta(\phi)\|_{L^4(\mathbb{R}^2)} + \|\nabla \gamma\|_{L^4(\mathbb{R}^2)}) \|\partial_y \phi\|_{L^8(\mathbb{R})}^2 \\
 &\quad + C(\|\partial_y \phi\|_{L^4(\mathbb{R})}^2 + \|\partial_y \phi\|_{L^6(\mathbb{R})}^3) \\
 &\quad + C\|\partial_y \phi\|_{L^4(\mathbb{R})} \|\partial_{yy} \phi\|_{L^4(\mathbb{R})} + C(\|\nabla \beta(\phi)\|_{L^4(\mathbb{R}^2)} \\
 &\quad + \|\nabla \gamma\|_{L^4(\mathbb{R}^2)}) \|\partial_y \gamma\|_{L^\infty(\mathbb{R}^2)} \|\partial_y \phi\|_{L^4(\mathbb{R})} \\
 &\quad + C(\|\nabla \beta(\phi)\|_{L^2(\mathbb{R}^2)} + \|\nabla \gamma\|_{L^2(\mathbb{R}^2)}) \|\partial_y \gamma\|_{L^\infty(\mathbb{R}^2)} \\
 &\quad + C\|\partial_y \phi\|_{L^\infty(\mathbb{R})} \|\nabla \partial_y \gamma\|_{L^2(\mathbb{R}^2)} \\
 &\quad + C\|\partial_y \gamma\|_{L^\infty(\mathbb{R}^2)} (\|\partial_y \phi\|_{L^2(\mathbb{R})} + \|\partial_y \phi\|_{L^4(\mathbb{R})}^2 + \|\partial_{yy} \phi\|_{L^2(\mathbb{R})} \\
 &\quad + \|\nabla \partial_y \gamma\|_{L^2(\mathbb{R}^2)}) \\
 &\leq C\|(\phi, \psi, \mathcal{N})\|_{\mathcal{H}^2}(\|\partial_y \phi\|_{H^2(\mathbb{R})} + \|\psi\|_{H^3(\mathbb{R})} \\
 &\quad + \|\mathcal{N}\|_{H^3(\mathbb{R}^2)}).
 \end{aligned}$$

Using the Sobolev embedding of  $H^2(\mathbb{R})$  into  $L^\infty(\mathbb{R})$  and Lemmas 5.1, 5.2 and 5.3, we obtain the required estimate.

- We have

$$\Lambda_4 = \mathcal{R}(x, \beta(\phi))(\partial_x \gamma) + \tilde{\mathcal{R}}(x, \beta(\phi), \gamma)(\gamma)(\partial_x \beta(\phi)) + \tilde{\mathcal{R}}(x, \beta(\phi), \gamma)(\gamma)(\partial_x \gamma).$$

from Proposition (5.2), we have for  $|\zeta| \leq 1/2$ ,

$$|\mathcal{R}(x, \zeta)| + |\partial_x \mathcal{R}(x, \zeta)| \leq \frac{C}{\text{ch } x} |\zeta|,$$

and

$$|\partial_\zeta \mathcal{R}(x, \zeta)| + |\partial_x \partial_\zeta \mathcal{R}(x, \zeta)| + |\partial_{\zeta \zeta} \mathcal{R}(x, \zeta)| \leq \frac{C}{\text{ch } x},$$

for  $|u| \leq 1/2$  and  $|v| \leq 1/2$ ,

$$|\tilde{\mathcal{R}}(x, u, v)| + |\partial_u \tilde{\mathcal{R}}(x, u, v)| + |\partial_v \tilde{\mathcal{R}}(x, u, v)| \leq \frac{C}{\text{ch } x}.$$

Thus

$$|\Lambda_4| \leq \frac{C}{\text{ch } x} (|\beta(\phi)| |\partial_x \gamma| + |\gamma| |\partial_x \beta(\phi)| + |\gamma| |\partial_x \gamma|)$$

implies

$$\begin{aligned} \|\Lambda_4\|_{L^2(\mathbb{R}^2)} &\leq C \|\phi\|_{L^4(\mathbb{R})} (\|\partial_x \gamma\|_{L^4(\mathbb{R}^2)} + \|\gamma\|_{L^4(\mathbb{R}^2)}) + C \|\gamma\|_{L^\infty(\mathbb{R}^2)} \|\partial_x \gamma\|_{L^2(\mathbb{R}^2)} \\ &\leq C \|(\phi, \psi, \mathcal{N})\|_{\mathcal{H}^2} (\|\partial_y \phi\|_{H^2(\mathbb{R})} + \|\psi\|_{H^3(\mathbb{R})} + \|\mathcal{N}\|_{H^3(\mathbb{R}^2)}) \end{aligned}$$

and

$$\begin{aligned} |\nabla \Lambda_4| &\leq \frac{C}{\text{ch } x} (|\beta(\phi)| |\partial_x \gamma| + |\nabla \beta(\phi)| |\partial_x \gamma| + |\beta(\phi)| |\nabla \partial_x \gamma|) \\ &\quad + \frac{C}{\text{ch } x} (|\beta(\phi)| |\gamma| |\partial_x \gamma| + |\nabla \beta(\phi)| |\gamma| |\partial_x \gamma| \\ &\quad + |\nabla \gamma| |\gamma| |\partial_x \gamma| + |\nabla \gamma| |\partial_x \gamma| + |\gamma| |\nabla \partial_x \gamma|) \\ &\quad + \frac{C}{\text{ch } x} (|\beta(\phi)| |\gamma| |\partial_x \beta(\phi)| \\ &\quad + |\nabla \beta(\phi)| |\gamma| |\partial_x \beta(\phi)| + |\nabla \gamma| |\gamma| |\partial_x \beta(\phi)| \\ &\quad + |\nabla \gamma| |\partial_x \beta(\phi)| + |\gamma| |\nabla \partial_x \beta(\phi)|) \end{aligned}$$

gives

$$\begin{aligned} \|\nabla \Lambda_4\|_{L^2(\mathbb{R}^2)} &\leq C \|\partial_x \gamma\|_{L^4(\mathbb{R}^2)} (\|\phi\|_{L^4(\mathbb{R})} \\ &\quad + \|\nabla \beta(\phi)\|_{L^4(\mathbb{R})}) + C \|\phi\|_{L^\infty(\mathbb{R})} \|\nabla \partial_x \gamma\|_{L^2(\mathbb{R}^2)} \\ &\quad + C \|\beta(\phi)\|_{L^\infty(\mathbb{R})} \|\phi\|_{L^4(\mathbb{R})} \|\gamma\|_{L^4(\mathbb{R}^2)} + C \|\nabla \partial_x \beta(\phi)\|_{L^4(\mathbb{R})} \|\gamma\|_{L^4(\mathbb{R}^2)} \\ &\quad + C \|\gamma\|_{L^\infty(\mathbb{R}^2)} \|\phi\|_{L^4(\mathbb{R})} (\|\nabla \beta(\phi)\|_{L^4(\mathbb{R})} + \|\nabla \gamma\|_{L^4(\mathbb{R}^2)}) \\ &\quad + C \|\gamma\|_{L^\infty(\mathbb{R}^2)} \|\phi\|_{L^4(\mathbb{R})} \|\partial_x \gamma\|_{L^4(\mathbb{R}^2)} + C \|\nabla \gamma\|_{L^4(\mathbb{R}^2)} \|\partial_x \gamma\|_{L^4(\mathbb{R}^2)} \\ &\quad + C \|\gamma\|_{L^\infty(\mathbb{R}^2)} \|\partial_x \gamma\|_{L^4(\mathbb{R}^2)} (\|\nabla \beta(\phi)\|_{L^4(\mathbb{R})} + \|\nabla \gamma\|_{L^4(\mathbb{R}^2)}) \\ &\quad + C \|\phi\|_{L^4(\mathbb{R})} \|\nabla \gamma\|_{L^4(\mathbb{R}^2)} + C \|\gamma\|_{L^\infty(\mathbb{R}^2)} \|\nabla \partial_x \gamma\|_{L^2(\mathbb{R}^2)} \\ &\leq C \|(\phi, \psi, \mathcal{N})\|_{\mathcal{H}^2} (\|\partial_y \phi\|_{H^2(\mathbb{R})} + \|\psi\|_{H^3(\mathbb{R})} + \|\mathcal{N}\|_{H^3(\mathbb{R}^2)}). \end{aligned}$$

Using Lemmas 5.2 and 5.3.

- The last term  $\Lambda_5$  is given by

$$\Lambda_5 = \tilde{\mathcal{S}}(x, \beta(\phi), \gamma)(\gamma).$$

There exists  $C$  such that for  $u$  and  $v$  in  $B(0, 1/2)$ , we have

$$\begin{aligned} |\tilde{\mathcal{S}}(x, u, v)| + |\partial_x \tilde{\mathcal{S}}(x, u, v)| &\leq C(|u| + |v|) \\ \text{and } |\partial_u \tilde{\mathcal{S}}(x, u, v)| + |\partial_v \tilde{\mathcal{S}}(x, u, v)| &\leq C. \end{aligned}$$

Therefore,

$$|\Lambda_5| \leq C(|\beta(\phi)| + |\gamma|)|\gamma|$$

yields

$$\begin{aligned} \|\Lambda_5\|_{L^2(\mathbb{R}^2)} &\leq C(\|\beta(\phi)\|_{L^\infty(\mathbb{R}^2)} + \|\gamma\|_{L^\infty(\mathbb{R}^2)})\|\gamma\|_{L^2(\mathbb{R}^2)} \\ &\leq C\|(\phi, \psi, \mathcal{N})\|_{\mathcal{H}^2}(\|\partial_y\phi\|_{H^2(\mathbb{R})} + \|\psi\|_{H^3(\mathbb{R})} + \|\mathcal{N}\|_{H^3(\mathbb{R}^2)}) \end{aligned}$$

also

$$|\nabla\Lambda_5| \leq C(|\beta(\phi)| + |\gamma|)(|\gamma| + |\nabla\gamma|) + C|\gamma|(|\nabla\beta(\phi)| + |\nabla\gamma|).$$

Using Lemma 5.2 and 5.3, we obtain

$$\begin{aligned} \|\nabla\Lambda_5\|_{L^2(\mathbb{R}^2)} &\leq C(\|\beta(\phi)\|_{L^\infty(\mathbb{R}^2)} + \|\gamma\|_{L^\infty(\mathbb{R}^2)})(\|\gamma\|_{L^2(\mathbb{R}^2)} + \|\nabla\gamma\|_{L^2(\mathbb{R}^2)}) \\ &\quad + C\|\gamma\|_{L^\infty(\mathbb{R}^2)}(\|\nabla\beta(\phi)\|_{L^2(\mathbb{R}^2)} + \|\nabla\gamma\|_{L^2(\mathbb{R}^2)}) \\ &\leq C\|(\phi, \psi, \mathcal{N})\|_{\mathcal{H}^2}(\|\partial_y\phi\|_{H^2(\mathbb{R})} + \|\psi\|_{H^3(\mathbb{R})} + \|\mathcal{N}\|_{H^3(\mathbb{R}^2)}). \end{aligned}$$

This concludes the proof of Proposition 5.3.

□

To fill up the proof of estimate (5.1), we use the boundedness of the linear transformations  $f$  and  $g$  and remark that for  $s \in \mathbb{N}$ , there exists  $C$  such that if  $u \in H^s(\mathbb{R}^2; \mathbb{R}^2)$  then  $\ell^i(u) \in H^s(\mathbb{R})$  ( $i = 1, 2$ ), which leads to

$$\|\ell^i(u)\|_{H^s(\mathbb{R})} \leq C\|u\|_{H^s(\mathbb{R}^2)}.$$

This estimate together with Proposition 5.3 establish the desired estimates on  $\mathcal{F}_1$  and  $\mathcal{F}_2$ . We obtain the claimed estimate on  $\mathcal{F}_3$  using (4.14).

### 5.1 Proof of estimate (5.2) and (5.3)

From (4.10) and (4.12), we recall that

$$\begin{aligned} \mathcal{F}_1(\phi, \psi, \mathcal{N}) &= v(\phi) (\partial_{yy}\psi + 2\psi) + \frac{\mathcal{B}(\phi)}{\mathcal{A}(\phi)}|\partial_y\phi|^2 + \frac{1}{\mathcal{A}(\phi)}f(\Lambda), \\ \mathcal{F}_2(\phi, \psi, \mathcal{N}) &= (1 - \mathcal{A}(\phi)) \partial_{yy}\phi - \mathcal{B}(\phi)|\partial_y\phi|^2 + g(\Lambda), \end{aligned}$$

where  $v(\sigma) = O(\sigma)$ ,  $\mathcal{A}(\sigma) = 1 + O(\sigma)$  and  $\mathcal{B}(\sigma) = O(\sigma)$ .

Our aim is to split  $\mathcal{F}_1 - \mathcal{F}_2$  on the form:  $\mathcal{F}_1 - \mathcal{F}_2 = \tilde{\mathcal{F}}_a + \tilde{\mathcal{F}}_b$ , where  $\tilde{\mathcal{F}}_a$  and  $\tilde{\mathcal{F}}_b$  satisfy the estimates (5.2) and (5.3), respectively. We do this splitting in the following manner:

We define  $\tilde{\mathcal{F}}_a$  and  $\tilde{\mathcal{F}}_b$  as,

$$\tilde{\mathcal{F}}_a = \tilde{\mathcal{F}}_a^1 + \tilde{\mathcal{F}}_a^2 \quad \text{and} \quad \tilde{\mathcal{F}}_b = \tilde{\mathcal{F}}_b^1 + \tilde{\mathcal{F}}_b^2.$$

We denote by:

$$\begin{aligned} \tilde{\mathcal{F}}_a^1 &= \left( \frac{\mathcal{B}(\phi)}{\mathcal{A}(\phi)} + \mathcal{B}(\phi) \right) |\partial_y\phi|^2 \quad \text{and} \quad \tilde{\mathcal{F}}_b^1 = v(\phi)(\partial_{yy}\psi + 2\psi) - (1 - \mathcal{A}(\phi)) \partial_{yy}\phi, \\ \tilde{\mathcal{F}}_a^2 &= \frac{1}{\mathcal{A}(\phi)}f(\Lambda^a) - g(\Lambda^a) \quad \text{and} \quad \tilde{\mathcal{F}}_b^2 = \frac{1}{\mathcal{A}(\phi)}f(\Lambda^b) - g(\Lambda^b), \end{aligned}$$

where we split  $\Lambda$  on the form  $\Lambda = \Lambda^a + \Lambda^b$ .

Using  $\mathcal{A}(\sigma) = 1 + O(\sigma)$  and  $\mathcal{B}(\sigma) = O(\sigma)$ , we obtain

$$\begin{aligned} \|\tilde{\mathcal{F}}_a^1\|_{L^1(\mathbb{R})} &\leq C\|\phi\|_{L^\infty(\mathbb{R})}\|\partial_y\phi\|_{L^2(\mathbb{R})}^2 \\ &\leq C(\|\partial_y\phi\|_{H^2(\mathbb{R})} + \|\psi\|_{H^3(\mathbb{R})} + \|\mathcal{N}\|_{H^3(\mathbb{R}^2)})^2, \end{aligned}$$

we have

$$|\tilde{\mathcal{F}}_b^1| \leq C|\phi|\partial_{yy}\psi + 2\psi| + C|\phi|\partial_{yy}\phi|,$$

implies

$$\begin{aligned} \|\tilde{\mathcal{F}}_b^1\|_{L^{4/3}(\mathbb{R})} &\leq C\|\phi\|_{L^4(\mathbb{R})}(\|\partial_{yy}\psi + 2\psi\|_{L^2(\mathbb{R})} + \|\partial_{yy}\phi\|_{L^2(\mathbb{R})}) \\ &\leq C(\|\partial_y\phi\|_{H^2(\mathbb{R})} + \|\psi\|_{H^3(\mathbb{R})} + \|\mathcal{N}\|_{H^3(\mathbb{R}^2)})\|\phi\|_{L^4(\mathbb{R})}. \end{aligned}$$

In order to get the expected estimates of  $\tilde{\mathcal{F}}_a^2$  and  $\tilde{\mathcal{F}}_b^2$ , we will split  $\Lambda$  on the form  $\Lambda = \Lambda^a + \Lambda^b$ . We recall the expression of  $\Lambda$  in (4.8) and describe this splitting for each term  $\Lambda_i (i = 1, \dots, 5)$ .

- We recall that

$$\begin{aligned} \Lambda_1 &= \mathcal{P}(\beta(\phi))\partial_{yy}\beta(\phi) + \tilde{\mathcal{P}}(\beta(\phi), \gamma)(\gamma)(\partial_{xx}\beta(\phi)) \\ &\quad + \tilde{\mathcal{P}}(\beta(\phi), \gamma)(\gamma)(\partial_{yy}\beta(\phi)) + \mathcal{P}(\beta(\phi) + \gamma)\Delta\gamma, \end{aligned}$$

notice that

$$\begin{aligned} \partial_{yy}\beta(\phi) &= \partial_\sigma\beta(\phi)\partial_{yy}\phi + \partial_{\sigma\sigma}\beta(\phi)|\partial_y\phi|^2 \text{ and } \mathcal{P}(\xi) = \mathcal{P}(\beta(\phi) + \gamma) \\ &= \mathcal{P}(\beta(\phi)) + \tilde{\mathcal{P}}(\beta(\phi), \gamma)(\gamma), \end{aligned}$$

where

$$\tilde{\mathcal{P}}(u, v) = \int_0^1 \mathcal{P}'(u + sv)ds.$$

We define the decomposition of  $\Lambda_1$  as  $\Lambda_1 = \Lambda_1^a + \Lambda_1^b$  with:

$$\begin{aligned} \Lambda_1^a &= \mathcal{P}(\beta(\phi))(\partial_{\sigma\sigma}\beta(\phi)|\partial_y\phi|^2) + \tilde{\mathcal{P}}(\beta(\phi), \gamma)(\gamma)(\partial_{xx}\beta(\phi)) \\ &\quad + \tilde{\mathcal{P}}(\beta(\phi), \gamma)(\gamma)(\partial_\sigma\beta(\phi)\partial_{yy}\phi) \\ &\quad + \tilde{\mathcal{P}}(\beta(\phi), \gamma)(\gamma)(\partial_{\sigma\sigma}\beta(\phi)|\partial_y\phi|^2) + \tilde{\mathcal{P}}(\beta(\phi), \gamma)(\gamma)(\Delta\gamma), \\ \Lambda_1^b &= \mathcal{P}(\beta(\phi))(\partial_\sigma\beta(\phi)\partial_{yy}\phi) + \mathcal{P}(\beta(\phi))(\Delta\gamma). \end{aligned}$$

We have

$$\begin{aligned} |\Lambda_1^a| &\leq \frac{C}{\text{ch}^2x}|\phi|\partial_y\phi|^2 + \frac{C}{\text{ch}x}|\gamma|\phi| + \frac{C}{\text{ch}x}|\gamma|\partial_{yy}\phi| \\ &\quad + \frac{C}{\text{ch}x}|\gamma|\partial_y\phi|^2 + C|\gamma|\Delta\gamma|, \end{aligned}$$

Using Lemma 5.1, we get

$$\begin{aligned} \|\Lambda_1^a\|_{L^1(\mathbb{R}^2)} &\leq C(\|\phi\|_{L^2(\mathbb{R})} + \|\gamma\|_{L^2(\mathbb{R}^2)})\|\partial_{yy}\phi\|_{L^2(\mathbb{R})} \\ &\quad + C(\|\phi\|_{L^2(\mathbb{R})} + \|\Delta\gamma\|_{L^2(\mathbb{R}^2)})\|\gamma\|_{L^2(\mathbb{R}^2)} \\ &\leq C(\|\partial_y\phi\|_{H^2(\mathbb{R})} + \|\psi\|_{H^3(\mathbb{R})} + \|\mathcal{N}\|_{H^3(\mathbb{R}^2)})^2. \end{aligned}$$

Furthermore,

$$|\Lambda_1^b| \leq \frac{C}{\text{ch}^2x}|\phi|\partial_{yy}\phi| + \frac{C}{\text{ch}x}|\phi|\Delta\gamma|,$$

and thus,

$$\|\Lambda_1^b\|_{L^{4/3}(\mathbb{R}^2)} \leq C(\|\partial_y \phi\|_{H^2(\mathbb{R})} + \|\psi\|_{H^3(\mathbb{R})} + \|\mathcal{N}\|_{H^3(\mathbb{R}^2)})\|\phi\|_{L^4(\mathbb{R})}.$$

- The decomposition of  $\Lambda_2$  is the following:  $\Lambda_2 = \Lambda_2^a + \Lambda_2^b$ , where

$$\begin{aligned} \Lambda_2^a &= \mathcal{Q}(\beta(\phi))(\partial_x \gamma, \partial_x \gamma) \\ &\quad + 2\tilde{\mathcal{Q}}(\beta(\phi), \gamma)(\gamma)(\partial_x \beta(\phi), \partial_x \gamma) + \tilde{\mathcal{Q}}(\beta(\phi), \gamma)(\gamma)(\partial_x \gamma, \partial_x \gamma), \\ \Lambda_2^b &= 2\mathcal{Q}(\beta(\phi))(\partial_x \beta(\phi), \partial_x \gamma) + \tilde{\mathcal{Q}}(\beta(\phi), \gamma)(\gamma)(\partial_x \beta(\phi), \partial_x \beta(\phi)) \end{aligned}$$

yields

$$|\Lambda_2^a| \leq \frac{C}{\text{ch } x} |\phi| |\partial_x \gamma|^2 + \frac{C}{\text{ch } x} |\phi| |\gamma| |\partial_x \gamma| + C |\gamma| |\partial_x \gamma|^2,$$

and

$$\begin{aligned} \|\Lambda_2^a\|_{L^1(\mathbb{R}^2)} &\leq C(\|\phi\|_{L^2(\mathbb{R})} + \|\gamma\|_{L^2(\mathbb{R}^2)})\|\partial_x \gamma\|_{L^4(\mathbb{R}^2)}^2 \\ &\quad + C\|\gamma\|_{L^\infty(\mathbb{R}^2)}\|\phi\|_{L^2(\mathbb{R})}\|\partial_x \gamma\|_{L^2(\mathbb{R}^2)} \\ &\leq C(\|\partial_y \phi\|_{H^2(\mathbb{R})} + \|\psi\|_{H^3(\mathbb{R})} + \|\mathcal{N}\|_{H^3(\mathbb{R}^2)})^2. \end{aligned}$$

On the other hand,

$$|\Lambda_2^b| \leq \frac{C}{\text{ch } x} |\phi| (|\partial_x \gamma| + |\gamma|),$$

implies

$$\begin{aligned} \|\Lambda_2^b\|_{L^{4/3}(\mathbb{R}^2)} &\leq C\|\gamma\|_{H^1(\mathbb{R}^2)}\|\phi\|_{L^4(\mathbb{R})} \\ &\leq C(\|\partial_y \phi\|_{H^2(\mathbb{R})} + \|\psi\|_{H^3(\mathbb{R})} + \|\mathcal{N}\|_{H^3(\mathbb{R}^2)})\|\phi\|_{L^4(\mathbb{R})}. \end{aligned}$$

- Concerning  $\Lambda_3$ , we recall:

$$\Lambda_3 = \mathcal{Q}(\xi)(\partial_\sigma \beta(\phi), \partial_\sigma \beta(\phi))|\partial_y \phi|^2 + 2\mathcal{Q}(\xi)(\partial_\sigma \beta(\phi), \partial_y \gamma)(\partial_y \phi) + \mathcal{Q}(\xi)(\partial_y \gamma, \partial_y \gamma).$$

Splitting of  $\Lambda_3$  is as follows:

$$\begin{aligned} \Lambda_3^a &= \tilde{\mathcal{Q}}(\beta(\phi), \gamma)(\gamma)(\partial_\sigma \beta(\phi), \partial_\sigma \beta(\phi))|\partial_y \phi|^2 + 2\tilde{\mathcal{Q}}(\beta(\phi), \gamma)(\gamma)(\partial_\sigma \beta(\phi), \partial_y \gamma)(\partial_y \phi), \\ &\quad + \tilde{\mathcal{Q}}(\beta(\phi), \gamma)(\gamma)(\partial_y \gamma, \partial_y \gamma), \\ \Lambda_3^b &= \mathcal{Q}(\beta(\phi))(\partial_\sigma \beta(\phi), \partial_\sigma \beta(\phi))|\partial_y \phi|^2 + 2\mathcal{Q}(\beta(\phi))(\partial_\sigma \beta(\phi), \partial_y \gamma)(\partial_y \phi) \\ &\quad + \mathcal{Q}(\beta(\phi))(\partial_y \gamma, \partial_y \gamma), \end{aligned}$$

we have

$$|\Lambda_3^a| \leq \frac{C}{\text{ch}^2 x} |\gamma| |\partial_y \phi|^2 + \frac{C}{\text{ch } x} |\gamma| |\partial_y \phi| |\partial_y \gamma| + C |\gamma| |\partial_y \gamma|^2,$$

and thus,

$$\begin{aligned} \|\Lambda_3^a\|_{L^1(\mathbb{R}^2)} &\leq C\|\gamma\|_{L^2(\mathbb{R}^2)}\|\partial_y \phi\|_{L^4(\mathbb{R})}^2 \\ &\quad + C\|\gamma\|_{L^\infty(\mathbb{R}^2)}\|\partial_y \gamma\|_{L^2(\mathbb{R}^2)}(\|\partial_y \phi\|_{L^2(\mathbb{R})} + \|\partial_y \gamma\|_{L^2(\mathbb{R}^2)}) \\ &\leq C(\|\partial_y \phi\|_{H^2(\mathbb{R})} + \|\psi\|_{H^3(\mathbb{R})} + \|\mathcal{N}\|_{H^3(\mathbb{R}^2)})^2. \end{aligned}$$

In addition,

$$|\Lambda_3^b| \leq \frac{C}{\text{ch}^3 x} |\phi| |\partial_y \phi|^2 + \frac{C}{\text{ch}^2 x} |\phi| |\partial_y \phi| |\partial_y \gamma| + \frac{C}{\text{ch } x} |\phi| |\partial_y \gamma|^2$$

gives

$$\begin{aligned} \|\Lambda_3^b\|_{L^{4/3}(\mathbb{R}^2)} &\leq C(\|\partial_y\phi\|_{L^4(\mathbb{R})}^2 + \|\partial_y\phi\|_{L^\infty(\mathbb{R})}\|\partial_y\gamma\|_{L^2(\mathbb{R}^2)} + \|\partial_y\gamma\|_{L^4(\mathbb{R}^2)}^2)\|\phi\|_{L^4(\mathbb{R})} \\ &\leq C(\|\partial_y\phi\|_{H^2(\mathbb{R})} + \|\psi\|_{H^3(\mathbb{R})} + \|\mathcal{N}\|_{H^3(\mathbb{R}^2)})\|\phi\|_{L^4(\mathbb{R})}. \end{aligned}$$

- We define the decomposition of  $\Lambda_4$  as

$$\begin{aligned} \Lambda_4^a &= \tilde{\mathcal{R}}(x, \beta(\phi), \gamma)(\gamma)(\partial_x\gamma), \\ \Lambda_4^b &= \mathcal{R}(x, \beta(\phi))(\partial_x\gamma) + \tilde{\mathcal{R}}(x, \beta(\phi), \gamma)(\gamma)(\partial_x\beta(\phi)). \end{aligned}$$

Therefore,

$$\begin{aligned} \|\Lambda_4^a\|_{L^1(\mathbb{R}^2)} &\leq C\|\gamma\|_{L^2(\mathbb{R}^2)}\|\partial_x\gamma\|_{L^2(\mathbb{R}^2)} \\ &\leq C(\|\partial_y\phi\|_{H^2(\mathbb{R})} + \|\psi\|_{H^3(\mathbb{R})} + \|\mathcal{N}\|_{H^3(\mathbb{R}^2)})^2 \end{aligned}$$

and

$$\begin{aligned} \|\Lambda_4^b\|_{L^{4/3}(\mathbb{R}^2)} &\leq C(\|\partial_x\gamma\|_{L^2(\mathbb{R}^2)} + \|\gamma\|_{L^2(\mathbb{R}^2)})\|\phi\|_{L^4(\mathbb{R})} \\ &\leq C(\|\partial_y\phi\|_{H^2(\mathbb{R})} + \|\psi\|_{H^3(\mathbb{R})} + \|\mathcal{N}\|_{H^3(\mathbb{R}^2)})\|\phi\|_{L^4(\mathbb{R})}. \end{aligned}$$

- The last term  $\Lambda_5$  is given by

$$\Lambda_5 = \tilde{\mathcal{S}}(x, \beta(\phi), \gamma)(\gamma).$$

Using the Taylor expansion, we obtain

$$\tilde{\mathcal{S}}(x, \beta(\phi), \gamma)(\gamma) = \partial_\zeta \tilde{\mathcal{S}}(x, \beta(\phi), \gamma)(\gamma, \gamma),$$

where

$$\tilde{\mathcal{S}}(x, u, v) = \frac{1}{2} \int_0^1 (1-s) \partial_{\zeta\zeta} \tilde{\mathcal{S}}(x, u + sv) ds.$$

Decomposition of  $\Lambda_5$  is the following:

$$\Lambda_5^a = \tilde{\mathcal{S}}(x, \beta(\phi), \gamma)(\gamma, \gamma) \text{ and } \Lambda_5^b = \partial_\zeta \tilde{\mathcal{S}}(x, \beta(\phi), \gamma)(\gamma).$$

yields

$$\|\Lambda_5^a\|_{L^1(\mathbb{R}^2)} \leq C\|\gamma\|_{L^2(\mathbb{R}^2)}^2 \leq C(\|\partial_y\phi\|_{H^2(\mathbb{R})} + \|\psi\|_{H^3(\mathbb{R})} + \|\mathcal{N}\|_{H^3(\mathbb{R}^2)})^2$$

and

$$\begin{aligned} \|\Lambda_5^b\|_{L^{4/3}(\mathbb{R}^2)} &\leq C\|\gamma\|_{L^2(\mathbb{R}^2)}\|\phi\|_{L^4(\mathbb{R})} \\ &\leq C(\|\partial_y\phi\|_{H^2(\mathbb{R})} + \|\psi\|_{H^3(\mathbb{R})} + \|\mathcal{N}\|_{H^3(\mathbb{R}^2)})\|\phi\|_{L^4(\mathbb{R})}. \end{aligned}$$

Denoting  $\Lambda^a = \sum_{i=1}^5 \Lambda_i^a$  and  $\Lambda^b = \sum_{i=1}^5 \Lambda_i^b$ , we have obtained that  $\Lambda = \Lambda^a + \Lambda^b$ , with

$$\|\Lambda^a\|_{L^1(\mathbb{R}^2)} \leq C(\|\partial_y\phi\|_{H^2(\mathbb{R})} + \|\psi\|_{H^3(\mathbb{R})} + \|\mathcal{N}\|_{H^3(\mathbb{R}^2)})^2, \tag{5.4}$$

$$\|\Lambda^b\|_{L^{4/3}(\mathbb{R}^2)} \leq C(\|\partial_y\phi\|_{H^2(\mathbb{R})} + \|\psi\|_{H^3(\mathbb{R})} + \|\mathcal{N}\|_{H^3(\mathbb{R}^2)})\|\phi\|_{L^4(\mathbb{R})}. \tag{5.5}$$

We use the boundedness of the linear transformations  $f$  and  $g$  together with (5.4) and (5.5), we obtain



$$\begin{aligned} \|\tilde{\mathcal{F}}_a^2\|_{L^1(\mathbb{R})} &\leq C(\|\partial_y\phi\|_{H^2(\mathbb{R})} + \|\psi\|_{H^3(\mathbb{R})} + \|\mathcal{N}\|_{H^3(\mathbb{R}^2)})^2, \\ \|\tilde{\mathcal{F}}_b^2\|_{L^{4/3}(\mathbb{R})} &\leq C(\|\partial_y\phi\|_{H^2(\mathbb{R})} + \|\psi\|_{H^3(\mathbb{R})} + \|\mathcal{N}\|_{H^3(\mathbb{R}^2)})\|\phi\|_{L^4(\mathbb{R})}. \end{aligned}$$

With this, we have obtained the expected decomposition. This concludes the proof of Proposition 5.1.  $\square$

### 6 Proof of Theorem 1.1

We proof our main result using the variational estimates. We recall that in new coordinates, we deal with the following system:

$$\partial_t\phi = \partial_{yy}\phi + (\partial_{yy}\psi + 2\psi) + \mathcal{F}_1(\phi, \psi, \mathcal{N}), \tag{6.1}$$

$$\partial_t\psi = -\partial_{yy}\phi + (\partial_{yy}\psi + 2\psi) + \mathcal{F}_2(\phi, \psi, \mathcal{N}), \tag{6.2}$$

$$\partial_t\mathcal{N} = \begin{pmatrix} -T\mathcal{N}_1 - (T-2)\mathcal{N}_2 \\ T\mathcal{N}_1 - (T-2)\mathcal{N}_2 \end{pmatrix} + \mathcal{F}_3(x, \phi, \psi, \mathcal{N}). \tag{6.3}$$

The unknown  $(\phi, \psi, \mathcal{N}_1, \mathcal{N}_2) \in \Pi$  defined in (4.2). The nonlinear terms  $\mathcal{F}_1, \mathcal{F}_2$  and  $\mathcal{F}_3$  are defined in (4.10), (4.12) and (4.14), respectively.

By variational estimates, we prove that if the initial data are small then the solution of (6.1)–(6.2)–(6.3) remains small which is essentially the stability result for the transformed equation. We use energy estimate technique to absorb the linear and nonlinear terms. When we multiply the equation by the unknowns or their space derivatives, the linear part shows good sign absorbing terms. To estimate the nonlinear terms, we have to control them by the absorbing terms. Before moving to the variational estimates, we establish a Sobolev-type inequality in the following lemma:

**Lemma 6.1** *There exist a constant  $C$  such that for all  $u \in H^2(\mathbb{R})$ ,*

$$\|u\|_{L^4(\mathbb{R})} \leq C\|u\|_{L^2(\mathbb{R})}^{1/2}\|u'\|_{L^2(\mathbb{R})}^{1/2}, \quad \text{and} \quad \|u\|_{L^4(\mathbb{R})} \leq C\|u\|_{L^2(\mathbb{R})}^{1/2}\|u\|_{H^1(\mathbb{R})}^{1/2}.$$

*Proof* From Sobolev embedding,  $W^{1,1}(\mathbb{R}) \hookrightarrow L^2(\mathbb{R})$  and there exists  $C$  such that

$$\|u\|_{L^2(\mathbb{R})} \leq C\|u'\|_{L^1(\mathbb{R})}.$$

We replace  $u$  by  $u^2$  in the previous inequality to get the first estimate. A straightforward calculation yields the second estimate. This conclude the proof of Lemma 6.1.  $\square$

#### 6.1 $H^1$ and $H^2$ estimates

Taking the inner product of (6.1) with  $-\partial_{yy}\phi$ , we get

$$\frac{1}{2} \frac{d}{dt} \left( \|\partial_y\phi\|_{L^2(\mathbb{R})}^2 \right) + \|\partial_{yy}\phi\|_{L^2(\mathbb{R})}^2 = - \int_{\mathbb{R}} (\partial_{yy}\psi + 2\psi)\partial_{yy}\phi - \int_{\mathbb{R}} \mathcal{F}_1(\phi, \psi, \mathcal{N})\partial_{yy}\phi.$$

Taking the inner product of (6.2) with  $-\partial_{yy}\psi + 2\psi$ , we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left( \|\partial_y\psi\|_{L^2(\mathbb{R})}^2 + 2\|\psi\|_{L^2(\mathbb{R})}^2 \right) + \|\partial_{yy}\psi\|_{L^2(\mathbb{R})}^2 \\ = 4\|\psi\|_{L^2(\mathbb{R})}^2 - \int_{\mathbb{R}} (-\partial_{yy}\psi + 2\psi)\partial_{yy}\phi \\ + \int_{\mathbb{R}} \mathcal{F}_2(\phi, \psi, \mathcal{N})(-\partial_{yy}\psi + 2\psi). \end{aligned}$$

Adding the previous equations, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left( \|\partial_y \phi\|_{L^2(\mathbb{R})}^2 + \|\partial_y \psi\|_{L^2(\mathbb{R})}^2 + 2\|\psi\|_{L^2(\mathbb{R})}^2 \right) + \left( \|\partial_{yy} \phi\|_{L^2(\mathbb{R})}^2 + \|\partial_{yy} \psi\|_{L^2(\mathbb{R})}^2 \right) \\ &= 4\|\psi\|_{L^2(\mathbb{R})}^2 - 4 \int_{\mathbb{R}} \psi \partial_{yy} \phi - \int_{\mathbb{R}} \mathcal{F}_1(\phi, \psi, \mathcal{N}) \partial_{yy} \phi \\ & \quad + \int_{\mathbb{R}} \mathcal{F}_2(\phi, \psi, \mathcal{N}) (-\partial_{yy} \psi + 2\psi). \end{aligned} \tag{6.4}$$

Taking the inner product of (6.1) with  $\partial_{yy}(\partial_{yy} \phi)$  and the inner product of (6.2) with  $\partial_{yy}(\partial_{yy} \psi - 2\psi)$ , we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left( \|\partial_{yy} \phi\|_{L^2(\mathbb{R})}^2 + \|\partial_{yy} \psi\|_{L^2(\mathbb{R})}^2 + 2\|\partial_y \psi\|_{L^2(\mathbb{R})}^2 \right) + \left( \|\partial_{yyy} \phi\|_{L^2(\mathbb{R})}^2 + \|\partial_{yyy} \psi\|_{L^2(\mathbb{R})}^2 \right) \\ &= 4\|\partial_y \psi\|_{L^2(\mathbb{R})}^2 + 4 \int_{\mathbb{R}} \psi \partial_{yy}(\partial_{yy} \phi) - \int_{\mathbb{R}} \partial_y (\mathcal{F}_1(\phi, \psi, \mathcal{N})) \partial_{yyy} \phi \\ & \quad - \int_{\mathbb{R}} \partial_y (\mathcal{F}_2(\phi, \psi, \mathcal{N})) \partial_y(\partial_{yy} \psi - 2\psi). \end{aligned} \tag{6.5}$$

Using Young’s inequality and estimating (5.1) in Proposition 5.1 together with (6.4) and (6.5) yield that while  $\|(\phi, \psi, \mathcal{N})\|_{\mathcal{H}^2} \leq \kappa$ , then

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left( \|\partial_y \phi\|_{L^2(\mathbb{R})}^2 + \|\partial_{yy} \phi\|_{L^2(\mathbb{R})}^2 + 2\|\psi\|_{L^2(\mathbb{R})}^2 + 3\|\partial_y \psi\|_{L^2(\mathbb{R})}^2 + \|\partial_{yy} \psi\|_{L^2(\mathbb{R})}^2 \right) \\ & \quad + \left( \|\partial_{yy} \phi\|_{L^2(\mathbb{R})}^2 + \|\partial_{yyy} \phi\|_{L^2(\mathbb{R})}^2 + \|\partial_{yy} \psi\|_{L^2(\mathbb{R})}^2 + \|\partial_{yyy} \psi\|_{L^2(\mathbb{R})}^2 \right) \\ & \leq 2 \left( 2\|\psi\|_{L^2(\mathbb{R})}^2 + 3\|\partial_y \psi\|_{L^2(\mathbb{R})}^2 + \|\partial_y \phi\|_{L^2(\mathbb{R})}^2 + \|\partial_{yy} \phi\|_{L^2(\mathbb{R})}^2 + \|\partial_{yy} \psi\|_{L^2(\mathbb{R})}^2 \right) \\ & \quad + C\|(\phi, \psi, \mathcal{N})\|_{\mathcal{H}^2} \left( \|\partial_y \phi\|_{H^2(\mathbb{R})}^2 + \|\psi\|_{H^3(\mathbb{R})}^2 + \|\mathcal{N}\|_{H^3(\mathbb{R}^2)}^2 \right). \end{aligned} \tag{6.6}$$

Taking the inner product of (6.3) with  $\begin{pmatrix} T^2 \mathcal{N}_1 \\ T(T-2)\mathcal{N}_2 \end{pmatrix}$ , we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left( \|T \mathcal{N}_1\|_{L^2(\mathbb{R}^2)}^2 + \|(T-2Id)\mathcal{N}_2\|_{L^2(\mathbb{R}^2)}^2 \right) \\ & \quad + \left( \|T^{3/2} \mathcal{N}_1\|_{L^2(\mathbb{R}^2)}^2 + \|T^{1/2}(T-2Id)\mathcal{N}_2\|_{L^2(\mathbb{R}^2)}^2 \right) \\ & \leq \|T^{1/2} \mathcal{F}_3\|_{L^2(\mathbb{R}^2)} \left( \|T^{3/2} \mathcal{N}_1\|_{L^2(\mathbb{R}^2)} + \|T^{1/2}(T-2Id)\mathcal{N}_2\|_{L^2(\mathbb{R}^2)} \right) \\ & \leq C\|(\phi, \psi, \mathcal{N})\|_{\mathcal{H}^2} \left( \|\partial_y \phi\|_{H^2(\mathbb{R})}^2 + \|\psi\|_{H^3(\mathbb{R})}^2 + \|\mathcal{N}\|_{H^3(\mathbb{R}^2)}^2 \right). \end{aligned} \tag{6.7}$$

We remark that from Proposition (3.1), on  $\mathcal{G}$ , we have the equivalence of norms:  $\|T^{3/2} \mathcal{N}_1\|_{L^2(\mathbb{R}^2)} \sim \|\mathcal{N}_1\|_{H^3(\mathbb{R}^2)}$  and  $\|T^{1/2}(T-2Id)\mathcal{N}_2\|_{L^2(\mathbb{R}^2)} \sim \|\mathcal{N}_2\|_{H^3(\mathbb{R}^2)}$ , while  $\|(\phi, \psi, \mathcal{N})\|_{\mathcal{H}^2} \leq \kappa$  and by Proposition (5.1), we obtain the previous estimate.

### 6.2 $L^2$ estimates

Subtracting (6.1) to (6.2) yields

$$\partial_t(\phi - \psi) = 2\partial_{yy} \phi + \mathcal{F}_1(\phi, \psi, \mathcal{N}) - \mathcal{F}_2(\phi, \psi, \mathcal{N}). \tag{6.8}$$

Taking the inner product of (6.8) with  $\phi - \psi$ , we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left( \|\phi - \psi\|_{L^2(\mathbb{R})}^2 \right) + 2\|\partial_y \phi\|_{L^2(\mathbb{R})}^2 &= 2 \int_{\mathbb{R}} \partial_y \phi \partial_y \psi + \int_{\mathbb{R}} (\mathcal{F}_1 - \mathcal{F}_2) \phi \\ &\quad - \int_{\mathbb{R}} (\mathcal{F}_1 - \mathcal{F}_2) \psi. \end{aligned}$$

By Young’s inequality and from Proposition 5.1, we get:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left( \|\phi - \psi\|_{L^2(\mathbb{R})}^2 \right) + 2\|\partial_y \phi\|_{L^2(\mathbb{R})}^2 &\leq \|\partial_y \phi\|_{L^2(\mathbb{R})}^2 + \|\partial_y \psi\|_{L^2(\mathbb{R})}^2 \\ &\quad + \|\tilde{\mathcal{F}}_a\|_{L^1(\mathbb{R})} \|\phi\|_{L^\infty(\mathbb{R})} \\ &\quad + \|\tilde{\mathcal{F}}_b\|_{L^{4/3}(\mathbb{R})} \|\phi\|_{L^4(\mathbb{R})} \\ &\quad + (\|\mathcal{F}_1\|_{L^2(\mathbb{R})} + \|\mathcal{F}_2\|_{L^2(\mathbb{R})}) \|\psi\|_{L^2(\mathbb{R})}. \end{aligned} \tag{6.9}$$

Furthermore, using estimates (5.2) and (5.3), while  $\|(\phi, \psi, \mathcal{N})\|_{\mathcal{H}^2} \leq \kappa$ , yields:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left( \|\phi - \psi\|_{L^2(\mathbb{R})}^2 \right) + \|\partial_y \phi\|_{L^2(\mathbb{R})}^2 &\leq \|\partial_y \psi\|_{L^2(\mathbb{R})}^2 + C\|\phi\|_{L^\infty(\mathbb{R})} (\|\partial_y \phi\|_{H^2(\mathbb{R})} + \|\psi\|_{H^3(\mathbb{R})} + \|\mathcal{N}\|_{H^3(\mathbb{R}^2)})^2 \\ &\quad + C(\|\partial_y \phi\|_{H^2(\mathbb{R})} + \|\psi\|_{H^3(\mathbb{R})} + \|\mathcal{N}\|_{H^3(\mathbb{R}^2)}) \|\phi\|_{L^4(\mathbb{R})}^2 \\ &\quad + C\|(\phi, \psi, \mathcal{N})\|_{\mathcal{H}^2} (\|\partial_y \phi\|_{H^2(\mathbb{R})} + \|\psi\|_{H^3(\mathbb{R})} + \|\mathcal{N}\|_{H^3(\mathbb{R}^2)}) \|\psi\|_{L^2(\mathbb{R})}. \end{aligned}$$

From Lemma 6.1, we have

$$\|\phi\|_{L^4(\mathbb{R})}^2 \leq C\|\phi\|_{L^2(\mathbb{R})} \|\partial_y \phi\|_{L^2(\mathbb{R})}.$$

Therefore, while  $\|(\phi, \psi, \mathcal{N})\|_{\mathcal{H}^2} \leq \kappa$ ,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left( \|\phi - \psi\|_{L^2(\mathbb{R})}^2 \right) + \|\partial_y \phi\|_{L^2(\mathbb{R})}^2 &\leq \|\partial_y \psi\|_{L^2(\mathbb{R})}^2 + C\|(\phi, \psi, \mathcal{N})\|_{\mathcal{H}^2} \left( \|\partial_y \phi\|_{H^2(\mathbb{R})}^2 + \|\psi\|_{H^3(\mathbb{R})}^2 + \|\mathcal{N}\|_{H^3(\mathbb{R}^2)}^2 \right). \end{aligned} \tag{6.10}$$

Adding up (6.6), (6.7), and (6.10), we obtain:

$$\frac{1}{2} \frac{d\tilde{\Lambda}(t)}{dt} + \mathcal{K}(t) \leq C\|(\phi, \psi, \mathcal{N})\|_{\mathcal{H}^2} \left( \|\partial_y \phi\|_{H^2(\mathbb{R})}^2 + \|\psi\|_{H^3(\mathbb{R})}^2 + \|\mathcal{N}\|_{H^3(\mathbb{R}^2)}^2 \right). \tag{6.11}$$

We define  $\tilde{\Lambda}$  and  $\mathcal{K}$  by

$$\begin{aligned} \tilde{\Lambda}(t) &= [\|\phi - \psi\|_{L^2(\mathbb{R})}^2 + \|\partial_y \phi\|_{L^2(\mathbb{R})}^2 + \|\partial_{yy} \phi\|_{L^2(\mathbb{R})}^2 + 2\|\psi\|_{L^2(\mathbb{R})}^2 + 3\|\partial_y \psi\|_{L^2(\mathbb{R})}^2 \\ &\quad + \|\partial_{yy} \psi\|_{L^2(\mathbb{R})}^2 + \|T\mathcal{N}_1\|_{L^2(\mathbb{R}^2)}^2 + \|(T - 2Id)\mathcal{N}_2\|_{L^2(\mathbb{R}^2)}^2](t) \end{aligned}$$

and

$$\begin{aligned} \mathcal{K}(t) &= [-\|\partial_y \phi\|_{L^2(\mathbb{R})}^2 - \|\partial_{yy} \phi\|_{L^2(\mathbb{R})}^2 + \|\partial_{yyy} \phi\|_{L^2(\mathbb{R})}^2 - 4\|\psi\|_{L^2(\mathbb{R})}^2 - 7\|\partial_y \psi\|_{L^2(\mathbb{R})}^2 \\ &\quad - \|\partial_{yy} \psi\|_{L^2(\mathbb{R})}^2 + \|\partial_{yyy} \psi\|_{L^2(\mathbb{R})}^2 + \|T^{3/2}\mathcal{N}_1\|_{L^2(\mathbb{R}^2)}^2 \\ &\quad + \|T^{1/2}(T - 2Id)\mathcal{N}_2\|_{L^2(\mathbb{R}^2)}^2](t). \end{aligned}$$

We recall that from Proposition (3.1), on  $\mathcal{G}$ , we have the equivalence of norms:  $\|T^{3/2}\mathcal{N}_1\|_{L^2(\mathbb{R}^2)} \sim \|\mathcal{N}_1\|_{H^3(\mathbb{R}^2)}$  and  $\|T^{1/2}(T - 2Id)\mathcal{N}_2\|_{L^2(\mathbb{R}^2)} \sim \|\mathcal{N}_2\|_{H^3(\mathbb{R}^2)}$ . So there exists a constant  $K_1$  such that

$$\mathcal{K} \geq K_1 \left( \|\partial_y \phi\|_{H^2(\mathbb{R})}^2 + \|\psi\|_{H^3(\mathbb{R})}^2 + \|\mathcal{N}\|_{H^3(\mathbb{R}^2)}^2 \right).$$

Moreover,  $\|\phi\|_{L^2(\mathbb{R})} \leq C\|\phi - \psi\|_{L^2(\mathbb{R})} + \|\psi\|_{L^2(\mathbb{R})}$ , by Proposition 3.1, there exists a constant  $K_2$  such that

$$\frac{1}{K_2} \|(\phi, \psi, \mathcal{N})\|_{\mathcal{H}^2} \leq \tilde{\Lambda}(t) \leq K_2 \|(\phi, \psi, \mathcal{N})\|_{\mathcal{H}^2}.$$

Hence while  $\|(\phi, \psi, \mathcal{N})\|_{\mathcal{H}^2} \leq \kappa$ , we obtain

$$\frac{1}{2} \frac{d\tilde{\Lambda}(t)}{dt} + \left( \|\partial_y \phi\|_{H^2(\mathbb{R})}^2 + \|\psi\|_{H^3(\mathbb{R})}^2 + \|\mathcal{N}\|_{H^3(\mathbb{R}^2)}^2 \right) (K_1 - CK_2 \tilde{\Lambda}(t)) \leq 0. \quad (6.12)$$

We introduce  $\kappa_0 = \min\left\{\frac{\kappa}{K_2}, \frac{K_1}{CK_2}\right\}$ . If  $\tilde{\Lambda}(0) \leq \kappa_0$ , then with (6.12),  $\tilde{\Lambda}(t)$  remains smaller than  $\frac{K_1}{CK_2}$ , which in turn implies that  $\tilde{\Lambda}(t)$  decreases and remains smaller than  $\kappa_0$ , so that  $\|(\phi, \psi, \mathcal{N})\|_{\mathcal{H}^2}$  remains smaller than  $\kappa$ . Therefore, from (6.11), we have

$$\forall t \geq 0, \quad \|(\phi, \psi, \mathcal{N})\|_{\mathcal{H}^2} \leq K_2 \tilde{\Lambda}(t) \leq K_2 \tilde{\Lambda}(0) \leq \kappa.$$

So we are always in the validity domain of our estimates. Hence we derived the stability result of  $(0, 0, 0)$  for (4.9)–(4.11)–(4.13). This concludes the proof of our main Theorem 1.1 using Propositions 2.1 and 4.2.  $\square$

## 7 Conclusion

In this article, we have analyzed the stability of static domain wall profile for a two-dimensional model of ferromagnetic material with a simplified expression of the demagnetizing field. The linear part of the perturbed equation admits 0 as a simple eigenvalue which prevented us to get the proof directly. The most crucial part is to separate the variables in the new coordinate system to get the transformed system of equations in which the linear part are almost independent. The separation of variables taken place due to the simplified expression of the demagnetizing field which helps us to perform the successful variational estimates. However, the stability of the static walls with the generalized demagnetizing field remains an open problem. Also, from the application point of view, it is very essential to study the wall dynamics in presence of external magnetic source, *i.e.*, basically the behavior of traveling wave solutions. In future work, we intend to analyze the stability and controllability features of traveling wave solutions with the complete model. These problems seem to be technically more complex and we need to have a different approach to apply the variational estimate technique.

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### References

- Adams, R.A., Fournier, J.J.F.: Sobolev Spaces, Pure and Applied Mathematics, vol. 140. Academic Press, Cambridge (Access Online via Elsevier) (2003)
- Agarwal, S., et al.: Control of a network of magnetic ellipsoidal samples. *Math. Control Relat. Fields* **1**(2), 129–147 (2011)
- Aharoni, A.: Introduction to the Theory of Ferromagnetism, vol. 109. Oxford University Press, Oxford (2000)
- Alouges, F., et al.: Néel and cross-tie wall energies for planar micromagnetic configurations, A tribute to J.L. Lions. *ESAIM Control Optim. Calc. Var.* **8**, 31–68 (2002)
- Alouges, F., Beauchard, K.: Magnetization switching on small ferromagnetic ellipsoidal samples. *ESAIM Control Optim. Calc. Var.* **15**, 676–711 (2009)
- Alouges, F., Soyeur, A.: On global weak solutions for Landau–Lifshitz equations: existence and nonuniqueness. *Nonlinear Anal.* **18**(11), 1071–1084 (1992)
- Bañas, L., et al.: A convergent implicit finite element discretization of the Maxwell–Landau–Lifshitz–Gilbert equation. *SIAM J. Numer. Anal.* **46**(3), 1399–1422 (2008)

8. Boust, F., et al.: 3D dynamic micromagnetic simulations of susceptibility spectra in soft ferromagnetic particles. *ESAIM Proc.* **22**, 127–131 (2008)
9. Brown, W.F.: *Micromagnetics*. Wiley, New York (1963)
10. Carbou, G., et al.: Control of travelling walls in a ferromagnetic nanowire. *Discrete Contin. Dyn. Syst. Ser. S* **1**(1), 51–59 (2008)
11. Carbou, G.: Stability of static walls for a three-dimensional model of ferromagnetic material. *J. Math. Pures Appl.* **93**, 183–203 (2010)
12. Carbou, G., et al.: Global weak solutions for the Landau–Lifschitz equation with magnetostriction. *Math. Methods Appl. Sci.* **34**(10), 1274–1288 (2011)
13. Carbou, G., Fabrie, P.: Time average in micromagnetism. *J. Differ. Equ.* **147**(2), 383–409 (1998)
14. Carbou, G., Fabrie, P.: Regular solutions for Landau–Lifschitz equation in a bounded domain. *Differ. Integral Equ.* **14**(2), 213–229 (2001)
15. Carbou, G., Fabrie, P.: Regular solutions for Landau–Lifschitz equation in  $\mathbb{R}^3$ . *Commun. Appl. Anal.* **5**(1), 17–30 (2001)
16. Carbou, G., Labbé, S.: Stability for static walls in ferromagnetic nanowires. *Discrete Contin. Dyn. Syst. Ser. B* **6**(2), 273–290 (2006)
17. DeSimone, A., et al.: Two-dimensional modelling of soft ferromagnetic films. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* **457**(2016), 2983–2991 (2001)
18. Ding, S., et al.: Global existence of weak solutions for Landau–Lifshitz–Maxwell equations. *Discrete Contin. Dyn. Syst.* **17**(4), 867–890 (2007)
19. Guo, B., Su, F.: Global weak solution for the Landau–Lifshitz–Maxwell equation in three space dimensions. *J. Math. Anal. Appl.* **211**(1), 326–346 (1997)
20. Hubert, A., Schäfer, R.: *Magnetic Domains: The Analysis of Magnetic Microstructures*. Springer, Berlin (1998)
21. Kapitula, T.: On the stability of travelling waves in weighted  $L^\infty$  spaces. *J. Differ. Equ.* **112**(1), 179–215 (1994)
22. Kapitula, T.: Multidimensional stability of planar travelling waves. *Trans. Am. Math. Soc.* **349**(1), 257–269 (1997)
23. Labbé, S.: Fast computation for large magnetostatic systems adapted for micromagnetism. *SISC SIAM J. Sci. Comput.* **26**(6), 2160–2175 (2005)
24. Landau, L., Lifschitz, E.: *Electrodynamique des Milieux Continus, Cours de Physique Théorique, (French) [Electrodynamics of continuous Media, Theoretical Physics Course], vol. VIII*. Mir, Moscow (1969)
25. Podio-Guidugli, P., Tomassetti, G.: On the steady motions of a flat domain wall in a ferromagnet. *Eur. Phys. J. B Condens. Matter Complex Syst.* **26**(2), 191–198 (2002)
26. Reed, M., Simon, B.: *Methods of Modern Mathematical Physics. IV. Analysis of Operators*. Academic Press, Inc., New York (1978)
27. Schryer, N.L., Walker, L.R.: The motion of  $180^\circ$  domain walls in uniform dc magnetic fields. *J. Appl. Phys.* **45**(12), 5406–5412 (1974)
28. Visintin, A.: On Landau Lifschitz equation for ferromagnetism. *Jpn. J. Appl. Math.* **1**(2), 69–84 (1985)
29. Walker, L.R.: Bell telephone laboratories memorandum, 1956, unpublished; cf Dillon Jr., J.F. in *A Treatise on Magnetism*, vol. VIII, pp. 450–453, Rado, G.T., Suhl, H. (eds.). Academic Press (1963)