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A class of non-holomorphic modular forms III: real analytic cusp forms for $\mathrm{SL}_2(\mathbb{Z})$

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Abstract

We define canonical real analytic versions of modular forms of integral weight for the full modular group, generalising real analytic Eisenstein series. They are harmonic Maass waveforms with poles at the cusp, whose Fourier coefficients involve periods and quasi-periods of cusp forms, which are conjecturally transcendental. In particular, we settle the question of finding explicit 'weak harmonic lifts' for every eigenform of integral weight k and level one. We show that mock modular forms of integral weight are algebro-geometric and have Fourier coefficients proportional to $n^{1-k}(a'_n + \rho a_n)$ for $n \neq 0$, where ρ is the normalised permanent of the period matrix of the corresponding motive, and a_n, a'_n are the Fourier coefficients of a Hecke eigenform and a weakly holomorphic Hecke eigenform, respectively. More generally, this framework provides a conceptual explanation for the algebraicity of the coefficients of mock modular forms in the CM case.

1 Introduction

Let $\mathfrak{H}$ denote the upper-half plane with the usual left action by $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. This paper is the third in a series [2,3] studying subspaces of the vector space $\mathcal{M}^!$ of real analytic functions $f : \mathfrak{H} \to \mathbb{C}$ which are modular of weights (r, s) for $r, s \in \mathbb{Z}$, i.e.

$$f(\gamma z) = (cz+d)^r (c\overline{z}+d)^s f(z) \quad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma\,, z \in \mathfrak{H}, \tag{1.1}$$

which furthermore admit an expansion of the form

$$f = \sum_{|k| \leq M} \mathbb{L}^k \Big(\sum_{m,n \geq -N} a^{(k)}_{m,n} q^m \overline{q}^n \Big) \quad \text{where } a^{(k)}_{m,n} \in \mathbb{C}\,, \tag{1.2}$$

for $M, N \in \mathbb{Z}$, where $q = \exp(2\pi i z)$ and $\mathbb{L} = \log|q| = -2\pi \mathrm{Im}(z)$. The space $\mathcal{M}^!$ is equipped with differential operators $\partial, \overline{\partial}$ closely related to Maass' raising and lowering operators [17], and a Laplacian Δ. In [2], we defined a subspace $\mathcal{MI}^! \subset \mathcal{M}^!$ of *modular iterated integrals*, generated from weakly holomorphic modular forms by repeatedly taking primitives with respect to ∂ and $\overline{\partial}$. In this instalment, we describe the subspace $\mathcal{MI}^!_1 \subset \mathcal{MI}^!$ of modular iterated integrals of *length one*. These correspond to a modular incarnation of the abelian quotient of the relative completion of the fundamental group [4,15] of the moduli stack of elliptic curves $\mathcal{M}_{1,1}$. They span the first level in an infinite tower of non-abelian or 'mixed' modular functions whose general definition was given in

[4, Sect. 18.5]. In [3] we worked out the Eisenstein part of this construction; here we spell out the length one subspace of the general case.

Examples of functions in the class $\mathcal{MI}_1^!$ are given by real analytic Eisenstein series, which are well known. Let $r, s \geq 0$ such that $w = r + s \geq 2$ is even, and define

$$\mathcal{E}_{r,s} = \frac{w!}{(2\pi i)^{w+2}} \frac{1}{2} \sum_{(m,n)\in\mathbb{Z}^2\setminus\{0,0\}} \frac{\mathbb{L}}{(mz+n)^{r+1}(m\overline{z}+n)^{s+1}} \, .$$

These functions are real analytic and modular of weights (r, s) and admit an expansion of the form (1.2) (with $N = 0$). Following the presentation given in [2, Sect. 4], they are the unique solutions to the following system of differential equations:

$$\partial\mathcal{E}_{r,s} = (r+1)\mathcal{E}_{r+1,s-1} \quad \text{for } s \geq 1,$$
$$\overline{\partial}\mathcal{E}_{r,s} = (s+1)\mathcal{E}_{r-1,s+1} \quad \text{for } r \geq 1,$$

where the definition of $\partial, \overline{\partial}$ is given in Sect. 3.1 and

$$\partial\mathcal{E}_{w,0} = \mathbb{L}\mathbb{G}_{w+2}, \quad \overline{\partial}\mathcal{E}_{0,w} = \mathbb{L}\overline{\mathbb{G}_{w+2}},$$

where $\mathbb{G}_{2k}$ are the Hecke normalised holomorphic Eisenstein series:

$$\mathbb{G}_{2k} = -\frac{b_{2k}}{4k} + \sum_{n\geq 1} \sigma_{2k-1}(n) q^n \, , \quad k \geq 1 \, . \tag{1.3}$$

Since $\partial\mathbb{L}^{-1}\mathcal{E}_{w,0} = \mathbb{G}_{w+2}$, the functions $\mathbb{L}^{-1}\mathcal{E}_{w,0}$ are modular primitives (with respect to ∂) of holomorphic Eisenstein series, and are annihilated by the Laplacian.

In this paper, we shall construct *real analytic cusp forms* $\mathcal{H}(f)_{r,s}$ which are canonically associated with any Hecke cusp form, and satisfy an analogous system of differential equations. It is clear from their construction that they are 'motivic', in that their coefficients only involve the periods of pure motives associated with cusp forms [21]. The functions $\mathcal{H}(f)_{r,s}$ generate $\mathcal{MI}_1^!$, and furthermore, they generate the subspace of $\mathcal{HM}^! \subset \mathcal{M}^!$ of eigenfunctions of the Laplacian. In other words, the overlap between the space $\mathcal{M}^!$ and the set of Maass waveforms is exactly described by the functions studied in this paper.

1.1 Real Frobenius

The essential ingredient in our construction is the real Frobenius, also known as complex conjugation. For all $n \in \mathbb{Z}$ let $M_n^!$ denote the space of weakly holomorphic modular forms of weight n.

They admit a Fourier expansion

$$f = \sum_{m\geq -N} a_m(f) q^m, \quad \text{where} \quad a_m(f) \in \mathbb{C} \tag{1.4}$$

for $N \in \mathbb{Z}$. Although the differential operator $D = q\frac{d}{dq}$ does not preserve modularity, a well-known result due to Bol implies that its powers define linear maps

$$D^{n+1} : M_{-n}^! \longrightarrow M_{n+2}^!$$

for all $n \geq 0$. The quotient $M_{n+2}^!/D^{n+1}M_{-n}^!$ can be interpreted as a space of modular forms of the second kind [6,11,23]. Indeed, it is canonically isomorphic to the algebraic de Rham cohomology of the moduli stack of elliptic curves with certain coefficients, and in particular, admits an action by Hecke operators. Furthermore, one shows [14] that every element $f \in M_{n+2}^!/D^{n+1}M_{-n}^!$ has a *unique representative* $f \in M_{n+2}^!$ such that f has a pole at the cusp of order at most $\dim S_{n+2}$:

$$\mathrm{ord}_\infty f \geq -\dim S_{n+2} \, .$$

This provides a splitting $M^!_{n+2} = D^{n+1}M^!_{-n} \oplus M^!_{n+2}/D^{n+1}M^!_{-n}$ which is possibly unnatural, but is canonical. We shall use this splitting to provide canonical constructions and uniqueness statements in the theorems below. A purist may prefer to avoid using this splitting at the expense of working modulo $D^{n+1}M^!_{-n}$.

The '*single-valued involution*' is a canonical Hecke-equivariant map

$$\mathbf{s} : M^!_{n+2}/D^{n+1}M^!_{-n} \xrightarrow{\sim} M^!_{n+2}/D^{n+1}M^!_{-n} \, .$$

It exists in much greater generality [5, Sect. 4.1] and is induced, via the comparison isomorphism, by complex conjugation on Betti cohomology. By the previous remarks, it lifts to an involution on $M^!_{n+2}$, which acts by zero on $D^{n+1}M^!_{-n}$. In fact, it can be written down explicitly on each cuspidal Hecke eigenspace in terms of a period matrix

$$\mathrm{P}_f = \begin{pmatrix} \eta_f^+ & \omega_f^+ \\ i\eta_f^- & i\omega_f^- \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}), \tag{1.5}$$

where $\omega_f^+, i\omega_f^-$ are the periods and $\eta_f^+, i\eta_f^-$ the quasi-periods with respect to a basis f, f' of a cuspidal Hecke eigenspace. More precisely, we show that

$$\mathbf{s}(f) = \Big(\frac{\eta_f^+\omega_f^- + \eta_f^-\omega_f^+}{\eta_f^-\omega_f^+ - \eta_f^+\omega_f^-}\Big) f + \Big(\frac{2\,\omega_f^+\omega_f^-}{\eta_f^+\omega_f^- - \eta_f^-\omega_f^+}\Big) f' \, . \tag{1.6}$$

It does not depend on the choice of basis f, f'. The quantity $\omega_f^+\omega_f^-$ is related to the Petersson norm of f. The determinant of the period matrix $\det(\mathrm{P}_f)$ is proportional to a power of $2\pi i$. The quantity $i(\eta_f^+\omega_f^- + \eta_f^-\omega_f^+)$ is the *permanent* of the period matrix, hence:

$$\mathbf{s}(f) = -\frac{\mathrm{perm}(\mathrm{P}_f)}{\det(\mathrm{P}_f)} f + \frac{2i\omega_f^+\omega_f^-}{\det(\mathrm{P}_f)} f'.$$

1.2 Summary of results

Theorem 1.1 *Let $n \geq 0$. Let f be a cuspidal Hecke eigenform of weight $n+2$ for $\mathrm{SL}_2(\mathbb{Z})$. There exists a unique family of real analytic modular functions*

$$\mathcal{H}(f)_{r,s} \quad \in \quad \mathcal{M}^!_{r,s}$$

for all $r + s = n$ and $r, s \geq 0$, satisfying the system of differential equations

$$\begin{aligned} \partial \mathcal{H}(f)_{r,s} &= (r+1)\, \mathcal{H}(f)_{r+1,s-1} \quad \text{if } s \geq 1, \\ \overline{\partial} \mathcal{H}(f)_{r,s} &= (s+1)\, \mathcal{H}(f)_{r-1,s+1} \quad \text{if } r \geq 1 \end{aligned}$$

and

$$\partial \mathcal{H}(f)_{n,0} = \mathbb{L} f, \quad \overline{\partial} \mathcal{H}(f)_{0,n} = \mathbb{L}\, \overline{\mathbf{s}(f)} \, .$$

The $\mathcal{H}(f)_{r,s}$ are eigenfunctions of the Laplace operator with eigenvalue $-n$. Equivalently, the functions $\mathbb{L}^{-1}\mathcal{H}(f)_{r,s}$ are harmonic: $\Delta \mathbb{L}^{-1}\mathcal{H}(f)_{r,s} = 0$.

The theorem holds also for weak cusp forms, defining a canonical map

$$\mathcal{H}_{r,s} : S^!_{n+2}/D^{n+1}M^!_{-n} \longrightarrow \mathcal{M}^!_{r,s}$$

for all $r + s = n$, with $r, s \geq 0$. Since $\mathbf{s}(\mathbb{G}_{n+2}) = -\mathbb{G}_{n+2}$, the real analytic Eisenstein series satisfy identical equations except with a difference of sign (for $\overline{\partial}\mathcal{E}_{0,n}$, which satisfies $\overline{\partial}\mathcal{E}_{0,n} = -\mathbb{L}\, \overline{\mathbf{s}(G_{n+2})}$). This justifies calling the $\mathcal{H}(f)_{r,s}$ *real analytic cusp forms.*

The theorem can be rephrased as follows. Consider the real analytic vector-valued function $\mathcal{H}(f) : \mathfrak{H} \to \mathbb{C}[X, Y]$ defined by $\mathcal{H}(f) = \sum_{r+s=n} \mathcal{H}(f)_{r,s}(X - zY)^r(X - \overline{z}Y)^s$. It is equivariant for the standard right action of Γ on $\mathbb{C}[X, Y]$ and satisfies

$$d\mathcal{H}(f) = \pi i f(z)(X - zY)^n dz + \pi i \overline{\mathbf{s}(f)}(X - \overline{z}Y)^n d\overline{z} .$$

The functions $\mathcal{H}(f)_{r,s}$ are given by the following explicit formula. First, for any weakly holomorphic modular form (1.4), write for all $k \geq 0$

$$f^{(k)} = \sum_{n \in \mathbb{Z} \setminus 0} \frac{a_n(f)}{(2n)^k} q^n . \tag{1.7}$$

It is an iterated primitive of f for $q \frac{d}{dq}$. For all $r, s \geq 0$ with $r + s = n$ define

$$R_{r,s}(f) = (-1)^r \binom{n}{r} \sum_{k=s}^{n} \binom{r}{k-s} (-1)^k \frac{k!}{\mathbb{L}^k} f^{(k+1)} . \tag{1.8}$$

Theorem 1.2 *The functions $\mathcal{H}(f)$ have the following form:*

$$\mathcal{H}(f)_{r,s} = \frac{a_0(f)}{n+1}\mathbb{L} + \alpha_f(-1)^r \binom{n}{r} \mathbb{L}^{-n} + R_{r,s}(f) + \overline{R_{s,r}(\mathbf{s}(f))}$$

for some uniquely determined $\alpha_f \in \mathbb{C}$.

The constant term α_f can be computed (Sect. 6.6) from the Fourier coefficients of f and $\mathbf{s}(f)$ in the case when f is cuspidal, and is given by an odd zeta value in the case when f is an Eisenstein series. It is a pure period in the cuspidal case; and a mixed period in the Eisenstein case. This dichotomy is due to the fact that the Tate twists of the Tate motive have non-trivial extensions, but the Tate twists of the motive of a cusp form do not (in the relevant range). When f is holomorphic, the constant α_f is proportional to the Petersson norm of f.

When f is a Hecke cuspidal eigenform with coefficients in a number field K_f, the coefficients in the expansion of $\mathcal{H}(f)_{r,s}$ lie in a K_f-vector space of dimension at most 3 which is spanned by periods. We show furthermore:

(1) If f is a Hecke eigenfunction with eigenvalues λ_m, then the functions $\mathcal{H}(f)_{r,s}$ satisfy an inhomogeneous Hecke eigenvalue equation with eigenvalues $m^{-1}\lambda_m$. See Sect. 6.5 for precise statements.
(2) The action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on Hecke eigenfunctions extends to an action on the functions $\mathcal{H}(f)_{r,s}$, for every r, s. In fact, this action extends to an action of a 'motivic' Galois group on a larger class of modular forms which acts on the coefficients in the expansion (1.2). This will be discussed elsewhere.

The main ingredient in this paper is the single-valued involution $\mathbf{s}$, which is derived from the real Frobenius. It would be interesting to replace it with a p-adic crystalline Frobenius to define p-adic versions of real analytic cusp forms (see [12]).

1.3 Weak harmonic lifts and mock modular forms of integral weight

Consider the special case $r = n, s = 0$. For the sole purposes of this introduction set

$$\widetilde{f} = \mathbb{L}^{-1}\mathcal{H}(f)_{n,0} .$$

Corollary 1.3 *For every (weakly holomorphic) cusp form f of weight $n+2$, the function $\widetilde{f}$ is a canonical weak harmonic lift of f. More precisely, using the notation (3.3),*

$$\partial \widetilde{f} = f \quad \text{and} \quad \Delta_{n+1,1} \widetilde{f} = 0\,.$$

In particular, $\widetilde{f}$ is a weak Maass waveform. It is given explicitly by

$$\widetilde{f} = \frac{\alpha_f}{\mathbb{L}^{n+1}} + \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{k!}{\mathbb{L}^{k+1}} f^{(k+1)} + \frac{n!}{\mathbb{L}^{n+1}} \overline{\mathbf{s}(f)^{(n+1).}}$$

The problem of constructing weak harmonic lifts has a long history, but an explicit construction has remained elusive. The existence of weak harmonic lifts in a much more general setting was proved in [9]. Having established existence, the general shape of the Fourier expansion is easily deduced—the only issue is to determine the unknown Fourier coefficients. On the other hand a direct, but highly transcendental, construction using Poincaré series was given in [8,20], involving complicated special functions. This procedure is potentially ill-defined: when the space of cusp forms has dimension greater than one, it involves choices, since there are relations between Poincaré series. The question of whether weak harmonic lifts have irrational coefficients or not has been raised in [10,11,20]. Our results imply that these functions, despite appearances, are in fact of geometric, and indeed, motivic, origin.

The 'mock' modular form associated with $\widetilde{f}$ is the complex conjugate of the antiholomorphic part of $\widetilde{f}$ times $\mathbb{L}^{n+1}$. It is harmonic and given by

$$M_f = \alpha_f + n!\, \mathbf{s}(f)^{(n+1)}.$$

When f is a Hecke eigenform, $\mathbf{s}(f)$ is given by (1.6), which leads to a very simple and explicit construction of mock modular forms of integral weights for $\mathrm{SL}_2(\mathbb{Z})$. In the literature, it is customary to rescale the mock modular forms by the Petersson norm. This gives

$$M'_f = \alpha'_f + (n-1)! \sum_{m \in \mathbb{Z}\setminus\{0\}} \frac{a'_m + \rho\, a_m}{m^{n-1}} q^m\,, \tag{1.9}$$

where a_m, a'_m are the Fourier coefficients of f, f', respectively, and

$$\rho = -\frac{1}{2}\Big(\frac{\eta^+}{\omega^+} + \frac{\eta^-}{\omega^-}\Big)\,.$$

The quantity α'_f is in the field of definition of the a_m, a'_m.

In Sect. 8, we compute this explicitly in the case of Ramanujan's Δ function. Let

$$\Delta = q - 24\,q^2 + 252\,q^3 - 1472\,q^4 + 4830 q^5 + \cdots,$$
$$\Delta' = q^{-1} + 47709536\,q^2 + 39862705122\,q^3 + \cdots,$$

where $\Delta' \in M^!_{12}$ is the unique normalised weakly holomorphic modular form which has a pole of order 1 at the cusp, and whose Fourier coefficients a_0, a_1 vanish. In this case $a_n, a'_n \in \mathbb{Z}$, and a_n is the Ramanujan τ-function. The functions Δ, Δ' are a basis for the de Rham realisation of the motive [21] of Δ. If ρ is irrational (as expected), then the nth Fourier coefficient of M'_f is irrational if and only if $a_n \neq 0$.

Since the space of cusp forms of weight 12 is one-dimensional, the method of Poincaré series [8] also yields in this case an explicit expression for this mock modular form in terms of special functions. Comparing the Fourier coefficients of the two gives:

Corollary 1.4 *For all $n > 0$,*

$$2\pi\, n^{\frac{11}{2}} \sum_{c=1}^{\infty} \frac{K(-1,n,c)}{c} I_{11}\Big(\frac{4\pi\sqrt{n}}{c}\Big) = a'_n + \rho a_n,$$

where K denotes a Kloosterman sum and I a Bessel function [20].

Since modular forms of level one do not have complex multiplication, Grothendieck's period conjecture, applied to the motives of cusp forms, would imply that its Fourier coefficients are transcendental. The reader will easily be able to generalise the results of this paper to the case of a general congruence subgroup using the results of [23].[1] In an "Appendix", we explain how the existence of a complex multiplication on the motive of a cusp form implies an algebraicity constraint on the single-valued involution. This explains the phenomena studied in the recent paper [10] which observed algebraicity of the Fourier coefficients of suitably normalised Maass waveforms associated with modular forms with complex multiplication.

1.4 Contents

In Sect. 2 we review the theory of weakly holomorphic modular forms. Much of this material is standard, but many aspects are not widely known and may be of independent interest. In Sect. 3 we review some properties of the space $\mathcal{M}^!$ of real analytic modular forms from [2], and its subspaces $\mathcal{HM}^!$ (Sect. 4) of Laplace eigenfunctions and $\mathcal{MI}^!$ (Sect. 5) of modular integrals. In Sect. 6 we describe the action of Hecke operators on $\mathcal{HM}^!$. Much of this material is well known. In Sect. 7 we prove the existence of weak modular lifts, and in Sect. 8 we discuss Ramanujan's function Δ.

2 Background on weakly holomorphic modular forms

2.1 Weakly holomorphic modular forms

The vector space $M^!_n$ of weakly holomorphic modular forms of weight $n \in \mathbb{Z}$ is the vector space of holomorphic functions $f : \mathfrak{H} \to \mathbb{C}$ with possible poles at the cusp, which are modular of weight n. They admit a Fourier expansion of the form

$$f = \sum_{n \geq -N} a_n q^n, \quad \text{where } a_n \in \mathbb{C}\,. \tag{2.1}$$

The space $S^!_n \subset M^!_n$ of weakly holomorphic cusp forms are those with $a_0 = 0$. The subspace of functions with Fourier coefficients a_n in a subring $R \subset \mathbb{C}$ will be denoted by $M^!_n(R)$.

Consider the following operator, which does not in general preserve modularity:

$$D = q\frac{\mathrm{d}}{\mathrm{d}q}\,. \tag{2.2}$$

An identity due to Bol [1] (see also Lemma 3.3) implies, however, that

$$D^{n+1} : M^!_{-n} \longrightarrow M^!_{n+2}\,.$$

Its image is contained in the space of cusp forms $S^!_{n+2}$. Elements in the cokernel of this map can be viewed as modular forms 'of the second kind', and can be interpreted as

[1] After we had written this paper, K. Ono and N. Diamantis kindly pointed out the recent work of Candelori [11], which is closely related to our construction and applies for modular forms of level ≥ 5. His formula (48) for the Fourier coefficients in the case $n \neq 0$ is very similar to (1.9). The case $n = 0$ requires an additional argument, which we provide in this paper using Hecke operators.

algebraic de Rham cohomology. Surprisingly this fact is not well known. It appeared for the first time implicitly in the work of Coleman [13] on p-adic modular forms, and later in [11,21–23]. A direct proof in the case of level one was given in [6].

Theorem 2.1 *Let $\mathcal{M}_{1,1}$ denote the moduli stack of elliptic curves over $\mathbb{Q}$, and $\mathcal{V}$ the algebraic vector bundle defined by the de Rham cohomology $H^1_{dR}(\mathcal{E}/\mathcal{M}_{1,1})$ of the universal elliptic curve $\mathcal{E}$ over $\mathcal{M}_{1,1}$, equipped with the Gauss–Manin connection. Set $\mathcal{V}_n = \mathrm{Sym}^n\,\mathcal{V}$. For all $n \geq 0$, there is a canonical isomorphism of $\mathbb{Q}$ vector spaces*

$$M^!_{n+2}(\mathbb{Q})/D^{n+1}M^!_{-n}(\mathbb{Q}) \overset{\sim}{\longrightarrow} H^1_{dR}(\mathcal{M}_{1,1};\mathcal{V}_n)\,. \tag{2.3}$$

The right-hand side vanishes if n is zero or odd.

This theorem has a number of consequences that we shall spell out below. Many of these have been known for some time, others apparently not.

There is a canonical decomposition into Eisenstein series and cusp forms

$$H^1_{dR}(\mathcal{M}_{1,1};\mathcal{V}_n) \;=\; H^1_{\mathrm{cusp},dR}(\mathcal{M}_{1,1};\mathcal{V}_n) \oplus H^1_{\mathrm{eis},dR}(\mathcal{M}_{1,1};\mathcal{V}_n)\,.$$

Via the isomorphism (2.3), the latter is generated by Eisenstein series (1.3)

$$H^1_{\mathrm{eis},dR}(\mathcal{M}_{1,1};\mathcal{V}_n) = \mathbb{Q}\,\mathbb{G}_{n+2}$$

for all $n \geq 2$, and the former is isomorphic to the space of cusp forms

$$H^1_{\mathrm{cusp},dR}(\mathcal{M}_{1,1};\mathcal{V}_n) = S^!_{n+2}(\mathbb{Q})/D^{n+1}M^!_{-n}(\mathbb{Q})\,.$$

Serre duality induces a pairing on the latter space. Explicitly, if $f, g \in S^!_{n+2}$ are weakly holomorphic cusp forms of weight $n+2$ with Fourier coefficients $a_k(f)$, $a_k(g)$, respectively, it is given by [6,14, Sect. 5]

$$\{f, g\} = \sum_{k\in\mathbb{Z}} \frac{a_k(f)a_{-k}(g)}{k^{n+1}}\,. \tag{2.4}$$

It vanishes if f or g is in the image of the Bol operator D^{n+1}. We have

$$\dim_{\mathbb{Q}} H^1_{\mathrm{cusp},dR}(\mathcal{M}_{1,1};\mathcal{V}_n) = 2\,\dim_{\mathbb{C}} S_{n+2}\,.$$

One shows [14] that every equivalence class

$$[f] \in M^!_{n+2}/D^{n+1}M^!_{-n}$$

has a unique representative $f \in M^!_{n+2}$ such that the order of the zero satisfies

$$\mathrm{ord}_{\infty} f \geq -\dim S_{n+2}\,.$$

Thus, we have a canonical isomorphism

$$M^!_{n+2} = D^{n+1}M^!_{-n} \oplus H^1_{dR}(\mathcal{M}_{1,1};\mathcal{V}_n)\,.$$

2.1.1 Hecke operators

The isomorphism (2.3) is equivariant with respect to the action of Hecke operators T_m, for $m \geq 1$, which act via the formula (6.2) (which we shall re-derive, in a more general context, in Sect. 6). If a formal power series (2.1) has a pole of order p at the cusp, then $T_m f$ has a pole of order mp at the cusp.

The Hecke operators commute with the Bol operator:

$$[T_m, D^n] = 0 \quad \text{for all } n\,,$$

which implies that there is an action of the Hecke algebra for all n

$$T_m : M^!_{n+2}/D^{n+1}M^!_{-n} \quad \longrightarrow \quad M^!_{n+2}/D^{n+1}M^!_{-n}\,.$$

The action of Hecke operators respects the decomposition into Eisenstein series and cusp forms. In particular, the Eisenstein series $\mathbb{G}_{2k}$ are normalised Hecke eigenfunctions: for all $n \geq 2$ and $m \geq 1$,

$$T_m \mathbb{G}_{n+2} = \sigma_{n+1}(m)\mathbb{G}_{n+2}. \tag{2.5}$$

The pairing (2.4) is orthogonal with respect to the action of T_m [14]

$$\{T_m f, g\} = \{f, T_m g\} \quad \text{for all}\, f, g \in S^!_{n+2}\,.$$

The space of cusp forms decomposes over $\overline{\mathbb{Q}}$ into Hecke eigenspaces

$$H^1_{\mathrm{cusp},dR}(\mathcal{M}_{1,1}; \mathcal{V}_n) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} = \bigoplus_{\underline{\lambda}} H^{dR}_{\underline{\lambda}} \otimes_{K_{\underline{\lambda}}} \overline{\mathbb{Q}},$$

where $\underline{\lambda} = (\lambda_m)_{m\geq 1}$ and $H^{dR}_{\underline{\lambda}}$ is a two-dimensional $K_{\underline{\lambda}}$ vector space, where $K_{\underline{\lambda}} \subset \mathbb{R}$ is the number field generated by the λ_m. $H^{dR}_{\underline{\lambda}}$ is generated by a normalised Hecke eigenform

$$f_{\underline{\lambda}} \quad \in \quad M_{n+2}(K_{\underline{\lambda}}),$$

which satisfies $T_m f_{\underline{\lambda}} = \lambda_m f_{\underline{\lambda}}$ for all m, and a *weak Hecke eigenform*

$$f'_{\underline{\lambda}} \quad \in \quad M^!_{n+2}(K_{\underline{\lambda}}),$$

which satisfies for all $m \geq 1$:

$$T_m f'_{\underline{\lambda}} = \lambda_m f'_{\underline{\lambda}} \pmod{D^{n+1}M^!_{-n}(K_{\underline{\lambda}})}\,. \tag{2.6}$$

We can assume as a consequence of [6, Proposition 5.6], that $f_{\underline{\lambda}}, f'_{\underline{\lambda}}$ satisfy:

$$\{f'_{\underline{\lambda}}, f_{\underline{\lambda}}\} = 1\,,$$

and furthermore, that $f'_{\underline{\lambda}}$ has poles at the cusp of order at most $\dim S_{n+2}$. With these conventions, $H^{dR}_{\underline{\lambda}}$ has a basis

$$H^{dR}_{\underline{\lambda}} = f'_{\underline{\lambda}} K_{\underline{\lambda}} \oplus f_{\underline{\lambda}} K_{\underline{\lambda}}, \tag{2.7}$$

which is well defined up to transformations $f'_{\underline{\lambda}} \mapsto f'_{\underline{\lambda}} + a f_{\underline{\lambda}}$, for $a \in K_{\underline{\lambda}}$.

Remark 2.2 One could fix a 'canonical' basis of $H^{dR}_{\underline{\lambda}}$ either by assuming that the Fourier coefficient a_1 of f' is equal to 1, or by demanding that $\{f'_{\underline{\lambda}}, f'_{\underline{\lambda}}\} = 0$ (note that $\{f_{\underline{\lambda}}, f_{\underline{\lambda}}\} = 0$ holds automatically). This will not be required in this paper. The latter condition holds for the basis chosen in Sect. 8.

2.1.2 Group cohomology and cocycles

Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Let $\mathbb{V}_n$ denote the local system $\mathrm{Sym}^n R^1\pi_*\mathbb{Q}$ on $\mathcal{M}_{1,1}(\mathbb{C})$ where $\pi : \mathcal{E} \to \mathcal{M}_{1,1}$ is the universal elliptic curve and $\mathbb{Q}$ is the constant sheaf on $\mathcal{E}(\mathbb{C})$. Its fibre at the tangent vector $\partial/\partial q$ on the q-disc ([4], Sect. 4.1) is the vector space

$$V_n = \bigoplus_{i+j=n} \mathbb{Q}X^iY^j$$

of homogeneous polynomials in variables X, Y, corresponding to the standard homology basis of the fibre of the universal elliptic curve. It admits a right action by Γ

$$(X, Y)\big|_{\gamma} = (aX + bY, cX + dY)$$

for γ of the form (1.1). Recall that the space of cocycles $Z^1(\Gamma; V_n)$ is the $\mathbb{Q}$-vector space generated by functions $\gamma \mapsto C_\gamma : \Gamma \to V_n$ satisfying the cocycle equation

$$C_{gh} = C_g\big|_h + C_h \quad \text{for all} \quad g, h \in \Gamma\,.$$

Such a cocycle is uniquely determined by C_S, C_T, where

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{2.8}$$

The polynomials C_S, C_T satisfy a system of equations called the cocycle equations. A cocycle is called cuspidal if $C_T = 0$. The subspace of coboundaries $B^1(\Gamma; V_n)$ is the $\mathbb{Q}$-vector space generated by cocycles of the form

$$C_\gamma = P\big|_\gamma - P$$

for some $P \in V_n$. The cohomology group is defined by

$$H^1(\Gamma; V_n) = Z^1(\Gamma; V_n)/B^1(\Gamma; V_n)\,.$$

There is a natural action of Hecke operators on $H^1(\Gamma; V_n)$. In fact, this action lifts (via the Eichler–Shimura isomorphism [25,26], see below) to an action on the space of cocycles $Z^1(\Gamma; V_n)$ which preserves $B^1(\Gamma; V_n)$ [19].

Complex conjugation on $\mathcal{M}_{1,1}(\mathbb{C})$ induces an involution called the real Frobenius F_∞ upon $H^1(\Gamma; V_n)$ (and in fact $Z^1(\Gamma; V_n)$). It acts on Γ by conjugation by

$$\epsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and on V_n by right action by ϵ, i.e. $(X, Y) \mapsto (X, -Y)$ (see [4] Sect. 5.4). In particular, there is a canonical decomposition

$$H^1(\Gamma; V_n) = H^1(\Gamma; V_n)^+ \oplus H^1(\Gamma; V_n)^- \tag{2.9}$$

into F_∞-eigenspaces. The first is spanned by classes of cocycles C such that C_S is ϵ-invariant (even), the second by cocycles which are anti-invariant (odd).

Finally, there is an inner product on $H^1_{\mathrm{cusp}}(\Gamma; V_n)$ induced by a pairing between cocycles and compactly supported cocycles [4, Sect. 8.3]:

$$\{\ ,\ \} : Z^1(\Gamma; V_n) \times Z^1_{\mathrm{cusp}}(\Gamma; V_n) \longrightarrow \mathbb{Q}\,,$$

a formula for which was given by Haberland, e.g. [4, 2.11].

2.1.3 Eichler–Shimura isomorphism

The following corollary is a consequence of a mild extension [6] of Grothendieck's algebraic de Rham theorem.

Corollary 2.3 *There is a canonical isomorphism*

$$\mathrm{comp}_{B,dR} : H^1_{dR}(\mathcal{M}_{1,1}; \mathcal{V}_n) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\ \sim\ } H^1(\Gamma; V_n) \otimes_{\mathbb{Q}} \mathbb{C}\,. \tag{2.10}$$

It respects the action of Hecke operators on both sides.

In particular, the comparison isomorphism respects the decomposition into Eisenstein and cuspidal parts. It can be computed as follows. Fix a point $z_0 \in \mathfrak{H}$.

Definition 2.4 For every $f \in M^!_{n+2}$, where $n \geq 0$, let us write

$$F_f(z) = \int_z^{z_0} (2\pi i)^{n+1} f(\tau)(X - \tau Y)^n \mathrm{d}\tau \,. \tag{2.11}$$

The integral converges since z_0 is finite. It defines a 1-cocycle

$$C^f_\gamma(X, Y) = F_f(\gamma z)\big|_\gamma - F_f(z) \in Z^1(\Gamma; V_n \otimes \mathbb{C}) \,. \tag{2.12}$$

which is independent of z.

Changing z_0 modifies this cocycle by a coboundary. We deduce a linear map

$$\begin{aligned} M^!_{n+2} &\longrightarrow H^1(\Gamma; V_n \otimes \mathbb{C}) \\ f &\mapsto [C^f] \end{aligned}$$

which is well defined, i.e. independent of the choice of point τ_0, and Hecke equivariant. One easily shows (see [7] or version 1 of [2]) that $f \in D^{n+1} M^!_{-n}$ if and only if $[C^f] \in B^1(\Gamma; V_n) \otimes \mathbb{C}$, and hence the previous map descends to an isomorphism

$$M^!_{n+2}/D^{n+1} M^!_{-n} \overset{\sim}{\longrightarrow} H^1(\Gamma; V_n) \otimes \mathbb{C},$$

which corresponds via Theorem 2.1 to the comparison isomorphism $\mathrm{comp}_{B,dR}$.

2.1.4 Period matrix

Since the comparison isomorphism is Hecke equivariant, it respects the decomposition into Hecke eigenspaces.

Let $H^B_{\underline{\lambda}}$ denote the Hecke eigenspace of $H^1(\Gamma; V_n)$ corresponding to the eigenvalues $\underline{\lambda}$. It is a $K_{\underline{\lambda}}$-vector space of dimension 2 and admits a decomposition

$$H^B_{\underline{\lambda}} = H^{B,+}_{\underline{\lambda}} \oplus H^{B,-}_{\underline{\lambda}}$$

into $\pm$ eigenspaces with respect to the real Frobenius F_∞.

The comparison isomorphism induces a canonical isomorphism

$$\mathrm{comp}_{B,dR} : H^{dR}_{\underline{\lambda}} \otimes_{\mathbb{Q}} \mathbb{C} \overset{\sim}{\longrightarrow} H^B_{\underline{\lambda}} \otimes_{\mathbb{Q}} \mathbb{C} \,.$$

Definition 2.5 Let us choose generators $P^\pm_{\underline{\lambda}}$ of $H^{B,\pm}_{\underline{\lambda}}$, respectively, and a basis (2.7) for $H^{dR}_{\underline{\lambda}}$. A *period matrix* is the comparison isomorphism $\mathrm{comp}_{B,dR}$ written with respect to these bases:

$$\mathrm{P}_{\underline{\lambda}} = \begin{pmatrix} \eta^+_{\underline{\lambda}} & \omega^+_{\underline{\lambda}} \\ i\eta^-_{\underline{\lambda}} & i\omega^-_{\underline{\lambda}} \end{pmatrix}. \tag{2.13}$$

It is well defined up to multiplication on the left by a diagonal matrix with entries in $K^\times_{\underline{\lambda}}$, which reflects the ambiguity in the choices of $P^\pm_{\underline{\lambda}}$ up to scalar, and multiplication on the right by a lower triangular matrix with 1's on the diagonal.

From the compatibility of the period isomorphism with complex conjugation and real Frobenius (Sect. 2.1.6), the numbers $\omega^\pm_{\underline{\lambda}}, \eta^\pm_{\underline{\lambda}}$ are real. The $\omega^+_{\underline{\lambda}}, i\omega^-_{\underline{\lambda}}$ are the usual periods of $f_{\underline{\lambda}}$, the numbers $\eta^+_{\underline{\lambda}}, i\eta^-_{\underline{\lambda}}$ could be called its 'quasi-periods' and seem not to have been considered in the literature. It was proved in [6, Theorem 1.7] that

$$\det(\mathrm{P}_{\underline{\lambda}}) \in (2\pi i)^{n+1} K^\times_{\underline{\lambda}} \,.$$

2.1.5 Hodge theory

The de Rham cohomology group $H^1_{dR}(\mathcal{M}_{1,1};\mathcal{V}_n)$ admits an increasing weight filtration W and a decreasing Hodge filtration F by $\mathbb{Q}$-vector spaces. The basis (2.7) is compatible with the Hodge filtration.

Similarly, $H^1(\Gamma;V_n)$ is equipped with an increasing filtration W compatible with the weight filtration on de Rham cohomology via the comparison isomorphism.

Thus, $H^1(\Gamma;V_n)$ defines a mixed Hodge structure and is in fact the Betti realisation of a motive [21]. The latter admits a decomposition (as motives)

$$H^1(\mathcal{M}_{1,1};\mathcal{V}_n)=H^1_{\mathrm{cusp}}(\mathcal{M}_{1,1};\mathcal{V}_n)\oplus H^1_{\mathrm{eis}}(\mathcal{M}_{1,1};\mathcal{V}_n),$$

where $H^1_{\mathrm{eis},dR}(\mathcal{M}_{1,1};\mathcal{V}_n)\cong\mathbb{Q}(-n-1)$ and $H^1_{\mathrm{cusp}}(\mathcal{M}_{1,1};\mathcal{V}_n)$ decomposes, over $\overline{\mathbb{Q}}$, as a direct sum of motives $V_{\underline{\lambda}}$ of rank 2 of type $(n+1,0)$ and $(0,n+1)$.

2.1.6 Real Frobenius and single-valued map

The constructions in this paper are simply a consequence of complex conjugation. The comparison isomorphism fits in the following commuting diagram

$$\begin{array}{ccc}\mathrm{comp}_{B,dR}:H^1_{dR}(\mathcal{M}_{1,1};\mathcal{V}_n)\otimes_{\mathbb{Q}}\mathbb{C} & \xrightarrow{\sim} & H^1(\Gamma;V_n)\otimes_{\mathbb{Q}}\mathbb{C}\\ \downarrow & & \downarrow\\ \mathrm{comp}_{B,dR}:H^1_{dR}(\mathcal{M}_{1,1};\mathcal{V}_n)\otimes_{\mathbb{Q}}\mathbb{C} & \xrightarrow{\sim} & H^1(\Gamma;V_n)\otimes_{\mathbb{Q}}\mathbb{C}\end{array}$$

where the vertical map on the left is the $\mathbb{C}$-anti-linear isomorphism c_{dR} which is the identity on $H^1_{dR}(\mathcal{M}_{1,1};\mathcal{V}_n)$ and complex conjugation on $\mathbb{C}$; and the vertical map on the right is $F_\infty\otimes c_B$ where c_B is complex conjugation on the coefficients.

It follows that the real Frobenius F_∞ induces an isomorphism which we have had occasion to call the 'single-valued map' [5, Sect. 4.1]:

$$\mathbf{s}:H^1_{dR}(\mathcal{M}_{1,1};\mathcal{V}_n)\otimes_{\mathbb{Q}}\mathbb{C}\xrightarrow{\sim}H^1_{dR}(\mathcal{M}_{1,1};\mathcal{V}_n)\otimes_{\mathbb{Q}}\mathbb{C}\,.$$

It is none other than the composition

$$\mathbf{s}=\mathrm{comp}^{-1}_{B,dR}\circ(F_\infty\otimes\mathrm{id})\circ\mathrm{comp}_{B,dR}\,.$$

It induces an isomorphism on every Hecke eigenspace

$$\mathbf{s}:H^{dR}_{\underline{\lambda}}\otimes\mathbb{C}\xrightarrow{\sim}H^{dR}_{\underline{\lambda}}\otimes\mathbb{C}\,.$$

Written in the basis (2.7), it is given explicitly by the 'single-valued' period matrix

$$\overline{P_{\underline{\lambda}}}^{-1}P_{\underline{\lambda}}=\frac{i}{\det\mathrm{P}_f}\begin{pmatrix}\eta^+_{\underline{\lambda}}\omega^-_{\underline{\lambda}}+\omega^+_{\underline{\lambda}}\eta^-_{\underline{\lambda}} & 2\omega^+_{\underline{\lambda}}\omega^-_{\underline{\lambda}}\\ -2\eta^+_{\underline{\lambda}}\eta^-_{\underline{\lambda}} & -\eta^+_{\underline{\lambda}}\omega^-_{\underline{\lambda}}-\omega^+_{\underline{\lambda}}\eta^-_{\underline{\lambda}}\end{pmatrix}.$$

On the Hecke eigenspace corresponding to Eisenstein series, which is a pure Tate motive $\mathbb{Q}(-n-1)$, $\mathbf{s}$ is multiplication by -1 and $\mathbf{s}(\mathbb{G}_{2n+2})=-\mathbb{G}_{2n+2}$. For cusp forms,

$$\mathbf{s}(f)=\Big(\frac{\eta^+_f\omega^-_f+\eta^-_f\omega^+_f}{\eta^-_f\omega^+_f-\eta^+_f\omega^-_f}\Big)f+\Big(\frac{2\,\omega^+_f\omega^-_f}{\eta^+_f\omega^-_f-\eta^-_f\omega^+_f}\Big)f'\,.$$

From this formula for $\mathbf{s}(f)$ and the equation $\{f,f'\}=1$ we find that

$$\{f,\mathbf{s}(f)\}=\frac{2i\omega^+_{\underline{\lambda}}\omega^-_{\underline{\lambda}}}{\det(\mathrm{P}_f)},\tag{2.14}$$

which by Proposition 5.6 of [6] is proportional (depending on one's choice of normalisation) to the Petersson norm of f. One could define the Petersson norm of f' to be $\{f',\mathbf{s}(f')\}$, which, in the case when f' is normalised by $\{f',f'\}=0$, gives

$$\{f',\mathbf{s}(f')\}=\frac{2i\eta^+_{\underline{\lambda}}\eta^-_{\underline{\lambda}}}{\det(\mathrm{P}_f)}.$$

The diagonal entries of the single-valued period matrix are proportional to the permanent

$$\mathrm{perm}\,(\mathrm{P}_{\underline{\lambda}}) = i(\eta^+_{\underline{\lambda}}\omega^-_{\underline{\lambda}} + \omega^+_{\underline{\lambda}}\eta^-_{\underline{\lambda}})\,,$$

and the coefficient of f in $\mathbf{s}(f)$ is the quantity

$$\frac{\mathrm{perm}\,(\mathrm{P}_{\underline{\lambda}})}{\det\,(\mathrm{P}_{\underline{\lambda}})} = \frac{\eta^+_{\underline{\lambda}}/\eta^-_{\underline{\lambda}} + \omega^+_{\underline{\lambda}}/\omega^-_{\underline{\lambda}}}{\eta^+_{\underline{\lambda}}/\eta^-_{\underline{\lambda}} - \omega^+_{\underline{\lambda}}/\omega^-_{\underline{\lambda}}}. \tag{2.15}$$

The constructions above clearly work for the motives [21] of any cuspidal eigenforms of integral weight for congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$. The preceding formula has implications in the case when the motive admits complex multiplication (see "Appendix").

3 The space $\mathcal{M}^!$ of non-holomorphic modular forms

We recall some definitions from [2]. Let

$$\mathbb{L} = \log|q| = i\pi(z-\overline{z}) = -2\pi y, \tag{3.1}$$

which is modular of weights $(-1,-1)$. Recall that $\mathcal{M}^!$ is the complex vector space of real analytic modular functions (1.1) admitting an expansion of the form (1.2). Let $\mathcal{M} \subset \mathcal{M}^!$ denote the subspace of functions for which N is zero, i.e. such that $a^{(k)}_{m,n}$ vanishes if m or n is negative. If $\mathcal{M}^!_{r,s}$ denotes the subspace of functions of modular weight (r,s), then

$$\mathcal{M}^! = \bigoplus_{r,s} \mathcal{M}^!_{r,s}$$

is a bigraded algebra over $\mathbb{C}$. The *constant part* of f is defined to be

$$f^0 = \sum_{|k|\leq M} \mathbb{L}^k a^{(k)}_{0,0} \in \mathbb{C}[\mathbb{L}^{\pm}]\,.$$

We say that f is a cusp form if $f^0 = 0$. The subspace of cusp forms is denoted $\mathcal{S}^! \subset \mathcal{M}^!$, and its component of weights (r,s) is denoted $\mathcal{S}^!_{r,s}$.

3.1 Differential operators

There exist bigraded derivations

$$\partial, \overline{\partial} : \mathcal{M}^! \longrightarrow \mathcal{M}^!$$

of bidegrees $(1,-1)$ and $(-1,1)$, whose restrictions to a component $\mathcal{M}^!_{r,s}$ are

$$\partial_r = (z-\overline{z})\frac{\partial}{\partial z} + r \quad \text{and} \quad \partial_s = (\overline{z}-z)\frac{\partial}{\partial \overline{z}} + s$$

respectively. The following is a straightforward consequence:

$$\begin{aligned} \partial_r \mathbb{L}^k q^m \overline{q}^n &= (2m\mathbb{L} + r + k)\mathbb{L}^k q^m \overline{q}^n, \\ \overline{\partial}_s \mathbb{L}^k q^m \overline{q}^n &= (2n\mathbb{L} + s + k)\mathbb{L}^k q^m \overline{q}^n\,. \end{aligned} \tag{3.2}$$

It is valid for any integers k, m, n, r, s.

Lemma 3.1 *For all r, s, the kernels of $\partial, \overline{\partial}$ are given by*

$$\begin{aligned} (\mathcal{M}^!_{r,s} \cap \ker \partial_r) &\cong \mathbb{L}^{-r}\overline{M}^!_{s-r}, \\ (\mathcal{M}^!_{r,s} \cap \ker \overline{\partial}_s) &\cong \mathbb{L}^{-s} M^!_{r-s}\,. \end{aligned}$$

In particular, $(\ker \partial) \cap (\ker \overline{\partial}) = \mathbb{C}[\mathbb{L}^{\pm}]$.

Since there exist weakly holomorphic modular forms of negative weight, it follows that primitives in $\mathcal{M}^!_{r,s}$, unlike the space $\mathcal{M}_{r,s}$, are never unique.

The bigraded Laplace operator is the linear map

$$\Delta : \mathcal{M}^! \longrightarrow \mathcal{M}^!$$

of bidegree (0, 0), which acts on $\mathcal{M}^!_{r,s}$ by

$$\Delta_{r,s} = -\overline{\partial}_{s-1}\partial_r + r(s-1) = -\partial_{r-1}\overline{\partial}_s + s(r-1)\,. \tag{3.3}$$

Define linear operators

$$\mathsf{h}, \mathsf{w} : \mathcal{M}^! \longrightarrow \mathcal{M}^!$$

by $\mathsf{h}(f) = (r-s)f$ and $\mathsf{w}(f) = (r+s)f$ for all $f \in \mathcal{M}^!_{r,s}$.

Lemma 3.2 *These operators satisfy the equations*

$$[\partial, \overline{\partial}] = \mathsf{h}, \quad [\mathsf{h}, \partial] = 2\partial, \quad [\mathsf{h}, \overline{\partial}] = -2\overline{\partial},$$

i.e. $\partial, \overline{\partial}$ *generate a copy of* $\mathfrak{sl}_2$. *Furthermore,*

$$[\partial, \mathbb{L}] = [\overline{\partial}, \mathbb{L}] = [\partial, \Delta] = [\overline{\partial}, \Delta] = 0\,, \text{ and}$$
$$[\mathbb{L}, \Delta] = \mathsf{w}\,\mathbb{L}, [\mathbb{L}, \mathsf{w}] = 2\mathbb{L}, [\mathbb{L}, \mathsf{h}] = [\Delta, \mathsf{w}] = 0\,,$$

The equations $[\partial, \mathbb{L}] = [\overline{\partial}, \mathbb{L}]$ imply that $\mathbb{L}$ is constant for the differential operators $\partial, \overline{\partial}$, and justify calling f^0 the 'constant' part.

3.2 Bol's operator

Recall the operator

$$D = q\frac{\mathrm{d}}{\mathrm{d}q} = \frac{1}{2\pi i}\frac{\partial}{\partial z}\,.$$

Lemma 3.3 *For all* $n \geq 0$, *the following identity of operators holds:*

$$\mathbb{L}^{n+1}\left(\frac{1}{\pi i}\frac{\partial}{\partial z}\right)^{n+1} = \partial_0\partial_{-1}\ldots\partial_{-n}\,. \tag{3.4}$$

Proof Consider the Weyl ring $\mathbb{Q}[x, \frac{\partial}{\partial x}]$ and write $\theta = x\frac{\partial}{\partial x}$. Then the following identity is easily verified for all $n \geq 1$:

$$\theta(\theta-1)\ldots(\theta-n) = x^{n+1}\left(\tfrac{\partial}{\partial x}\right)^{n+1}\,. \tag{3.5}$$

For example, it can be tested on x^m for $m \geq 0$. Set $d_z = (\pi i)^{-1}\partial/\partial z$ and observe that $\partial_r = \mathbb{L}d_z + r$. Since $d_z\mathbb{L} = 1$, there is an isomorphism $\mathbb{C}[x, \partial/\partial x] \overset{\sim}{\rightarrow} \mathbb{C}[\mathbb{L}, d_z]$ sending x to $\mathbb{L}$ and $\partial/\partial x$ to d_z. The image of $\theta + r$ is ∂_r, so (3.4) is equivalent to (3.5). □

Since ∂ commutes with $\mathbb{L}$, we can write

$$D^{n+1}\Big|_{\mathcal{M}^!_{-n,\bullet}} = \left(\frac{\partial}{2\mathbb{L}}\right)^{n+1}\Big|_{\mathcal{M}^!_{-n,\bullet}}\,. \tag{3.6}$$

This defines for all $s \in \mathbb{Z}$ a linear map

$$D^{n+1} : \mathcal{M}^!_{-n,s} \longrightarrow \mathcal{M}^!_{n+2,s}\,.$$

Its complex conjugate defines a map $\overline{D}^{n+1} : \mathcal{M}^!_{r,-n} \rightarrow \mathcal{M}^!_{r,n+2}$ for all r.

3.3 Vector-valued modular forms

Call a real analytic function $F:\mathfrak{H}\to V_n\otimes\mathbb{C}$ *equivariant* if for every $\gamma\in\mathrm{SL}_2(\mathbb{Z})$ and all $z\in\mathfrak{H}$ it satisfies (Sect. 2.1.2):

$$F(\gamma z)\big|_{\gamma}=F(z)\,.$$

There is a correspondence [2, Sect. 7.2], between sections of the trivial bundle $V_n\otimes\mathbb{C}$ on $\mathfrak{H}$ and families of functions $F_{r,s}:\mathfrak{H}\to\mathbb{C}$ for $r+s=n$ with $r,s\geq 0$. It is given by writing

$$F(z)=\sum_{r+s=n}F_{r,s}(X-zY)^r(X-\overline{z}Y)^s\,. \tag{3.7}$$

Then F is equivariant if and only if each $F_{r,s}$ is modular of weights (r,s). Furthermore, F admits an expansion in $\mathbb{C}[q^{-1},\overline{q}^{-1},q,\overline{q}]][z,\overline{z}]$ if and only if each $F_{r,s}\in\mathcal{M}^!_{r,s}$.

A special case of [2, Proposition 7.2] implies that

$$\mathrm{d}F=\pi i\,f(z)(X-zY)^n\mathrm{d}z+\pi i\,\overline{g(z)}(X-\overline{z}Y)^n\mathrm{d}\overline{z} \tag{3.8}$$

holds if and only if the following system of equations is true:

$$\begin{aligned}\partial F_{r,s}&=(r+1)F_{r+1,s-1}\quad\text{for all } s\geq 1,\\ \overline{\partial}F_{r,s}&=(s+1)F_{r-1,s+1}\quad\text{for all } r\geq 1,\\ \partial F_{n,0}=\mathbb{L}f\quad&,\quad\overline{\partial}F_{0,n}=\mathbb{L}\overline{g}\,.\end{aligned} \tag{3.9}$$

In the present paper, we only consider the case $f,g\in M^!_{n+2}$.

3.4 Some useful lemmas

Lemma 3.4 *Let $f\in\mathcal{M}^!_{r,s}$, and write $h=r-s$. Suppose that $\partial f=0$. Then*

$$\partial\overline{\partial}^k f=k(h-k+1)\,\overline{\partial}^{k-1}f$$

for all integers $k\geq 0$.

Proof It follows from $[\partial,\overline{\partial}]=\mathsf{h}$ and induction that

$$\partial\overline{\partial}^k-\overline{\partial}^k\partial=\sum_{i+j=k-1,i,j\geq 0}\overline{\partial}^i\,\mathsf{h}\,\overline{\partial}^j \tag{3.10}$$

Applying this to f gives the stated formula. □

Corollary 3.5 *Let $f\in\mathcal{M}^!_{r,s}$ with $r\geq s$. Let $h=r-s\geq 0$. Then if*

$$\partial f=0\qquad\textit{and}\qquad\overline{\partial}^{h+1}f=0,$$

then $f\in\mathbb{C}\mathbb{L}^{-r}$ if $r=s$ and f vanishes if $h>0$.

Proof By Lemma 3.1, $\partial f=0$ implies that $f\in\mathbb{L}^{-r}\overline{M}^!_{s-r}$. In particular, the coefficients in its expansion (1.2) satisfy $a^{(k)}_{m,n}(f)=0$ if $m\neq 0$. This property is stable under $\overline{\partial}$, so the same holds for all $\overline{\partial}^n f$. Again by Lemma 3.1, $\overline{\partial}^{h+1}f=0$ implies that $\overline{\partial}^h f\in\mathbb{L}^{-s}M^!$, and its coefficients satisfy $a^{(k)}_{m,n}(\overline{\partial}^h f)=0$ if either $m\neq 0$ or $n\neq 0$. It follows that $\overline{\partial}^h f\in\mathbb{C}[\mathbb{L}^{\pm}]$. If $h=0$, then $f\in\mathcal{M}^!_{r,r}$ and we have shown that $f\in\mathbb{C}\mathbb{L}^{-r}$. Now if $h>0$, $\overline{\partial}^h f\in\mathcal{M}^!_{s,r}$, and it follows that $\overline{\partial}^h f=0$ since all powers of $\mathbb{L}$ lie on the diagonal $\mathsf{h}=0$. Applying the previous lemma to f, we find that

$$\partial\overline{\partial}^k f=k(h-k+1)\,\overline{\partial}^{k-1}f$$

and so by decreasing induction on k, for $1\leq k\leq h$, we deduce that $\overline{\partial}^{k-1}f$ vanishes for all $k\geq 1$. This completes the proof. □

4 The space $\mathcal{HM}^!$ of harmonic functions

Definition 4.1 Let $\mathcal{HM}^! \subset \mathcal{M}^!$ (respectively, $\mathcal{HM} \subset \mathcal{M}$) denote the space of functions which are eigenfunctions of the Laplacian. For any $\lambda \in \mathbb{C}$ let

$$\mathcal{HM}^!(\lambda) = \ker\left(\Delta - \lambda : \mathcal{M}^! \longrightarrow \mathcal{M}^!\right)$$

denote the eigenspace with eigenvalue λ.

Lemma 4.2 *The space $\mathcal{HM}^!(\lambda)$ is stable under the action of $\mathfrak{sl}_2$:*

$$\partial, \overline{\partial} : \mathcal{HM}^!(\lambda) \longrightarrow \mathcal{HM}^!(\lambda)$$

and furthermore, multiplication by $\mathbb{L}$ is an isomorphism

$$\mathbb{L} : \mathcal{HM}^!_{r+1,s+1}(\lambda) \longrightarrow \mathcal{HM}^!_{r,s}(\lambda - r - s)\,. \tag{4.1}$$

Proof The first equation follows since $[\nabla, \partial] = [\nabla, \overline{\partial}] = 0$ by Lemma 3.2. For the second, $[\mathbb{L}, \Delta] = w\mathbb{L}$ implies that if $\Delta F = \lambda F$, then $\Delta(\mathbb{L}F) = (\lambda - w)\mathbb{L}F$. □

The lemma remains true on replacing $\mathcal{HM}^!(\lambda)$ by $\mathcal{HM}(\lambda) = \mathcal{HM}^!(\lambda) \cap \mathcal{M}$.

Lemma 4.3 *Every Laplace eigenvalue is an integer:*

$$\mathcal{HM}^! = \bigoplus_{n \in \mathbb{Z}} \mathcal{HM}^!(n)\,.$$

Every element $F \in \mathcal{HM}^!(\lambda)$ has a unique decomposition

$$F = F^h + F^0 + F^a\,, \tag{4.2}$$

where $F^0 \in \mathbb{C}[\mathbb{L}^{\pm}]$ is the constant part of F, and

$$F^h \in \mathbb{C}[q^{-1}, q]][\mathbb{L}^{\pm}],$$
$$F^a \in \mathbb{C}[\overline{q}^{-1}, \overline{q}]][\mathbb{L}^{\pm}]$$

are the (weakly) 'holomorphic' and 'antiholomorphic' parts of F and have no constant terms. Furthermore, each piece is an eigenfunction: $\Delta F^\bullet = \lambda F^\bullet$ for $\bullet \in \{h, 0, a\}$.

Proof This was proved for the space $\mathcal{HM}$ in [2, lemma 5.2]. The proof is more or less identical for $\mathcal{M}^!$. □

One can be more precise ([2], Sect. 5.1). Let $F \in \mathcal{HM}^!_{r,s}$ with eigenvalue $\lambda \in \mathbb{Z}$. Let $w = r + s$ be the total weight. Then there exists a $k_0 \in \mathbb{Z}$ such that

$$F^0 \quad \in \quad \mathbb{C}\mathbb{L}^{k_0} \oplus \mathbb{C}\mathbb{L}^{1-w-k_0}, \tag{4.3}$$

where $k_0 < 1 - w - k_0$ and $\lambda = k_0(1 - w - k_0)$, and furthermore:

$$F^h \quad \in \quad \bigoplus_{k=k_0}^{-s} \mathbb{C}[q^{-1}, q]]\mathbb{L}^k, \quad F^a \quad \in \quad \bigoplus_{k=k_0}^{-r} \mathbb{C}[\overline{q}^{-1}, \overline{q}]]\mathbb{L}^k\,. \tag{4.4}$$

5 The space $\mathcal{MI}^!_1$ of weak modular primitives

The subspace $\mathcal{MI}^! \subset \mathcal{M}^!$ of modular iterated integrals was defined in [2].

Definition 5.1 Let $\mathcal{MI}^!_{-1} = 0$. For every $k \geq 0$, let

$$\mathcal{MI}^!_k \subset \bigoplus_{r,s \geq 0} \mathcal{M}^!_{r,s}$$

be the largest subspace which is concentrated in the positive quadrant of $\mathcal{M}^!$ (with modular weights (r, s) with $r, s \geq 0$) with the property that

$$\begin{aligned} \partial \mathcal{MI}^!_k &\subset \mathcal{MI}^!_k + M^![\mathbb{L}] \otimes \mathcal{MI}^!_{k-1}, \\ \overline{\partial} \mathcal{MI}^!_k &\subset \mathcal{MI}^!_k + \overline{M^!}[\mathbb{L}] \otimes \mathcal{MI}^!_{k-1} \end{aligned} \tag{5.1}$$

for all $k \geq 0$. We define $\mathcal{MI}^! = \sum_k \mathcal{MI}^!_k$. It is closed under complex conjugation.

We call the increasing filtration $\mathcal{MI}^!_k \subset \mathcal{MI}^!$ the length. In this paper we shall focus only on length ≤ 1. We first dispense with the subspace of length 0.

Proposition 5.2 $\mathcal{MI}^!_0 = \mathbb{C}[\mathbb{L}^{-1}]$.

Proof Firstly, the space $\mathbb{C}[\mathbb{L}^{-1}]$ satisfies the conditions of the definition since $[\partial, \mathbb{L}] = [\overline{\partial}, \mathbb{L}] = 0$, and so $\mathbb{C}[\mathbb{L}^{-1}] \subset \mathcal{MI}^!_0$. Now let $F \in \mathcal{MI}^!_0$ be of modular weights $(n, 0)$, where $n \geq 0$. Since ∂F has weights $(n+1, -1)$, which lies outside the positive quadrant, we must by (5.1) and $\mathcal{MI}^!_{-1} = 0$ have $\partial F = 0$. Similarly, the element $F' = \overline{\partial}^n F$ has weights $(0, n)$ and so $\overline{\partial} F' = 0$ since it also lies outside the positive quadrant. By Corollary 3.5, F vanishes if $n > 0$ and $F \in \mathbb{C}$ if $n = 0$. By complex conjugation, it follows that $\mathcal{MI}^!_0$ vanishes in modular weights $(0, n)$ and $(n, 0)$ for all $n \geq 1$ and is contained in $\mathbb{C}$ in weights $(0, 0)$. We can now repeat the argument for any $F \in \mathcal{MI}^!_0$ of modular weights $(n, 1)$ by replacing F with $\mathbb{L}F$ and arguing as above. We deduce that $\mathcal{MI}^!_0$ vanishes in all weights $(n, 1)$ and $(1, n)$ for $n \geq 2$ and is contained in $\mathbb{C}\mathbb{L}^{-1}$ in weights $(1, 1)$. Continuing in this manner, we conclude that $\mathcal{MI}^!_0 \subset \mathbb{C}[\mathbb{L}^{-1}]$. □

5.1 Modular iterated integrals of length one

It follows from the previous proposition that $\mathcal{MI}^!_1$ is the largest subspace of $\mathcal{M}^!$ which satisfies

$$\begin{aligned} \partial \mathcal{MI}^!_1 &\subset \mathcal{MI}^!_1 + M^![\mathbb{L}^{\pm}], \\ \overline{\partial} \mathcal{MI}^!_1 &\subset \mathcal{MI}^!_1 + \overline{M^!}[\mathbb{L}^{\pm}]. \end{aligned} \tag{5.2}$$

In particular, any element $F \in \mathcal{MI}^!_1$ of weights $(n, 0)$, with $n \geq 0$, satisfies

$$\partial F = \mathbb{L} f$$

for some $f \in M^!_{n+2}$ weakly holomorphic of weight $n+2$. We call such an element a *modular primitive* of $\mathbb{L}f$. It is necessarily a Laplace eigenfunction with eigenvalue $-n$ since $(\Delta + n)F = -\overline{\partial}\partial F = 0$ by (3.3).

Remark 5.3 As a consequence, $\mathbb{L}^{-1}F$ satisfies $\partial \mathbb{L}^{-1} F = f$ and $\Delta \mathbb{L}^{-1} F = 0$. It is therefore what is known as a weak harmonic lift of f.

Proposition 5.4 *Let $n \geq 0$. Let f be a weakly holomorphic modular form of weight $n+2$, and let $X_{n,0} \in \mathcal{M}^!$ be a primitive of $\mathbb{L}f$:*

$$\partial X_{n,0} = \mathbb{L} f\ .$$

Then there exist unique elements $X_{r,s} \in \mathcal{M}^!_{r,s}$, for $r, s \geq 0$ and $r + s = n$ such that

$$\begin{aligned} \partial X_{r,s} &= (r+1) X_{r+1,s-1} \quad \textit{for } s \geq 1, \\ \overline{\partial} X_{r,s} &= (s+1) X_{r-1,s+1} \quad \textit{for } r \geq 1 \end{aligned} \tag{5.3}$$

and

$$\overline{\partial} X_{0,n} = \mathbb{L}\overline{g}$$

for some $g \in M^!_{n+2}$ a weakly holomorphic modular form of weight $n+2$. It follows that $(\Delta + n)X_{r,s} = 0$ for all $r+s=n$, i.e. $X_{r,s} \in \mathcal{HM}^!(-n)$.

Proof Suppose that $X_{n,0}$ is a primitive of $\mathbb{L}f$. Define $X_{r,s}$ by the formula

$$X_{r,s} = \frac{\overline{\partial}^s}{s!} X_{n,0} \tag{5.4}$$

for all $r+s=n$, $r,s \geq 0$. The second equation of (5.3) holds for all r,s. For the first equation, apply identity (3.10) to $X_{n,0}$ to obtain

$$\partial \overline{\partial}^k X_{n,0} - \overline{\partial}^k \partial X_{n,0} = k(n-k+1)\overline{\partial}^{k-1} X_{n,0}\,.$$

For $k \geq 1$ the second term is $\mathbb{L}\,\overline{\partial}^k f$, which vanishes. Therefore, by (5.4),

$$k!\,\partial X_{n-k,k} = k(n-k+1)(k-1)! X_{n-k+1,k-1}$$

which is exactly the first equation of (5.3). Applying $\mathsf{h} = [\partial, \overline{\partial}]$ to $X_{0,n}$, and using the equations (5.3), one finds that $\partial\overline{\partial} X_{0,n} = 0$. Therefore,

$$\overline{\partial} X_{0,n} \in \mathcal{M}^!_{-1,n+1} \cap \ker \partial_{-1},$$

and by Lemma 3.1, it follows that $\overline{\partial} X_{0,n} = \mathbb{L}\overline{g}$ for some $g \in M^!_{n+2}$ as claimed. Finally, the fact that the $X_{r,s}$ are Laplace eigenfunctions with eigenvalue $-n$ follows easily from (3.3), (5.3) and, when $n=0$, the equations $\partial X_{n,0} = \mathbb{L}f$, $\overline{\partial} X_{0,n} = \mathbb{L}\overline{g}$. ◻

Remark 5.5 If we define $\mathfrak{X} : \mathfrak{H} \to V_n \otimes \mathbb{C}$ by

$$\mathfrak{X} = \sum_{r+s=n} X_{r,s}(X - zY)^r (X - \overline{z}Y)^s$$

then $\mathfrak{X}$ is modular equivariant, and equations (5.3) are equivalent to

$$\mathrm{d}\mathfrak{X} = \frac{2\pi i}{2}\Big(f(z)(X-zY)^n \mathrm{d}z + \overline{g(z)}(X-\overline{z}Y)^n \mathrm{d}\overline{z}\Big)\,.$$

The fact that the coefficients $X_{r,s}$ are eigenfunctions is equivalent to the identity

$$\frac{\partial^2}{\partial z \partial \overline{z}} \mathfrak{X} = 0\,.$$

We now turn to uniqueness.

Lemma 5.6 *Let $X_{n,0}$ (respectively, $X'_{n,0}$) be modular primitives of $\mathbb{L}f$, and let $X_{r,s}, g$ (resp. $X'_{r,s}, g'$) be the functions in $\mathcal{M}^!$ defined in Proposition 5.4. Then there exists a weakly holomorphic modular form $\xi \in M^!_{-n}$ such that for all $r+s=n$ and $r,s \geq 0$*

$$X'_{r,s} - X_{r,s} = \mathbb{L}^{-n} \frac{\overline{\partial}^s}{s!} \overline{\xi}\,,$$

and

$$g' - g = \frac{1}{n!} \overline{\partial}^{n+1} \mathbb{L}^{-n-1} \xi\,.$$

In other words, g and g' are equivalent modulo $D^{n+1} M^!_{-n}$.

Proof By Lemma 3.1, $X'_{n,0} - X_{n,0} \in \mathbb{L}^{-n}\overline{M^!}_{-n}$. Apply (5.4) and (3.6) to conclude. ◻

Corollary 5.7 *If $X_{n,0}$ is a primitive of $\mathbb{L}f$, and $X_{r,s}$, g are as defined in Proposition 5.4, then $Y_{r,s} = \overline{X}_{s,r}$ is a system of solutions to the equations (5.3) and satisfies*

$$\partial Y_{n,0} = \mathbb{L}g \quad \text{and} \quad \overline{\partial} Y_{0,n} = \mathbb{L}\overline{f}\,.$$

Therefore, complex conjugation reverses the roles of f and g.

5.2 Harmonic functions and structure of $\mathcal{MI}_1^!$

We show that the modular primitives of Proposition 5.4 generate $\mathcal{MI}_1^!$ under multiplication by $\mathbb{L}^{-1}$. This section can be skipped and is not required for the rest of the paper.

Proposition 5.8 *Modular integrals of length one lie in the harmonic subspace of $\mathcal{M}^!$:*

$$\mathcal{MI}_1^! \subset \mathcal{HM}^!\,.$$

More precisely, any element $F \in \mathcal{MI}_1^!$ of modular weights (r,s) can be uniquely decomposed as a linear combination of elements

$$F = \sum_{0\leq k\leq \min\{r,s\}} F_k,$$

where $F_k \in \mathcal{MI}_1^!$ also has modular weights (r,s) and satisfies:

$$\Delta F_k = (k-1)(r+s-k)\,F_k\,.$$

Specifically, if $r \geq s$, each F_k is of the form $F_k = \mathbb{L}^{-k}\overline{\partial}^{s-k}X_k$, for some X_k a modular primitive of $\mathbb{L}f_k$, where $f_k \in M^!_{r+s+2-2k}$ is weakly holomorphic.

In the case $s \leq r$, we can take $F_k = \mathbb{L}^{-k}\partial^{r-k}\overline{X}_k$, with X_k a modular primitive of $\mathbb{L}g_k$, where $g_k \in M^!_{r+s+2-2k}$ is weakly holomorphic.

Proof Suppose that F is in $\mathcal{MI}_1^!$ of modular weights (r,s) with $r \geq s$. We show by induction on s that it is a linear combination:

$$F = \sum_{0\leq k\leq s} F_k \quad \text{where } F_k = \mathbb{L}^{-k}\frac{\overline{\partial}^{s-k}}{(s-k)!}X_k \in \mathcal{M}^!_{r,s}, \tag{5.5}$$

where $\partial X_k \in \mathbb{L}M^!$, and hence, X_k is a modular primitive of total weight $r+s-2k$. By Proposition 5.4, X_k is a Laplace eigenfunction with eigenvalue $2k-r-s$, and it follows from $[\Delta, \overline{\partial}] = 0$ and (4.1) that F_k is also an eigenfunction with eigenvalue

$$(2k-w) + (w-2k) + (w-2k+2) + \cdots + (w-2) = (k-1)(w-k),$$

where we write $w = r+s$. Since these eigenvalues are distinct for distinct values of $0 \leq k \leq w/2$, the F_k are linearly independent and the decomposition is unique.

The statement (5.5) is true for F of modular weights $(n,0)$: in that case (5.2), together with the fact that ∂F lies outside the positive quadrant, implies that

$$\partial F \in M^![\mathbb{L}^{\pm}]$$

and hence $\partial F = \mathbb{L}f$, for some $f \in M^!_{n+2}$. Therefore, F is a modular primitive of $\mathbb{L}f$, and by Proposition 5.4, an eigenfunction of the Laplacian with eigenvalue $-n$. Now suppose that $F \in \mathcal{MI}_1^!$ of modular weights (r,s) with $r \geq s \geq 0$ and suppose that (5.5) is true for all smaller values of s. Then since

$$\partial F \in \mathcal{MI}_1^! + M^![\mathbb{L}^{\pm}]$$

has modular weights $(r+1, s-1)$, the induction hypothesis implies that

$$\partial F = \mathbb{L}^{1-s} f + \sum_{0 \leq k \leq s-1} \mathbb{L}^{-k} \frac{\overline{\partial}^{s-1-k}}{(s-1-k)!} X_k$$

for some $f \in M^!_{r-s+2}$. From the proof of Proposition 5.4, each term $\frac{\overline{\partial}^{s-1-k}}{(s-1-k)!} X_k$ has a modular primitive $\frac{\overline{\partial}^{s-k}}{(s-k)!} X_k$. Define X_s via the formula

$$\mathbb{L}^{-s} X_s = F - \sum_{0 \leq k \leq s-1} \mathbb{L}^{-k} \frac{\overline{\partial}^{s-k}}{(s-k)!} X_k \,.$$

Then X_s is a modular primitive of $\mathbb{L}f$ and F is of the required form, completing the induction step. The case where $s \geq r$ follows by complex conjugating, which reverses the roles of r and s. Taking both cases together implies the first statement. □

In particular,

- an element $F \in \mathcal{MI}^!_1$ of modular weights $(n, 0)$ is necessarily an eigenfunction of the Laplacian with eigenvalue $-n$.
- an element $F \in \mathcal{MI}^!_1$ of modular weights $(n-1, 1)$ is a linear combination of two eigenfunctions of the Laplacian with possible eigenvalues $\{-n, 0\}$.
- an element $F \in \mathcal{MI}^!_1$ of total weight w can have eigenvalues in the set

$$\{-w\,,\ 0\,,\ w-2\,,\ 2(w-3)\,,\ 3(w-4)\,,\ \ldots\,,\ \tfrac{w}{2}(1-\tfrac{w}{2})\}.$$

Remark 5.9 Elements in $\mathcal{MI}^!_k$ for $k \geq 2$ are no longer harmonic and satisfy a more complicated structure with respect to the Laplace operator. See, for example, [3, Sect. 11.3-4].

5.3 Ansatz for primitives

Recall that for $f \in M^!_{n+2}$ a weakly holomorphic modular form, the functions $f^{(k)}$ and $R_{r,s}(f)$ were defined in (1.7) and (1.8). In particular,

$$\begin{aligned} R_{n,0}(f) &= (-1)^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{k!}{\mathbb{L}^k} f^{(k+1)}, \\ R_{0,n}(f) &= (-1)^n \frac{n!}{\mathbb{L}^n} f^{(n+1)}. \end{aligned} \tag{5.6}$$

We shall write $R_{r,s}$ instead of $R_{r,s}(f)$ when f is understood.

Proposition 5.10 *The functions $R_{r,s}$ satisfy*

$$\begin{aligned} \partial_r R_{r,s} &= (r+1) R_{r+1,s-1} \quad \textit{for all } s \geq 1 \\ \partial_n R_{n,0} &= (-1)^n \mathbb{L} f^{(0)}\,. \end{aligned}$$

Furthermore,

$$\overline{\partial}_s R_{r,s} = \begin{cases} (s+1) R_{r-1,s+1} & \textit{if } r \geq 1, \\ 0 & \textit{if } r = 0. \end{cases}$$

Proof Let us write

$$S_{r,s} = \sum_{k=s}^{n} \binom{r}{k-s} (-1)^k \frac{k!}{\mathbb{L}^k} f^{(k+1)}\,,$$

and show that for all $s \geq 1$,

$$\partial_r S_{r,s} + sS_{r+1,s-1} = 0\,.$$

We first verify using (3.2) that

$$\begin{aligned}\partial_r \mathbb{L}^{-k} f^{(k+1)} &= \left(\sum_{m \in \mathbb{Z}\setminus 0} \frac{2m}{(2m)^{k+1}} a_m \mathbb{L}^{1-k} q^m \right) + (r-k)\, \mathbb{L}^{-k} f^{(k+1)} \\ &= \mathbb{L}^{1-k} f^{(k)} + (r-k)\mathbb{L}^{-k} f^{(k+1)}\,.\end{aligned}$$

It follows that

$$\begin{aligned}\partial_r S_{r,s} + sS_{r+1,s-1} = \sum_{k\leq n} &\binom{r}{k-s} (-1)^k k! \Big(\mathbb{L}^{1-k} f^{(k)} + (r-k)\mathbb{L}^{-k} f^{(k+1)} \Big) \\ &+ s\Big(\binom{r+1}{k-s+1} (-1)^k k! \mathbb{L}^{-k} f^{(k+1)} \Big)\end{aligned}$$

Using $r+s=n$, the right-hand side reduces to

$$\sum_{k\leq n} \binom{r}{k-s} (-1)^k \frac{k!}{\mathbb{L}^k} f^{(k+1)} \Big[\frac{-(k+1)(n-k)}{k-s+1} + (r-k) + \frac{s(r+1)}{k-s+1} \Big] = 0$$

since the term in square brackets simplifies to zero. Finally, since

$$R_{r,s} = (-1)^r \binom{n}{r} S_{r,s}\,,$$

we find that for all $s \geq 1$,

$$\begin{aligned}\partial_r R_{r,s} - (r+1)R_{r+1,s-1} &= (-1)^r \frac{(r+s)!}{r!s!} \partial_r S_{r,s} - (-1)^{r+1} \frac{(r+s)!(r+1)}{(r+1)!(s-1)!} S_{r+1,s-1} \\ &= (-1)^r \frac{(r+s)!}{r!s!} \Big(\partial_r S_{r,s} + sS_{r+1,s-1} \Big)\end{aligned}$$

which vanishes. This proves the first equation. For the second, by (5.6), we have

$$\partial_n R_{n,0} = (-1)^n \sum_{k=0}^{n} \frac{n!}{(n-k)!} (-1)^k \big[\mathbb{L}^{1-k} f^{(k)} + (n-k)\mathbb{L}^{-k} f^{(k+1)} \big]$$

By telescoping, only the first term in square brackets (for $k=0$), and the second term (for $k=n$) survive. The latter is zero, and the former is exactly $(-1)^n \mathbb{L} f^{(0)}$.

For the last part, compare $\overline{\partial}_s R_{r,s}$ and $(s+1)R_{r-1,s+1}$ using:

$$(-1)^r \binom{n}{r} \binom{r}{k-s} (s-k) = (-1)^{r-1} \binom{n}{r-1} \binom{r-1}{k-s-1} (n-r+1).$$

where $n = r+s$. The case $r=0$ is immediate from Lemma 3.1. □

Lemma 5.11 *Let $E : \mathfrak{H} \to V_n \otimes \mathbb{C}$ be real analytic and T-equivariant such that*

$$\frac{\partial E}{\partial z} = 0, \quad \frac{\partial E}{\partial \overline{z}} = c(X - \overline{z}Y)^n,$$

where $c \in \mathbb{C}$. Then $c = 0$ and $E = \frac{\alpha}{(\pi i)^n} Y^n$ for some $\alpha \in \mathbb{C}$. Writing

$$E = \sum_{r+s=n} E_{r,s} (X - zY)^r (X - \overline{z}Y)^s,$$

we find that

$$E_{r,s} = \alpha(-1)^r \binom{n}{r} \mathbb{L}^{-n}\,.$$

If E is modular equivariant and $n > 0$ then α vanishes.

Proof Consider the function $e : \mathfrak{H} \to \mathbb{C}$ obtained by composing E with $V_n \otimes \mathbb{C} \to (V_n \otimes \mathbb{C})/Y\mathbb{C} \cong \mathbb{C}$. It is the coefficient of X^n in E. It satisfies $\frac{\partial e}{\partial z} = 0$ and $\frac{\partial e}{\partial \overline{z}} = c$ and therefore $e = c\overline{z} + \beta$ for some $\beta \in \mathbb{C}$. Since T fixes Y and acts on X by $T(X) = X + Y$, the condition of T-invariance implies that $e(z+1) = e(z)$. This forces $c = 0$. It follows that $\frac{\partial}{\partial z}E = \frac{\partial}{\partial \overline{z}}E = 0$ and so E is constant. By T-invariance, E lies in $V_n^T = \mathbb{C}Y^n$, and hence $E = \frac{\alpha}{(\pi i)^n}Y^n$ for some $\alpha \in \mathbb{C}$. But

$$\begin{aligned} E &= \frac{\alpha}{(\pi i)^n}Y^n = \frac{\alpha}{(\pi i)^n}\frac{1}{(z-\overline{z})^n}\Big((X-\overline{z}Y)-(X-zY)\Big)^n \\ &= \alpha\mathbb{L}^{-n}\sum_{r+s=n}(-1)^r\binom{n}{r}(X-zY)^r(X-\overline{z}Y)^s \end{aligned}$$

since $\mathbb{L} = \pi i(z-\overline{z})$, which proves the formula for $E_{r,s}$.

Finally, if E is modular equivariant, $E_{r,s} \in \mathbb{C}\mathbb{L}^{-n}$ is modular of weights (r,s) with $r+s = n > 0$. But $\mathbb{L}^{-n}$ is modular of weights (n,n), which implies that $E_{r,s} = 0$. □

Corollary 5.12 *Let $f \in M^!_{n+2}$ be a weakly holomorphic modular form. Let $X_{n,0}$ be a modular primitive of $\mathbb{L}f$, and let $X_{r,s}$ and $g \in M^!_{n+2}$ be as determined by Proposition 5.4. Then the zeroth Fourier coefficients of f and g are conjugate:*

$$a = a_0(f) = \overline{a_0(g)}$$

and there exists some $\alpha \in \mathbb{C}$ such that

$$X_{r,s} = \frac{a}{n+1}\mathbb{L} + \alpha(-1)^r\binom{n}{r}\mathbb{L}^{-n} + R_{r,s}(f) + \overline{R_{s,r}(g)} \tag{5.7}$$

for all $r,s \geq 0$ and $r+s = n$.

Proof Let $a = a_0(f)$. Define

$$Y_{r,s} = \frac{a}{n+1}\mathbb{L} + R_{r,s}(f) + \overline{R_{s,r}(g)}\,.$$

We first check that the expression for $Y_{r,s}$ satisfies the equations (5.3). By (3.2), we have $\partial_r\mathbb{L} = (r+1)\mathbb{L}$, and by Proposition 5.10, we deduce that $\partial_r Y_{r,s} = (r+1)Y_{r+1,s-1}$ for all $s \geq 1$. Similarly, using the fact that n is even, we check that

$$\partial_n Y_{n,0} = a\,\mathbb{L} + \partial_n R_{n,0}(f) = a\,\mathbb{L} + (-1)^n f^{(0)}\,\mathbb{L} = \mathbb{L}f\ .$$

By complex conjugating, $\overline{\partial}_s Y_{r,s} = (s+1)Y_{r-1,s+1}$ for all $r \geq 1$, and

$$\overline{\partial}_n Y_{0,n} = a\,\mathbb{L} + \overline{\partial}_n\overline{R_{n,0}(g)} = \mathbb{L}(a + \overline{g}^{(0)})\,.$$

Define $E_{r,s} = X_{r,s} - Y_{r,s}$. The function $E = \sum_{r+s=n}E_{r,s}(X-zY)^r(X-\overline{z}Y)^s$ satisfies

$$\frac{\partial E}{\partial z} = 0 \quad \text{and} \quad \frac{\partial E}{\partial \overline{z}} = \pi i(\overline{a_0(g)} - a)(X-\overline{z}Y)^n$$

by (3.8). It is a real analytic and T-invariant section of $V_n \otimes \mathbb{C}$, since $X_{r,s}$ and $Y_{r,s}$ are T-invariant. By the previous lemma we conclude that there exists an $\alpha \in \mathbb{C}$ such that

$$X_{r,s} = \alpha(-1)^r\binom{n}{r}\mathbb{L}^{-n} + Y_{r,s}$$

for all $r+s = n$, and furthermore, that $\overline{a_0(g)} = a$. □

We shall determine the unknown coefficient α using Hecke operators. Another way to prove the corollary is to use the fact that $X_{r,s}$ are eigenfunctions of the Laplacian (Proposition 5.8) and the explicit shape (4.3) and (4.4) for the latter. We chose the approach above since it explains the origin of the indeterminate coefficient α, and since functions in $\mathcal{MI}$ are not harmonic in general.

Corollary 5.13 *A modular primitive of $\mathbb{L}f$, if it exists, is of the form:*

$$X_{n,0} = \frac{a}{n+1}\mathbb{L} + \frac{\alpha}{\mathbb{L}^n} + \frac{n!}{\mathbb{L}^n}\overline{g^{(n+1)}} + \sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{k!}{\mathbb{L}^k}f^{(k+1)}\,. \tag{5.8}$$

5.4 Example: real analytic Eisenstein series

Let $\mathcal{E}_{r,s}$ denote the functions defined in the introduction. By [2, Proposition 4.3], and equation (5.7), we have

$$\mathcal{E}_{r,s} = \mathcal{E}^0_{r,s} + R_{r,s}(\mathbb{G}_{w+2}) + R_{s,r}(\overline{\mathbb{G}_{w+2}})\,,$$

where

$$\mathcal{E}^0_{r,s} = -\frac{B_{w+2}}{2(w+1)(w+2)}\mathbb{L} + (-1)^r\binom{w}{r}\frac{w!}{2^{w+1}}\zeta(w+1)\mathbb{L}^{-w}\,.$$

In this example the coefficient α is an odd zeta value, which is the period of a non-trivial extension of Tate motives, and is conjecturally transcendental. It can be obtained as a special value of a suitably defined L-function of $\mathcal{E}_{r,s}$ (see [2], Sect. 9.4).

6 Hecke operators

We review some basic properties of Hecke operators. For any $\alpha \in \mathrm{GL}_2(\mathbb{R})$ write

$$\alpha = \begin{pmatrix} a_\alpha & b_\alpha \\ c_\alpha & d_\alpha \end{pmatrix}, \tag{6.1}$$

i.e. a, b, c, d are the standard generators on the affine ring $\mathcal{O}(\mathrm{GL}_2)$.

6.1 Definition

Let $f : \mathfrak{H} \to V_n \otimes \mathbb{C}$ be real analytic and equivariant. Let $m \geq 1$ be an integer, and let M_m denote the set of 2×2 matrices with integer entries which have determinant m. The Hecke operator is defined by the formula[2]

$$T_m f(z) = \frac{1}{m}\sum_{\alpha\in\Gamma\backslash M_m} f(\alpha z)\big|_\alpha\,.$$

Since f is equivariant, it follows that for all $\gamma \in \Gamma$,

$$f(\gamma\alpha z)\big|_{\gamma\alpha} = f(\gamma(\alpha z))\big|_\gamma\big|_\alpha = f(\alpha z)\big|_\alpha$$

and hence the formula for $T_m f$ is well defined. The set of cosets $\Gamma\backslash M_m$ is finite and are described below. Since right multiplication by any $\gamma \in \Gamma$ defines a bijection of cosets

[2]The reason for the factor m^{-1} is that f is a function; the usual formula for Hecke operators involves one-forms: for α as in (6.1),

$$\mathrm{d}(\alpha z) = \frac{\det(\alpha)}{(c_\alpha z + \mathrm{d}_\alpha)^2}\mathrm{d}z\,,$$

and the $\det(\alpha)$ accounts for an extra multiple of m in the formula for T_m.

$\Gamma\backslash M_m \overset{\sim}{\longrightarrow} \Gamma\backslash M_m$, we deduce from the calculation

$$(T_m f)(\gamma z)\big|_\gamma = \sum_{\alpha\in\Gamma\backslash M_m} f(\alpha\gamma z)\big|_{\alpha\gamma} = \sum_{\alpha'\in\Gamma\backslash M_m} f(\alpha' z)\big|_{\alpha'} = T_m f(z)$$

that $T_m f : \mathfrak{H} \to V_n \otimes \mathbb{C}$ is equivariant. Via the dictionary Sect. 3.3 between equivariant vector-valued modular forms and modular forms of weights (r, s), we deduce an action of T_m on the latter. It is given by the following formula.

Lemma 6.1 *If f is real analytic modular of weights (r, s), then*

$$T_m f = \sum_{\alpha\in\Gamma\backslash M_m} \frac{m^{r+s-1}}{(c_\alpha z + d_\alpha)^r (c_\alpha \overline{z} + d_\alpha)^s} f(\alpha z)$$

and is real analytic modular of weights (r, s).

Proof For any α as in (6.1),

$$(X - \alpha z Y)\big|_\alpha = \frac{\det(\alpha)}{(c_\alpha z + d_\alpha)}(X - zY)\,.$$

Writing f in the form $f = \sum_{r+s=n} f_{r,s}(X - zY)^r (X - \overline{z}Y)^s$, we find that

$$T_m f = \frac{1}{m} \sum_{\alpha\in\Gamma\backslash M_m} \sum_{r+s=n} f_{r,s}(\alpha z) \frac{\det(\alpha)^n}{(c_\alpha z + d_\alpha)^r (c_\alpha \overline{z} + d_\alpha)^s} (X - zY)^r (X - \overline{z}Y)^s\,.$$

Reading off the coefficients gives the stated formula. □

6.2 Properties

Lemma 6.2 *View T_m, multiplication by $\mathbb{L}$, and ∂, $\overline{\partial}$, Δ as operators acting on real analytic modular functions. Then they satisfy*

$$m\, T_m\, \mathbb{L} = \mathbb{L}\, T_m\,,$$
$$[T_m, \partial] = [T_m, \overline{\partial}] = 0\,.$$

The second equation implies that $[T_m, \Delta] = 0$.

Proof For any α as in (6.1),

$$\mathrm{Im}(\alpha z) = \frac{\det(\alpha)}{(c_\alpha z + d_\alpha)(c_\alpha \overline{z} + d_\alpha)} \mathrm{Im}(z)\,.$$

If f is modular of weights (r, s), then $\mathrm{Im}(z) f$ is modular of weights $(r-1, s-1)$ and

$$\begin{aligned} T_m(\mathrm{Im}(z) f) &= \sum_{\alpha\in\Gamma\backslash M_m} \frac{m^{r+s-3}}{(c_\alpha z + d_\alpha)^{r-1} (c_\alpha \overline{z} + d_\alpha)^{s-1}} \mathrm{Im}(\alpha z) f(\alpha z) \\ &= \mathrm{Im}(z) \sum_{\alpha\in\Gamma\backslash M_m} \frac{m^{r+s-2}}{(c_\alpha z + d_\alpha)^r (c_\alpha \overline{z} + d_\alpha)^s} f(\alpha z) = \frac{1}{m} \mathrm{Im}(z) T_m f(z)\,. \end{aligned}$$

The first equation follows from $\mathbb{L} = -2\pi\, \mathrm{Im}(z)$. One verifies for any α of the form (6.1) (dropping the subscripts α for convenience):

$$\partial_r\Big((cz + d)^{-r} f(\alpha z)\Big) = (cz + d)^{-r-1}(c\overline{z} + d)\Big(\partial_r f\Big)(\alpha z),$$
$$\overline{\partial}_s\Big((c\overline{z} + d)^{-s} f(\alpha z)\Big) = (cz + d)(c\overline{z} + d)^{-s-1}\Big(\overline{\partial}_s f\Big)(\alpha z).$$

Since $\partial_r f$ is modular of weights $(r+1, s-1)$

$$T_m(\partial_r f) = \sum_{\alpha \in \Gamma \backslash M_m} \frac{m^{r+s-1}}{(c_\alpha z + d_\alpha)^{r+1} (c_\alpha \overline{z} + d_\alpha)^{s-1}} (\partial_r f)(\alpha z)$$

$$= \sum_{\alpha \in \Gamma \backslash M_m} \frac{m^{r+s-1}}{(c_\alpha \overline{z} + d_\alpha)^{s}} \partial_r \Big(\frac{f(\alpha z)}{(c_\alpha z + d_\alpha)^r} \Big) = \partial_r T_m(f),$$

which proves that $[T_m, \partial]f = 0$. The statement for $\overline{\partial}$ follows by complex conjugation. The equation $[T_m, \Delta]f = 0$ follows from the definition of the Laplacian (Sect. 3.1). □

By [24, Sect. 5.2 Lemma 2], a complete set of representatives for the set of cosets $\Gamma \backslash M_m$ are given by the $\sigma_1(m) = \sum_{d|m} d$ integer matrices

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \quad \text{where} \quad ad = m\,,\ a \geq 1\,,\ 0 \leq b < d\,.$$

It follows from Lemma 6.1 that for any f modular of weights (r, s), we have

$$T_m f(z) = m^{w-1} \sum_{ad=m, a,d>0} \frac{1}{d^w} \sum_{0 \leq b < d} f\Big(\frac{az+b}{d}\Big), \tag{6.2}$$

where $w = r + s$ is the total weight of f, which is the usual formula. The operators T_m commute and satisfy the following relations [24, Sect. 5.1]:

$$\begin{aligned} T_m T_n &= T_{mn} && \text{if } (m,n) \text{ coprime}, \\ T_p T_{p^n} &= T_{p^{n+1}} + p^{w-1} T_{p^{n-1}} && \text{if } p \text{ prime}, n \geq 1 \end{aligned}$$

viewed as operators acting on modular forms of total weight w.

6.3 *q*-expansions

The Hecke operators do not preserve the spaces $\mathcal{M}$ and $\mathcal{M}^!$. Indeed, it follows from the definitions that the map $f(z) \mapsto f(\frac{az+b}{d})$ acts via

$$\mathbb{L}^k q^m \overline{q}^n \quad \mapsto \quad \Big(\frac{a}{d}\Big)^k e^{2\pi i (m-n) \frac{b}{d}} \, \mathbb{L}^k q^{\frac{ma}{d}} \overline{q}^{\frac{na}{d}}$$

The following corollary is a consequence of formula (6.2) and continuity.

Corollary 6.3 *Let $R \subset \mathbb{C}$. The Hecke operator T_N defines a linear map*

$$T_N : \mathcal{M}^!(R) \longrightarrow \mathcal{M}^{[N],!}(R[e^{2\pi i/N}])$$

where $\mathcal{M}^{[N],!}(S)$ is the space of real analytic modular forms which admit an expansion in $S[q^{-1/N}, \overline{q}^{-1/N}, q^{1/N}, \overline{q}^{1/N}]][\mathbb{L}]$.

It is well known that for $r \in \mathbb{Z}$,

$$\sum_{0 \leq b < d} e^{2\pi i r \frac{b}{d}} = \begin{cases} 0 & \text{if } d \nmid r, \\ d & \text{if } d | r. \end{cases} \tag{6.3}$$

Corollary 6.4 *Let $f \in \mathcal{M}^!_{r,s}$ with an expansion*

$$f = \sum a^{(k)}_{m,n} \mathbb{L}^k q^m \overline{q}^n \tag{6.4}$$

satisfying the property that for all $d|N$, $d > 1$,

$$a^{(k)}_{m,n} = 0 \quad \textit{whenever } 0 \not\equiv m \equiv n \pmod{d}\,. \tag{6.5}$$

Then $T_N f \in \mathcal{M}^!_{r,s}$. More precisely, one has the formula

$$T_N f = \sum_{k,\mu,\nu} \alpha^{(k)}_{\mu,\nu} \mathbb{L}^k q^\mu \overline{q}^\nu \tag{6.6}$$

where

$$\alpha^{(k)}_{\mu,\nu} = \sum_{a|(N,\mu,\nu), a\geq 1} a^{w-1} \left(\frac{a^2}{N}\right)^k a^{(k)}_{\frac{\mu N}{a^2}, \frac{\nu N}{a^2}} .$$

In particular, if $f \in \mathcal{M}_{r,s}$ and satisfies (6.5), then $T_N f \in \mathcal{M}_{r,s}$.

Proof Apply T_N to the expansion of f via formula (6.2) to deduce that

$$T_N f = \sum_{k,m,n} N^{w-1} \sum_{ad=N, a,d>0} \left(\frac{a}{d}\right)^k \frac{1}{d^w} \sum_{0\leq b<d} a^{(k)}_{m,n} e^{2\pi i(m-n)\frac{b}{d}} \mathbb{L}^k q^{\frac{ma}{d}} \overline{q}^{\frac{na}{d}} .$$

This reduces using (6.3) to

$$T_N f = \sum_{k,m,n} \sum_{ad=N,a,d>0} \left(\frac{a}{d}\right)^k a^{w-1} a^{(k)}_{m,n} \mathbb{L}^k q^{\frac{ma}{d}} \overline{q}^{\frac{na}{d}} .$$

By assumption (6.5), replace m, n with $m' = m/d$ and $n' = n/d$ to obtain

$$T_N f = \sum_{k,m',n'} \sum_{ad=N,a,d>0} \left(\frac{a}{d}\right)^k a^{w-1} a^{(k)}_{m'd,n'd} \mathbb{L}^k q^{m'a} \overline{q}^{n'a} .$$

Comparing with (6.6) and collecting terms in $q^\mu \overline{q}^\nu$ gives

$$\alpha^{(k)}_{\mu,\nu} = \sum_{a|(N,\mu,\nu), a\geq 1} a^{w-1} \left(\frac{a}{d}\right)^k a^{(k)}_{\frac{\mu d}{a}, \frac{\nu d}{a}} .$$

where in the sum, d denotes N/a. □

Condition (6.5) holds in particular if $a^{(k)}_{m,n} = 0$ for all $mn \neq 0$.

Corollary 6.5 *The Hecke algebra acts on $\mathcal{HM}^!$.*

If $f = f^a + f^0 + f^h$ as in (4.2) then $(T_N f)^\bullet = T_N(f^\bullet)$ for $\bullet \in \{a, 0, h\}$. It follows from the formula that if $f^\bullet$ has a pole of order at most p at the cusp, then $T_N f^\bullet$ has a pole of order at most Np at the cusp, for $\bullet = a, h$.

Corollary 6.6 *Let f be as in Corollary 6.4. Let $w = r + s$. Then*

$$\alpha^{(k)}_{0,0} = \sigma_{2k+w-1}(N) N^{-k} a^{(k)}_{0,0} . \tag{6.7}$$

Corollary 6.7 *Let $N = p$ be prime. Then for all k, μ, ν,*

$$\alpha^{(k)}_{\mu,\nu} = p^{-k} a^{(k)}_{\mu p, \nu p} + p^{w+k-1} a^{(k)}_{\mu/p, \nu/p}$$

where the second term arises only if p divides μ and ν, and is absent otherwise.

The space of almost weakly holomorphic modular forms $M^![\mathbb{G}^*_2, \mathbb{L}^\pm]$ consists of harmonic functions. It is preserved by the Hecke operators.

Example 6.8 The modified Eisenstein series $\mathbb{G}^*_2 = \mathbb{G}_2 - \frac{1}{4\mathbb{L}}$ is modular of weights $(2, 0)$ and lies in $\mathcal{M}_{2,0}$, where

$$\mathbb{G}_2(q) = -\frac{1}{24} + \sum_{n=1}^\infty \sigma_1(n) q^n = -\frac{1}{24} + q + 3q^2 + 4q^3 + 7q^4 + \cdots$$

By formula (6.7) with $w = 2$, we find that $T_n(\mathbb{L}^{-1}) = n^{-1}\sigma_{-1}(n)\mathbb{L}^{-1} = \sigma_1(n)\mathbb{L}^{-1}$, and hence, $\mathbb{G}_2^*$ is a Hecke eigenform. For all $n \geq 1$,

$$T_n\mathbb{G}_2^* = \sigma_1(n)\mathbb{G}_2^* .$$

Remark 6.9 The quotient $\mathcal{HM}^!/\partial(\mathcal{HM}^!)$ also admits an action of the Hecke algebra.

6.4 Hecke operators on weakly holomorphic modular forms

Let $f \in M^!_{k+2}$ be a weak Hecke eigenform. Then for all m,

$$(T_m - \lambda_m)f = \psi_m$$

for some λ_m, where ψ_m is a weakly holomorphic modular form

$$\psi_m \in D^{k+1}M^!_{-k} .$$

Since the operators T_m, T_n commute, they satisfy

$$(T_m - \lambda_m)\psi_n = (T_n - \lambda_n)\psi_m \tag{6.8}$$

for all m, n. From the standard relations between Hecke operators:

$$\psi_{mn} = \lambda_n\psi_m + T_m\,\psi_n$$

for all (m, n) coprime. For all p prime and $n \geq 1$,

$$\psi_{p^{n+1}} = T_p\psi_{p^n} - p^{k+1}\psi_{p^{n-1}} + \lambda_{p^n}\psi_p .$$

6.5 Hecke action on modular primitives

Let $f, g, X_{r,s}$ be as in Proposition 5.4.

Proposition 6.10 *f is a weak Hecke eigenform with eigenvalues λ_m if and only if g is a weak Hecke eigenform with eigenvalues λ_m. In this case,*

$$\Big(T_m - \frac{\lambda_m}{m}\Big)X_{r,s} = \frac{1}{m}\mathbb{L}^{-n}\Big(\frac{\partial^r}{r!}\psi_m + \frac{\overline{\partial}^s}{s!}\overline{\phi_m}\Big) \tag{6.9}$$

for some weakly holomorphic functions $\psi_m, \phi_m \in M^!_{-n}$ satisfying

$$\begin{aligned}(T_m - \lambda_m)f &= \frac{1}{n!}\,\mathbb{L}^{-n-1}\partial^{n+1}\psi_m,\\ (T_m - \lambda_m)g &= \frac{1}{n!}\,\mathbb{L}^{-n-1}\partial^{n+1}\phi_m .\end{aligned} \tag{6.10}$$

Proof Suppose that f is a weak Hecke eigenform with eigenvalues λ_m. Therefore, by (3.4), there exists for every $m \geq 1$ a $\psi_m \in M^!_{-n}$ such that

$$(T_m - \lambda_m)f = \frac{1}{n!}\,\mathbb{L}^{-n-1}\partial^{n+1}\psi_m .$$

Since $\partial X_{n,0} = \mathbb{L}f$, it follows from $[T_m, \partial] = 0$ (Lemma 6.2) that

$$\partial\Big(T_m - \frac{\lambda_m}{m}\Big)X_{n,0} = \frac{1}{mn!}\mathbb{L}^{-n}\partial^{n+1}\psi_m.$$

Hence, by Lemma 3.1 there exists a $\phi_m \in M^!_{-n}$ such that

$$\Big(T_m - \frac{\lambda_m}{m}\Big)X_{n,0} = \frac{1}{m}\mathbb{L}^{-n}\Big(\frac{\partial^n}{n!}\psi_m + \overline{\phi_m}\Big) . \tag{6.11}$$

This proves the case $(r,s)=(n,0)$ of (6.9). By taking the complex conjugate of Lemma 3.4 we find, using the fact that $\overline{\partial}\psi_m=0$, that

$$\overline{\partial}\partial^k\psi_m=\overline{\partial}\partial^k\psi_m-\partial^k\overline{\partial}\psi_m=k(n-k+1)\partial^{k-1}\psi_m\,.$$

By induction on s, this in turn implies that

$$\frac{\overline{\partial}^s}{s!}\frac{\partial^n}{n!}\psi_m=\frac{\partial^{n-s}}{(n-s)!}\psi_m\,.$$

From the definition $X_{r,s}=\frac{\overline{\partial}^s}{s!}X_{n,0}$, we apply $\frac{\overline{\partial}^s}{s!}$ to (6.11) and use $[T_m,\overline{\partial}]=0$ (Lemma 6.2) and the previous equation to deduce that

$$\Big(T_m-\frac{\lambda_m}{m}\Big)X_{r,s}=\frac{1}{m}\mathbb{L}^{-n}\Big(\frac{\partial^r}{r!}\psi_m+\frac{\overline{\partial}^s}{s!}\overline{\phi_m}\Big)\ .$$

This proves (6.9). Now apply $\overline{\partial}$ to this expression in the case $(r,s)=(0,n)$. We find, since $\overline{\partial}X_{0,n}=\mathbb{L}\overline{g}$ and $\overline{\partial}\psi_m=0$ that,

$$\Big(T_m-\frac{\lambda_m}{m}\Big)\mathbb{L}\overline{g}=\frac{1}{m}\mathbb{L}^{-n}\frac{\overline{\partial}^{n+1}}{n!}\overline{\phi_m}$$

which is equivalent by Lemma 6.2 to the second line of (6.10). By (3.4), g is a weak Hecke eigenform with eigenvalues λ_m, and completes the proof. The converse result, where we assume that g is a weak Hecke eigenform and deduce the same for f, holds by complex conjugation. □

Remark 6.11 Remark 5.5 implies an equality on the Betti image under $\mathrm{comp}_{B,dR}$ of the de Rham cohomology classes in $H^1_{dR}(\mathcal{M}_{1,1}(\mathbb{C});\mathcal{V}_n)$:

$$[2\pi i f(z)(X-zY)^n\mathrm{d}z]=\overline{[2\pi i\,g(z)(X-zY)^n dz]}\,.$$

Since the Hecke operators act on cohomology, it follows that f is a weak Hecke eigenform if and only if g is, and that they have the same eigenvalues. Incidentally, this argument also proves that $g=\mathbf{s}(f)$.

6.6 Determination of the coefficient of $\mathbb{L}^{-n}$

Corollary 6.12 *Let $f,g\in M^!_{n+2}$ and $X_{r,s}$ be as in the previous proposition. Then*

$$X^0_{r,s}=\alpha(-1)^r\binom{n}{r}\mathbb{L}^{-n}\,,$$

where the constant $\alpha\in\mathbb{C}$ satisfies for all $m\geq 1$

$$(\sigma_{n+1}(m)-\lambda_m)\,\alpha=a_0(\psi_m)+\overline{a_0(\phi_m)},\tag{6.12}$$

where a_0 denotes the zeroth Fourier coefficient.

If f is cuspidal, $\sigma_{n+1}(m)-\lambda_m\neq 0$ for all m sufficiently large, in which case

$$\alpha=\frac{a_0(\psi_m)+\overline{a_0(\phi_m)}}{\sigma_{n+1}(m)-\lambda_m}\,.\tag{6.13}$$

In particular, if f,g have real Fourier coefficients, then α is real.

Proof By Proposition 6.10,

$$(mT_m-\lambda_m)X_{n,0}=\mathbb{L}^{-n}\Big(\frac{\partial^n}{n!}\psi_m+\overline{\phi_m}\Big)$$

for all $m \geq 1$. By equation (5.7), the constant term of $X_{n,0}$ is $X^0_{n,0} = \alpha\, \mathbb{L}^{-n}$, since f is a cusp form. On the other hand, equation (6.7) in weight n implies that

$$mT_m\, \mathbb{L}^{-n} = \sigma_{-n-1}(m)m^{n+1}\, \mathbb{L}^{-n} = \sum_{d|m}\left(\frac{m}{d}\right)^{n+1} \mathbb{L}^{-n} = \sigma_{n+1}(m)\, \mathbb{L}^{-n}\,.$$

Putting the pieces together yields

$$(mT_m - \lambda_m)X^0_{n,0} = (\sigma_{n+1}(m) - \lambda_m)\alpha\, \mathbb{L}^{-n} = \mathbb{L}^{-n}\left(a^{(0)}_{0,0}\Big(\frac{\partial^n}{n!}\psi_m\Big) + a^{(0)}_{0,0}(\overline{\phi_m})\right),$$

where $a^{(0)}_{0,0}$ denotes the coefficient in the expansion (1.2). Since $\psi_m \in M^!_{-n}$,

$$a^{(0)}_{0,0}(\partial^n\psi_m) = a^{(0)}_{0,0}(\partial_{-1}\ldots\partial_{-n}\psi_m) = (-1)^n n! a^{(0)}_{0,0}(\psi_m)$$

by successive application of (3.2), which never decreases the powers of $\mathbb{L}$.

The λ_m are the eigenvalues of a normalised holomorphic Hecke eigenform $g \in S_{n+2}$. Then $\lambda_m = a_m(g)$ and an elementary estimate [16, Lemma 2], implies that $|a_m(g)|$ grows at most like $m^{n/2+1}$. Since $\sigma_{n+1}(m) \geq m^{n+1}$, it follows that $(\sigma_{n+1}(m) - \lambda_m)$ is nonzero for sufficiently large m. □

The consistency of equations (6.12) for different values of m follows from (6.8). Equation (6.13) would have poles for every n if f were an Eisenstein series by (2.5).

7 Existence of modular primitives

Having determined the form of modular primitives, we now turn to their existence.

7.1 Cocycles and periods

Let us fix a system $\underline{\lambda}$ of Hecke eigenvalues corresponding to a cuspidal eigenform $f \in S_{n+2}$, and let $H^{dR}_{\underline{\lambda}}$, $H^B_{\underline{\lambda}}$ be as defined in Sect. 2. For simplicity, we shall drop the subscripts $\underline{\lambda}$ from now on, and set $K = K_{\underline{\lambda}}$. Let

$$f \in M_{n+2} \qquad \text{and} \qquad f' \in M^!_{n+2}$$

denote a K-basis for H^{dR} of the form (2.7). Likewise, choose a K-basis P^+ of $H^{B,+}$ and P^- of $H^{B,-}$. We have $P^{\pm}_T = 0$. The polynomials $P^{\pm}_S$ are known, respectively, as the even and odd period polynomials of f.

Let us choose a basepoint $z_0 \in \mathfrak{H}$ and let $C, C' \in Z^1(\Gamma; V_n)$ denote the cocycles associated with f and f', respectively. The comparison isomorphism (2.10) implies that

$$\begin{aligned} [C'] &= \eta^+\,[P^+] + i\eta^-\,[P^-], \\ [C] &= \omega^+\,[P^+] + i\omega^-\,[P^-], \end{aligned} \tag{7.1}$$

where $\omega^+, i\omega^-, \eta^+, i\eta^-$ are the entries of the period matrix in these bases.

Lemma 7.1 *There exists a canonical Hecke equivariant splitting over $\mathbb{Q}$:*

$$s : H^1_{\text{cusp}}(\Gamma; V_n) \longrightarrow Z^1_{\text{cusp}}(\Gamma; V_n)\,.$$

Proof See [4, Lemma 7.3]. □

We can assume that $P^+, P^- \in Z^1_{\text{cusp}}(\Gamma; V_n \otimes K)$ are the unique Hecke-equivariant lifts of the cohomology classes chosen earlier. They satisfy $P^{\pm}_T = 0$.

Corollary 7.2 *There exist polynomials $Q, Q' \in V_n \otimes \mathbb{C}$ such that for all $\gamma \in \Gamma$,*

$$C'_\gamma = \eta^+ P^+_\gamma + i\eta^- P^-_\gamma + Q'(X,Y)\big|_{\gamma - \mathrm{id}},$$
$$C_\gamma = \omega^+ P^+_\gamma + i\omega^- P^-_\gamma + Q(X,Y)\big|_{\gamma - \mathrm{id}}.$$

The polynomials Q, Q' depend on the choice of basepoint z_0.

7.2 Real and imaginary analytic cusp forms

We shall construct explicit modular primitives of cusp forms in two steps.

Recall that the integrals $F_f(z)$ were defined in (2.11) relative to the basepoint $z_0 \in \mathfrak{H}$.

Definition 7.3 Define real analytic functions $\mathfrak{H} \to V_n \otimes \mathbb{C}$ by

$$\mathcal{I}_f(z) = (2\pi i)^{-2n}\Big(\omega^+ \mathrm{Re}\left(\mathcal{F}_{f'}(z) - Q'\right) - \eta^+ \mathrm{Re}\left(\mathcal{F}_f(z) - Q\right)\Big),$$
$$\mathcal{R}_f(z) = (2\pi i)^{-2n}\Big(\omega^- \mathrm{Im}\left(\mathcal{F}_{f'}(z) - Q'\right) - \eta^- \mathrm{Im}\left(\mathcal{F}_f(z) - Q\right)\Big).$$

Note that (2.11) involves an odd power of $2\pi i$, which explains why ‘real’ and ‘imaginary’ are apparently interchanged.

These functions satisfy the differential equations

$$\begin{aligned} \mathrm{d}\mathcal{I}_f(z) &= \omega^+ \mathrm{Re}\left(2\pi i f'(z)(X-zY)^n \mathrm{d}z\right) - \eta^+ \mathrm{Re}\left(2\pi i f(z)(X-zY)^n \mathrm{d}z\right) \\ &= \pi i\left(\omega^+ f' - \eta^+ f\right)(X-zY)^n \mathrm{d}z + \pi i\Big(\eta^+ \overline{f} - \omega^+ \overline{f'}\Big)(X-\overline{z}Y)^n \mathrm{d}\overline{z} \end{aligned}$$

and similarly

$$\mathrm{d}\mathcal{R}_f(z) = \pi i\left(\omega^- f' - \eta^- f\right)(X-zY)^n \mathrm{d}z + \pi i\Big(\omega^- \overline{f'} - \eta^- \overline{f}\Big)(X-\overline{z}Y)^n d\overline{z}$$

Theorem 7.4 *The functions $\mathcal{I}_f(z)$ and $\mathcal{R}_f(z)$ are well defined (independent of the choice of basepoint z_0), and Γ-equivariant.*

Proof The Γ-equivariance of $\mathcal{I}_f(z)$ follows from Corollary 7.2:

$$\mathcal{I}_f(\gamma z)\big|_\gamma - \mathcal{I}_f(z) = \omega^+\left(\eta^+ P^+_\gamma\right) - \eta^+\left(\omega^+ P^+_\gamma\right) = 0.$$

Changing base point z_0 yields a modular equivariant solution to the same differential equation for $\mathrm{d}\mathcal{I}_f$ given above (which is independent of the basepoint). By Lemma 5.11, any modular equivariant solution is unique. The argument for $\mathcal{R}_f(z)$ is similar. □

Extract the coefficients of $\mathcal{I}_f$ and $\mathcal{R}_f$ via

$$\mathcal{I}_f = \sum_{r+s=n} I_{r,s}(X-zY)^r(X-\overline{z}Y)^s,$$
$$\mathcal{R}_f = \sum_{r+s=n} R_{r,s}(X-zY)^r(X-\overline{z}Y)^s.$$

They define weakly holomorphic modular forms in $\mathcal{M}^!$.

Corollary 7.5 *There exists a family $I_{r,s} \in \mathcal{M}^!_{r,s}$, for $r+s=n$, such that*

$$\partial I_{r,s} = (r+1)I_{r+1,s-1} \quad \textit{for all } 1 \leq s \leq n,$$
$$\overline{\partial} I_{r,s} = (s+1)I_{r-1,s+1} \quad \textit{for all } 1 \leq r \leq n,$$

and

$$\partial I_{n,0} = \mathbb{L}\Big(\omega^+ f' - \eta^+ f\Big), \qquad \overline{\partial} I_{0,n} = \mathbb{L}\Big(\eta^+ \overline{f} - \omega^+ \overline{f'}\Big).$$

They are ‘imaginary’ in the sense that $\overline{I}_{r,s} = -I_{s,r}$.

Similarly, there exists a family of elements $R_{r,s} \in \mathcal{M}^!_{r,s}$ *for* $r+s = n$ *and* $r, s \geq 0$*, satisfying the identical equations, except that the last line is replaced by*

$$\partial\, R_{n,0} = \mathbb{L}\Big(\omega^- f' - \eta^- f\Big) \qquad \text{and} \qquad \overline{\partial}\, R_{n,0} = \mathbb{L}\Big(\omega^- \overline{f'} - \eta^- \overline{f}\Big).$$

They are ‘real’ in the sense that $\overline{R}_{r,s} = R_{s,r}$.

Proof This is a straightforward application of (3.9) to the previous discussion. □

7.3 Modular primitives of cusp forms

Since the period isomorphism is invertible, we can change basis, to deduce the existence of modular primitives for all cusp forms.

Definition 7.6 For any basis f, f' of (2.7) define

$$\mathcal{H}(f) = p^{-1}\Big(\omega_f^- \mathcal{I}_f - \omega_f^+ \mathcal{R}_f\Big),$$
$$\mathcal{H}(f') = p^{-1}\Big(\eta_f^- \mathcal{I}_f - \eta_f^+ \mathcal{R}_f\Big),$$

where $p = (\omega_f^+ \eta_f^- - \omega_f^- \eta_f^+) = -i\ \det(P_f) \neq 0$, and $\eta_f^+, i\eta_f^-, \omega_f^+, i\omega_f^-$ are entries of the period matrix P_f with respect to this basis.

Write $\mathcal{H}(f) = \sum_{r+s=n} \mathcal{H}(f)_{r,s}(X - zY)^r(X - \overline{z}Y)^s$ as usual.

Theorem 7.7 *The family of functions* $\mathcal{H}(f)_{r,s}$ *satisfy the equations*

$$\partial\, \mathcal{H}(f)_{r,s} = (r+1)\mathcal{H}(f)_{r+1,s-1} \quad \text{for all } 1 \leq s \leq n$$
$$\overline{\partial}\, \mathcal{H}(f)_{r,s} = (s+1)\mathcal{H}(f)_{r-1,s+1} \quad \text{for all } 1 \leq r \leq n$$

and

$$\partial\, \mathcal{H}(f)_{n,0} = \mathbb{L}f, \qquad \overline{\partial}\, \mathcal{H}(f)_{0,n} = \mathbb{L}\overline{\mathbf{s}(f)}.$$

The family of functions $\mathcal{H}(f')_{r,s}$ *satisfy the same equations with* f *interchanged everywhere with* f'*, and* ω *interchanged with* η*. In particular,* f *admits a canonical weak harmonic lift (see Sect. 1.3).*

Proof Straightforward consequence of the previous corollary using:

$$\mathbf{s}\left(\omega^+ f' - \eta^+ f\right) = -\left(\omega^+ f' - \eta^+ f\right) \text{ and } \mathbf{s}\left(\omega^- f' - \eta^- f\right) = \left(\omega^- f' - \eta^- f\right).$$

This is immediate from the definition of the single-valued period matrix $\overline{P_f}^{-1} P_f$ on noting that $P_f^{-1}(f) = ip^{-1}(\omega^+ f' - \eta^+ f)$ and $P_f^{-1}(f') = p^{-1}(\omega^- f' - \eta^- f)$. □

It follows from uniqueness (lemma 5.11) that $\mathcal{H}(f)_{r,s}$ is well defined (only depends on f and not the choice of basis f, f'), since it only depends on f and its image under the single-valued involution $\mathbf{s}(f)$, which is canonical.

In this manner we have defined a canonical modular primitive (compare discussion of Sect. 1.3):

$$\begin{aligned} H^{dR}_{\underline{\lambda}} &\longrightarrow \mathcal{M}^!_{n,0}(K_{\underline{\lambda}}[\omega^{\pm}_{\underline{\lambda}}, \eta^{\pm}_{\underline{\lambda}}])\,. \\ f &\mapsto \mathcal{H}(f)_{n,0} \end{aligned}$$

This map is injective since $\mathbb{L}^{-1}\partial\mathcal{H}(f)_{n,0} = f$.

Corollary 7.8 *For all $r+s=n$, complex conjugation acts via:*

$$\overline{\mathcal{H}(f)}_{r,s} = \mathcal{H}(\mathbf{s}(f))_{s,r}\,.$$

Corollary 7.9 *Suppose that $f \in S_n$ is a cuspidal Hecke eigenform. The constant term in $\mathcal{H}(f)_{r,s}$ is proportional to the Petersson norm of f times $\mathbb{L}^{-n}$*

$$\mathcal{H}(f)^0 \in \{f, \mathbf{s}(f)\}\, K_{\underline{\lambda}} \mathbb{L}^{-n}\,.$$

Proof This follows from (6.13): the term ψ_m vanishes since f has no pole, and the sole contribution to α comes, via ϕ_m, from the action of Hecke operators on the f' term in $g = \mathbf{s}(f) \in \mathbb{C}f \oplus \mathbb{C}f'$. But the coefficient of f' in $\mathbf{s}(f)$ is proportional to $\{f, \mathbf{s}(f)\}$, since $\{f, f\} = 0$ and $\{f, f'\} \in K_{\underline{\lambda}}$. The quantity $\{f, \mathbf{s}(f)\}$ can be interpreted as the Petersson norm via (2.14) and the comments which follow. □

Corollary 7.10 *Every modular form admits a modular primitive in $\mathcal{M}^!$.*

Proof Every modular form of integral weight is a linear combination of Eisenstein series and cuspidal Hecke eigenforms. □

7.4 Vanishing constant term

The space $H^{dR} \otimes \mathbb{C}$ decomposes into eigenspaces with respect to the map $\mathbf{s}$:

$$H^{dR} \otimes \mathbb{C} = (H^{dR} \otimes \mathbb{C})^+ \oplus (H^{dR} \otimes \mathbb{C})^-\,.$$

They are, respectively, the preimages of $H_B^{\pm} \otimes \mathbb{C}$ under the comparison isomorphism.

An element $f \in (H^{dR} \otimes \mathbb{C})^+$ satisfies $\mathbf{s}(f) = f$, and hence, $\mathcal{H}(f)_{r,s}$ is proportional to the 'real' function $\mathcal{R}(f)_{r,s}$.

An element $f \in (H^{dR} \otimes \mathbb{C})^-$ satisfies $\mathbf{s}(f) = -f$, and hence, $\mathcal{H}(f)_{r,s}$ is proportional to the 'imaginary' function $\mathcal{I}(f)_{r,s}$. The latter satisfies $\mathcal{I}^0_{r,s} = 0$ since by Corollary 7.5 and (6.13), the constant term α is real and hence vanishes since $\mathcal{I}(f)_{r,r} = -\overline{\mathcal{I}}(f)_{r,r}$. It is therefore cuspidal: $\mathcal{I}(f)_{r,s} \in \mathcal{S}^!$.

8 Example: Real analytic version of Ramanujan's function Δ

8.1 Weakly holomorphic cusp forms in weight 12

Let $n = 10$. Let Δ denote Ramanujan's cusp form of weight 12

$$\Delta = q \prod_{n\geq 1} (1-q^n)^{24} = q - 24\,q^2 + 252\,q^3 + 1472\,q^4 + 4830q^5 - 6048q^6 + \cdots$$

Since $\dim S_{12} = 1$, it is a Hecke eigenform with eigenvalues in $\mathbb{Z}$. There exists a unique weakly holomorphic modular form $\Delta' \in M^!_{12}$ which has a pole of order at most 1 at the cusp, and whose Fourier coefficients a_0, a_1 vanish. Explicitly,

$$\Delta' = q^{-1} + 47709536\,q^2 + 39862705122\,q^3 + 7552626810624\,q^4 + \cdots$$

It satisfies $\{\Delta, \Delta'\} = 1$. It follows that there is a single cuspidal Hecke eigenspace, and that it has the de Rham basis:

$$H^{dR} = H^1_{\mathrm{cusp},dR}(\mathcal{M}_{1,1}; \mathcal{V}_{10}) = \mathbb{Q}\Delta' \oplus \mathbb{Q}\Delta\,.$$

The function Δ' is a weak Hecke eigenform with the same eigenvalues as Δ:

$$(T_m - \lambda_m)\Delta' = D^{11} p_m \qquad \text{for all } m \geq 1$$

for some $p_m \in M^!_{-10}$. For example, $\lambda_2 = -24$ and hence $(T_2 - \lambda_2)\Delta' = D^{11}\,\psi_2$, where

$$p_2 = 24\,\mathbb{G}_{14}\Delta^{-2} = -q^{-2} - 24q^{-1} + 196560 + 47709536q + \cdots .$$

In the notations of Proposition 6.10, we have $\psi_2 = 10!\,2^{-11}p_2$ by (3.4).

8.2 Cocycles

Let $P^{\pm} \in Z^1(\Gamma; V_{10})$ be the Hecke-invariant cocycles $P^{\pm} : \Gamma \to V_{10}$ which are uniquely determined by $P^{\pm}_T = 0$ and

$$\begin{aligned} P^+_S &= \frac{36}{691}(Y^{10} - X^{10}) + X^2Y^2(X^2 - Y^2)^3, \\ P^-_S &= 4X^9Y - 25X^7Y^3 + 42X^5Y^5 - 25X^3Y^7 + 4XY^9. \end{aligned}$$

Their Haberland inner product is $\{P^+, P^-\} = 1$. They provide a Betti basis

$$H^B = H^1_{\mathrm{cusp},B}(\mathcal{M}_{1,1}; \mathcal{V}_{10}) = \mathbb{Q}P^+ \oplus \mathbb{Q}P^- .$$

8.3 Periods

Following the method given in Sect. 2, we can easily compute the period matrix (2.13) in this basis. We find that for all $\gamma \in \Gamma$,

$$\int_\gamma (2\pi i)^{11}\Delta(z)(X - zY)^{10}\mathrm{d}z = \omega^+P^+_\gamma + \omega^-P^-_\gamma,$$

where

$$\omega_+ = -68916772.809595194754\ldots, \quad \omega_- = -5585015.3793104018668\ldots$$

which agree with the numerical values for the periods of Δ given in the literature. The periods of Δ', on the other hand, are $\eta_+, i\eta_-$ where

$$\eta_+ = 127202100647.17709477\ldots, \quad \eta_- = 10276732343.649132750\ldots$$

I could find no reference for these values for comparison. In accordance with proposition 5.6 of [6], we can indeed verify numerically that

$$\det\begin{pmatrix} \eta_+ & \omega_+ \\ i\eta_- & i\omega_- \end{pmatrix} = 10! \times (2\pi i)^{11} .$$

The Petersson norm of Δ, in its standard normalisation, is

$$\frac{-2\omega_+\omega_-}{2^{11}(2\pi i)^{22}} = 0.00000103536205\ldots > 0$$

8.4 Single-valued involution

The single-valued period matrix is

$$\frac{i}{10!(2\pi i)^{11}}\begin{pmatrix} \eta^+\omega^- + \eta^-\omega+ & 2\omega^+\omega^- \\ 2\eta^+\eta^- & -(\eta^+\omega^- + \eta^-\omega+) \end{pmatrix} = \begin{pmatrix} 648.84093\ldots & -0.3520770\ldots \\ 1195742.7\ldots & -648.84093\ldots \end{pmatrix}$$

in the basis Δ, Δ'. It does not depend on the choice of Betti basis. Therefore,

$$\mathbf{s}(\Delta) = \sigma\,\Delta' + \tau\,\Delta \qquad \text{where } \sigma = -0.35207\ldots, \tau = -648.84\ldots$$

For convenience, we evaluate the ratio

$$\rho = \frac{\tau}{\sigma} = -\frac{\eta^+\omega^- + \eta^-\omega^+}{2\omega^+\omega^-} = -\frac{1}{2}\Big(\frac{\eta^+}{\omega^+} + \frac{\eta^-}{\omega^-}\Big) = 1842.8947269\ldots$$

8.5 The constant term

Since $\sigma_{11}(2) = 2049$, we check that

$$\sigma_{11}(2) - \lambda_2 = 2073 = 3.691$$

is nonzero and therefore since $\psi_2 = 10!\, 2^{-11} p_2$,

$$\frac{a_0(\psi_2)}{\sigma_{11}(2) - \lambda_2} = \frac{10!}{2^{11}} \frac{196560}{3.691} = \frac{10!}{2^{11}} \frac{7!\, 13}{691} .$$

The 691 in the denominator is a consequence of the congruence $\Delta \equiv \mathbb{G}_{12} \pmod{691}$. Formula (6.13) therefore implies that

$$\alpha = \frac{7!\, 13}{691} \frac{10!}{2^{11}} \sigma = \frac{7!\, 13}{691} \frac{2i\omega^+\omega^-}{(4\pi i)^{11}} .$$

8.6 Real analytic cusp forms

The real analytic cusp forms $\mathcal{H}(\Delta)_{r,s}$ for $r + s = 10$ can be written down explicitly from the formulae given in Theorem 1.2.

8.7 The mock modular form M_Δ

Denote the Fourier coefficients of Δ, Δ' by a_n, a'_n. Our formula for the 'mock' modular form defined in Sect. 1.3 is

$$M_\Delta = \alpha + \frac{10!}{2^{11}} \sum_n \frac{\sigma a'_n + \tau a_n}{n^{11}} q^n,$$

where σ, τ are the periods given above. In order to compare more directly with Ono's normalisation [20], let us rescale by setting

$$M'_\Delta = -\frac{11 \times 2^{11}}{\sigma} M_\Delta = 11! \Big(-\frac{7!\, 13}{691} + \sum_n \frac{a'_n + \rho a_n}{n^{11}} q^n \Big) .$$

Its first five Fourier coefficients are given exactly by

$$11! \Big(q^{-1} - \frac{65520}{691} - \rho q + \Big(\frac{3}{256}\rho - \frac{1490923}{64}\Big) q^2 + \Big(-\frac{28}{19683}\rho - \frac{164044054}{729}\Big) q^3 + \cdots \Big) .$$

By uniqueness, this function coincides with the mock modular form for Δ given in [20] and discussed in [10]. We have verified, by substituting the above numerical value of ρ, that this agrees with the computation in [20] (1.7) to the accuracy given in that paper.

Ono's formula [20] for its nth Fourier coefficient, for $n > 0$, is:

$$-2\pi\, \Gamma(12) n^{-\frac{11}{2}} \sum_{c=1}^{\infty} \frac{K(-1, n, c)}{c} I_{11}\Big(\frac{4\pi\sqrt{n}}{c}\Big)$$

where K is a Kloosterman sum and I is a Bessel function. Combining this with our expression for its Fourier coefficients proves Corollary 1.4.

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Appendix: remark on complex multiplication

Both referees specifically asked for clarification of some remarks concerning complex multiplication. Complex multiplication does not arise for modular forms of full level and is mostly irrelevant to the present paper. Therefore, we shall remain brief. Nonetheless, it seems not to be widely known that complex multiplication induces relations between both the periods *and* the quasi-periods of motives.

Let $k \subset \mathbb{C}$ be a number field and consider the category $\mathcal{T}$ whose objects are triples $M = (M_{dR}, M_B, c)$ where M_{dR} is a k-vector space, M_B a $\mathbb{Q}$-vector space and c is an isomorphism $M_{dR} \otimes_k \mathbb{C} \xrightarrow{\sim} M_B \otimes_{\mathbb{Q}} \mathbb{C}$. The morphisms in $\mathcal{T}$ are linear maps on the components M_B, M_{dR} compatible with c. A pure motive over k defines an object in $\mathcal{T}$, where M_{dR}, M_B are its de Rham and Betti realisations (relative to the embedding $k \subset \mathbb{C}$), and c the comparison isomorphism. A period matrix for M is a representation of c as a matrix with respect to a choice of bases for M_{dR} and M_B. Let M be an object of $\mathcal{T}$ of rank 2, equipped with an isomorphism

$$\mu : M \longrightarrow M$$

such that $\mu^2 + a\mu + b = 0$ for some $a, b \in k$, for some irreducible polynomial $x^2 + ax + b$ with zeros in an extension field $K \subset \mathbb{C}$. Then $\mu \otimes \mathrm{id}$ induces an automorphism of $M_{dR} \otimes K$, which splits into eigenspaces for $\mu \otimes \mathrm{id}$. Let σ_1, σ_2 be a $\mathbb{Q}$-basis of M_B and choose a basis of eigenvectors f_1, f_2 of $M_{dR} \otimes K$, with eigenvalues λ_1, λ_2, respectively. They are the zeros of the quadratic polynomial $x^2 + ax + b$. The period matrix of $M \otimes K$, in this basis, has entries

$$\omega_{i,j} = \sigma_i(cf_j) \qquad \text{for } 1 \leq i, j \leq 2\,.$$

The map μ induces equivalences of matrix coefficients [5, Sect. 2]

$$\lambda_j[M, f_j, \sigma_1] = [M, \mu_{dR} f_j, \sigma_1] = [M, f_j, \mu_B^\vee \sigma_1]$$

which, on applying the period homomorphism, induces a relation between periods

$$\lambda_j \omega_{1,j} = \alpha_1 \omega_{1,j} + \alpha_2 \omega_{2,j}$$

where $\mu_B^\vee \sigma_1 = \alpha_1 \sigma_1 + \alpha_2 \sigma_2$, and $\alpha_i \in \mathbb{Q}$. Since the period matrix has non-vanishing determinant, and since $\lambda_j \notin \mathbb{Q}$, it follows from $(\lambda_j - \alpha_1)\omega_{1,j} = \alpha_2 \omega_{2,j}$ that all $\omega_{i,j}$ are nonzero and satisfy

$$\omega_{1,j}/\omega_{2,j} \in K^\times$$

for $j = 1, 2$. In conclusion, for a suitably chosen de Rham basis of eigenforms for the complex multiplication, the ratio of both periods and quasi-periods (with respect to this basis) is algebraic. Since complex multiplication induces a morphism of mixed Hodge structure, it preserves the Hodge filtration on the de Rham cohomology, and so we can assume the above de Rham basis is adapted to the Hodge filtration.

Example 8.1 Suppose that M are the realisations of the motive of a CM modular form. Then $M_B = M_B^+ \oplus M_B^-$ has a decomposition into Frobenius-invariant and anti-invariant subspaces, and write $\omega^+, i\omega^-, \eta^+, i\eta^-$ for the entries of the period matrix P with respect to a choice of de Rham basis of $M_{dR} \otimes K$ which are μ-eigenvectors, and Betti basis which are eigenvectors for real Frobenius. These notations are consistent with our earlier notations (2.13). With this choice of de Rham basis, the quantity (2.15) is algebraic. More precisely, if we write $\omega^+ = \alpha i \omega_-$ and $\eta^+ = \beta i \eta_-$, for $\alpha, \beta \in K^\times$, then the single-valued period matrix takes the form

$$\overline{P}^{-1} P = \begin{pmatrix} \frac{\alpha+\beta}{\alpha-\beta} & \left(\frac{2\beta}{\alpha-\beta}\right)\frac{\omega_-}{\eta_-} \\ \left(\frac{2\alpha}{\beta-\alpha}\right)\frac{\omega_-}{\eta_-} & \frac{\alpha+\beta}{\beta-\alpha} \end{pmatrix}$$

and we see that its diagonal entries are algebraic. Therefore, with respect to any basis of M_{dR}, the single-valued period matrix is obtained from the above by conjugation by a matrix with entries in K. In particular, the single-valued map $\mathbf{s}$ only involves the Petersson norm of f (see Sect. 2.1.6), and algebraic numbers in K.

Example 8.2 (The case of an elliptic curve). We use the notations of [18, Chapter 3]. Many thanks to J. Fresán for bringing this reference to our attention. For $\tau \in \mathfrak{H}$, satisfying $A + B\tau + C\tau^2 = 0$, where $A, B, C \in \mathbb{Z}$, lemma 3.1 of *loc. cit.* implies that there exists an algebraic κ in the field of complex multiplication, such that

$$\begin{aligned} \omega_2 &= \tau\omega_1 \\ A\eta_1 - C\tau\eta_2 &= \kappa\omega_2 \, . \end{aligned}$$

The number κ is not always zero, and so the ratio of quasi-periods η_2/η_1 is not algebraic in these cases. However, changing de Rham basis by adding to the differential of the second kind a multiple r of the holomorphic differential changes the quasi-periods by

$$\eta_i' = \eta_i + r\omega_i \quad \text{for } i = 1, 2$$

and we find that

$$A\eta_1' - C\tau\eta_2' = ((A - C\tau^2)r + \kappa\tau)\omega_1$$

which vanishes precisely when

$$r = \frac{-\kappa\,\tau}{2A + B\tau} \, .$$

Note that $2A + B\tau \neq 0$, since $\mathrm{Im}(\tau) > 0$. The quasi-periods defined with respect to the new de Rham basis are indeed proportional by an algebraic number:

$$A\eta_1' = C\tau\eta_2' \, ,$$

and κ vanishes exactly when the new de Rham basis agrees with the original one.

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