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RESEARCH

On the theory of generalized quasiconformal mappings

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Abstract
We study generalized quasiconformal mappings in the context of the inverse Poletsky inequality. We consider the local behavior and the boundary behavior of mappings with the inverse Poletsky inequality. In particular, we obtain logarithmic Hölder continuity for such classes of mappings.

Keywords Quasiconformal mappings · Sobolev spaces

Mathematics Subject Classification 30C65 · 46E35

1 Introduction

This article is devoted to the Hölder continuity of generalized quasiconformal mappings $f : D \to D'$ which are defined by capacity (moduli) inequalities. The method of capacity (moduli) inequalities arises to the Grötzsch problem and was introduced in [1]. In subsequent works (see, for example, [13, 15] and [21]), the conformal modulus method was used in the theory of quasiconformal (quasiregular) mappings and its generalizations. The classes of mappings generating bounded composition operators on Sobolev spaces [32, 33] arise in the geometric analysis of PDE [7, 17]. These mappings are called weak (p, q)-quasiconformal mappings [3, 30] and can be characterized by the inverse capacitory (moduli) Poletsky inequality [27]

$$\operatorname{cap}_q^{1/q}(f^{-1}(E), f^{-1}(F); D) \leqslant K_{p,q}(\varphi; \Omega)\operatorname{cap}_p^{1/p}(E, F; D'),\ \ 1 < q \leqslant p < \infty.$$

The detailed study of the mappings with the inverse conformal Poletsky inequality for modulus of paths was given in [24, 26] and [23]. In this case, $p = q = n$,the Hölder continuity, the continuous boundary extension, and the behavior on the closure of domains were obtained.

In the recent works,[6, 20] were considered connections between weak (p, q)-quasiconformal mappings and Q-homeomorphisms. In the present article, we suggest an approach to the generalized quasiconformal mappings which are based on the following integral inequality

$$\int\limits_D |\nabla(u \circ f(x))|^q \, dm(x) \leqslant \int\limits_{D'} |\nabla u(y)|^q Q_q(y)\, dm(y),\ \ u \in C^1(D').$$

Depending on the properties of the function Q_q,we obtain various classes of the generalized quasiconformal mappings: BMO-quasiconformal mappings, weak (p, q)-quasiconformal mappings, Q-mappings, and so on.

The weak (p, q)-quasiconformal mappings have significant applications in the spectral theory of elliptic operators [8, 9]. The Hölder continuity of weak (p, q)-quasiconformal mappings was considered in [30]. In the recent article,[5] was considered the boundary behavior of the weak (p, q)-quasiconformal mappings. In this article,we study the log-

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arithmic Hölder continuity, the continuous boundary extension, and the behavior in the closure of domains of non-homeomorphic generalizations of quasiconformal mappings.

Let us give the basic definitions. Let Γ be a family of paths γ in $\mathbb{R}^n$. A Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is called *admissible* for Γ if

$$\int\limits_{\gamma} \rho(x)|dx| \geqslant 1 \qquad (1.1)$$

for all (locally rectifiable) paths $\gamma \in \Gamma$. In this case, we write: $\rho \in \operatorname{adm} \Gamma$. Given a number $q \geqslant 1$, *q-modulus* of the family of paths Γ is defined as

$$M_q(\Gamma) = \inf_{\rho \in \operatorname{adm} \Gamma} \int\limits_{D} \rho^q(x)\, dm(x). \qquad (1.2)$$

Let $x_0 \in \overline{D}$, $x_0 \neq \infty$, then

$$B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}, \quad \mathbb{B}^n = B(0, 1), \quad (1.3)$$

$$S(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| = r\}, S_i = S(x_0, r_i), \quad i = 1, 2,$$
$$A = A(x_0, r_1, r_2) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\}.$$

Given sets $E, F \subset \overline{\mathbb{R}^n}$ and a domain $D \subset \mathbb{R}^n$, we denote $\Gamma(E, F, D)$ a family of all paths $\gamma : [a, b] \to \overline{\mathbb{R}^n}$ such that $\gamma(a) \in E, \gamma(b) \in F$ and $\gamma(t) \in D$ for all $t \in (a, b)$.

Let $Q : \mathbb{R}^n \to [0, \infty]$ be a Lebesgue measurable function. We say that f *satisfies the Poletsky inverse inequality with respect to q-modulus* at a point $y_0 \in f(D)$, $1 < q < \infty$, if the moduli inequality

$$M_q(\Gamma(E, F, D)) \leqslant \int\limits_{A(y_0,r_1,r_2)\cap f(D)} Q(y) \cdot \eta^q(|y - y_0|)\, dm(y) \qquad (1.4)$$

holds for any continua $E \subset f^{-1}(\overline{B(y_0, r_1)})$, $F \subset f^{-1}(f(D) \setminus B(y_0, r_2))$, $0 < r_1 < r_2 < r_0 = \sup\limits_{y \in f(D)} |y - y_0|$, and any Lebesgue measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)\, dr \geqslant 1. \qquad (1.5)$$

The case $q = n$ was studied in details in [26], cf. [24] and [23]. The present article is dedicated to the case $q \neq n$.

Let us formulate the main results of this manuscript. Recall that a mapping $f : D \to \mathbb{R}^n$ is called *discrete* if a pre-image $\{f^{-1}(y)\}$ of each point $y \in \mathbb{R}^n$ consists of isolated points, and *open* if the image of any open set $U \subset D$ is an open set in $\mathbb{R}^n$. The mapping f of the domain D onto D' is called *closed* if $f(E)$ is closed in D' for any of the closed $E \subset D$ (see, e.g., [34, Section 3]).

In the extended Euclidean n-dimensional space $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$, a *spherical (chordal) metric* is defined as $h(x, y) = |\pi(x) - \pi(y)|$, where π is a stereographic projection of $\overline{\mathbb{R}^n}$ onto the sphere $S^n(\frac{1}{2}e_{n+1}, \frac{1}{2})$ in $\mathbb{R}^{n+1}$. Namely:

$$h(x, y) = \frac{|x - y|}{\sqrt{1 + |x|^2}\sqrt{1 + |y|^2}},$$
$$x \neq \infty \neq y,\ h(x, \infty) = \frac{1}{\sqrt{1 + |x|^2}}. \qquad (1.6)$$

(see, e.g., [28, definition 12.1]). Given sets $A, B \subset \overline{\mathbb{R}^n}$, we put

$$h(A, B) = \inf_{x \in A, y \in B} h(x, y), \quad h(A) = \sup_{x, y \in A} h(x, y),$$

where h is defined in (1.6). In addition, we put

$$\operatorname{dist}(A, B) = \inf_{x \in A, y \in B} |x - y|, \quad \operatorname{diam}(A) = \sup_{x, y \in A} |x - y|.$$

Let $D \subset \mathbb{R}^n$, $n \geqslant 2$, be a domain. For a number $1 \leqslant q < \infty$ and a Lebesgue measurable function $Q : \mathbb{R}^n \to [0, \infty]$, we denote by $\mathfrak{F}^q_Q(D)$ a family of all open discrete mappings $f : D \to \mathbb{R}^n$ such that relation (1.4) holds for any $y_0 \in f(D)$, for any continua

$$E \subset f^{-1}(\overline{B(y_0, r_1)}),\ \ F \subset f^{-1}(f(D) \setminus B(y_0, r_2)),$$
$$0 < r_1 < r_2 < r_0 = \sup_{y \in f(D)} |y - y_0|,$$

and any Lebesgue measurable function $\eta : (r_1, r_2) \to [0, \infty]$ with condition (1.5).

The following theorem holds.

Theorem 1.1 *Let $f \in \mathfrak{F}^q_Q(\mathbb{B}^n)$, $q \geqslant n$. Suppose that $Q \in L^1(\mathbb{R}^n)$ and K is a compact set in $\mathbb{B}^n$. Then the inequality*

$$|f(x) - f(y)| \leqslant C_n \cdot \frac{(\|Q\|_1)^{\frac{1}{q}}}{\log^{\frac{1}{n}}\left(1 + \frac{r_0}{2|x-y|}\right)}, \ r_0 = d(K, \partial\mathbb{B}^n), \qquad (1.7)$$

holds for all $x, y \in K$, where $\|Q\|_1$ denotes the L^1-norm of the function Q in $\mathbb{R}^n$ and a constant $C_n > 0$ depends on n and q only.

Let $D \subset \mathbb{R}^n$ be a domain. Then D is called *locally connected at the point* $x_0 \in \partial D$, if for any neighborhood U of x_0,there is a neighborhood $V \subset U$ of this point such that

$V \cap D$ is connected. The domain D is locally connected on ∂D, if D is locally connected at every point $x_0 \in \partial D$. The domain $D \subset \mathbb{R}^n$ is called *finitely connected at the point* $x_0 \in \partial D$, if for any neighborhood U of x_0, there is a neighborhood $V \subset U$ of this point such that the set $V \cap D$ consists of a finite number of components (see, e.g., [34]). The domain D is finitely connected on ∂D, if D is finitely connected at every point $x_0 \in \partial D$.

Let ∂D be a boundary of the domain $D \subset \mathbb{R}^n$. Then the boundary ∂D is called *weakly flat* at the point $x_0 \in \partial D$, if for each $P > 0$ and for any neighborhood U of this point, there is a neighborhood $V \subset U$ of the same point such that $M(\Gamma(E, F, D)) > P$ for any continua $E, F \subset D$ that intersect ∂U and ∂V. The boundary of a domain D is called weakly flat if the corresponding property holds at any point of ∂D.

Let D, D' be domains in $\mathbb{R}^n$. For given numbers $n \leqslant q < \infty$, $\delta > 0$, a continuum $A \subset D'$, and an arbitrary Lebesgue measurable function $Q : D' \to [0, \infty]$, we denote by $\mathfrak{S}^q_{\delta,A,Q}(D, D')$ a family of all open discrete and closed mappings f of D onto D' satisfying the condition (1.4) for any $y_0 \in D'$, any compacts

$$E \subset f^{-1}(\overline{B(y_0, r_1)}),\ \ F \subset f^{-1}(D' \setminus B(y_0, r_2)),\\ 0 < r_1 < r_2 < r_0 = \sup_{y \in D'} |y - y_0|,$$

and any Lebesgue measurable function $\eta : (r_1, r_2) \to [0, \infty]$ with the condition (1.5), such that $h(f^{-1}(A), \partial D) \geqslant \delta$. The following statement holds.

Theorem 1.2 *Let $D \subset \mathbb{R}^n$ be a bounded with a weakly flat boundary. Suppose that, for any point $y_0 \in \overline{D'}$ and $0 < r_1 < r_2 < r_0 := \sup_{y \in D'} |y - y_0|$, there is a set $E \subset [r_1, r_2]$ of a positive linear Lebesgue measure such that the function Q is integrable on $S(y_0, r)$ for every $r \in E$. If D' is locally connected on its boundary, then any $f \in \mathfrak{S}^q_{\delta,A,Q}(D, D')$ has a continuous extension $\overline{f} : \overline{D} \to \overline{D'}$, $\overline{f}(\overline{D}) = \overline{D'}$, and the family $\mathfrak{S}^q_{\delta,A,Q}(\overline{D}, \overline{D'})$, which consists of all extended mappings $\overline{f} : \overline{D} \to \overline{D'}$, is equicontinuous in $\overline{D}$.*

In particular, the statement of Theorem 1.2 is fulfilled if the above condition on Q is replaced by a simpler one: $Q \in L^1(D')$.

Remark 1.3 In Theorem 1.2, the equicontinuity must be understood with respect to the Euclidean metric in the preimage under the mapping, and the chordal metric in the image, i.e., for any $\varepsilon > 0$, there is $\delta = \delta(\varepsilon, x_0) > 0$ such that the condition $|x - x_0| < \delta$, $x \in D$, implies that inequality $h(\overline{f}(x, \overline{f}(x_0)) < \varepsilon$ holds for any $\overline{f} \in \mathfrak{S}^q_{\delta,A,Q}(\overline{D}, \overline{D'})$.

2 On the integral inverse Poletsky inequality

In this section,we suggest an approach to the generalized quasiconformal mappings which is based on the following integral inequality

$$\int\limits_D |\nabla(u \circ f(x))|^q \, dm(x)\\ \leqslant \int\limits_{D'} |\nabla u(y)|^q Q_q(y)\, dm(y),\ \ u \in C^1(D').$$

This approach allows to unify various generalizations of quasiconformal mappings, such as mappings which generate bounded composition operators on seminormed Sobolev spaces and Q-mappings. We explain that both concepts of generalizations are very close one to another and, in some sense, represent similar classes. Of course, it is a subject of more deep study. We are trying to put attention of readers to this useful interplay.

Let D be a domain in the Euclidean space $\mathbb{R}^n$, $n \geqslant 2$. The Sobolev space $W^1_p(D)$, $1 \leqslant p \leqslant \infty$, is defined as a Banach space of locally integrable weakly differentiable functions $u : D \to \mathbb{R}$ equipped with the following norm:

$$\|u \mid W^1_p(D)\| = \|u \mid L_p(D)\| + \|\nabla u \mid L_p(D)\|,$$

where ∇u is the weak gradient of the function u.

The homogeneous seminormed Sobolev space $L^1_p(D)$, $1 \leqslant p \leqslant \infty$, is defined as a space of locally integrable weakly differentiable functions $u : D \to \mathbb{R}$ equipped with the following seminorm:

$$\|u \mid L^1_p(D\| = \|\nabla u \mid L_p(D)\|.$$

In accordance with the non-linear potential theory [19],we consider the elements of Sobolev spaces $W^1_p(\Omega)$ as equivalence classes up to a set of p-capacity zero [18].

Suppose $f : D \to \mathbb{R}^n$ is a mapping of the Sobolev class $W^1_{1,\mathrm{loc}}(D; \mathbb{R}^n)$. Then the formal Jacobi matrix $Df(x)$ and its determinant (Jacobian) $J(x, f)$ are well defined at almost all points $x \in D$. The norm $|Df(x)|$ is the operator norm of $Df(x)$.

Recall the change of variable formula for the Lebesgue integral [10]. Let a mapping $f : D \to \mathbb{R}^n$ belongs to $W^1_{1,\mathrm{loc}}(D; \mathbb{R}^n)$. Then there exists a measurable set $S \subset D$, $|S| = 0$ such that the mapping $f : D \setminus S \to \mathbb{R}^n$ has the Luzin N-property and the change of variable formula

$$\int_E u\circ f(x)|J(x,f)|\,dm(x) = \int_{\mathbb{R}^n\setminus\varphi(S)} u(y)N_f(E,y)\,dm(y) \tag{2.1}$$

holds for every measurable set $E \subset D$ and every non-negative measurable function $u:\mathbb{R}^n\to\mathbb{R}$. Here $N_f(E,y)$ is the multiplicity function (or the Banach indicatrix) of f.

Now let D and D' be domains in Euclidean space $\mathbb{R}^n$, $n\geqslant 2$. We consider a homeomorphism $f:D\to D'$ of the class $W^1_{1,\mathrm{loc}}(D;D')$ which has finite distortion. Recall that the mapping f is called the mapping of finite distortion if $|Df(x)|=0$ for almost all $x\in Z=\{z\in D : J(x,f)=0\}$.

By using the composition of functions $u\in C^1(D)$ with this homeomorphism $f:D\to D'$,we obtain the following inequality

$$\|u\circ f \mid L^1_q(D)\|^q := \int_D |\nabla(u\circ f(x))|^q\,dm(x) \leqslant \int_D |\nabla u(f(x))|^q|Df(x))|^q\,dm(x) = \int_{D\setminus Z} |\nabla u(f(x))|^q|J(x,f)||Df(x))|^q|J(x,f)|^{-1}\,dm(x).$$

By the change of variables formula [10], we have the following *integral inverse Poletsky inequality*

$$\int_D |\nabla(u\circ f(x))|^q\,dm(x) \leqslant \int_{D'} |\nabla u(y)|^q Q_q(y)\,dm(y), \tag{2.2}$$

where

$$Q_q(y) := \begin{cases} \frac{|Df(x)|^q}{|J(x,f)|}, & x=f^{-1}(y)\in \mathrm{D}\setminus(S\cup Z),\\ 0, & x=f^{-1}(y)\in S\cup Z.\end{cases}$$

The characterization of mappings which generate bounded composition operators on Sobolev spaces in terms of integrability of this distortion function Q_q was given in [31] (see, also, [32, 33]).

Depending on the properties of the distortion function $Q_q(y)$ we obtain different classes of generalized quasiconformal mappings. Let us recall the notion of the variational p-capacity [4]. The condenser in the domain $D\subset\mathbb{R}^n$ is the pair (E,F) of connected closed relatively to D sets $E,F\subset D$. Recall that a continuous function $u\in L^1_p(D)$ is called an admissible function for the condenser (E,F), denoted $u\in W_0(E,F)$, if the set $E\cap D$ is contained in some connected component of the set $\mathrm{Int}\{x: u(x)=0\}$, the set $F\cap D$ is contained in some to the connected component of the set $\mathrm{Int}\{x: u(x)=1\}$. Then we call as a p-capacity of the condenser (E,F) relatively to a domain D the value

$$\mathrm{cap}_p(E,F;\Omega) = \inf \|u|L^1_p(D)\|^p,$$

where the greatest lower bond is taken over all admissible for the condenser $(E,F)\subset D$ functions. If the condenser has no admissible functions, we put the capacity equal to infinity.

The case of K-quasiconformal mappings. Let $q=n$ and

$$\operatorname*{ess\,sup}_{y\in D'} Q_n(y) = \operatorname*{ess\,sup}_{y\in D'} \frac{|Df(f^{-1}(y))|^n}{|J(f^{-1}(y),f)|} = K_n < \infty.$$

Then by the inequality (2.2) for any condenser $(E,F)\subset D'$,the inequality

$$\mathrm{cap}_n(f^{-1}(E),f^{-1}(F);D) \leqslant K_n\,\mathrm{cap}_n(E,F;D')$$

holds. Hence f is a K_n-quasiconformal mapping [28]. From another side, quasiconformal mappings generate bounded composition operators on Sobolev spaces $L^1_n(D')$ and $L^1_n(D)$ [29].

The special case represents conformal mappings that correspond to the case $q=n=2$ and $K=1$. In this case, we have isometries of Sobolev spaces $L^1_2(D')$ and $L^1_2(D)$.

The case of q-quasiconformal mappings. Let $1<q<\infty$ and

$$\operatorname*{ess\,sup}_{y\in D'} Q_q(y) = \operatorname*{ess\,sup}_{y\in D'} \frac{|Df(f^{-1}(y))|^q}{|J(f^{-1}(y),f)|} = K_q < \infty.$$

Then by the inequality (2.2) for any condenser $(E,F)\subset D'$ the inequality

$$\mathrm{cap}_q(f^{-1}(E),f^{-1}(F);D) \leqslant K_q\,\mathrm{cap}_q(E,F;D')$$

holds. Hence f is a q-quasiconformal mapping [30]. From another side by [3, 30] q-quasiconformal mappings generate bounded composition operators on Sobolev spaces $L^1_q(D')$ and $L^1_q(D)$.

The case of (p,q)-quasiconformal mappings. Let $1<q<p<\infty$ and $Q_q\in L_s(\Omega)$, $s>1$. Then by the Hölder inequality

$$\left(\int_D |\nabla(u\circ f)|^q\,dm(x)\right)^{\frac{1}{q}} \leqslant \left(\int_{D'} |\nabla u(y)|^q Q_q(y)\,dm(y)\right)^{\frac{1}{q}} \leqslant \left(\int_{D'} Q_q^s(y)\,dm(y)\right)^{\frac{1}{qs}} \left(\int_{D'} |\nabla u(y)|^{q\frac{s}{s-1}}\,dm(y)\right)^{\frac{s-1}{qs}}.$$

Denote $p = q\frac{s}{s-1}$. Then $s = p/(p-q)$ and we obtain

$$\left(\int\limits_D |\nabla(u \circ f)|^q \, dm(x)\right)^{\frac{1}{q}} \leqslant \left(\int\limits_{D'} Q_q^{\frac{p}{p-q}}(y)\, dm(y)\right)^{\frac{p-q}{pq}} \left(\int\limits_{D'} |\nabla u(y)|^p \, dm(y)\right)^{\frac{1}{p}}.$$

Hence [27] for any condenser $(E, F) \subset D'$,the inequality

$$\mathrm{cap}_q^{\frac{1}{q}}(f^{-1}(E), f^{-1}(F); D) \leqslant \left(\Phi(D' \setminus (E \cup F))\right)^{\frac{p-q}{pq}} \mathrm{cap}_p^{\frac{1}{p}}(E, F; D')$$

holds, where

$$\Phi(D' \setminus (E \cup F)) = \int\limits_{D' \setminus (E \cup F)} Q_q^{\frac{p}{p-q}}(y)\, dm(y).$$

So f is a (p, q)-quasiconformal mapping [27]. From another side by [27],(p, q)-quasiconformal mappings generate bounded composition operators on Sobolev spaces $L^1_p(D')$ and $L^1_q(D)$.

The case of Q-mappings. Let $q = n$ and $Q_q \in L_1(D)$. Then,we have the class of mappings with capacitory inverse Poletsky inequality which was intensively studied recently [24, 26] and [23].

So we can conclude that the integral inequality

$$\int\limits_D |\nabla(u \circ f)|^q \, dm(x) \leqslant \int\limits_{D'} |\nabla u(y)|^q Q(y)\, dm(y)$$

is the basic tool for generalizations of quasiconformal mappings. In the present work, we consider the Hölder continuity and the continuous boundary extension of continuous mappings $f : D : \mathbb{R}^n$ in the case $q \neq n$ and $Q_q \in L_1(D)$. This class of mappings derives properties of mappings (p, q)-quasiconformal mappings which are important in the spectral theory of elliptic operators.

In the case of connected closed relatively to D sets $E, F \subset D$,the notions of the capacity and the modulus coincide, but in view of suggested techniques,we will use the notion of the modulus.

3 On the Hölder continuity of mappings

Let us first formulate the important topological statement, which is repeatedly used later (see, for example, [12, theorem 1.I.5.46]).

Proposition 3.1 *Let A be a set in a topological space X. If the set C is connected, $C \cap A \neq \varnothing$ and $C \setminus A \neq \varnothing$, then $C \cap \partial A \neq \varnothing$.*

Let $D \subset \mathbb{R}^n$, $f : D \to \mathbb{R}^n$ be a discrete open mapping, $\beta : [a, b) \to \mathbb{R}^n$ be a path, and $x \in f^{-1}(\beta(a))$. A path $\alpha : [a, c) \to D$ is called a *maximal f-lifting* of β starting at x, if (1) $\alpha(a) = x$; (2) $f \circ \alpha = \beta|_{[a, c)}$; (3) for $c < c' \leqslant b$, there is no a path $\alpha' : [a, c') \to D$ such that $\alpha = \alpha'|_{[a, c)}$ and $f \circ \alpha' = \beta|_{[a, c')}$. Similarly, we may define a maximal f-lifting $\alpha : (c, b] \to D$ of a path $\beta : (a, b] \to \mathbb{R}^n$ ending at $x \in f^{-1}(\beta(b))$. The maximal lifting $\alpha : [a, c) \to D$ of the path $\beta : [a, b) \to \mathbb{R}^n$ at the mapping f with the origin at the point x is called *whole (total)* if, in the above definition, $c = b$. The following assertion holds (see [14, Lemma 3.12]).

Proposition 3.2 *Let $f : D \to \mathbb{R}^n$, $n \geqslant 2$, be an open discrete mapping, let $x_0 \in D$, and let $\beta : [a, b) \to \mathbb{R}^n$ be a path such that $\beta(a) = f(x_0)$ and such that either $\lim\limits_{t\to b} \beta(t)$ exists, or $\beta(t) \to \partial f(D)$ as $t \to b$. Then β has a maximal f-lifting $\alpha : [a, c) \to D$ starting at x_0. If $\alpha(t) \to x_1 \in D$ as $t \to c$, then $c = b$ and $f(x_1) = \lim\limits_{t\to b} \beta(t)$. Otherwise $\alpha(t) \to \partial D$ as $t \to c$.*

Given a path $\gamma : [a, b] \to \mathbb{R}^n$, we use the notation

$$|\gamma| := \{x \in \mathbb{R}^n : \exists t \in [a, b] : \gamma(t) = x\}$$

for the *locus* of γ, see, e.g., [28, Section 1.1], [21, Section II.1].

Proof of Theorem 1.1 In general, we follow the logic of the proof of Theorem 1.2 in [24], see also Theorem 1.2 in [26] and Theorems 1–2 in [23]. Let us fix $x, y \in K \subset \mathbb{B}^n$ and $f \in \mathfrak{F}_Q(\mathbb{B}^n)$. We put

$$|f(x) - f(y)| := \varepsilon_0. \tag{3.1}$$

If $\varepsilon_0 = 0$, there is nothing to prove. Let $\varepsilon_0 > 0$. Let us give a straight line through the points $f(x)$ and $f(y)$: $r = r(t) = f(x) + (f(x) - f(y))t$, $-\infty < t < \infty$. Let $\gamma^1 : [1, c) \to \mathbb{B}^n$, $1 < c \leqslant \infty$ be a maximum f-lifting of the ray $r = r(t)$, $t \geqslant 1$, with the origin at the point x, which exists due to Proposition 3.2. Let us prove that, the case $\gamma^1(t) \to x_1 \in \mathbb{B}^n$ as $t \to c$ is impossible. Indeed, in this case, by Proposition 3.2, we obtain that $c = \infty$ and $f(x_1) = \lim\limits_{t\to+\infty} r(t)$. Due to the openness of f, $f(x_1) \in f(\mathbb{B}^n)$, but on the other hand, $f(x_1) = \infty$ by the definition of $r = r(t)$. Since $\infty \notin f(\mathbb{B}^n)$, we obtain a contradiction. Therefore, $\gamma^1(t) \to x_1 \in \mathbb{B}^n$ as $t \to c$, is impossible, as required. By Proposition 3.2

$$h(\gamma^1(t), \partial\mathbb{B}^n) \to 0 \tag{3.2}$$

as $t \to c-0$. Similarly, denote by $\gamma^2 : (d, 0] \to \mathbb{B}^n$, $-\infty \leqslant d < 0$, the maximal f-lifting of a ray $r = r(t)$, $t \leqslant 0$, with the end at the point y, which exists by Proposition 3.2. Similarly to (3.2), we obtain that

$$h(\gamma^2(t), \partial\mathbb{B}^n) \to 0$$

as $t \to d+0$. Let $z = \gamma^1(t_1)$ be some point on γ^1, which is located at the distance $r_0/2$ from the unit sphere, where $r_0 := d(K, \partial\mathbb{B}^n)$ and let $w = \gamma^2(t_2)$ be some point on γ^2, located at the distance $r_0/2$ from the unit sphere. Put $\gamma^* := \gamma^1|_{[1,t_1]}$ and $\gamma_* := \gamma^1|_{[t_2,0]}$. By the triangle inequality, diam $(|\gamma^*|) \geqslant r_0/2$ and diam $(|\gamma_*|) \geqslant r_0/2$. Let $\Gamma := \Gamma(|\gamma^*|, |\gamma_*|, \mathbb{B}^n)$. Now, by using [35, lemma 4.3], we obtain that

$$M(\Gamma) \geqslant (1/2) \cdot M(\Gamma(|\gamma^*|, |\gamma_*|, \mathbb{R}^n)), \tag{3.3}$$

and, on the other hand, by [36, Lemma 7.38]

$$M(\Gamma(|\gamma^*|, |\gamma_*|, \mathbb{R}^n)) \geqslant c_n \cdot \log\left(1 + \frac{1}{m}\right), \tag{3.4}$$

where $c_n > 0$ is some constant depends on n only and

$$m = \frac{\mathrm{dist}(|\gamma^*|, |\gamma_*|)}{\min\{\mathrm{diam}\,(|\gamma^*|), \mathrm{diam}\,(|\gamma_*|)\}}.$$

Note that, $\mathrm{diam}\,(|\gamma^i|) = \sup\limits_{\omega, w \in |\gamma^i|} |\omega - w| \geqslant r_0/2$, $i = 1, 2$. Then, by (3.3) and (3.4) and taking into account that $\mathrm{dist}\,(|\gamma^*|, |\gamma_*|) \leqslant |x - y|$, we obtain

$$\begin{aligned} M(\Gamma) &\geqslant \widetilde{c_n} \cdot \log\left(1 + \frac{r_0}{2\mathrm{dist}(|\gamma^*|, |\gamma_*|)}\right) \\ &\geqslant \widetilde{c_n} \cdot \log\left(1 + \frac{r_0}{2|x-y|}\right), \end{aligned} \tag{3.5}$$

where $\widetilde{c_n} > 0$ is some constant depends on n only. By the Hölder inequality, for any function $\rho \in \mathrm{adm}\,\Gamma$,we have

$$M(\Gamma) \leqslant \int\limits_{\mathbb{B}^n} \rho^n(x)\,dm(x) \leqslant \left(\int\limits_{\mathbb{B}^n} \rho^q(x)\,dm(x)\right)^{\frac{n}{q}} \cdot (\Omega_n)^{\frac{q-n}{n}}. \tag{3.6}$$

Taking in the right side of the inequality (3.6) the infimum over all $\rho \in \mathrm{adm}\,\Gamma$, we obtain that

$$M(\Gamma) \leqslant \inf \int\limits_{\mathbb{B}^n} \rho^n(x)\,dm(x) \leqslant \left(M_q(\Gamma)\right)^{\frac{n}{q}} \cdot (\Omega_n)^{\frac{q-n}{n}}. \tag{3.7}$$

Combining (3.5) and (3.6), we obtain that

$$M_q(\Gamma) \geqslant (m(\Omega_n))^{(n-q)q} (\widetilde{c_n})^{\frac{q}{n}} \cdot \log^{\frac{q}{n}}\left(1 + \frac{r_0}{2|x-y|}\right). \tag{3.8}$$

Let $z_1 := f(z)$, $\varepsilon^{(1)} := |f(x) - z^1|$ and $\varepsilon^{(2)} := |f(y) - z^1|$. Note that

$$\begin{aligned} &|f(y) - f(x)| + \varepsilon^{(1)} = \\ &\quad = |f(y) - f(x)| + |f(x) - z^1| = |z^1 - f(y)| = \varepsilon^{(2)}, \end{aligned} \tag{3.9}$$

therefore, $\varepsilon^{(1)} < \varepsilon^{(2)}$.

Now let us to obtain an upper estimate for $M_q(\Gamma)$. We put $\mathbf{P} = |f(\gamma^*)|$, $\mathbf{Q} = |f(\gamma^2)|$, and

$$A := A(z^1, \varepsilon^{(1)}, \varepsilon^{(2)}) = \{x \in \mathbb{R}^n : \varepsilon^{(1)} < |x - z^1| < \varepsilon^{(2)}\}.$$

Note that, $E := \gamma^*$ and $F := \gamma_*$ are continua in $\mathbb{B}^n$. Let us to prove that

$$|\gamma^*| \subset f^{-1}(\overline{B(z^1, \varepsilon^{(1)})}), \qquad |\gamma_*| \subset f^{-1}(f(\mathbb{B}^n) \setminus B(z^1, \varepsilon^{(2)})).$$

Indeed, let $x_* \in |\gamma^*|$. Then $f(x_*) \in \mathbf{P}$, therefore, there exist numbers $1 \leqslant t \leqslant s$ such that $f(x_*) = f(y) + (f(x) - f(y))t$, where $z^1 = f(y) + (f(x) - f(y))s$. Thus,

$$\begin{aligned} |f(x_*) - z^1| &= |(f(x) - f(y))(s - t)| \\ &\leqslant |(f(x) - f(y))(s - 1)| \\ &= |(f(x) - f(y))s + f(y) - f(x))| \\ &= |f(x) - z^1| = \varepsilon^{(1)}. \end{aligned} \tag{3.10}$$

By (3.10), it follows that $|\gamma^*| \subset f^{-1}(\overline{B(z^1, \varepsilon^{(1)})})$. The inclusion $|\gamma_*| \subset f^{-1}(f(\mathbb{B}^n) \setminus B(z^1, \varepsilon^{(2)}))$ may be proved similarly.

Let us put

$$\eta(t) = \begin{cases} \frac{1}{\varepsilon_0}, & t \in [\varepsilon^{(1)}, \varepsilon^{(2)}], \\ 0, & t \notin [\varepsilon^{(1)}, \varepsilon^{(2)}], \end{cases}$$

where ε_0 is a number from (3.1). Note that the function η satisfies the relation (1.5) for $r_1 = \varepsilon^{(1)}$ and $r_2 = \varepsilon^{(2)}$. Indeed, by (3.1) and (3.9),we obtain that

$$r_1 - r_2 = \varepsilon^{(2)} - \varepsilon^{(1)} = |f(y) - z^1| - |f(x) - z^1| =$$

$$= |f(x) - f(y)| = \varepsilon_0.$$

Then $\int\limits_{\varepsilon^{(1)}}^{\varepsilon^{(2)}} \eta(t)\,dt = (1/\varepsilon_0)\cdot(\varepsilon^{(2)}-\varepsilon^{(1)}) \geqslant 1$. Applying the moduli inequality (1.4) for the point z^1, we obtain that

$$M_q(\Gamma) \leqslant \frac{1}{\varepsilon_0^q}\int\limits_{\mathbb{R}^n} Q(z)\,dm(z) = \frac{\|Q\|_1}{|f(x)-f(y)|^q}. \tag{3.11}$$

Finally, from (3.8) and (3.11), we obtain that

$$(\Omega_n)^{(n-q)q}(\widetilde{c_n})^{\frac{q}{n}}\cdot\log^{\frac{q}{n}}\left(1+\frac{r_0}{2|x-y|}\right) \leqslant \frac{\|Q\|_1}{|f(x)-f(y)|^q}.$$

Hence, it follows that

$$|f(x)-f(y)| \leqslant C_n\cdot\frac{(\|Q\|_1)^{\frac{1}{q}}}{\log^{\frac{1}{n}}\left(1+\frac{r_0}{2|x-y|}\right)},$$

where $C_n := (\Omega_n)^{\frac{(q-n)q}{q}}(\widetilde{c_n})^{-\frac{1}{n}}$. The theorem is proved. □

4 Hölder continuity in arbitrary domains

Let D, D' be domains in $\mathbb{R}^n$, $n \geqslant 2$. For numbers $1 \leqslant q < \infty$ and a Lebesgue measurable function $Q : \mathbb{R}^n \to [0,\infty]$, $Q = 0$ a.e. on $\mathbb{R}^n \setminus D'$, we denote be $\mathfrak{R}^q_Q(D, D')$ the family of all open and discrete mappings $f : D \to D'$ such that the moduli inequality (1.4) holds at any point $y_0 \in D'$. The following theorem generalizes [24, Theorem 4.1].

Theorem 4.1 *Let $Q \in L^1(\mathbb{R}^n)$ and $q \geqslant n$. Suppose that, K is compact in D, and D' is bounded. Then there exists a constant $C = C(n, q, K, \|Q\|_1, D, D') > 0$ such that the inequality*

$$|f(x)-f(y)| \leqslant C_n\cdot\frac{(\|Q\|_1)^{\frac{1}{q}}}{\log^{\frac{1}{n}}\left(1+\frac{r_0}{2|x-y|}\right)},\quad r_0 = d(K,\partial D), \tag{4.1}$$

holds for any $x, y \in K$ and $f \in \mathfrak{R}_Q(D, D')$, where $\|Q\|_1$ denotes the L^1-norm of the function Q in $\mathbb{R}^n$.

Proof It is sufficient to find an upper bound for the value

$$|f(x)-f(y)|\cdot\log^{\frac{1}{n}}\left(1+\frac{r_0}{2|x-y|}\right) \tag{4.2}$$

over all $x, y \in K$ and $f \in \mathfrak{R}_Q(D, D')$.

We fix $x, y \in K$ and $f \in \mathfrak{R}_Q(D, D')$. If $|x-y| \geqslant r_0/2$, the expression in (4.2) is trivially bounded. Indeed, by the triangle inequality,

$$|f(x)-f(y)| \leqslant |f(x)|+|f(y)| \leqslant 2M_0, \tag{4.3}$$

where $M_0 = \sup\limits_{z\in D'}|z|$. Since D' is bounded, $M_0 < \infty$. By (4.3), we obtain that

$$|f(x)-f(y)|\cdot\log^{\frac{1}{n}}\left(1+\frac{r_0}{2|x-y|}\right) \leqslant M_0\cdot\log^{\frac{1}{n}}2, \tag{4.4}$$

as required.

Now let $|x-y| < r_0/2$. In this case, $y \in B(x, r_0)$. Let ψ be a conformal mapping of the unit ball $\mathbb{B}^n$ onto the ball $B(x, r_0)$, exactly, $\psi(z) = zr_0 + x$, $z \in \mathbb{B}^n$. In particular, $\psi^{-1}(B(x, r_0/2)) = B(0, 1/2)$. Applying the restriction $\widetilde{f} := f|_{B(x,r_0)}$ and considering the auxiliary mapping $F := \widetilde{f}\circ\psi$, $F : \mathbb{B}^n \to D'$, we conclude that the relation (1.4) also holds for F with the same function Q. Then by Theorem 1.1

$$|F(\psi^{-1}(x)) - F(\psi^{-1}(y))| \leqslant \frac{C_2\cdot(\|Q\|_1)^{1/q}}{\log^{\frac{1}{n}}\left(1+\frac{1}{4|\psi^{-1}(x)-\psi^{-1}(y)|}\right)}. \tag{4.5}$$

Since $F(\psi^{-1}(x)) = f(x)$ and $F(\psi^{-1}(y)) = f(y)$, we may rewrite (4.5) in the form

$$|f(x)-f(y)| \leqslant \frac{C_2\cdot(\|Q\|_1)^{1/q}}{\log^{\frac{1}{n}}\left(1+\frac{1}{4|\psi^{-1}(x)-\psi^{-1}(y)|}\right)}. \tag{4.6}$$

Note that, the mapping $\psi^{-1}(y)$ is Lipschitz with the Lipschitz constant $\frac{1}{r_0}$. In this case, due to (4.6), we obtain that

$$|f(x)-f(y)| \leqslant \frac{C_2\cdot(\|Q\|_1)^{1/q}}{\log^{\frac{1}{n}}\left(1+\frac{r_0}{4|x-y|}\right)}. \tag{4.7}$$

Finally, by the L'Hôpital rule, $\log^{\frac{1}{n}}\left(1+\frac{1}{nt}\right) \sim \log^{\frac{1}{n}}\left(1+\frac{1}{kt}\right)$ as $t \to +0$ and any fixed $k, n > 0$. It follows that

$$\frac{C_2\cdot(\|Q\|_1)^{1/q}}{\log^{\frac{1}{n}}\left(1+\frac{r_0}{4|x-y|}\right)} \leqslant \frac{C_1\cdot(\|Q\|_1)^{1/q}}{\log^{\frac{1}{n}}\left(1+\frac{r_0}{2|x-y|}\right)}$$

for some constant $C_1 = C_1(r_0) > 0$. Then, from (4.7) it follows that

$$|f(x)-f(y)| \leqslant \frac{C_1\cdot(\|Q\|_1)^{1/q}}{\log^{\frac{1}{n}}\left(1+\frac{r_0}{2|x-y|}\right)}. \tag{4.8}$$

Finally, from (4.4) and (4.8), it follows the desired inequality (4.1) with some constant

$$C := \max\{C_1\cdot(\|Q\|_1)^{1/q}, M_0\cdot\log^{\frac{1}{n}}2\}. \qquad \square$$

□

5 Boundary behavior of mappings

The following result in the case $q = n$ was proved in [26, Theorem 3.1], [23, Theorem 4].

Theorem 5.1 *Let $n \leqslant q < \infty$, $D \subset \mathbb{R}^n$, $n \geqslant 2$, be a bounded domain with a weakly flat boundary, and let $D' \subset \mathbb{R}^n$ be a domain which is finitely connected on its boundary. Suppose f is open discrete and closed mapping of D onto D' satisfying the relation (1.4) at any point $y_0 \in \partial D'$, and the following condition holds: for any $y_0 \in \partial D'$ and $0 < r_1 < r_2 < r_0 := \sup\limits_{y\in D'} |y - y_0|$, there is some set $E \subset [r_1, r_2]$ of positive linear Lebesgue measure such that the function Q is integrable on $S(y_0, r)$ for each $r \in E$. Then f has a continuous extension $\overline{f} : \overline{D} \to \overline{D'}$, moreover, $\overline{f}(\overline{D}) = \overline{D'}$.*

In particular, the statement of the theorem 5.1 holds if $Q \in L^1(D')$.

Proof Let $x_0 \in \partial D$. We should prove the possibility of continuous extension of mapping f to point x_0. Let us prove it from the opposite, namely, suppose that f does not have a continuous extension to x_0. Then, there are sequences $x_i, y_i \in D$, $i = 1, 2, \ldots$, such that $x_i, y_i \to x_0$ as $i \to \infty$, and there is $a > 0$ such that

$$h(f(x_i), f(y_i)) \geqslant a > 0 \tag{5.1}$$

for any $i \in \mathbb{N}$, where h is a chordal (spherical) metric, defined in (1.6). Since the space $\overline{\mathbb{R}^n}$ is compact, we may assume that $f(x_i)$ and $f(y_i)$ converge as $i \to \infty$ to z_1 and z_2, respectively, and $z_1 \neq \infty$.

Since f is closed, it preserves the boundary of the domain see [34, theorem 3.3], therefore $z_1, z_2 \in \partial D'$. Since D' is finitely connected on its boundary, there are paths $\alpha : [0, 1) \to D'$ and $\beta : [0, 1) \to D'$ such that $\alpha \to z_1$ and $\beta \to z_2$ as $t \to 1 - 0$ such that $|\alpha|$ contains some subsequence of the sequence $f(x_i)$ and β contains some subsequence of the sequence $f(y_i)$, $i = 1, 2, \ldots$ (see [34, lemma 3.10]). Without loss of generality, we may assume that the paths α and β contain sequences $f(x_i)$ and $f(y_i)$, respectively. Due to the definition of finite connectedness of the domain D' on the boundary, we may assume that

$$|\alpha| \subset B(z_1, R_*),\ |\beta| \subset \mathbb{R}^n \setminus B(z_1, R_0),\quad 0 < R_* < R_0 < \infty. \tag{5.2}$$

We denote by α_i a subpath of α with the origin at a point $f(x_i)$ and end at $f(x_1)$ and, similarly, by β_i a subpath of β starting at $f(y_i)$ and ending at $f(y_1)$. By the change of a parameter, we may consider that, the paths α_i and β_i are parameterized so that $\alpha_i : [0, 1] \to D'$ and $\beta_i : [0, 1] \to D'$. Let $\widetilde{\alpha_i} : [0, 1) \to D$ and $\widetilde{\beta_i} : [0, 1) \to D$ be whole f-liftings of α_i and β_i starting at points x_i and y_i, respectively (these lifts exist by [34, lemma 3.7]). By Proposition 3.2, paths $\widetilde{\alpha_i}$ and $\widetilde{\beta_i}$ can be extended to closed paths $\widetilde{\alpha_i} : [0, 1] \to D$ and $\widetilde{\beta_i} : [0, 1] \to D$. Note that, the points $f(x_1)$ and $f(y_1)$ may not have more than a finite number of pre-images under f in D, see [16, Theorem 2.8]. Then, there is $r_0 > 0$ such that $\widetilde{\alpha_i}(1), \widetilde{\beta_i}(1) \in D \backslash B(x_0, r_0)$ for all $i = 1, 2, \ldots$. Since the boundary of the domain D is weakly flat, for any $P > 0$, there exists $i = i_P \geqslant 1$ such that

$$M(\Gamma(|\widetilde{\alpha_i}|, |\widetilde{\beta_i}|, D)) > P \qquad \forall\, i \geqslant i_P. \tag{5.3}$$

By Hölder inequality, for any function $\rho \in \operatorname{adm} \Gamma$,

$$M(\Gamma) \leqslant \int\limits_D \rho^n(x)\, dm(x) \leqslant \left(\int\limits_D \rho^q(x)\, dm(x)\right)^{\frac{n}{q}} \cdot m^{\frac{q-n}{n}}(D). \tag{5.4}$$

Letting (5.4) to inf over all $\rho \in \operatorname{adm} \Gamma$, we obtain that

$$M(\Gamma) \leqslant \int\limits_D \rho^n(x)\, dm(x) \leqslant \big(M_q(\Gamma)\big)^{\frac{n}{q}} \cdot m^{\frac{q-n}{n}}(D). \tag{5.5}$$

Using (5.3) and (5.5), we obtain that

$$M_q(\Gamma(|\widetilde{\alpha_i}|, |\widetilde{\beta_i}|, D)) > P \cdot m^{-\frac{q-n}{n}}(D) \qquad \forall\, i \geqslant i_P. \tag{5.6}$$

Let us to show that, the condition (5.3) contradicts the definition of mapping f in (1.4). Indeed, using (5.2) and applying (1.4) for $E = |\widetilde{\alpha_i}|$, $F = |\widetilde{\beta_i}|$, $r_1 = R_*$ and $r_2 = R_0$, we obtain that

$$M_q(\Gamma(|\widetilde{\alpha_i}|, |\widetilde{\beta_i}|, D)) \leqslant \int\limits_{A(z_1, R_*, R_0)\cap D'} Q(y) \cdot \eta^q(|y - z_1|)\, dm(y), \tag{5.7}$$

where $\eta : (R_*, R_0) \to [0, \infty]$ is any Lebesgue measurable function such that

$$\int\limits_{R_*}^{R_0} \eta(r)\, dr \geqslant 1. \tag{5.8}$$

Below, we use the standard conventions: $a/\infty = 0$ for $a \neq \infty$, $a/0 = \infty$ for $a > 0$ and $0 \cdot \infty = 0$ (see, e.g., [22, 3.I]). Let us put $\widetilde{Q}(y) = \max\{Q(y), 1\}$,

$$\widetilde{q}_{y_0}(r) = \frac{1}{\omega_{n-1} r^{n-1}} \int\limits_{S(y_0, r)} \widetilde{Q}(y)\, d\mathcal{H}^{n-1}(y) \tag{5.9}$$

and

$$I = \int\limits_{R_*}^{R_0} \frac{dt}{t^{\frac{n-1}{q-1}} \widetilde{q}_{z_1}^{1/(q-1)}(t)}. \tag{5.10}$$

By assumption of the theorem, for any $y_0 \in \partial D'$ and $0 < r_1 < r_2 < r_0 := \sup\limits_{y \in D'} |y - y_0|$, there is a set $E \subset [r_1, r_2]$ of a positive Lebesgue linear measure such that Q is integrable on $S(y_0, r)$ for any $r \in E$. Then $0 \neq I \neq \infty$. In this case, the function $\eta_0(t) = \frac{1}{It^{\frac{n-1}{q-1}} \widetilde{q}_{z_1}^{1/(q-1)}(t)}$ satisfies the relation (5.8). Substituting this function into the right-hand side of (5.7) and applying Fubini theorem (see [22, theorem 8.1, Ch. III]), we obtain that

$$M_q(\Gamma(|\widetilde{\alpha_i}|, |\widetilde{\beta_i}|, D)) \leqslant \int\limits_{A(z_1, R_*, R_0)\cap D'} Q(y) \cdot \eta^q(|y - z_1|)\, dm(y)$$

$$= \int\limits_{R_*}^{R_0} \int\limits_{S(z_1, t)} Q(y) \cdot \eta^q(|y - z_1|)\, d\mathcal{H}^{n-1}\, dt = \frac{\omega_{n-1}}{I^{q-1}} < \infty. \tag{5.11}$$

The relation (5.11) contradicts (5.3), which disproves the assumption made in (5.1). The resulting contradiction disproves the assumption that there is no a limit of f at the point x_0.

It remains to check the equality $\overline{f}(\overline{D}) = \overline{D'}$. It is obvious that $\overline{f}(\overline{D}) \subset \overline{D'}$. Let us show that $\overline{D'} \subset \overline{f}(\overline{D})$. Indeed, let $y_0 \in \overline{D'}$, then either $y_0 \in D'$, or $y_0 \in \partial D'$. If $y_0 \in D'$, then $y_0 = f(x_0)$ and $y_0 \in \overline{f}(\overline{D})$, since by condition f is the mapping of D onto D'. Finally, let $y_0 \in \partial D'$, then there is a sequence $y_k \in D'$ such that $y_k = f(x_k) \to y_0$ as $k \to \infty$, $x_k \in D$. Due to the compactness of $\overline{\mathbb{R}^n}$, we may assume that $x_k \to x_0$, where $x_0 \in \overline{D}$. Note that, $x_0 \in \partial D$, since f is open. Then $\overline{f}(x_0) = y_0 \in \overline{f}(\partial D) \subset \overline{f}(\overline{D})$. In the whole, Theorem 5.1 is proved, excluding the discussion of the situation $Q \in L^1(D')$.

If $Q \in L^1(D')$, by the Fubini theorem,

$$\int\limits_{B(y_0, r_0)} Q(y)\, dm(y) = \int\limits_0^{r_0} \int\limits_{S(y_0, t)} Q(y)\, d\mathcal{H}^{n-1}\, dt < \infty,$$

whence it follows that $q_{y_0}(t) < \infty$ for all $y_0 \in \partial D'$ and almost all $t \in \mathbb{R}$ (here, of course, we extend the function Q by an identical zero outside D'). Thus, the case $Q \in L^1(D')$ is a special case of the conditions on Q mentioned above. The theorem is completely proved. □

6 The equicontinuity of some family of mappings in the closure of domains

Proof of Theorem 1.2 Let $f \in \mathfrak{S}^q_{\delta, A, Q}(D, D')$. By Theorem 5.1, f has a continuous extension $\overline{f} : \overline{D} \to \overline{D'}$, moreover, $\overline{f}(\overline{D}) = \overline{D'}$. The equicontinuity of the family $\mathfrak{S}^q_{\delta, A, Q}(\overline{D}, \overline{D'})$ in D is a statement of Theorem 4.1. It remains to establish its equicontinuity on ∂D.

We will carry out a proof from the opposite (cf. [26, Theorem 1.2], [23, Theorem 5]). Assume that, there is $x_0 \in \partial D$, a number $\varepsilon_0 > 0$, a sequence $x_m \in \overline{D}$, which converges to x_0 as $m \to \infty$,and a sequence of mappings $\overline{f}_m \in \mathfrak{S}^q_{\delta, A, Q}(\overline{D}, \overline{D})$ such that

$$h(\overline{f}_m(x_m), \overline{f}_m(x_0)) \geqslant \varepsilon_0, \quad m = 1, 2, \ldots. \tag{6.1}$$

Let us put $f_m := \overline{f}_m|_D$. Since f_m has a continuous extension on ∂D, we may assume that $x_m \in D$. Therefore, $\overline{f}_m(x_m) = f_m(x_m)$. In addition, there exists a sequence $x'_m \in D$ such that $x'_m \to x_0$ as $m \to \infty$ and $h(f_m(x'_m), \overline{f}_m(x_0)) \to 0$ as $m \to \infty$. Since the space $\overline{\mathbb{R}^n}$ is compact, we may assume that the sequences $f_m(x_m)$ and $\overline{f}_m(x_0)$ converge as $m \to \infty$. Let $f_m(x_m) \to \overline{x_1}$ and $\overline{f}_m(x_0) \to \overline{x_2}$ as $m \to \infty$. By the continuity of the metric in (6.1), $\overline{x_1} \neq \overline{x_2}$. Since f_m is closed, it preserves the boundary (see [34, theorem 3.3]). It follows that $\overline{x_2} \in \partial D'$. Let $\widetilde{x_1}$ and $\widetilde{x_2}$ be arbitrary distinct points of the continuum A, none of which coincides with $\overline{x_1}$. Due to [24, Lemma 2.1], we may join two pairs of points $\widetilde{x_1}, \overline{x_1}$ and $\widetilde{x_2}, \overline{x_2}$ using paths $\gamma_1 : [0, 1] \to \overline{D'}$ and $\gamma_2 : [0, 1] \to \overline{D'}$ such that $|\gamma_1| \cap |\gamma_2| = \varnothing$, $\gamma_1(t), \gamma_2(t) \in D$ for $t \in (0, 1)$, $\gamma_1(0) = \widetilde{x_1}$, $\gamma_1(1) = \overline{x_1}$, $\gamma_2(0) = \widetilde{x_2}$ and $\gamma_2(1) = \overline{x_2}$. Since D' is locally connected on $\partial D'$, there are disjoint neighborhoods U_1 and U_2 containing the points $\overline{x_1}$ and $\overline{x_2}$, such that the sets $W_i := D' \cap U_i$ are path connected. Without loss of generality, we may assume that $\overline{U_1} \subset B(\overline{x_1}, \delta_0)$ and

$$\overline{B(\overline{x_1}, \delta_0)} \cap |\gamma_2| = \varnothing = \overline{U_2} \cap |\gamma_1|, \quad \overline{B(\overline{x_1}, \delta_0)} \cap \overline{U_2} = \varnothing. \tag{6.2}$$

Due to (6.2), there is $\sigma_0 > \delta_0 > 0$ such that

$$\overline{B(\overline{x_1}, \sigma_0)} \cap |\gamma_2| = \varnothing = \overline{U_2} \cap |\gamma_1|, \quad \overline{B(\overline{x_1}, \sigma_0)} \cap \overline{U_2} = \varnothing.$$

We also may assume that $f_m(x_m) \in W_1$ and $f_m(x'_m) \in W_2$ for all $m \in \mathbb{N}$. Let a_1 and a_2 be two different points belonging to $|\gamma_1| \cap W_1$ and $|\gamma_2| \cap W_2$, in addition, let $0 < t_1, t_2 < 1$ be such that $\gamma_1(t_1) = a_1$ and $\gamma_2(t_2) = a_2$. Join the points a_1 and $f_m(x_m)$ with a path $\alpha_m : [t_1, 1] \to W_1$ such that $\alpha_m(t_1) = a_1$

and $\alpha_m(1) = f_m(x_m)$. Similarly, let us join a_2 and $f_m(x'_m)$ by a path $\beta_m : [t_2, 1] \to W_2$, such that $\beta_m(t_2) = a_2$ and $\beta_m(1) = f_m(x'_m)$. Set

$$C^1_m(t) = \begin{cases} \gamma_1(t), & t \in [0, t_1], \\ \alpha_m(t), & t \in [t_1, 1] \end{cases}, \quad C^2_m(t) = \begin{cases} \gamma_2(t), & t \in [0, t_2], \\ \beta_m(t), & t \in [t_2, 1] \end{cases}.$$

Let D^1_m and D^2_m be total f_m-liftings of the paths $|C^1_m|$ and $|C^2_m|$ starting at points x_m and x'_m, respectively (such lifts exist by [34, Lemma 3.7]). In particular, under the condition $h(f_m^{-1}(A), \partial D) \geqslant \delta > 0$, which is part of the definition of the class $\mathfrak{S}^q_{\delta,A,Q}(D, D')$, the ends of b^1_m and b^2_m of paths D^1_m and D^2_m, respectively, distant from ∂D at a distance not less than δ.

Denote by $|C^1_m|$ and $|C^2_m|$ the loci of the paths C^1_m and C^2_m, respectively. Let us put

$$l_0 = \min\{\mathrm{dist}\,(|\gamma_1|, |\gamma_2|), \mathrm{dist}\,(|\gamma_1|, U_2 \setminus \{\infty\})\}$$

and consider the coverage $A_0 := \bigcup\limits_{x\in|\gamma_1|} B(x, l_0/4)$ of the path $|\gamma_1|$ using balls. Since $|\gamma_1|$ is a compact set, we may choose a finite number of indices $1 \leqslant N_0 < \infty$ and corresponding points $z_1, \ldots, z_{N_0} \in |\gamma_1|$ such that $|\gamma_1| \subset B_0 := \bigcup\limits_{i=1}^{N_0} B(z_i, l_0/4)$. In this case,

$$|C^1_m| \subset U_1 \cup |\gamma_1| \subset B(\overline{x_1}, \delta_0) \cup \bigcup_{i=1}^{N_0} B(z_i, l_0/4).$$

Let us put

$$D_{mi} = f_m^{-1}\left(|C^1_m| \cap \overline{B(z_i, l_0/4)}\right), \quad 1 \leqslant i \leqslant N_0,$$
$$D_{m0} = f_m^{-1}\left(|C^1_m| \cap \overline{B(\overline{x_1}, \delta_0)}\right), \quad i = 0.$$

Since f_m is a closed mapping, the pre-image of an arbitrary compact set in D' is a compact set in D (see, e.g., [34, Theorem 3.3 (4)]). Then, the sets D_{mi} are compact in D, and by the definition, $D_{mi} \subset f_m^{-1}(\overline{B(z_i, l_0/4)})$ for $i > 0$ and $D_{m0} \subset f_m^{-1}(\overline{B(\overline{x_1}, \delta_0)})$.

Let Γ^*_m be the family of all paths joining $|D^1_m|$ and $|D^2_m|$ in D, and let Γ_{mi} be a subfamily of paths $\gamma : [0, 1] \to D$ in Γ_m such that $f(\gamma(0)) \in \overline{B(z_i, l_0/4)}$ for $1 \leqslant i \leqslant N_0$ and $f(\gamma(0)) \in \overline{B(\overline{x_1}, \delta_0)}$ for $i = 0$. In this case,

$$\Gamma^*_m = \bigcup_{i=0}^{N_0} \Gamma_{mi}, \tag{6.3}$$

where Γ_{mi} is a family of all paths $\gamma : [0, 1] \to D$ such that $\gamma(0) \in D_{mi}$ and $\gamma(1) \in |D^2_m|$, $0 \leqslant i \leqslant N_0$. Due to the definition of l_0 and σ_0,

$$|D^2_m| \subset f_m^{-1}\left(D' \setminus \left(\bigcup_{i=1}^{N_0} B(z_i, l_0/2) \cup B(\overline{x_1}, \sigma)\right)\right).$$

Then, we may apply the definition of the class of mappings in (1.4) to any family Γ_{mi}. Let us put $\widetilde{Q}(y) = \max\{Q(y), 1\}$ and

$$\widetilde{q}_{z_i}(r) = \frac{1}{\omega_{n-1} r^{n-1}} \int\limits_{S(z_i,r)} \widetilde{Q}(y)\, d\mathcal{H}^{n-1}.$$

Note that, $\widetilde{q}_{z_i}(r) \ne \infty$ for $r \in E \subset [l_0/4, l_0/2]$, $m_1(E) > 0$ (this follows from the condition of the theorem). Let us put

$$I_i = I_i(z_i, l_0/4, l_0/2) = \int\limits_{l_0/4}^{l_0/2} \frac{dr}{r^{\frac{n-1}{q-1}} \widetilde{q}_{z_i}^{\frac{1}{q-1}}(r)}, \quad 1 \leqslant i \leqslant N_0,$$

$$I_0 = I_0(\overline{x_1}, \delta_0, \sigma_0) = \int\limits_{\delta_0}^{\sigma_0} \frac{dr}{r^{\frac{n-1}{q-1}} \widetilde{q}_{\overline{x_1}}^{\frac{1}{q-1}}(r)}.$$

Note that, $I_i \ne 0$, since $\widetilde{q}_{z_i}(r) \ne \infty$ for $r \in E \subset [l_0/4, l_0/2]$, $m_1(E) > 0$. In addition, $I_i \ne \infty$, $i = 0, 1, 2, \ldots, N_0$. In this case, we put

$$\eta_i(r) = \begin{cases} \dfrac{1}{I_i r^{\frac{n-1}{q-1}} \widetilde{q}_{z_i}^{\frac{1}{q-1}}(r)}, & r \in [l_0/4, l_0/2], \\ 0, & r \notin [l_0/4, l_0/2], \end{cases}$$

$$\eta_0(r) = \begin{cases} \dfrac{1}{I_0 r^{\frac{n-1}{q-1}} \widetilde{q}_{\overline{x_1}}^{\frac{1}{q-1}}(r)}, & r \in [\delta_0, \sigma_0], \\ 0, & r \notin [\delta_0, \sigma_0]. \end{cases}$$

Note that, the functions η_i and η_0 satisfy (1.5). Substituting these functions into the definition (1.4), and using the Fubini theorem with a ratio (6.3), we obtain that

$$M_q(\Gamma^*_m) \leqslant \sum_{i=0}^{N_0} M_q(\Gamma_{im}) \leqslant \sum_{i=1}^{N_0} \frac{\omega_{n-1}}{I_i^{q-1}} + \frac{\omega_{n-1}}{I_0^{p-1}} := C_0, \quad m = 1, 2, \ldots. \tag{6.4}$$

Let us show that, the relation (6.4) contradicts the weak flatness of the boundary of the domain D'. Indeed, by construction

$$h(|D^1_m|) \geqslant h(x_m, b^1_m) \geqslant (1/2) \cdot h(f_m^{-1}(A), \partial D) > \delta/2,$$
$$h(|D^2_m|) \geqslant h(x'_m, b^2_m) \geqslant (1/2) \cdot h(f_m^{-1}(A), \partial D) > \delta/2 \tag{6.5}$$

for any $m \geqslant M_0$ and some $M_0 \in \mathbb{N}$. Put $U := B_h(x_0, r_0) = \{y \in \overline{\mathbb{R}^n} : h(y, x_0) < r_0\}$, where $0 < r_0 < \delta/4$ and the number δ refers to ratio (6.5). Note that, $|D_m^1| \cap U \neq \varnothing \neq |D_m^1| \cap (D \setminus U)$ for any $m \in \mathbb{N}$, because $h(|D_m^1|) \geqslant \delta/2$ and $x_m \in |D_m^1|$, $x_m \to x_0$ at $m \to \infty$. Similarly, $|D_m^2| \cap U \neq \varnothing \neq |D_m^2| \cap (D \setminus U)$. Since $|D_m^1|$ and $|D_m^2|$ are continua, by Proposition 3.1

$$|D_m^1| \cap \partial U \neq \varnothing, \quad |D_m^2| \cap \partial U \neq \varnothing. \tag{6.6}$$

Let C_0 be the number from the relation (6.4). Since ∂D is weakly flat, for the number $P := C_0 \cdot m^{\frac{q-n}{n}}(D) > 0$, there is a neighborhood $V \subset U$ of the point x_0 such that

$$M(\Gamma(E, F, D)) > C_0 \cdot m^{\frac{q-n}{n}}(D) \tag{6.7}$$

for any continua $E, F \subset D$ such that $E \cap \partial U \neq \varnothing \neq E \cap \partial V$ and $F \cap \partial U \neq \varnothing \neq F \cap \partial V$. Let us show that,

$$|D_m^1| \cap \partial V \neq \varnothing, \quad |D_m^2| \cap \partial V \neq \varnothing \tag{6.8}$$

for sufficiently large $m \in \mathbb{N}$. Indeed, $x_m \in |D_m^1|$ and $x_m' \in |D_m^2|$, where $x_m, x_m' \to x_0 \in V$ as $m \to \infty$. In this case, $|D_m^1| \cap V \neq \varnothing \neq |D_m^2| \cap V$ for sufficiently large $m \in \mathbb{N}$. Note that $h(V) \leqslant h(U) \leqslant 2r_0 < \delta/2$. By (6.5), $h(|D_m^1|) > \delta/2$. Therefore, $|D_m^1| \cap (D \setminus V) \neq \varnothing$ and, therefore, $|D_m^1| \cap \partial V \neq \varnothing$ (see Proposition 3.1). Similarly, $h(V) \leqslant h(U) \leqslant 2r_0 < \delta/2$. It follows from (6.5) that, $h(|D_m^2|) > \delta/2$. Therefore, $|D_m^2| \cap (D \setminus V) \neq \varnothing$. By Proposition 3.1, we obtain that $|D_m^2| \cap \partial V \neq \varnothing$. Thus, the ratio (6.8) is established. Combining relations (6.6), (6.7),and (6.8), we obtain that $M(\Gamma_m^*) = M(\Gamma(|D_m^1|, |D_m^2|, D)) > C_0 \cdot m^{\frac{q-n}{n}}(D)$. Finally, by the Hölder inequality, taking into account the last condition, we obtain that

$$M_q(\Gamma_m^*) \geqslant C_0 \cdot m^{\frac{q-n}{n}}(D) \cdot m^{-\frac{q-n}{n}}(D) = C_0. \tag{6.9}$$

The latter relation contradicts with (6.4), which proves theorem in the case of functions Q integrable over spheres. The case $Q \in L^1(D')$ can be considered by analogy with the last one part of the proof of Theorem 5.1. □

7 Consequences for mappings with other modulus and capacity conditions

First of all, consider the relation

$$M_q(\Gamma(E, F, D)) \leqslant \int\limits_{f(D)} Q(y) \cdot \rho_*^q(y)\, dm(y) \qquad \forall\ \rho_* \in \operatorname{adm}(f(\Gamma(E, F, D))). \tag{7.1}$$

The following statement holds.

Theorem 7.1 *Let $y_0 \in f(D)$, $q < \infty$ and let $Q : \mathbb{R}^n \to [0, \infty]$ be a Lebesgue measurable function. If f is a mapping that satisfies relation (7.1) for any disjoint nondegenerate compact sets $E, F \subset D$, then f also satisfies condition (1.4) for arbitrary compact sets $E \subset f^{-1}(\overline{B(y_0, r_1)})$, $F \subset f^{-1}(f(D) \setminus B(y_0, r_2))$, $0 < r_1 < r_2 < r_0 = \sup\limits_{y \in D'} |y - y_0|$, and an arbitrary Lebesgue measurable function $\eta : (r_1, r_2) \to [0, \infty]$ with the condition (1.5).*

Proof Let $E \subset f^{-1}(\overline{B(y_0, r_1)})$, $F \subset f^{-1}(f(D) \setminus B(y_0, r_2))$, $0 < r_1 < r_2 < r_0 = \sup\limits_{y \in D'} |y - y_0|$, be arbitrary nondegenerate compacta. Also, let $\eta : (r_1, r_2) \to [0, \infty]$ be an arbitrary Lebesgue measurable function that satisfies condition (1.5). Let us put $\rho_*(y) := \eta(|y - y_0|)$ for $y \in A \cap f(D)$ and $\rho_*(y) = 0$ otherwise, where $A = A(y_0, r_1, r_2) = \{y \in \mathbb{R}^n : r_1 < |y - y_0| < r_2\}$. By Luzin theorem, we may assume that the function ρ_* is Borel measurable (see e.g., [2, Section 2.3.6]). By [28, theorem 5.7]

$$\int\limits_{\gamma_*} \rho_*(y)\, |dy| \geqslant \int\limits_{r_1}^{r_2} \eta(r)\, dr \geqslant 1$$

for any (rectifiable) path $\gamma_* \in \Gamma(f(E), f(F), f(D))$. Then, by (7.1), we obtain that

$$M_q(\Gamma(E, F, D)) \leqslant \int\limits_{A \cap f(D)} Q(y) \cdot \rho_*^q(y)\, dm(y) = \int\limits_{A \cap f(D)} Q(y) \cdot \eta^q(|y - y_0|)\, dm(y).$$

□

Given a Lebesgue measurable function $Q : \mathbb{R}^n \to [0, \infty]$, a *$q$-capacity of (E, F) with a weight Q and with a respect to D* is defined by

$$\operatorname{cap}_{q,Q}(E, F, D) = \inf_{u \in W_0(E,F)} \int\limits_D Q(x) \cdot |\nabla u|^q\, dm(x). \tag{7.2}$$

The following statement holds.

Theorem 7.2 *Let $y_0 \in f(D)$, $q < \infty$ and let $Q : \mathbb{R}^n \to [0, \infty]$ be Lebesgue measurable function. If f is a homeomorphism that satisfies the relation*

$$\operatorname{cap}_q(E, F, D) \leqslant \operatorname{cap}_{q,Q}(f(E), f(F), f(D)), \tag{7.3}$$

for arbitrary compacts (continua) $E, F \subset D$, and

$$\mathrm{cap}_{q,Q}(f(E), f(F), f(D)) = M_{q,Q}(f(E), f(F), f(D)), \tag{7.4}$$

where

$$M_{q,Q}(f(E), f(F), f(D)) = \inf_{\rho_* \in \mathrm{adm}\, \Gamma(f(E), f(F), f(D))} \int\limits_{f(D)} \rho_*^q(y) \cdot Q(y)\, dm(y),$$

then f satisfies the condition (1.4) for arbitrary compacts (continua) sets $E \subset f^{-1}(\overline{B(y_0, r_1)})$, $F \subset f^{-1}(f(D) \setminus B(y_0, r_2))$, $0 < r_1 < r_2 < r_0 = \sup\limits_{y \in D'} |y - y_0|$, and an arbitrary Lebesgue measurable function $\eta : (r_1, r_2) \to [0, \infty]$ with the condition (1.5).

Proof Let $E \subset f^{-1}(\overline{B(y_0, r_1)})$, $F \subset f^{-1}(f(D) \setminus B(y_0, r_2))$, $0 < r_1 < r_2 < r_0 = \sup\limits_{y \in D'} |y - y_0|$, be arbitrary non-degenerate compacta. Also, let $\eta : (r_1, r_2) \to [0, \infty]$ be an arbitrary Lebesgue measurable function that satisfies the condition (1.5). By Hesse equality (see [11, Theorem 5.5]), $\mathrm{cap}_q(E, F, D) = M_q(\Gamma(E, F, D))$. Since f is a homeomorphism, $f(\Gamma(F, E, D)) = \Gamma(f(E), f(F), f(D))$. Then, by (7.3),we obtain that

$$M_q(\Gamma(E, F, D)) \leqslant \mathrm{cap}_{q,Q}(f(E), f(F), f(D)) \leqslant \int\limits_{f(D)} Q(y) \cdot \rho_*^q(y)\, dm(y) \tag{7.5}$$

for any function $\rho_* \in \mathrm{adm}\, f(\Gamma(E, F, D)) = \mathrm{adm}\, \Gamma(f(E), f(F), f(D))$. The desired conclusion follows by Theorem 7.1. □

Due to Theorem 7.2, all results of this paper hold for homeomorphisms with (7.5), the corresponding weight Q of which satisfies the relation (7.4).

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