Complex Analysis and its Synergies (2023) 9:3
https://doi.org/10.1007/s40627-023-00113-x

RESEARCH

Sharpening of Turán type inequalities for polar derivative of a polynomial

Thangjam Birkramjit Singh[1] · Robinson Soraisam[2] · Barchand Chanam[2]

Received: 26 April 2022 / Accepted: 27 December 2022 / Published online: 21 January 2023

Abstract
Let $p(z)$ be a polynomial of degree n. The polar derivative of $p(z)$ with respect to a complex number α is defined by

$$D_\alpha p(z) = np(z) + (\alpha - z)p'(z).$$

If $p(z) = z^s \sum_{j=0}^{n-s} c_j z^j$, $0 \le s \le n$, has all its zeros in $|z| \le k$, $k \ge 1$, then for $|\alpha| \ge k$, Kumar and Dhankhar [Bull, Math. Soc. Sci. Math., 63(4), 359-367 (2020)] proved

$$\max_{|z|=1} |D_\alpha p(z)| \ge \frac{n(|\alpha| - k)}{1 + k^{n-s}} \left(1 + \frac{(|c_{n-s}|k^n - |c_0|k^s)(k-1)}{2(|c_{n-s}|k^n + |c_0|k^{s+1})}\right) \max_{|z|=1} |p(z)|.$$

In this paper, we first improve the above inequality. Besides, we are able to prove an improvement of a result due to Govil and Mctume [Acta. Math. Hungar., 104, 115-126 (2004)] and also prove an inequality for a subclass of polynomials having no zero in $|z| < k$, $k \le 1$.

Keywords Polynomial · Polar derivative · Inequality

Mathematics Subject Classification 30A10 · 30C10 · 30C15

1 Introduction

Let $p(z) = \sum_{j=0}^{n} c_j z^j$ be a polynomial of degree n over the set of complex numbers. We will use $q(z)$ to represent the polynomial $z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$.

According to the famous Bernstein's inequality [6],

$$\max_{|z|=1} |p'(z)| \le n \max_{|z|=1} |p(z)|. \tag{1}$$

Equality in (1) holds for $p(z) = \alpha z^n$, $\alpha \ne 0$.

If we restrict the zeros of $p(z)$, inequality (1) can be refined. In this direction, Erdös conjectured and later Lax [19, p. 1] proved that if $p(z)$ is a polynomial of degree n having no zero in $|z| < 1$, then

$$\max_{|z|=1} |p'(z)| \le \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{2}$$

Robinson Soraisam and Barchand Chanam have contributed equally to this work.

✉ Robinson Soraisam
soraisam.robinson@gmail.com

Thangjam Birkramjit Singh
birkramth@gmail.com

Barchand Chanam
barchand_2004@yahoo.co.in

1 Department of Mathematics, St. Joseph University, Virgin Town, Ikishe Model Village, Dimapur, Nagaland 797115, India

2 Department of Mathematics, National Institute of Technology Manipur, Langol, Imphal, Manipur 795004, India

Inequality (2) is best possible for $p(z) = \alpha + \beta z^n$, where $|\alpha| = |\beta|$.

It was R. P. Boas who asked that if $p(z)$ is a polynomial of degree n not vanishing in $|z| < k$, $k > 0$, then how large

$$\left\{\max_{|z|=1}|p'(z)| \Big/ \max_{|z|=1}|p(z)|\right\} \quad \text{can be ?}$$

A partial answer to this problem was given by Malik [20, Theorem, p. 58], who proved that if $p(z)$ is a polynomial of degree n having no zeros in $|z| < k$, $k \geq 1$, then

$$\max_{|z|=1}|p'(z)| \leq \frac{n}{1+k}\max_{|z|=1}|p(z)|. \tag{3}$$

In the literature, there exist generalizations and improvements of inequality (3), for brief understanding one can refer to: Chan and Malik [8], Qazi [21], Bidkham and Dewan [7], Aziz and Zargar [4], Chanam and Dewan [9], Aziz and Shah [3] etc.

On the other hand, for the class of polynomials $p(z)$ such that $p(z) \neq 0$ for $|z| < k$, $k \leq 1$, the precise estimate for maximum of $|p'(z)|$ on $|z| = 1$ does not seem to be easily obtainable. For quit some time, it was believed that the inequality analogous to (3) for $p(z) \neq 0$ in $|z| < k$, $k \leq 1$, should be

$$\max_{|z|=1}|p'(z)| \leq \frac{n}{1+k^n}\max_{|z|=1}|p(z)|, \tag{4}$$

till E. B. Saff gave the example $p(z) = \left(z - \frac{1}{2}\right)\left(z + \frac{1}{3}\right)$ to counter this belief.

With extra assumption inequality (4) could be satisfied. In this direction, Govil [11] proved that if $p(z)$ is a polynomial of degree n having no zero in $|z| < k$, $k \leq 1$, with additional hypothesis that $|p'(z)|$ and $|q'(z)|$ attain their maxima at the same point on $|z| = 1$, then

$$\max_{|z|=1}|p'(z)| \leq \frac{n}{1+k^n}\max_{|z|=1}|p(z)|. \tag{5}$$

Under the same set of hypothesis, Kumar and Dhankar [18, Theorem 2] further improved inequality (5) by proving

$$\max_{|z|=1}|p'(z)| \leq \frac{n}{1+k^n}\left\{1 - \frac{k^n\left(|c_0| - |c_n|k^n\right)(1-k)}{2\left(|c_0|k + |c_n|k^n\right)}\right\}\max_{|z|=1}|p(z)|. \tag{6}$$

Another improvement of (5) was also recently obtained by Singh and Chanam [23, Theorem 3] by proving

$$\max_{|z|=1}|p'(z)| \leq \left[\frac{n}{1+k^n} - \frac{\left(\sqrt{|c_0|} - k^{\frac{n}{2}}\sqrt{|c_n|}\right)k^n}{(1+k^n)\sqrt{|c_0|}}\right]\max_{|z|=1}|p(z)|. \tag{7}$$

In 1939, Turán [26] provided a lower bound estimate of the derivative to the size of the polynomial by restricting its zeros, and proved that if $p(z)$ has all its zeros in $|z| \leq 1$, then

$$\max_{|z|=1}|p'(z)| \geq \frac{n}{2}\max_{|z|=1}|p(z)|. \tag{8}$$

Aziz and Dawood [1, Theorem 4] further refined inequality (8) by involving $\min_{|z|=1}|p(z)|$. In fact, they proved

$$\max_{|z|=1}|p'(z)| \geq \frac{n}{2}\left\{\max_{|z|=1}|p(z)| + \min_{|z|=1}|p(z)|\right\}. \tag{9}$$

Both the inequalities (8) and (9) are best possible and equality holds if $p(z)$ has all its zeros on $|z| = 1$.

Inequalities (8) and (9) have been extended and generalized in different directions (see [3,5,12–14]). For polynomial $p(z)$ having all its zeros in $|z| \leq k$, $k \geq 1$, Govil [12, Theorem, p. 544] proved that

$$\max_{|z|=1}|p'(z)| \geq \frac{n}{1+k^n}\max_{|z|=1}|p(z)|. \tag{10}$$

Further, as an improvement of (10) and a generalization of (9), Govil [13, Theorem 2] proved

$$\max_{|z|=1}|p'(z)| \geq \frac{n}{1+k^n}\max_{|z|=1}|p(z)| + \frac{n}{1+k^n}\min_{|z|=k}|p(z)|. \tag{11}$$

Inequalities (10) and (11) are sharp and equality holds for $p(z) = z^n + k^n$.

The concept of ordinary derivative of a polynomial has been generalized to polar derivative of a polynomial as follows:

If $p(z)$ is a polynomial of degree n and α be any real or complex number, the polar derivative of $p(z)$ with respect to α, denoted by $D_\alpha p(z)$, is defined as

$$D_\alpha p(z) = np(z) + (\alpha - z)p'(z).$$

It is easy to see that $D_\alpha p(z)$ is a polynomial of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty}\left[\frac{D_\alpha p(z)}{\alpha}\right]=p'(z).$$

Shah [22] extended inequality (8) to the polar derivative and proved that if $p(z)$ is a polynomial of degree n having all its zeros in $|z|\le 1$, then for any complex number α with $|\alpha|\ge 1$

$$\max_{|z|=1}|D_\alpha p(z)|\ge \frac{n(|\alpha|-1)}{2}\max_{|z|=1}|p(z)|. \tag{12}$$

Recently, Gulzar et al. [17, Theorem 2.1] refined inequality (12) and proved that if $p(z)=\sum_{j=0}^{n}c_jz^j$ is a polynomial of degree n having all its zeros in $|z|\le 1$, then for any complex number α with $|\alpha|\ge 1$ and $|z|=1$

$$|D_\alpha p(z)|\ge \frac{(|\alpha|-1)}{2}\left(n+\frac{\sqrt{|c_n|}-\sqrt{|c_0|}}{\sqrt{|c_n|}}\right)|p(z)|. \tag{13}$$

In 1998, Aziz and Rather [2, Theorem 2] extended inequality (10) to polar derivative by proving that if $p(z)$ is a polynomial of degree n having all its zeros in $|z|\le k, k\ge 1$, then for every complex number α with $|\alpha|\ge k$,

$$\max_{|z|=1}|D_\alpha p(z)|\ge n\left(\frac{|\alpha|-k}{1+k^n}\right)\max_{|z|=1}|p(z)|. \tag{14}$$

Recently, Kumar and Dhankhar [18, Theorem 3] obtained a generalization as well as improvement of (14) by establishing that if $p(z)=z^s\sum_{j=0}^{n-s}c_jz^j, 0\le s\le n$, is a polynomial of degree n having all its zeros in $|z|\le k, k\ge 1$, then for any complex number α with $|\alpha|\ge k$,

$$\begin{aligned}\max_{|z|=1}&|D_\alpha p(z)|\\ &\ge \frac{n(|\alpha|-k)}{1+k^{n-s}}\left(1+\frac{(|c_{n-s}|k^n-|c_0|k^s)(k-1)}{2(|c_{n-s}|k^n+|c_0|k^{s+1})}\right)\\ &\quad\times\max_{|z|=1}|p(z)|.\end{aligned} \tag{15}$$

With the same hypothesis, Singh and Chanam [23, Theorem 1] provided another improvement of (14) and a generalization of (13) and obtained

$$\begin{aligned}\max_{|z|=1}&|D_\alpha p(z)|\\ &\ge \frac{(|\alpha|-k)}{1+k^n}\left(n+s+\frac{k^{\frac{n-s}{2}}\sqrt{|c_{n-s}|}-\sqrt{|c_0|}}{k^{\frac{n-s}{2}}\sqrt{|c_{n-s}|}}\right)\\ &\quad\times\max_{|z|=1}|p(z)|.\end{aligned} \tag{16}$$

Govil and Mctume [15, Theorem 3] extended inequality (11) to polar derivative and proved

$$\begin{aligned}\max_{|z|=1}&|D_\alpha p(z)|\\ &\ge n\left(\frac{|\alpha|-k}{1+k^n}\right)\max_{|z|=1}|p(z)|+n\left(\frac{|\alpha|-(1+k+k^n)}{1+k^n}\right)\\ &\quad\times\min_{|z|=k}|p(z)|,\end{aligned} \tag{17}$$

where α is any complex number with $|\alpha|\ge 1+k+k^n$.

Improvements of inequality (17) by involving leading coefficient and constant term of the polynomial can be seen in recent works of Singh and Chanam [23, Theorem 2] and Singh et al. [24, Theorem 4].

2 Main results

We begin by presenting the following refinement of inequality (15) and inequality (16).

Theorem 1 *If $p(z)=z^s\sum_{j=0}^{n-s}c_jz^j, 0\le s\le n$, is a polynomial of degree n having all its zeros in $|z|\le k, k\ge 1$, then for any complex number α with $|\alpha|\ge k$,*

$$\begin{aligned}\max_{|z|=1}&|D_\alpha p(z)|\\ &\ge \left(\frac{|\alpha|-k}{1+k^{n-s}}\right)\left(n+s+\frac{k^{\frac{n-s}{2}}\sqrt{|c_{n-s}|}-\sqrt{|c_0|}}{k^{\frac{n-s}{2}}\sqrt{|c_{n-s}|}}\right)\\ &\quad\times\left\{1+\frac{(|c_{n-s}|k^n-|c_0|k^s)\,(k-1)}{2\left(|c_{n-s}|k^n+|c_0|k^{s+1}\right)}\right\}\max_{|z|=1}|p(z)|.\end{aligned} \tag{18}$$

Remark 1 Since the polynomial $h(z)=\frac{p(z)}{z^s}=\sum_{j=0}^{n-s}c_jz^j$ has all its zeros in $|z|\le k, k\ge 1$, we have

$$\left|\frac{c_0}{c_{n-s}}\right|\le k^{n-s},$$

which is equivalent to

$$|c_0|k^s\le |c_{n-s}|k^n,$$

and

$$k^{\frac{n-s}{2}}\sqrt{|c_{n-s}|}\ge\sqrt{|c_0|}.$$

Dividing both sides of (18) by $|\alpha|$ and taking limit as $|\alpha|\to\infty$, we get the following generalization and refinement of inequality (10) due to Govil [12].

Corollary 1 *If $p(z) = z^s\sum_{j=0}^{n-s} c_j z^j, 0 \le s \le n$, is a polynomial of degree n having all its zeros in $|z| \le k, k \ge 1$, then*

$$\max_{|z|=1}|p'(z)| \ge \left(\frac{1}{1+k^{n-s}}\right)\left(n+s+\frac{k^{\frac{n-s}{2}}\sqrt{|c_{n-s}|}-\sqrt{|c_0|}}{k^{\frac{n-s}{2}}\sqrt{|c_{n-s}|}}\right)\times\left\{1+\frac{(|c_{n-s}|k^n-|c_0|k^s)(k-1)}{2\left(|c_{n-s}|k^n+|c_0|k^{s+1}\right)}\right\}\max_{|z|=1}|p(z)|. \tag{19}$$

When $s = 0$, Theorem 1, in particular, gives the following improvement of inequality (14) proved by Aziz and Rather [2] and a generalization and an improvement of inequality (13) of Gulzar et al. [17].

Corollary 2 *If $p(z) = \sum_{j=0}^{n} c_j z^j$ is a polynomial of degree n having all its zeros in $|z| \le k, k \ge 1$, then for any complex number $|\alpha|$ with $|\alpha| \ge k$*

$$\max_{|z|=1}|D_\alpha p(z)| \ge \left(\frac{|\alpha|-k}{1+k^n}\right)\left(n+\frac{k^{\frac{n}{2}}\sqrt{|c_n|}-\sqrt{|c_0|}}{k^{\frac{n}{2}}\sqrt{|c_n|}}\right)\times\left\{1+\frac{(|c_n|k^n-|c_0|)(k-1)}{2\left(|c_n|k^n+|c_0|k\right)}\right\}\max_{|z|=1}|p(z)|. \tag{20}$$

Dividing both sides of (20) by $|\alpha|$ and taking limit as $|\alpha| \to \infty$, we get the following refinement of inequality (10) due to Govil [12].

Corollary 3 *If $p(z) = \sum_{j=0}^{n} c_j z^j$ is a polynomial of degree n having all its zeros in $|z| \le k, k \ge 1$, then*

$$\max_{|z|=1}|p'(z)| \ge \left(\frac{1}{1+k^n}\right)\left(n+\frac{k^{\frac{n}{2}}\sqrt{|c_n|}-\sqrt{|c_0|}}{k^{\frac{n}{2}}\sqrt{|c_n|}}\right)\times\left\{1+\frac{(|c_n|k^n-|c_0|)(k-1)}{2\left(|c_n|k^n+|c_0|k\right)}\right\}\max_{|z|=1}|p(z)|. \tag{21}$$

The inequality (21) is best possible for $p(z) = z^n + k^n$.

Remark 2 Taking $k = 1$ in Corollary 3, inequality (21) provides a refinement of inequality (8) due to Turán [26].

As an application of Theorem 1, we obtain the following result which is a refinement of inequality (17) due to Govil and Mctume [15] and a result recently proved by Singh and Chanam [23, Theorem 2].

Theorem 2 *If $p(z) = \sum_{j=0}^{n} c_j z^j$ is a polynomial of degree n having all its zeros in $|z| \le k, k \ge 1$, then for any complex number α with $|\alpha| \ge 1 + k + k^n$*

$$\begin{aligned}\max_{|z|=1}|D_\alpha p(z)| &\ge \frac{(|\alpha|-k)}{1+k^n}\left(n+\frac{k^{\frac{n}{2}}\sqrt{|c_n|}-\sqrt{|c_0+e^{i\theta_0}m|}}{k^{\frac{n}{2}}\sqrt{|c_n|}}\right)\\ &\times\left\{1+\frac{(|c_n|k^n-|c_0+e^{i\theta_0}m|)(k-1)}{2(|c_n|k^n+|c_0+e^{i\theta_0}m|k)}\right\}\max_{|z|=1}|p(z)|\\ &+\Bigg[n\left(\frac{|\alpha|-(1+k+k^n)}{1+k^n}\right)\\ &+\frac{|\alpha|-k}{1+k^n}\Bigg\{\frac{k^{\frac{n}{2}}\sqrt{|c_n|}-\sqrt{|c_0+e^{i\theta_0}m|}}{k^{\frac{n}{2}}\sqrt{|c_n|}}\\ &+\frac{(|c_n|k^n-|c_0+e^{i\theta_0}m|)(k-1)}{2(|c_n|k^n+|c_0+e^{i\theta_0}m|k)}\\ &\left(n+\frac{k^{\frac{n}{2}}\sqrt{|c_n|}-\sqrt{|c_0+e^{i\theta_0}m|}}{k^{\frac{n}{2}}\sqrt{|c_n|}}\right)\Bigg\}\Bigg]m,\end{aligned} \tag{22}$$

where $m = \min_{|z|=k}|p(z)|$ and $\theta_0 = \arg\left\{p(e^{i\phi_0})\right\}$ such that $|p(e^{i\phi_0})| = \max_{|z|=1}|p(z)|$.

Remark 3 If $p(z) = \sum_{j=0}^{n} c_j z^j$ is a polynomial of degree n having all its zeros in $|z| \le k, k \ge 1$, then for any complex number $|\lambda|e^{i\theta_0}$ with $|\lambda| < 1$, by Rouche's theorem it follows that the polynomial $p(z)+|\lambda|e^{i\theta_0}m = (c_0+|\lambda|e^{i\theta_0}m)+c_1z+\cdots+c_nz^n$ has all its zeros in $|z| \le k$, where $m = \min_{|z|=k}|p(z)|$, then

$$k^n \ge |\frac{c_0+|\lambda|e^{i\theta_0}m}{c_n}|,$$

which implies that

$$k^{\frac{n}{2}}\sqrt{|c_n|} \ge \sqrt{|c_0+|\lambda|e^{i\theta_0}m|}.$$

Taking $|\lambda| \to 1$, we get

$$k^{\frac{n}{2}}\sqrt{|c_n|} \ge \sqrt{|c_0+e^{i\theta_0}m|},$$

and

$$k^n|c_n| \ge |c_0+e^{i\theta_0}m|.$$

Remark 4 Dividing both sides of (22) by $|\alpha|$ and taking limit as $|\alpha| \to \infty$, we have the following refinement of inequality (11) due to Govil [13].

Corollary 4 *If* $p(z) = \sum_{j=0}^{n} c_j z^j$ *is a polynomial of degree n having all its zeros in* $|z| \le k, k \ge 1$, *then*

$$\max_{|z|=1} |p'(z)| \ge \frac{1}{1+k^n}\left(n + \frac{k^{\frac{n}{2}}\sqrt{|c_n|} - \sqrt{|c_0 + e^{i\theta_0}m|}}{k^{\frac{n}{2}}\sqrt{|c_n|}}\right) \times \left\{1 + \frac{(|c_n|k^n - |c_0 + e^{i\theta_0}m|)(k-1)}{2(|c_n|k^n + |c_0 + e^{i\theta_0}m|)}\right\} \max_{|z|=1}|p(z)| + \left[\frac{n}{1+k^n} + \frac{1}{1+k^n}\left\{\frac{k^{\frac{n}{2}}\sqrt{|c_n|} - \sqrt{|c_0 + e^{i\theta_0}m|}}{k^{\frac{n}{2}}\sqrt{|c_n|}} + \frac{(|c_n|k^n - |c_0 + e^{i\theta_0}m|)(k-1)}{2(|c_n|k^n + |c_0 + e^{i\theta_0}m|)}\left(n + \frac{k^{\frac{n}{2}}\sqrt{|c_n|} - \sqrt{|c_0 + e^{i\theta_0}m|}}{k^{\frac{n}{2}}\sqrt{|c_n|}}\right)\right\}\right]m, \tag{23}$$

where $m = \min_{|z|=k} p(z)$ *and* $\theta_0 = \arg\{p(e^{i\phi_0})\}$ *such that* $|p(e^{i\phi_0})| = \max_{|z|=1}|p(z)|$.

Inequality (23) is best possible for $p(z) = z^n + k^n$.

Remark 5 Taking $k = 1$ in Corollary 4, inequality (23) reduces to a refinement of inequality (9) due to Aziz and Dawood [1].

Corollary 5 *If* $p(z) = \sum_{j=0}^{n} c_j z^j$ *is a polynomial of degree n having all its zeros in* $|z| \le 1$, *then*

$$\max_{|z|=1} |p'(z)| \ge \frac{1}{2}\left(n + \frac{\sqrt{|c_n|} - \sqrt{|c_0 + e^{i\theta_0}m|}}{\sqrt{|c_n|}}\right)\max_{|z|=1}|p(z)| + \frac{1}{2}\left[n + \left(\frac{\sqrt{|c_n|} - \sqrt{|c_0 + e^{i\theta_0}m|}}{\sqrt{|c_n|}}\right)\right]m, \tag{24}$$

where $m = \min_{|z|=1}|p(z)|$ *and* $\theta_0 = \arg\{p(e^{i\phi_0})\}$ *such that* $|p(e^{i\phi_0})| = \max_{|z|=1}|p(z)|$.

Further, we are able to prove an improvement of inequalities (6) and (7).

Theorem 3 *If* $p(z) = \sum_{j=0}^{n} c_j z^j$ *is a polynomial of degree n having no zero in* $|z| < k, k \le 1$. *If* $|p'(z)|$ *and* $|q'(z)|$ *attain their maxima at the same point on* $|z| = 1$, *then*

$$\max_{|z|=1} |p'(z)| \le \frac{1}{1+k^n}\left[n - k^n\left\{\frac{\sqrt{|c_0|} - k^{\frac{n}{2}}\sqrt{|c_n|}}{\sqrt{|c_0|}} + \frac{(|c_0| - k^n|c_n|)(1-k)}{2(|c_0|k + k^n|c_n|)}\left(n + \frac{\sqrt{|c_0|} - k^{\frac{n}{2}}\sqrt{|c_n|}}{\sqrt{|c_0|}}\right)\right\}\right] \max_{|z|=1}|p(z)|. \tag{25}$$

The result is sharp and equality in (25) *holds for* $p(z) = z^n + k^n$.

Remark 6 Since $p(z) = \sum_{j=0}^{n} c_j z^j$ has all its zeros in $|z| \ge k, k \le 1$, $q(z)$ has all its zeros in $|z| \le \frac{1}{k}, \frac{1}{k} \ge 1$, then

$$|\frac{c_n}{c_0}| \le \frac{1}{k^n},$$

which equivalently gives

$$|c_0| \ge |c_n|k^n, \tag{26}$$

and

$$\sqrt{|c_0|} \ge k^{\frac{n}{2}}\sqrt{|c_n|}. \tag{27}$$

From inequalities (26) and (27), it is evident that the bound (25) improves both the bounds given by (6) and (7).

Remark 7 Taking $k = 1$ in Theorem 3, we get the following improvement of (2) due to Erdös and Lax for a subclass of polynomials.

Corollary 6 *If* $p(z) = \sum_{j=0}^{n} c_j z^j$ *is a polynomial of degree n having no zero in* $|z| < 1$. *If* $|p'(z)|$ *and* $|q'(z)|$ *attain their maxima at the same point on* $|z| = 1$, *then*

$$\max_{|z|=1}|p'(z)| \le \frac{1}{2}\left(n - \frac{\sqrt{|c_0|} - \sqrt{|c_n|}}{\sqrt{|c_0|}}\right)\max_{|z|=1}|p(z)|. \tag{28}$$

3 Lemmas

We need the following lemmas to prove our theorems.

Lemma 1 *If* $p(z) = \sum_{j=0}^{n} c_j z^j$ *is a polynomial of degree* $n \ge 1$ *having all its zeros in* $|z| \le 1$, *then for all z on* $|z| = 1$ *with* $p(z) \ne 0$.

$$\Re\left(z\frac{p'(z)}{p(z)}\right) \ge \frac{1}{2}\left(n + \frac{\sqrt{|c_n|} - \sqrt{|c_0|}}{\sqrt{|c_n|}}\right). \tag{29}$$

The above result is due to Dubin [10, Theorem 4](also see Singh and Chanam [23, Lemma 3] and Wali and Shah [25, Inequality 9]).

Lemma 2 *Let $p(z) = z^s\sum_{j=0}^{n-s} c_j z^j, 0 \le s \le n$ be a polynomial of degree n having all its zeros in $|z| \le k, k \ge 1$, then*

$$\max_{|z|=k}|p(z)| \ge \frac{2k^n}{1+k^{n-s}}\left(1+\frac{(|c_{n-s}|k^n-|c_0|k^s)(k-1)}{2(|c_{n-s}|k^n+|c_0|k^{s+1})}\right)\max_{|z|=1}|p(z)|. \quad (30)$$

The above result appears in Kumar and Dhankar [18, Lemma 4].

Lemma 3 *If $p(z) = z^s\sum_{j=0}^{n-s} c_j z^j, 0 \le s \le n$ is a polynomial of degree n having all its zeros in $|z| \le 1$, with $s-$fold zeros at the origin, then for any complex number α with $|\alpha| \ge 1$ and on $|z| = 1$*

$$|D_\alpha p(z)| \ge \frac{(|\alpha|-1)}{2}\left(n+s+\frac{\sqrt{|c_{n-s}|}-\sqrt{|c_0|}}{\sqrt{|c_{n-s}|}}\right)|p(z)|. \quad (31)$$

This result appears in Singh and Chanam [23, Lemma 5].

Lemma 4 *If $p(z)$ is a polynomial of degree n, then on $|z| = 1$*

$$|p'(z)|+|q'(z)| \le n\max_{|z|=1}|p(z)|. \quad (32)$$

The above result is a particular case of a result [16, Inequality 3.2] due to Govil and Rahman.

4 Proofs of the theorems

Proof of Theorem 1 Since $p(z) = z^s\sum_{j=0}^{n-s} c_j z^j$ has all its zeros in $|z| \le k, k \ge 1$, the polynomial $p(kz) = z^s\left(k^s c_0 + k^{s+1}c_1 z + \cdots k^n c_n z^{n-s}\right)$ has all its zeros in $|z| \le 1$. Using Lemma 3 to $p(kz)$, we get for $|\frac{\alpha}{k}| \ge 1$

$$\max_{|z|=1}|D_{\frac{\alpha}{k}} p(kz)| \ge \frac{|\alpha|-k}{2k}\left(n+s+\frac{k^{\frac{n-s}{2}}\sqrt{|c_{n-s}|}-\sqrt{|c_0|}}{k^{\frac{n-s}{2}}\sqrt{|c_{n-s}|}}\right)\times\max_{|z|=1}|p(kz)|,$$

that is

$$\max_{|z|=1}\left|np(kz)+\left(\frac{\alpha}{k}-z\right)kp'(kz)\right| \ge \frac{(|\alpha|-k)}{2k}\left(n+s+\frac{k^{\frac{n-s}{2}}\sqrt{|c_{n-s}|}-\sqrt{|c_0|}}{k^{\frac{n-s}{2}}\sqrt{|c_{n-s}|}}\right)\max_{|z|=k}|p(z)|. \quad (33)$$

Using Lemma 2 and the fact that $\max_{|z|=1}\left|np(kz)+\left(\frac{\alpha}{k}-z\right)kp'(kz)\right| = \max_{|z|=k}|D_\alpha p(z)|$, inequality (33) implies

$$\max_{|z|=k}|D_\alpha p(z)| \ge \frac{(|\alpha|-k)}{2k}\left(n+s+\frac{k^{\frac{n-s}{2}}\sqrt{|c_{n-s}|}-\sqrt{|c_0|}}{k^{\frac{n-s}{2}}\sqrt{|c_{n-s}|}}\right)\times\frac{2k^n}{1+k^{n-s}}\left\{1+\frac{(|c_{n-s}|k^n-|c_0|k^s)(k-1)}{2(|c_{n-s}|k^n+|c_0|k^{s+1})}\right\}\times\max_{|z|=1}|p(z)|. \quad (34)$$

As we can see that $D_\alpha p(z)$ is a polynomial of degree at most $n-1$ and $k \ge 1$, it is well-known that $\max_{|z|=k}|D_\alpha p(z)| \le k^{n-1}\max_{|z|=1}|D_\alpha p(z)|$. Using this fact, inequality (34) gives

$$k^{n-1}\max_{|z|=1}|D_\alpha p(z)| \ge (|\alpha|-k)\left(n+s+\frac{k^{\frac{n-s}{2}}\sqrt{|c_{n-s}|}-\sqrt{|c_0|}}{k^{\frac{n-s}{2}}\sqrt{|c_{n-s}|}}\right)\times\frac{k^{n-1}}{1+k^{n-s}}\left\{1+\frac{(|c_{n-s}|k^n-|c_0|k^s)(k-1)}{2(|c_{n-s}|k^n+|c_0|k^{s+1})}\right\}\times\max_{|z|=1}|p(z)|.$$

which gives inequality (18), and the proof of Theorem 1 is complete. □

Proof of Theorem 2 If $p(z)$ has a zero on $|z| = k$, then $m = 0$ and the result follows trivially from Theorem 1. So, without loss of generality, let us assume that $p(z)$ has all its zeros in $|z| < k, k \ge 1$, then it follows by Rouche's theorem that for any complex number λ with $|\lambda| < 1$, the polynomial $p(z)+\lambda m = (c_0+\lambda m)+c_1 z+\cdots+c_n z^n$ has all its zeros in $|z| < k, k \ge 1$. Therefore, applying Theorem 1 to $p(z)+\lambda m$ with $s = 0$, we get for $|\alpha| \ge 1+k+k^n$

$$\max_{|z|=1}|D_\alpha[p(z)+\lambda m]| \ge \left(\frac{|\alpha|-k}{1+k^n}\right)\left(n+\frac{k^{\frac{n}{2}}\sqrt{|c_n|}-\sqrt{|c_0+\lambda m|}}{k^{\frac{n}{2}}\sqrt{|c_n|}}\right)\times\left\{1+\frac{(|c_n|k^n-|c_0+\lambda m|)(k-1)}{2(|c_n|k^n+|c_0+\lambda m|k)}\right\}\times\max_{|z|=1}|p(z)+\lambda m|. \quad (35)$$

Let $0 \le \phi_0 < 2\pi$, be such that $|p(e^{i\phi_0})| = \max_{|z|=1}|p(z)|$. Then, inequality (35) takes

$$\max_{|z|=1}|D_\alpha p(z)+n\lambda m| \geq \left(\frac{|\alpha|-k}{1+k^n}\right)\left(n+\frac{k^{\frac{n}{2}}\sqrt{|c_n|}-\sqrt{|c_0+\lambda m|}}{k^{\frac{n}{2}}\sqrt{|c_n|}}\right) \tag{36}$$

$$\times\left\{1+\frac{(|c_n|k^n-|c_0+\lambda m|)(k-1)}{2(|c_n|k^n+|c_0+\lambda m|k)}\right\}|p(e^{i\phi_0})+\lambda m|. \tag{37}$$

Now,

$$\begin{aligned}|p(e^{i\phi_0})+\lambda m| &= ||p(e^{i\phi_0})|e^{i\theta_0}+|\lambda|e^{i\phi}m| \\ &= ||p(e^{i\phi_0})|+|\lambda|e^{i(\phi-\theta_0)}m|.\end{aligned}$$

Setting the argument ϕ such that $\phi=\theta_0$, then

$$|p(e^{i\phi_0})+\lambda m| = |p(e^{i\phi_0})|+|\lambda|m. \tag{38}$$

Using this fact in inequality (37), we have

$$\begin{aligned}&\max_{|z|=1}|D_\alpha p(z)|+n|\lambda|m \\ &\geq \left(\frac{|\alpha|-k}{1+k^n}\right)\left(n+\frac{k^{\frac{n}{2}}\sqrt{|c_n|}-\sqrt{|c_0+|\lambda|e^{i\theta_0}m|}}{k^{\frac{n}{2}}\sqrt{|c_n|}}\right) \\ &\times\left\{1+\frac{\left(|c_n|k^n-|c_0+|\lambda|e^{i\theta_0}m|\right)(k-1)}{2\left(|c_n|k^n+|c_0+|\lambda|e^{i\theta_0}m|k\right)}\right\} \\ &\left(|p(e^{i\phi_0})|+|\lambda|m\right).\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\max_{|z|=1}|D_\alpha p(z)| \\ &\geq \frac{(|\alpha|-k)}{1+k^n}\left(n+\frac{k^{\frac{n}{2}}\sqrt{|c_n|}-\sqrt{|c_0+|\lambda|e^{i\theta_0}m|}}{k^{\frac{n}{2}}\sqrt{|c_n|}}\right) \\ &\times\left\{1+\frac{(|c_n|k^n-|c_0+|\lambda|e^{i\theta_0}m|)(k-1)}{2(|c_n|k^n+|c_0+|\lambda|e^{i\theta_0}m|k)}\right\}\max_{|z|=1}|p(z)| \\ &+|\lambda|\left[n\left(\frac{|\alpha|-(1+k+k^n)}{1+k^n}\right)\right. \\ &+\frac{|\alpha|-k}{1+k^n}\left\{\frac{k^{\frac{n}{2}}\sqrt{|c_n|}-\sqrt{|c_0+|\lambda|e^{i\theta_0}m|}}{k^{\frac{n}{2}}\sqrt{|c_n|}}\right. \\ &+\frac{(|c_n|k^n-|c_0+|\lambda|e^{i\theta_0}m|)(k-1)}{2(|c_n|k^n+|c_0+|\lambda|e^{i\theta_0}m|k)} \\ &\left.\left.\left(n+\frac{k^{\frac{n}{2}}\sqrt{|c_n|}-\sqrt{|c_0+|\lambda|e^{i\theta_0}m|}}{k^{\frac{n}{2}}\sqrt{|c_n|}}\right)\right\}\right]m,\end{aligned}$$

Taking $|\lambda|\to 1$, the above inequality reduces to (22). This completes the proof of Theorem 2. □

Proof of Theorem 3 Since $p(z)$ has all its zeros in $|z|\geq k, k\leq 1$, $q(z)$ has all its zeros in $|z|\leq\frac{1}{k}, \frac{1}{k}\geq 1$. Then applying Corollary 3 to $q(z)$, we have

$$\max_{|z|=1}|q'(z)| \geq \left(\frac{k^n}{1+k^n}\right)\left(n+\frac{\left(\frac{1}{k}\right)^{\frac{n}{2}}\sqrt{|c_0|}-\sqrt{|c_n|}}{\left(\frac{1}{k}\right)^{\frac{n}{2}}\sqrt{|c_0|}}\right) \times\left\{1+\frac{\left(|c_0|\left(\frac{1}{k}\right)^n-|c_n|\right)\left(\frac{1}{k}-1\right)}{2\left(|c_0|\left(\frac{1}{k}\right)^n+|c_n|\frac{1}{k}\right)}\right\}\max_{|z|=1}|p(z)|. \tag{39}$$

By Lemma 4, we have on $|z|=1$,

$$|p'(z)|+|q'(z)|\leq n\max_{|z|=1}|p(z)|. \tag{40}$$

Since $|p'(z)|$ and $|q'(z)|$ attain their maxima at the same point on $|z|=1$, then

$$\max_{|z|=1}\left\{|p'(z)|+|q'(z)|\right\}=\max_{|z|=1}|p'(z)|+\max_{|z|=1}|q'(z)|. \tag{41}$$

Combining (39), (40) and (41), we have

$$\begin{aligned}&n\max_{|z|=1}|p(z)| \\ &\geq \left(\frac{k^n}{1+k^n}\right)\left(n+\frac{\sqrt{|c_0|}-k^{\frac{n}{2}}\sqrt{|c_n|}}{\sqrt{|c_0|}}\right) \\ &\times\left\{1+\frac{(|c_0|-k^n|c_n|)(1-k)}{2(|c_0|k+k^n|c_n|)}\right\}\max_{|z|=1}|p(z)|+\max_{|z|=1}|p'(z)|,\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\max_{|z|=1}|p'(z)| \\ &\leq \frac{1}{1+k^n}\left[n-k^n\left\{\frac{\sqrt{|c_0|}-k^{\frac{n}{2}}\sqrt{|c_n|}}{\sqrt{|c_0|}}\right.\right. \\ &\left.\left.+\frac{(|c_0|-k^n|c_n|)(1-k)}{2(|c_0|k+k^n|c_n|)}\left(n+\frac{\sqrt{|c_0|}-k^{\frac{n}{2}}\sqrt{|c_n|}}{\sqrt{|c_0|}}\right)\right\}\right] \\ &\max_{|z|=1}|p(z)|,\end{aligned}$$

□

Declarations

Conflict of interest The authors declares that they have no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

References

1. Aziz, A., Dawood, Q.M.: Inequalities for a polynomial and its derivative. J. Approx. Theory **54**(3), 306–313 (1988)
2. Aziz, A., Rather, N.A.: A refinement of a Theorem of Paul Turán concerning polynomials. J. Math. Inequal. Appl. **1**(2), 231–238 (1998)
3. Aziz, A., Shah, W.M.: Inequalities for a polynomial and its derivative. Math. Inequal. Appl. **7**(3), 379–391 (2004)
4. Aziz, A., Zargar, B.A.: Inequalities for a polynomial and its derivative. Math. Inequal. Appl. **1**(4), 543–550 (1998)
5. Aziz, A., Zargar, B.A.: Inequalities for the maximum modulus of the derivative of a polynomial. J. Inequal. Pure Appl. Math. **8**(2), 1–8 (2007)
6. Bernstein, S.: Lecons Sur Les Propriétés extrémales et la meilleure approximation desfunctions analytiques d'une fonctions reele. Gauthier-Villars, Paris (1926)
7. Bidkham, M., Dewan, K.K.: Inequalities for polynomial and its derivative. J. Math. Anal. Appl. **166**, 319–324 (1992)
8. Chan, T.N., Malik, M.A.: On Erdös-Lax Theorem. Proc. Indian Acad. Sci. **92**(3), 191–193 (1983)
9. Chanam, B., Dewan, K.K.: Inequalities for a polynomial and its derivative. J. Math. Anal. Appl. **336**, 171–179 (2007)
10. Dubinin, V.N.: Distortion theorems for polynomials on the circle. Math. Sb. **191**(12), 51–60 (2000)
11. Govil, N.K.: On a theorem of S. Bernstein. Proc. Natl. Acad. Sci. USA **50**, 50–52 (1980)
12. Govil, N.K.: On the derivative of a polynomial. Proc. Am. Math. Soc. **41**(2), 543–546 (1973)
13. Govil, N.K.: Some inequalities for derivative of polynomial. J. Approx. Theory **66**(1), 29–35 (1991)
14. Govil, N.K., Kumar, P.: On sharpening of an inequality of Turán. Appl. Anal. Discret. Math. **13**(3), 711–720 (2019)
15. Govil, N.K., Mctume, G.N.: Some generalizations involving the polar derivative for an inequality of Paul Turán. Acta Math. Hungar. **104**(1–2), 115–126 (2004)
16. Govil, N.K., Rahman, Q.I.: Functions of exponential type not vanishing in a half-plane and related polynomials. Trans. Am. Math. Soc. **137**, 501–517 (1969)
17. Gulzar, M.H., Zargar, B.A., Akhter, Rubia: Inequalities for the polar derivative of a polynomial. J. Anal. **28**, 923–929 (2020)
18. Kumar, P., Dhankhar, R.: Some refinement of inequalities for polynomials. Bull. Math. Soc. Sci. Math. Roumanie. **111**(4), 359–367 (2020)
19. Lax, P.D.: Proof of a conjecture of P. Erdös on the derivative of a polynomial. Bull. Am. Math. Soc **50**, 509–513 (1944)
20. Malik, M.A.: On the derivative of a polynomial. J. Lond. Math. Soc. **2**(1), 57–60 (1969)
21. Qazi, M.A.: On the maximum modulus of polynomials. Proc. Am. Math. Soc. **115**, 337–343 (1992)
22. Shah, W.M.: A generalization of a theorem of Paul Turán. J. Ramanujan Math. Soc. **1**, 29–35 (1996)
23. Singh, T.B., Chanam, B.: Generalisations and sharpenings of certain Bernstein and Turán types of inequalities for the polar derivative of a polynomial. J. Math. Inequal. **15**(4), 1663–1675 (2021)
24. Singh, T.B., Devi, M.T., Chanam, B.: Sharpening of Bernstein and Turán-type inequalities for polynomials. J. Class. Anal. **18**(2), 137–148 (2021)
25. Wali, S.L., Shah, W.M.: Some applications of Dubinin's lemma to rational functions with prescribed poles. J. Math. Anal. Appl. **50**(4), 769–779 (2017)
26. Turán, P.: Uber die ableitung von polynommen. Compos. Math. **7**, 89–95 (1939)