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RESEARCH

On an invariant distance induced by the Szegő Kernel

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Abstract
In this paper we introduce a new distance by means of the so-called Szegő kernel and examine some basic properties and its relationship with the so-called Skwarczyński distance. We also examine the relationship between this distance, and the so-called Bergman distance and Szegő distance.

Keywords Szegő kernel · Weighted Bergman kernel · Szegő distance · Bergman distance · Green's function

Mathematics Subject Classification Primary 32A36 · Secondary 32A25

1 Introduction

In this paper, we introduce and describe some new distance by means of the Szegő kernel, called here by the *Szegő projective distance* and denoted by ϱ^S_Ω. Since the Szegő kernel doesn't respect the transformation rule, we also consider the so-called Fefferman–Szegő kernel (described below) and the *Fefferman-Szegő projective distance* defined by it (denoted by $\varrho^{S_F}_\Omega$). Both are defined on the same way and in a fashion similar to the so-called Skwarczyński distance (denoted by ϱ_Ω) (see [1, p. 20], and the definition actually based on ideas from projective geometry. The Skwarczyński distance is given more explicitly than the so-called Bergman distance and this is also our motivation too. Since this is new, we list and prove properties of this distance like completeness. The above considerations are nothing but natural generalizations of theorems valid in the case for the Bergman kernel and the Skwarczyński distance. We decided, however, to enclose it here for the sake of completeness.

The new results can be found in Sect. 3.2.1 and at the end of the paper. The main results of the paper are Theorems 23 and 24. We examine the relationship between completeness in the Szegő projective distance and completeness in the Skwarczyński distance.

2 Definitions and notation

Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain with C^2-smooth boundary. Let $A(\Omega)$ be those functions on $\overline{\Omega}$, which are both continuous on $\overline{\Omega}$ and holomorphic in Ω. Denote by $H^2_E(\partial\Omega)$ the space consisting of the closure in the $L^2(\partial\Omega, d\sigma_E)$ topology of the restrictions to $\partial\Omega$ of elements of $A(\Omega)$ (here $d\sigma_E$ denotes the Euclidean surface area measure on $\partial\Omega$). Then $H^2_E(\partial\Omega)$ is a proper Hilbert subspace of $L^2(\partial\Omega, d\sigma_E)$. Recall that each element $f \in H^2_E(\partial\Omega)$ has a natural holomorphic extension to Ω given by its Poisson integral (see [2, p. 66]). The *Szegő kernel* $S(z, w)$ is the reproducing kernel for $H^2_E(\partial\Omega)$, that is

$$f(z) = \int_{\partial\Omega} S(z,w)f(w)d\sigma_E(w), \quad \forall f \in H^2_E(\partial\Omega),$$

The problem is that the Euclidean surface measure does not transform nicely under biholomorphic mappings. We deal with this problem by instead using the so-called *Fefferman surface area measure* σ_F (see [3]), which is given by:

$$d\sigma_F = c_n \sqrt[n+1]{-\det\begin{pmatrix} 0 & \rho_{\bar{k}} \\ \rho_j & \rho_{j\bar{k}} \end{pmatrix}_{1\le j,k\le n}} \frac{d\sigma_E}{\|d\rho\|},$$

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where $\rho_j \equiv \partial\rho/\partial z_j$, $\rho_{\bar{k}} \equiv \partial\rho/\partial\bar{z}_k$, $\rho_{j\bar{k}} \equiv \partial^2\rho/\partial z_j\partial\bar{z}_k$, and ρ is a defining function for Ω (here $\|\cdot\|$ denotes the usual Euclidean distance). The constant c_n is a dimensional constant (see [3]). We should consider the space $H^2_F(\partial\Omega)$ defined in the same way as $H^2_E(\partial\Omega)$ with $d\sigma_F$ instead of $d\sigma_E$. The space $H^2_F(\partial\Omega)$ is a Hilbert space with reproducing kernel in the sense of Aronszajn (see [4]). So it has the reproducing kernel $S_F(z,w)$. Of course, this new kernel is in general not the same as the usual Szegő kernel, but it certainly obeys the reproducing property (see [2, p. 66] and [5]):

$$f(z) = \int_{\partial\Omega} S_F(z,w)f(w)d\sigma_F(w), \quad \forall f \in H^2_F(\partial\Omega).$$

Throughout the paper we are working with both S_F and S. We always try to highlight what kernel is actually considered. When considering S (denoted also by S_Ω) we automatically assume that Ω is a bounded domain with C^2-smooth boundary in $\mathbb{C}^n$. If S_F (denoted also by $S_{F,\Omega}$) is considered, $\Omega \Subset \mathbb{C}^n$ is assumed to be strongly pseudoconvex with C^∞ -smooth boundary.

3 The Fefferman–Projective Szegő distance and some remarks

It turns out that, like the Bergman kernel, the Fefferman–Szegő kernel respects the so-called transformation rule (see ([5], Prop. 2 and also [6], Prop. 3.3):

Proposition 1 *Let $\Omega_1, \Omega_2 \subset \mathbb{C}^n$ and $\varphi : \Omega_1 \to \Omega_2$ be a biholomorphic mapping. Assume there exists a well-defined holomorphic branch of $(\det J_{\mathbb{C}}\varphi(z))^{n/(n+1)}$ on Ω_1. Then we have*

$$S_{F,\Omega_1}(z,w) = S_{F,\Omega_2}(\varphi(z), \varphi(w))(\det J_{\mathbb{C}}\varphi(z))^{n/n+1}\overline{(\det J_{\mathbb{C}}\varphi(w))}^{n/n+1},$$

where $S_{F,\Omega_j}(z,w)$ is the Fefferman-Szegő kernel on Ω_j for $j = 1, 2$.

This property leads us to the biholomorphically invariant distance induced by the Fefferman–Szegő kernel. We have to point out here that for $n > 1$ the classical Szegő kernel doesn't obey the above transformation rule.

In order to introduce the distance, we recall some ideas from the theory of Hilbert spaces.

Let $(H, \langle\cdot,\cdot\rangle)$ be an arbitrary separable Hilbert space. Let us consider the following relation between two nonzero elements: $x \sim y$ if and only if there exists a complex constant $c \neq 0$ such that $x = cy$. The set of equivalence classes forms the (generally infinite dimensional) projective Hilbert space $P(H)$. This is a complete metric space with respect to the distance

$$d_H([x],[y]) = \operatorname{dist}([x] \cap S_H, [y] \cap S_H),$$

where $S_H \subset H$ is the unit sphere, and as usual $\operatorname{dist}(A,B) = \inf\{d(x,y) | x \in A, y \in B\}$ for two nonempty subsets A, B of H. Explicitly,

$$\begin{aligned} d^2_H([x],[y]) &= \inf_{\varphi,\phi\in[0,2\pi]} \left\| \frac{e^{i\varphi}x}{\|x\|} - \frac{e^{i\phi}y}{\|y\|} \right\|^2 \\ &= \inf_{\varphi,\phi\in[0,2\pi]} \left[2 - 2\operatorname{Re} \frac{e^{i(\varphi-\phi)}\langle x,y\rangle}{\|x\|\|y\|} \right] \\ &= 2 - 2\left[\frac{\langle x,y\rangle\langle y,x\rangle}{\langle x,x\rangle\langle y,y\rangle} \right]^{1/2} \end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product of the Hilbert space H. Using this idea, M. Skwarczyński introduced in [1, p. 20] the biholomorphically invariant pseudodistance on domains in $\mathbb{C}^n$. It is directly based on the so-called Bergman kernel (see for example [7, p. 410] and [2, p. 49]). At first, we need an analogue of this idea for Szegő kernels.

Note that $S_F(z,z)$ does not vanish at any point $z \in \Omega$ (see [2, p. 66]). Define the map $\tau : \Omega \to P(H^2_F(\partial\Omega))$ by the formula

$$\tau(z) := [S_F(\cdot, z)].$$

This enables us to introduce the following continuous pseudodistance on $\Omega \times \Omega$:

$$\begin{aligned} \varrho^{S_F}_\Omega(z,w) &:= \frac{1}{\sqrt{2}} d_{H^2(\partial\Omega)}(\tau(z), \tau(w)) \\ &= \left(1 - \frac{|S_F(z,w)|}{\sqrt{S_F(z,z)}\sqrt{S_F(w,w)}}\right)^{1/2}. \end{aligned} \tag{1}$$

(we recall that the symbol ϱ_Ω is fixed for the so-called Skwarczyński distance (see [1, p. 20])).

Remark 2 Observe that the following conditions are equivalent:

(a) τ is injective;
(b) for each two distinct points $z, w \in \Omega$ the functions $S_F(\cdot,z)$, $S_F(\cdot,w)$ are linearly independent;
(c) $\varrho^{S_F}_\Omega$ is a distance.

Let us note the following:

Remark 3 Since Ω is bounded, $\varrho^{S_F}_\Omega$ is a distance.

Proof of the Remark 3 Let $w, t \in \Omega$, $w \neq t$. The points w and t differ by at least one coordinate, let us say the k th one. The polynomial $g(z) = z_k - w_k$ is an element of $H^2_F(\partial\Omega)$ and

$g(w) = 0, g(t) \neq 0$. Let us note now that point evaluations E_t and E_w are linearly independent. Indeed, if

$$\lambda E_t(f) + \beta E_w(f) = 0$$

for all $f \in H^2_F(\partial\Omega)$, then for $f = g$ we have that $\lambda = 0$. The choice $f \equiv 1$ implies that $\beta = 0$, which shows that E_t and E_w are linearly independent. Since the transformation given in the Riesz Representation Theorem, which assigns to any linear, continuous functional its representing vector, is an antilinear isometry, then the vectors $S_{F,\Omega}(\cdot, t)$ and $S_{F,\Omega}(\cdot, w)$ are linearly independent. The conclusion follows now from Remark 2. □

We call $\varrho^{S_F}_\Omega$ *the Fefferman–Szegő projective distance* (taking K_Ω—the regular Bergman kernel instead of S_Ω, we get the so-called Skwarczyński distance—see [1, p. 20]). The advantage of this distance is that, compared to the (regular) Szegő distance (given by the Szegő metric—see [5]), it is given in a more explicit way and thus seems to be advantageous from the computational point of view. Moreover, it is uniquely determined by the real analytic function

$$H(z, w) = \frac{S_F(z, w)S_F(w, z)}{S_F(z, z)S_F(w, w)}$$

on $\Omega \times \Omega$.

Remark 4 We define *the Szegő projective distance on the same way* just taking S_Ω instead of $S_{F,\Omega}$. We note, however, that in contrast to $\varrho^{S_F}_\Omega$, the distance ϱ^{S}_Ω is not biholomorphically invariant. We call $\varrho^{S_F}_\Omega$ the *Fefferman–Szegő projective distance*, and ϱ^{S}_Ω the Szegő projective distance.

Remark 5 We see that, for any biholomorphic mapping $\varphi : \Omega_1 \to \Omega_2$, we have

$$\varrho^{S_F}_{\Omega_1}(z, w) = \varrho^{S_F}_{\Omega_2}(\varphi(z), \varphi(w)) .$$

The proof follows from the transformation rule for the Szegő kernel (see Proposition 1).

3.1 The Fefferman–Szegő projective distance on the unit ball

Let $\mathbb{B}^n = \{z \in \mathbb{C}^n : \rho(z) := |z|^2 - 1 < 0\} \subset \mathbb{C}^n$. Then the Szegő kernel for the unit ball $\mathbb{B}^n$ is given by

$$S_F(z, w) = \frac{(n-1)!}{2\pi^n} \frac{1}{(1 - z \cdot \overline{w})^n} .$$

From the formula (1) it follows directly that

$$\left(\varrho^{S_F}_{\mathbb{B}^n}\right)^2(z, w) = 1 - \left(\frac{(1-|z|^2)(1-|w|^2)}{|1 - z \cdot \overline{w}|^2}\right)^{n/2} .$$

$$\varrho^{S_F}_{\mathbb{B}^2}(z, w) = \frac{|z - w|^2}{|1 - z \cdot \overline{w}|^2}$$

Recall now that the Skwarczyński distance for the unit disc in $\mathbb{C}$ is

$$\varrho_{\mathbb{D}}(z, w) = \left|\frac{z - w}{1 - z\overline{w}}\right|$$

(see [1, p. 21]). Thus

$$\varrho^{S_F}_{\mathbb{B}^2}((z_1, 0), (w_1, 0)) = \varrho^2_{\mathbb{D}}(z_1, w_1) .$$

Moreover, for $n = 1$, we have

$$\varrho_{\mathbb{D}}(z, w) = \varrho^{S_F}_{\mathbb{B}^1}(z, w)\sqrt{2 - (\varrho^{S_F}_{\mathbb{B}^1})^2(z, w)} .$$

Remark 6 The same formulas hold for ϱ^S.

3.2 Completeness with respect to the ϱ^S_Ω distance

In this subsection we are interested in the Szegő projective distance rather than Fefferman–Szegő projective distance. The reason is that arguments from this subsection repeated for the Fefferman–Szegő projective distance give that every strongly pseudoconvex domain with smooth boundary is automatically ϱ^{S_F}-complete, and also complete in the Szegő metric (introduced in [5]).

We list here some important theorems which are directly taken from the Bergman kernel theory (see [7, Theorem 12.9.6.] for instance). We are doing this for the sake of completeness of the paper.

Following ideas of [8] and particularly [1, p. 22] we can study completeness with respect to the invariant distance. Additionally, we will prove now that the so-called Kobayashi condition implies ϱ^S_Ω-completeness.

Theorem 7 *A sequence $(z_m) \in \Omega$, $m = 1, 2, \ldots$, is Cauchy with respect to the distance ϱ^S_Ω if and only if the sequence $\tau(z_m)$ is Cauchy in $P(H^2(\partial\Omega))$.*

Proof This is a direct consequence of the definition of ϱ^S_Ω. □

Theorem 8 *A sequence $z_m \in \Omega$, $m = 1, 2, \ldots$, is Cauchy with respect to ϱ^S_Ω if and only if there exists an $f \in H^2(\partial\Omega)$ such that $||f||_{H^2} = 1$ and*

$$\lim_{m\to\infty} \frac{|f(z_m)|^2}{S_\Omega(z_m,z_m)} = 1. \tag{2}$$

Proof By the previous theorem, a sequence (z_m) is Cauchy in Ω if and only if $(\tau(z_m))$ is Cauchy in $P(H^2(\partial\Omega))$. By completeness of $P(H^2(\partial\Omega))$, the sequence $(\tau(z_m))$ converges to some $[f]$. We may assume that $||f||_{H^2} = 1$. Thus

$$\lim_{m\to\infty} d_{H^2(\partial\Omega)}(\tau(z_m),[f]) = 0\,,$$

but this is equivalent (by the definition) to (2). The reverse implication is a direct consequence of the definition of ϱ^S_Ω. □

Theorem 9 (see p. 494 in [7]) *The Euclidean distance and ϱ^S_Ω induce the same topology in Ω.*

Proof Assume that $z_j \in \Omega$ converges to $z \in \Omega$ in the Euclidean norm. Then $\lim_{j\to\infty}\varrho^S_\Omega(z_j,z) = 0$ since the Szegő function is continuous. Conversely, $\lim_{j\to\infty}\varrho^S_\Omega(z_j,z) = 0$ implies that

$$\lim_{j\to\infty}\left|\left|\frac{e^{i\theta_j}S_\Omega(\cdot,z_j)}{\sqrt{S_\Omega(z_j,z_j)}} - \frac{S_\Omega(\cdot,z)}{\sqrt{S_\Omega(z,z)}}\right|\right|_{H^2} = 0\,,$$

where (θ_j) is a suitable sequence of real numbers. Thus there exist constants $c_j \neq 0$, $j = 1,2,\ldots,$ such that $c_jS_\Omega(\cdot,z_j)\stackrel{H^2}{\longrightarrow} S_\Omega(\cdot,z)$. Since $1 \in H^2(\partial\Omega)$, we see that

$$\lim_{j\to\infty}\overline{c}_j = \lim_{j\to\infty}(1,c_jS_\Omega(\cdot,z_j))_\mu = (1,S_\Omega(\cdot,z))_{H^2} = 1\,.$$

Let π_k denote the kth coordinate function. We have

$$\begin{aligned}&\lim_{j\to\infty}\pi_k(z_j)\\ &\quad= \lim_{j\to\infty}(\pi_k(\cdot),S_\Omega(\cdot,z_j))_\mu = \lim_{j\to\infty}\frac{1}{\overline{c}_j}(\pi_k(\cdot),c_jS_\Omega(\cdot,z_j))_{H^2}\\ &\quad= (\pi_k(\cdot),S_\Omega(\cdot,z))_{H^2} = \pi_k(z), \quad \text{i.e.} \quad \lim_{j\to\infty} z_j = z\,.\end{aligned}$$

□

Hence the two topologies coincide. Having this result in hand, we can prove (in a fashion similar to that for the Bergman kernels see [9, p. 93]) that the ϱ^S_Ω completeness is closely related to the dimension of $H^2(\partial\Omega)$ $(L^2_H(\partial\Omega))$.

Theorem 10 *If Ω is ϱ^S_Ω complete, then* $\dim H^2(\partial\Omega) = \infty$.

Proof We can adapt the proof in [9, p. 93]. Assume that $\dim H^2(\partial\Omega) < \infty$. Then the closed unit ball in $H^2(\partial\Omega)$ is compact. Let $g_z(\cdot) = \dfrac{S_\Omega(\cdot,z)}{\sqrt{S_\Omega(z,z)}}$, where $z \in \Omega$. Then

$$\begin{aligned}||g_z||^2 &= \int_\Omega g_z(w)\overline{g_z(w)}\mu(w)\mathrm{d}V\\ &= \int_\Omega \frac{S_\Omega(w,z)}{\sqrt{S_\Omega(z,z)}}\overline{\frac{S_\Omega(w,z)}{\sqrt{S_\Omega(z,z)}}}\mathrm{d}V\\ &= \frac{1}{S_\Omega(z,z)}\int_\Omega S_\Omega(w,z)\overline{S_\Omega(w,z)}\mathrm{d}V\\ &= \frac{1}{S_\Omega(z,z)}S_\Omega(z,z).\\ &= 1.\end{aligned}$$

If $(z_k)^\infty_{k=1} \to z_0 \in \partial\Omega$ (in the usual Euclidean topology), then (by compactness of the unit ball) $(g_{z_{k_j}})^\infty_{j=1}$ has a subsequence that is convergent to $g \in H^2(\partial\Omega)$, where $||g||_{H^2} = 1$. Denote this sequence by $(g_{z_k})^\infty_{k=1}$. Let us see that $(z_k)^\infty_{k=1}$ is ϱ^S_Ω–Cauchy. Indeed

$$\begin{aligned}&|\langle g_{z_m},g_{z_n}\rangle|\\ &= \left|\int_\Omega g_{z_m}(w)\overline{g_{z_n}(w)}\mu(w)\mathrm{d}V\right|\\ &= \left|\int_\Omega \frac{S_\Omega(w,z_m)}{\sqrt{S_{\Omega,\mu}(z_m,z_m)}}\overline{\frac{S_\Omega(w,z_n)}{\sqrt{S_\Omega(z_n,z_n)}}}\mu(w)\mathrm{d}V\right|\\ &= \frac{1}{\sqrt{S_\Omega(z_m,z_m)}\sqrt{S_{\Omega,\mu}(z_n,z_n)}}\left|\int_\Omega S_\Omega(w,z_m)\overline{S_\Omega(w,z_n)}\mu(w)\mathrm{d}V\right|\\ &= \frac{|S_\Omega(z_m,z_n)|}{\sqrt{S_\Omega(z_m,z_m)}\sqrt{S_\Omega(z_n,z_n)}},\end{aligned}$$

i.e.

$$(\varrho^S_\Omega)^2(z_m,z_n) = 1 - |\langle g_{z_m},g_{z_n}\rangle|$$

Since the term on the right hand side tends to 0 when $m,n \to \infty$ we conclude that $\varrho^S_\Omega(z_m,z_n) < \epsilon$ for m, n large enough. Thus we found a ϱ^S_Ω-sequence which has the limit $z_0 \in \partial\Omega$. This should not happen since Ω is assumed to be ϱ^S_Ω-complete. □

Some of the ideas below—particularly Theorems 11 and 12—follow upon the ones in [1, p. 23, 24] .

Theorem 11 (Szegő version of the Kobayashi theorem) *Assume that, for every sequence $(z_m) \in \Omega$ without an accumulation point in Ω and for every $f \in H^2(\partial\Omega)$,*

$$\lim_{m\to\infty}\frac{|f(z_m)|^2}{S_\Omega(z_m,z_m)} = 0. \tag{3}$$

Then Ω is ϱ^S_Ω-complete.

Proof Suppose that $(z_m) \in \Omega$ is a Cauchy sequence without limit in Ω. Thus (z_m) has no accumulation point in Ω, and

(3) holds. But (3) contradicts (2). Thus there is a limit point of (z_m) in Ω. □

The hypothesis of the above theorem applied to the Bergman kernel K instead of S, and to the Szegő space $H^2(\partial\Omega)$ instead of $L^2_H(\partial\Omega)$, is the so-called Kobayashi condition (see [8]). Kobayashi showed that this condition implies that the considered domain is Bergman complete. Skwarczyński has a proof that this condition implies ϱ_Ω-completeness—[10] (see Sect. 3.4 for the definition of ϱ_Ω).

Theorem 12 *Suppose that, for each boundary point $z_0 \in \partial\Omega$ (of a bounded domain Ω with C^2-smooth boundary), there is a function $h \in \mathcal{O}(\Omega)$ such that*

(a) $|h(z)| < 1$ *for* $z \in \Omega$,
(b) $\lim_{z\to z_0} |h(z)| = 1$.

Then Ω is complete with respect to ϱ^S_Ω.

Proof Let (z_m) be a sequence with no accumulation point in Ω. It suffices to show that, for any $f \in H^2(\partial\Omega)$,

$$\lim_{m\to\infty} \frac{|f(z_m)|^2}{S_\Omega(z_m, z_m)} = 0\,.$$

We may assume that $z_m \to z \in \partial\Omega$. For any $\epsilon > 0$, there is k such that $||h^k f||^2_{H^2} < \epsilon$ (by the Lebesgue dominated convergence theorem). If m is large enough, then

$$(1-\epsilon)|f(z_m)|^2 \le |h^k(z_m) f(z_m)|^2 \le S_\Omega(z_m, z_m)||h^k f||^2_{H^2}\,.$$

Thus

$$\frac{|f(z_m)|^2}{S_\Omega(z_m, z_m)} \le \frac{\epsilon}{1-\epsilon}\,.$$

□

Recall the definition of a *peak point with respect to* (some family) $\mathcal{F}$.

Definition 13 Let D be a bounded domain in $\mathbb{C}^n$. A boundary point $z_0 \in \partial D$ is called a *peak point* with respect to $\mathcal{F} \subset C(\overline{D})$ if there is $h \in \mathcal{F}$ such that

(a) $h(z_0) = 1$
(b) $|h(z)| < 1$ on $\overline{D} \setminus \{z_0\}$

Recall some classical results concerning peak points.

Theorem 14 (cf. [7, p. 802]; [11, 12]) *If D is a strongly pseudoconvex domain in $\mathbb{C}^n$ and $z_0 \in \partial D$, then z_0 is a peak point with respect to $\mathcal{O}(\overline{D})$. If D is a strongly pseudoconvex domain in $\mathbb{C}^n$ with a smooth boundary and $z_0 \in \partial D$ then z_0 is a peak point with respect to $\mathcal{O}(D) \cap C(\overline{D})$. Moreover, if D is a bounded pseudoconvex domain in $\mathbb{C}^2$ with real analytic boundary, then any boundary point $z_0 \in \partial D$ is a peak point with respect to $\mathcal{O}(D) \cap C(\overline{D})$.*

We infer from this and Theorem 12 the following:

Corollary 15 *Every strongly pseudoconvex domain Ω in $\mathbb{C}^n$ with C^2-smooth boundary is complete with respect to ϱ^S_Ω. Moreover, every bounded pseudoconvex domain Ω in $\mathbb{C}^2$ with real analytic boundary is complete with respect to ϱ^S_Ω.*

Remark 16 So now one can clearly note, that the above arguments, repeated for S_F rather than S provide every strongly pseudoconvex domain with C^∞-smooth boundary is automatically complete in the $\varrho^{S_F}_\Omega$ distance.

3.2.1 Comparison of the Bergman and Szegő kernels off the diagonal

Using recent estimates obtained in [13] we may estimate the quotient $|S_\Omega(z,w)/K_\Omega(z,w)|$ on domains which are not ϱ_Ω -complete (Skwarczyński distance). Note that, if $(z_n)_{n=1}^\infty$ is a ϱ_Ω-Cauchy sequence then, for any $\epsilon > 0$,

$$1-\epsilon \le \frac{|K_\Omega(z_n, z_m)|}{\sqrt{K_\Omega(z_n, z_n)}\sqrt{K_\Omega(z_m, z_m)}} \le 1$$

if only m, n are large enough. We now have the following:

Theorem 17 *If $\Omega \Subset \mathbb{C}^n$ is a pseudoconvex domain with C^2 -smooth boundary and which is not ϱ_Ω-complete then*

$$\lim_{n\to\infty} \left|\frac{S(z_n, z_m)}{K(z_n, z_m)}\right| = 0$$

for any ϱ_Ω-Cauchy sequence $(z_p)_{p=1}^\infty$, where m is large enough.

Proof Let $\epsilon > 0$. Then for n, m large enough we have

$$\left|\frac{K_\Omega(z_n, z_m)}{S_\Omega(z_n, z_m)}\right| = \frac{|K_\Omega(z_n, z_m)|}{\sqrt{K_\Omega(z_n, z_n)}\sqrt{K_\Omega(z_m, z_m)}} \frac{\sqrt{K_\Omega(z_n, z_n)}\sqrt{K_\Omega(z_m, z_m)}}{\sqrt{S_\Omega(z_n, z_n)}\sqrt{S_\Omega(z_m, z_m)}} \cdot \frac{\sqrt{S_\Omega(z_n, z_n)}\sqrt{S_\Omega(z_m, z_m)}}{|S_\Omega(z_m, z_n)|} \xrightarrow[n\to\infty]{} \infty$$

(since $\frac{S(z,z)}{K(z,z)} \leq c\,\delta(z)|\ln(\delta(z))|^{\alpha} \xrightarrow[z\to\partial\Omega]{} 0$—see [13]). ☐

Thus we have

Corollary 18 *If $\Omega \Subset \mathbb{C}^n$ is a pseudoconvex domain with C^2-smooth boundary such that its Bergman kernel $K_\Omega(z,w)$ satisfies $|K_\Omega(z,w)| \leq M$, $|S_\Omega(z,w)| > 0$ on $\overline{\Omega}\times\overline{\Omega}\setminus F$, where $F = \{(z,w) \in \partial\Omega\times\partial\Omega, z = w\}$ then Ω is ϱ_Ω-complete, and thus Bergman complete.*

In particular, every strongly pseudoconvex domain $\Omega \Subset \mathbb{C}^n$ with C^∞-boundary is ϱ_Ω-complete (see [14]) (we know this already from Corollary 15 applied to the Bergman kernel—see [1, p. 25]).

3.3 Relation of ϱ^S to the Skwarczyński distance

It turns out that ϱ^S_Ω is related to ϱ_Ω (Skwarczyński distance) by some biholomorphic invariants introduced below. Let us recall that ϱ^S_Ω, by its definition, is uniquely determined by the real analytic function

$$H_\Omega(z,w) = \frac{S_\Omega(z,w)S_\Omega(w,z)}{S_\Omega(z,z)S_\Omega(w,w)}.$$

Define $L_\Omega(z,w)$ a corresponding quotient for the Bergman kernel, i.e.

$$L_\Omega(z,w) = \frac{K_\Omega(z,w)K_\Omega(w,z)}{K_\Omega(z,z)K_\Omega(w,w)}.$$

Now define a new biholomorphically invariant $HL_\Omega(z,w)$ by

$$HL_\Omega(z,w) = \frac{L_\Omega(z,w)^n}{H_\Omega(z,w)^{n+1}} = \frac{|K_\Omega(z,w)|^{2n}}{|S_\Omega(z,w)|^{2n+2}}\frac{S_\Omega(z,z)^{n+1}}{K_\Omega(z,z)^n}\frac{S_\Omega(w,w)^{n+1}}{K_\Omega(w,w)^n}$$

We can write it by means of another biholomorphic invariant $SK_\Omega(z,w)$, where

$$SK_\Omega(z,w) = \frac{S_\Omega(z,w)^{n+1}}{K_\Omega(z,w)^n} \quad z,w \in \Omega$$

introduced (for $S_{F,\Omega}$ in fact) in [5, formula (3.1)]. See also [13]. Now

$$HL_\Omega(z,w) = \frac{1}{|SK_\Omega(z,w)|^2}SK_\Omega(z,z)SK_\Omega(w,w).$$

By its definition, HL_Ω is a symmetric, real analytic function on $\Omega\times\Omega$. Moreover

Lemma 19 *The following holds:*

(a) *For any $z \in \Omega$, $HL_\Omega(z,z) = 1$*
(b) $HL_\Omega(z,w) = \frac{(1-\varrho^2(z,w))^{2n}}{(1-(\varrho^S)^2(z,w))^{2n+2}}$
(c) *$HL_{\mathbb{B}^n}(z,w) = 1$ for all $z,w \in \mathbb{B}^n$.*

Proof Properties (a), (b), (c) follows directly from the definition of HL and from the formulas:

$$S_\Omega(z,w) = \frac{(n-1)!}{2\pi^n}\frac{1}{(1-z\cdot\overline{w})^n}, \quad K_\Omega(z,w) = \frac{1}{vol(\mathbb{B})}\frac{1}{(1-z\cdot\overline{w})^{n+1}}.$$

☐

However, the Szegő kernel $S_\Omega$itself blows up on the boundary. Indeed :

Remark 20 Similarly to the Bergman kernel K_Ω, the Szegő kernel S_Ω satisfies

$$\lim_{z\to\partial\Omega} S_\Omega(z,z) = \infty.$$

Proof By the definition (see [5]),

$$SK_\Omega(z,z) = \frac{S(z,z)^{n+1}}{K(z,z)^n} = S(z,z)\frac{S(z,z)^n}{K(z,z)^n}$$

But $\frac{S_\Omega(z,z)}{K_\Omega(z,z)} \leq c\,\delta(z)|\ln(\delta(z))|^{\alpha} \xrightarrow[z\to\partial\Omega]{} 0$—see [13], and (as previously) $\lim_{z\to\partial\Omega} SK_\Omega(z,z) = \text{constant} > 0$. ☐

One can note that both $\varrho_\Omega(z,w)$ and $\varrho^S_\Omega(z,w)$ tend to 1 for $w \to \partial\Omega, z \neq w$. But, in view of Lemma 19, the quantity ϱ has stronger boundary asymptotic properties than ϱ^S.

Remark 21 When considering $\{\Omega = \rho < 0\} \Subset \mathbb{C}^n$ a strongly pseudoconvex domain with C^∞ boundary, and $S_{F,\Omega}$ rather than S_Ω beside of the above properties one also has that

Proposition 22 *If $S_{F,\Omega}(z,w) \neq 0$ for $z \in \Omega$ and any $w \in \partial\Omega$ then $\lim_{w\to\partial\Omega} HL_\Omega(z,w)$ exists and is finite.*

Proof Note first that since Ω is smoothly bounded, strongly pseudoconvex domain the assumptions are always fulfilled for w in or near the boundary $\partial\Omega$ and z near to w. This is because $S_{F,\Omega}(z,w)$ (as well as $K_\Omega(z,w)$) does not vanish in this case (as follows from [15]). Recall (see [5]) that, for a strongly pseudoconvex domain Ω with the defining function suitably normalized,

$$S_F K_\Omega(z,z) = \begin{cases} (n-1)!/(c_n^{n+1}(n\pi)^n) + (n-3)! q_\Omega r^2/(c_n^{n+1} n^n) + O(r^3), & n \geq 4 \\ 2/(c_3^4 27\pi^3) + q_\Omega r^2/(9c_3^4) + O(r^3 |\ln r|), & n = 3 \\ 1/(c_2^3 4\pi^2) + \mu_2 r^2 + \mu_3 r^4 \ln r + 3\pi^2 \tilde{q}_\Omega^2 r^6 \ln^2 r/(16 c_2^3) + O(r^6 |\ln r|), & n = 2 \end{cases}$$

for z close to the boundary, where $\mu_2, \mu_3 \in C^\infty(\overline{\Omega})$ and $q_\Omega, \tilde{q}_\Omega$ are certain local geometric boundary invariants and $r := -\rho$. From the above follows

$$\lim_{z\to\partial\Omega} S_F K_\Omega(z,z) = \text{constant} > 0$$

Now, since the Szegő kernel S_F (like the Bergman kernel K) extends continuously outside diagonal on $\partial\Omega \times \partial\Omega$ then, of course, $|S_{F,\Omega}(z,w)| > M > 0$, $|K_\Omega(z,w)| < N$ (by the assumption from (c)) and so

$$\frac{1}{|S_F K_\Omega(z,w)|^2} = \frac{|K_\Omega(z,w)|^{2n}}{|S_{F,\Omega}(z,w)|^{2n+2}} \leq NM < \infty\,.$$

Putting this together and using the definition of HL_Ω we ends the proof. □

Theorem 23 *Assume that Ω is a ϱ_Ω^S-complete domain on which $HL_\Omega(z,w) \leq 1$ for all $z,w \in \Omega$. Then Ω is ϱ_Ω-complete (that is, complete in the Skwarczyński distance).*

Proof Let (z_n) be any ϱ-Cauchy sequence. Then, for any $\epsilon_1 > 0$, we have that $\varrho(z_k,z_p) < \epsilon_1$ if k,p large enough. In view of Lemma 19b we have

$$\epsilon_1^2 > \varrho_\Omega^2(z_k,z_p) = 1 - \sqrt[2n]{HL(z_k,z_p)}(1-(\varrho_\Omega^S)^2(z_k,z_p))^{\frac{2n+2}{2n}}\,,$$

which is equivalent to

$$\frac{(1-\epsilon_1^2)^{\frac{2n}{2n+2}}}{\sqrt[2n+2]{HL_\Omega(z_k,z_p)}} \leq 1 - (\varrho_\Omega^S)^2(z_k,z_p)$$

or

$$\left(\varrho_\Omega^S\right)^2(z_k,z_p) < 1 - \frac{\left(1-\epsilon_1^2\right)^{\frac{2n}{2n+2}}}{\sqrt[2n+2]{HL_\Omega(z_k,z_p)}}. \quad (\star)$$

We want to show that (z_n) is convergent by proving it is ϱ_Ω^S-Cauchy. So, for any $\epsilon > 0$, we want to have $\varrho_\Omega^S(z_k,z_p) < \epsilon$ for k,p large enough. Note that $(\star)$ implies $\varrho_\Omega^S(z_k,z_p) \leq 1$, thus for $\epsilon \geq 1$ one has $\varrho_\Omega^S(z_k,z_p) < \epsilon$. For $\epsilon \in (0,1)$ one can pick any ϵ_1 in $\left(0, \sqrt{1-(1-\epsilon^2)^{\frac{2n+2}{2n}} \sqrt[2n]{HL_\Omega(z_k,z_p)}}\right)$. Note that the quantity under the square root is nonnegative, since

$$HL_\Omega(z_k,z_p) < \frac{1}{(1-\epsilon^2)^{2n+2}},$$

as we assumed that $HL_\Omega \leq 1$ on $\Omega\times\Omega$. Now, after squaring and rearranging, one has

$$(1-\epsilon^2)\sqrt[2n+2]{HL_\Omega(z_k,z_p)} < (1-\epsilon_1^2)^{\frac{2n}{2n+2}}\,,$$

which implies by $(\star)$ that $\varrho_\Omega^S(z_k,z_p) < \epsilon$, for k,p sufficiently large. □

Theorem 24 *Assume Ω is a ϱ_Ω-complete domain on which $HL_\Omega(z,w) \geq 1$ for all $z,w \in \Omega$. Then Ω is ϱ_Ω^S-complete.*

Proof Let (z_n) be any ϱ^S-Cauchy sequence. Then, for any $\epsilon_1 > 0$, we have that $\varrho_\Omega^S(z_k,z_p) < \epsilon_1$ if k,p large enough. In view of Lemma 19b we have

$$\epsilon_1^2 > \left(\varrho_\Omega^S\right)^2(z_k,z_p) = 1 - \frac{\left(1-\varrho_\Omega^2(z_k,z_p)\right)^{\frac{2n}{2n+2}}}{\sqrt[2n+2]{HL_\Omega(z_k,z_p)}}$$

which is equivalent to

$$\left(1-\varrho_\Omega^2(z_k,z_p)\right)^{\frac{2n}{2n+2}} > \left(1-\epsilon_1^2\right)\sqrt[2n+2]{HL_\Omega(z_k,z_p)}$$

or

$$\varrho_\Omega^2(z_k,z_p) < 1 - \left(1-\epsilon_1^2\right)^{\frac{2n+2}{2n}} \sqrt[2n]{HL_\Omega(z_k,z_p)}. \quad (\star\star)$$

We want to show that (z_n) is convergent by proving it is ϱ_Ω-complete. So, for any $\epsilon > 0$, we want to have $\varrho_\Omega(z_k,z_p) < \epsilon$ for k,p large enough. Note that $(\star\star)$ implies that $\varrho_\Omega(z_k,z_p) \leq 1$, thus for $\epsilon \geq 1$ one has $\varrho_\Omega(z_k,z_p) < \epsilon$. For $\epsilon \in (0,1)$ one can pick any ϵ_1 in $\left(0, \sqrt{1-(1-\epsilon^2)^{\frac{2n}{2n+2}}}\right)$. Note that the quantity under the square root is nonnegative, since $\epsilon > 0$. That means

$$\frac{1-\epsilon^2}{(1-\epsilon_1^2)^{\frac{2n+2}{2n}}} < 1 \leq \sqrt[2n]{HL_\Omega(z_k,z_p)}\,,$$

which together with $(\star\star)$ yields $\varrho_\Omega(z_k,z_p) < \epsilon$. □

Corollary 25 *If Ω is a domain for which $HL_\Omega(z,w) = 1$ for all $z,w \in \Omega$, then Ω is ϱ-complete if and only if Ω is ϱ_Ω^S-complete.*

Thus we have derived a characterization of those domains on which ϱ-completeness is equivalent to ϱ^S-completeness.

Corollary 26 *For a domain Ω with $HL_\Omega \equiv 1$ on $\Omega \times \Omega$, consider the statements*:

(1) *Ω is ϱ-complete.*
(2) *Ω is ϱ_Ω^S-complete.*
(3) *Ω is Bergman complete.*

Then (1) $\iff$ (2), (2) $\Rightarrow$ (3).

The proof of (1) $\Rightarrow$ (3) is given in [16]. According to our knowledge it is still open question whether (3) $\Rightarrow$ (1).

3.4 Relation of the Fefferman–Szegő projective distance ϱ^{S_F} to the Bergman distance and the Szegő distance

Assume that now we do consider the Fefferman–Szegő kernel S_F. Let us recall that the Bergman metric F_B on Ω at z in the direction vector ξ based at z, $F_B(z,\xi)$ is related to the Fefferman-Szegő metric $F_{S_F}(z,\xi)$ by

$$m_\Omega F_{S_F}(z,\xi) \le F_B(z,\xi) < M_\Omega F_{S_F}(z,\xi) \tag{2}$$

where $0 < m_\Omega < M_\Omega < \infty$ and $z \in \Omega$ and $\xi \in T\Omega$ (see [5, Theorems 3–5], [15, 17]).

Denote by $s_F(z,w)$ and $b(z,w)$ the distances induced by the Szegő and Bergman metric respectively (on the standard way—see [7] (p.482) for instance).

Theorem 27 *There are some positive constants $c, \widetilde{m(\Omega)}, \widetilde{M(\Omega)}$, such that for every $z, w \in \Omega$ one has*:

$$\varrho_\Omega^{S_F}(z,w) \le c \cdot s_F(z,w) \le \widetilde{m(\Omega)} b(z,w) \le \widetilde{M(\Omega)} s_F(z,w).$$

Proof This clearly follows from the estimation (2) and techniques analogous to the ones used for the Bergman kernel in [16]. □

4 The relationship between *HL*, the Bergman metric and the Fefferman–Szegő metric

In this section we get an exact connection between the Bergman and Szegő metrics by means of the quantity given by HL_Ω. The key idea is a simple remark. Note that, by the definition, we have

$$L_\Omega(z,w) = (1 - \varrho_\Omega^2(z,w))^2$$

or just

$$\frac{|K_\Omega(z,w)|^2}{(1-\varrho_\Omega^2(z,w))^2} = K_\Omega(z,z)K_\Omega(w,w),$$

for $z, w \in \Omega$. Taking the natural logarithm ln on both sides, one gets

$$\ln K_\Omega(z,z) + \ln K_\Omega(w,w) = \ln K_\Omega(z,w) + \ln K_\Omega(w,z) - 2\ln(1-\varrho_\Omega^2(z,w)),$$

so

$$\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \ln K_\Omega(z,z) = -2 \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \ln(1-\varrho_\Omega^2(z,w)).$$

Now, we can do the same for $S_{F,\Omega}$ instead of K_Ω and H_Ω instead of L_Ω and thus

$$\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \ln S_{F,\Omega}(z,z) = -2 \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \ln(1-(\varrho_\Omega^{S_F})^2(z,w)).$$

Using Lemma 19 (*b*) one gets

$$\begin{aligned}\sum_{i,j=1}^n \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \ln HL_\Omega(z,w)\xi_i\bar{\xi}_j &= 2n \sum_{i,j=1}^n \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \ln(1-\varrho_\Omega^2(z,w))\xi_i\bar{\xi}_j \\ &\quad - (2n+2)\sum_{i,j=1}^n \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \ln(1-(\varrho_\Omega^{S_F})^2(z,w))\xi_i\bar{\xi}_j\end{aligned}$$

or just

$$\sum_{i,j=1}^n \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \ln HL_\Omega(z,w)\xi_i\bar{\xi}_j = -nF_B^2(z;\xi) + (n+1)F_{S_F}^2(z;\xi). \tag{4}$$

But this right hand side expression is the quantity $E(z,\xi)$ introduced in [5]. Thus we get

$$\begin{aligned}\sum_{i,j=1}^n \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \ln HL_\Omega(z,w)\xi_i\bar{\xi}_j &= E(z,\xi) \\ &= \sum_{i,j=1}^n \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \ln SK_\Omega(z,z)\xi_i\bar{\xi}_j\end{aligned} \tag{5}$$

Remark 28 In the case of the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$, $E(z,\xi) \equiv 0$, since $HL \equiv 1$ by the hypothesis of Lemma 19. In particular, we have a direct connection between the Bergman and Szegő metrics on the unit ball in $\mathbb{C}^n$, namely:

$$F_B = \sqrt{\frac{n+1}{n}} F_{S_F} .$$

This is also derived in [5]. Note that (4) clearly implies that $E(z;\xi)$ defines a semi-positive definite form on a set $\{\xi \in T\overline{\Omega}; F_B(z,\xi) \le \sqrt{\frac{n+1}{n}} F_{S_F}(z,\xi)\}$. So for example if Ω is simply connected, and biholomorphic to the ball (see properties of $E(z;\xi)$ in [5]).

5 Closing remarks

It has become increasingly clear that analysis on domains in $\mathbb{C}^n$ must be formulated in the language of invariant metrics. Thus it is worthwhile to develop and study new invariant metrics, and to compare them with the more familiar metrics that were developed in the twentieth century. This contribution is a step in that direction.

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Declarations

Conflict of interest The authors declare that they have no conflicts of interest.

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