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RESEARCH

Analyticity in partial differential equations

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Abstract

Here we shall discuss analyticity results for several important partial differential equations. This includes the analytic regularity of sub-Laplacians under the finite type condition; the analyticity of the solution in both variables to the Cauchy problem for the Camassa–Holm equation with analytic initial data by using the Ovsyannikov theorem, which is a Cauchy–Kowalevski type theorem for nonlocal equations; the Cauchy problem for BBM with analytic initial data; the Cauchy problem for KdV with analytic initial data examining the evolution of uniform radius of spatial analyticity; and finally the time regularity of KdV solutions, which is Gevrey 3.

Keywords Analyticity · Analytic hypoellipticity of sub-Laplacians · Cauchy problem with analytic data · Ovsyannikov theorem · Camassa–Holm equation · Benjamin–Bona–Mahony equation · Korteweg–de Vries equation · Approximate conservation law · Uniform radius of spatial analyticity

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1 Introduction

Analyticity in partial differential equations (PDE) appears naturally. For example, imagine that the temperature of a body occupying a region U in $\mathbb{R}^3$ is at a steady state, that is, it does not change with time. If we know the temperature at each point of its surface (boundary), then to find the temperature $u(x)$ at each point x inside the body we must solve the boundary value problem consisting of the Laplace equation

$$\Delta u \doteq \partial_{x_1}^2 u + \partial_{x_2}^2 u + \partial_{x_3}^2 u = 0, \quad \text{in} \quad U, \tag{1.1}$$

and the boundary condition $u = g$, where g are the known values (data) of u on the boundary ∂U. For smooth enough boundary ∂U and data g we can find a solution formula for this problem (see, for example, Evans [26]). However, independently of the derivation of this solution formula we can prove that the temperature distribution $u(x)$ inside the domain U is an analytic function. That is, at any point $x_0 \in U$ the solution u to the Laplace equation (1.1) can be represented by a power series in the variable x near x_0. In fact, this is a special case of a more general result stating that any solution to an elliptic linear equation $P(x, D)u = f(x)$, with analytic coefficients in a domain $U \subset \mathbb{R}^n$ and analytic forcing (output) $f(x)$ in U, is analytic in U (see, for example, Hörmander [45]).

In the direction of non-elliptic linear PDE there is an important class of operators, that look like the Laplacian and arise naturally in analysis, geometry and probability theory. These operators are called sub-Laplacians or "sums of squares of vector fields" and are of the form

$$\Delta_X \doteq X_1^2 + \cdots + X_m^2, \quad \text{in} \quad U, \tag{1.2}$$

where U is an open set in $\mathbb{R}^n$ and $X = \{X_1, \ldots, X_m\}$ are m real vector fields with C^∞ coefficients in U. When all points of U are of finite type, that is, the Lie algebra of the vector fields X has dimension n at every point of U, then all solutions (classical or weak) to the equation $\Delta_X u = f$, $f \in C^\infty(U)$, are in $C^\infty(U)$ (i.e., Δ_X is hypoelliptic). This is the celebrated sums of squares theorem of Hörmander [46]. If the coefficients

In memory of Nick Hanges.

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of the vector fields X are analytic, then the corresponding Hörmander type theorem in the analytic category is not true. Finding *necessary and sufficient conditions for the analytic regularity (hypoellipticity) of sub-Laplacians remains an open problem* that we will discuss here presenting some results, where Nick Hanges was involved.

Also, motivated by the work with Nick Hanges, here we shall present some results on the Cauchy problem of linear and nonlinear PDE with analytic initial data, starting with the classical Cauchy–Kowalevski theorem. Then, we will consider the Cauchy problem for nonlocal equations and state an abstract Cauchy–Kowalevski theorem, which is known as the Ovsyannikov theorem. A well known nonlocal equation that this theorem applies to is the so called Camassa–Holm (CH) equation

$$\partial_t u + u\partial_x u + \partial_x\left(1-\partial_x^2\right)^{-1}\left[u^2+\frac{1}{2}(\partial_x u)^2\right]=0, \tag{1.3}$$

which in the framework of water wave theory was derived by Camassa and Holm [16] starting from the Euler equations. This is an integrable equation, and in this framework was derived by Fokas and Fuchssteiner [28]. The solution to the Cauchy problem of CH with analytic initial data is analytic in both variables x and t (see [40] for the circle and [3] for the line). We shall discuss this result here. Also, here we shall present a recent similar result [43] for the Benjamin–Bona–Mahony (BBM) equation

$$\partial_t u + \partial_x\left(1-\partial_x^2\right)^{-1}\left[u^2+\frac{1}{2}(\partial_x u)^2\right]=0, \tag{1.4}$$

which unlike the CH equation can be thought as an abstract ODE.

Finally, we will discuss the periodic Cauchy problem for the celebrated Korteweg–de Vries equation, which was derived in [11] and [50] as a model of water waves (solitons),

$$\partial_t u + \partial_x^3 u + u\partial_x u = 0. \tag{1.5}$$

For initial data $u(x,0)$ that are analytic on the torus and have uniform radius of analyticity r_0, we will examine the evolution of the radius of spatial analyticity $r(t)$ of the solution $u(t)$ at any future time t. Following [44], we will show that the size of the radius of spatial analyticity persists for some time and after that it evolves in a such a way that its size at any time t is bounded below by ct^{-2}, for some $c>0$. It is worth mentioning that the optimality of this bound remains an open question.

2 Analytic regularity of sub-Laplacians

Here, following [38], we construct non-analytic solutions to a sub-Laplacian defined by three real vector fields in $\mathbb{R}^3$ satisfying the finite type condition everywhere. More precisely, we have the result.

Theorem 2.1 *Let k be an odd positive integer, and Δ_X be the sub-Laplacian in $\mathbb{R}^3$ defined by*

$$\Delta_X = \partial_{x_1}^2 + x_1^{k-1}\partial_{x_2}^2 + x_1^{2k}\partial_{x_3}^2. \tag{2.1}$$

Then one can construct non-analytic solutions to the equation $\Delta_X u = 0$ near the origin.

If $k=1$ then the operator Δ_X in (2.1) is the well known Baouendi–Goulaouic operator which provided the first counterexample to analytic hypoellipticity of a sum of squares operator satisfying the finite type condition [1]. The class of operators in (2.1) is contained in a class studied by Oleinik and Radkevic [58]. There, necessary and sufficient conditions for analytic regularity are given. The existence of singular solutions is proved by indirect methods. Here, following [38], we provide an explicit construction.

Proof of Theorem 2.1 First we observe that by using Hörmander's sum of squares theorem we see that all solutions to equation $\Delta_X u = 0$ are in C^∞. Furthermore, separation of variables suggests that we should look for non-analytic solution of the form

$$u(x) = \int_0^\infty e^{i\rho^{k+1}x_3} e^{\sqrt{\mu}x_2\rho^{(k+1)/2}} A(\rho x_1) w(\rho)\, d\rho, \tag{2.2}$$

where the functions A and w, and the positive constant μ are to be determined. By applying Δ_X formally to u we obtain

$$\begin{aligned}\Delta_X u(x) &= \int_0^\infty e^{i\rho^{k+1}x_3} e^{\sqrt{\mu}x_2\rho^{(k+1)/2}} \rho^2 \\ &\quad \left[A''(\rho x_1) - [(x_1\rho)^{2k} - \mu(x_1\rho)^{k-1}]A(\rho x_1)\right] w(\rho)\, d\rho.\end{aligned}$$

Thus u is a formal solution to $\Delta_X u = 0$ if A satisfies the following ordinary differential equation

$$\left(-\frac{d^2}{dt^2} + t^{2k}\right)A(t) = \mu t^{k-1} A(t). \tag{2.3}$$

For u to be well defined in x_1 we shall require that A is in the space of Schwartz functions, that is $A \in \mathcal{S}(\mathbb{R})$. Also, for u to be well defined in x_2 we shall choose

$$w(\rho) = e^{-\rho^{(k+1)/2}}. \tag{2.4}$$

Furthermore, the following lemma in [37] shows that the generalized eigenvalue problem (2.2)–(2.3) has infinitely many solutions, although only one will suffice.

Lemma 2.1 *The eigenvalue problem (2.2)–(2.3) has a non-zero solution if and only if $\mu \in M$, where*

$$M = \{\mu : \mu = 2j(k+1)+k \text{ or } \mu = 2j(k+1)+k+2, \quad j = 0,1,2,\dots\}.$$

Moreover the solution is unique up to a constant factor and is of the form

$$A_\mu(t) = B_\mu(t)e^{-\frac{1}{k+1}t^{k+1}},$$

where B_μ is a polynomial which can be computed explicitly.

Now by Lemmas 2.1 and (2.4) for any $\mu \in M$ the function

$$u_\mu(x) = \int_0^\infty e^{i\rho^{k+1}x_3 + (\sqrt{\mu}x_2 - 1)\rho^{(k+1)/2}} A_\mu(\rho x_1)d\rho, \tag{2.5}$$

is a well defined C^∞ function in the open set $\{x \in \mathbb{R}^3 : |x_2| < 1/\sqrt{\mu}\}$, and a solution to $\Delta_X u_\mu = 0$. It remains to show that u_μ is not analytic near the origin. For any $j \in \{0,1,2,\dots\}$ we have

$$\begin{aligned}\partial^j_{x_3} u_\mu(0) &= i^j A_\mu(0)\int_0^\infty \rho^{j(k+1)}e^{-\rho^{(k+1)/2}}d\rho \\ &= i^j A_\mu(0)\left(2j-1+\frac{2}{k+1}\right)\left(2j-2+\frac{2}{k+1}\right) \\ &\quad \cdots \left(1+\frac{2}{k+1}\right)\frac{2}{k+1}C_0,\end{aligned}$$

where $C_0 = \int_0^\infty e^{-\rho^{(k+1)/2}}d\rho$. Here we may assume that $A_\mu(0) \neq 0$. Therefore

$$|\partial^j_{x_3} u_\mu(0)| \geq C(2j)!, \tag{2.6}$$

for some $C > 0$ and independent of j. By (2.6) u_μ is not analytic near $0 \in \mathbb{R}^3$, and this completes the proof of Theorem 2.1. □

Remark 1 The function A_μ corresponding to $\mu = k$ is given by $A(t) = e^{-\frac{1}{k+1}t^{k+1}}$. Therefore by (2.5) the non-analytic solution corresponding to $\mu = k$ takes the following explicit form

$$u(x) = \int_0^\infty e^{i\rho^{k+1}x_3 + (\sqrt{k}x_2 - 1)\rho^{(k+1)/2} - \frac{1}{k+1}(\rho x_1)^{k+1}}d\rho. \tag{2.7}$$

Treves Conjecture. The characteristic set of the sub-Laplacian Δ_X defined by (2.1) is symplectic, and therefore contains no curves orthogonal to its tangent space with respect to the fundamental symplectic form. This shows that the existence of such curves is not the deciding factor which distinguishes hypoellipticity from analytic regularity. It also shows that a necessary condition for analytic regularity, conjectured by Treves [70], can not be sufficient. Next, we shall explain this for general sub-Laplacians. If (x,ξ) are the variables in T^*U then the principal symbol of Δ_X is $p(x,\xi) = X_1^2(x,\xi) + \cdots + X_m^2(x,\xi)$, and its characteristic set is $\Sigma = \{X_1(x,\xi) = \cdots = X_m(x,\xi) = 0\}$. We shall assume that Σ is a real analytic submanifold of T^*U. We recall that Σ is symplectic if the restriction of the fundamental symplectic form

$$\sigma = \sum_{j=1}^n d\xi_j \wedge dx_j$$

to $T\Sigma$ is non-degenerate. It has been proved by Treves [70] and Tartakoff [68], independently, that Δ_X is analytic hypoelliptic in U if Σ is symplectic and p vanishes to second order on Σ. The symplecticity of Σ does not allow the existence of Treves curves in it. A non-constant curve $\alpha(t)$ inside the characteristic set Σ is said to be a Treves curve for Σ if $\dot\alpha$ is orthogonal to $T\Sigma$ with respect to σ at every point of α. That is,

$$\sigma(\dot\alpha, \Theta) = 0, \quad \forall\Theta \in T\Sigma, \text{at every point of } \alpha.$$

If $k = 1$ then the sub-Laplacian Δ_X in Theorem 2.1 is the well known Baouendi-Goulaouic operator which provided the first counterexample to analytic regularity of a sum of squares operator satisfying the finite type condition [1]. In this case the principal symbol is $p(x,\xi) = -(\xi_1^2 + \xi_2^2 + x_1^2\xi_3^2)$ and the characteristic set is $\Sigma = \{x_1 = \xi_1 = \xi_2 = 0\}$. Moreover the curve $\alpha(t) = (0, t + x_2^0, x_3^0; 0, 0, \xi_3^0)$, $\xi_3^0 \neq 0$, is a Treves curve inside Σ. In fact in all counterexamples to analytic hypoellipticity for the operators Δ_X found in the literature there exists a Treves curve inside their characteristic set, [1, 18, 19, 25, 35, 36, 39, 55, 63]. This is consistent with the following conjecture of Treves [70].

Treves conjecture: *A necessary condition for the analytic hypoellipticity of Δ_X is the following condition (T): The characteristic set of Δ_X contains no Treves curves.*

Although this conjecture remains an open problem, Theorem 2.1 implies the following related result.

Corollary 2.1 *Condition (T) is not sufficient for the analytic hypoellipticity of Δ_X.*

Proof If $k = 3,5,7,\dots$, then the characteristic set of Δ_X in (1.1) is $\Sigma = \{x_1 = \xi_1 = 0\}$. Since Σ is symplectic it does not contain any Treves curves. Since by Theorem 2.1 Δ_X is not analytic hypoelliptic we conclude that condition (T) is not sufficient, which proves the corollary. □

We mention that Grigis and Sjöstrand [32] have shown analytic regularity for a class of operators Δ_X whose characteristic set is not symplectic but it does not contain any Treves curves. For refined versions of Treves conjecture and more recent results on the problem of analytic regularity for sub-Laplacians we refer the reader to the works by Cordaro and Hanges [21–24], Bove and Chinni [13, 14], Bove and Treves [12], Chinni [17], and the references therein.

3 The Cauchy–Kowalevski and the Ovsyannikov theorems

For a Cauchy problem (or initial value problem (ivp)) of k-th order in normal form, that is

$$\begin{cases} \partial_t^k u(x,t) = G\Big(x,t,(\partial_x^\alpha \partial_t^j u)_{|\alpha|+j\le k,\, j<k}\Big), \\ \partial_t^j u(x,0) = \varphi_j(x), \qquad 0 \le j < k, \end{cases} \tag{3.1}$$

the Cauchy–Kowalevski theorem reads as follows.

Theorem 3.1 (Cauchy 1842, special version—Kowalevski 1875, general version). *If $G, \varphi_0, \dots, \varphi_{k-1}$ are analytic near the origin, then the Cauchy problem (3.1) has a unique analytic solution defined in* **some** *neighborhood of the origin.*

We would like to make a few remarks concerning the Cauchy–Kowalevski theorem. First, it is a very general theorem with a very simple proof. It consists of reducing Cauchy problem (3.1) to a first order quasilinear system, which in the case of two variables reads as the following Burgers like ivp

$$\begin{cases} \partial_t u = a(x,u)\partial_x u + b(x,u), \\ u(x,0) = 0, \end{cases} \tag{3.2}$$

and then using power series yields a simpler Cauchy problem that majorizes it and which can be solved by the method of characteristics in a neighborhood of the origin (see, for example, Folland [29]). Second, it provides no information about the analytic lifespan of the solution and about its radius of spatial analyticity as time evolves. Third, it does not apply to important evolution equation, like the KdV, the Schrödinger and the heat equations.

Also, a major drawback of the Cauchy–Kowalevski theorem is that the data-to-solution map may be highly unstable, as Hadamard demonstrates it in the case of Laplacian in $\mathbb{R}^2$ [33], where he introduces the notion of well-posedness.

However, it can be extended so that it applies to some important nonlocal evolution equations, like the CH equation and the Euler equations.

Before describing this extension to nonlocal equations, we will recall the needed analytic spaces $G^{\delta,s}$ introduced by Foias and Temam [27] and which in the periodic case are defined as follows

$$\begin{aligned} G^{\delta,s}(\mathbb{T}) &= \Big\{ \varphi \in L^2(\mathbb{T}) : ||\varphi||^2_{G^{\delta,s}(\mathbb{T})} \doteq ||\varphi||^2_{\delta,s} \\ &= \sum_{k\in\mathbb{Z}} \langle k\rangle^{2s} e^{2\delta|k|} |\widehat{\varphi}(k)|^2 < \infty \Big\}, \end{aligned} \tag{3.3}$$

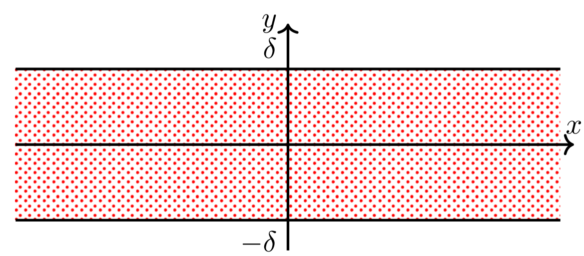
while in the non-periodic case are defined as

$$\begin{aligned} G^{\delta,s}(\mathbb{R}) = \Big\{ \varphi \in L^2(\mathbb{R}) &: ||\varphi||^2_{G^{\delta,s}(\mathbb{R})} \doteq ||\varphi||^2_{\delta,s} \\ &= \int_{\mathbb{R}} \langle \xi\rangle^{2s} e^{2\delta|\xi|} |\widehat{\varphi}(\xi)|^2 d\xi < \infty \Big\}, \end{aligned} \tag{3.4}$$

where $\delta > 0, s \ge 0, \langle k\rangle = \sqrt{1+k^2}$ and $\langle \xi\rangle = \sqrt{1+\xi^2}$. Here, when a result holds for both the periodic and non-periodic case then we will use the notation $||\cdot||_{\delta,s}$ and $G^{\delta,s}$ for the norm and the space in both cases.

Next, we recall an important property of a function $\varphi \in G^{\delta,s}$. For $\delta > 0$ and $s \in \mathbb{R}$, it is straightforward to check that a function $\varphi \in G^{\delta,s}$ is a restriction to the real axis of a function analytic on a symmetric strip of width 2δ.

Definition 1 This $\delta > 0$ is called the radius of spatial analyticity of φ.



In fact, the following Paley–Wiener theorem provides an alternative description of $G^{\delta,s}$ (see [48]).

Theorem 3.2 (Paley–Wiener). *$\varphi \in G^{\delta,s}$ iff $\varphi(x)$ is the restriction to the real line of a holomorphic function $\varphi(x+iy)$ in the strip*

$$S_\delta = \{x + iy : |y| < \delta\},$$

and satisfies $\sup_{|y|<\delta} \|\varphi(x+iy)\|_{H^s} < \infty$.

Next, we discus the initial value problem (ivp) of two simple models, one linear and one nonlinear, with data in $G^{\delta,s}$.

Example 1 (The transport equation). To solve the initial value problem for the transport equation

$$\begin{cases} u_t + cu_x = 0 \\ u(x,0) = g(x), \end{cases} \tag{3.5}$$

where the initial datum $g(x)$ is in $G^{\delta,s}$ using the power series method we find the solution

$$u(x,t)=\sum_{k=0}^{\infty}\frac{\partial_t^k u(x,0)}{k!}t^k$$
$$=\sum_{k=0}^{\infty}\frac{g^{(k)}(x)}{k!}(-ct)^k=g(x-ct),$$

which has the following properties:

- It exists for *all time*, i.e., its lifespan is infinite (*solution is global*).
- It is *analytic in both* the space and the time variables.
- At any time t it extends holomorhically to the same strip S_δ, i.e., its radius of spatial in analyticity persists!

Example 2 (The inviscid Burgers equation). Also, the Cauchy–Kowalevski theorem applies for the (inviscid) Burgers equation ivp

$$\begin{cases} u_t+uu_x=0\\ u(x,0)=g(x),\end{cases} \tag{3.6}$$

where the initial datum $g(x)$ is in $G^{\delta,s}$, which by Theorem 3.2 means that it is analytic on $\mathbb{R}$ and can be extended as a holomorphic functions in the strip

$$S_\delta=\{z=x+iy\in\mathbb{C}:|\mathrm{Im}z|<\delta\}.$$

Using the power series method we see that the solution has following properties:

- The solution may exist only for *finite time* T, i.e., $0\le t<T$.
- It is *analytic in both* the space and the time variables.
- The radius of spatial analyticity may shrink as time t goes on.

We note that in both examples above, the solution can be found with the method of characteristics.

3.1 Autonomous Ovsyannikov theorem

We consider the following initial value problem for a nonlocal autonomous equation

$$\frac{du}{dt}=F(u),\quad u(0)=u_0, \tag{3.7}$$

and prove existence and uniqueness of solution in a space of analytic functions under appropriate conditions on $F(u)$, which is defined on a scale of Banach spaces. Furthermore, we prove an estimate for the analytic lifespan. The motivation comes from the 2003 work in [40] about the Cauchy problem of the Camassa–Holm (CH) equation with analytic initial data on the circle $\mathbb{T}=\mathbb{R}/(2\pi\mathbb{Z})$,

$$\begin{aligned}&\frac{du}{dt}=-u\partial_x u-\left(1-\partial_x^2\right)^{-1}\partial_x\left[u^2+\frac{1}{2}(\partial_x u)^2\right]\doteq F(u)\\ &u(0)=u_0\in C^{\omega}(\mathbb{T}).\end{aligned} \tag{3.8}$$

There it was proved the following Cauchy–Kowalevski type result for CH. If $u_0(x)$ is analytic on $\mathbb{T}$, then there exist an $\varepsilon>0$ and a unique solution $u(x,t)$ of the CH Cauchy problem (3.8), which is analytic on $(-\varepsilon,\varepsilon)\times\mathbb{T}$.

While this result provides the analyticity of the solution in both the spatial and time variables (a phenomenon which does not hold for KdV, see [47] and [30]) it gives no estimate about the size of the analytic lifespan ε. Also, it provides no information about the evolution of the uniform radius of analyticity. Considering these to be important questions for CH and other nonlocal equations and systems, we shall discuss them here on both the circle and the line. Furthermore, we will study the stability of their solution map.

To do this in a unified way we shall need a refined version of the so called Ovsyannikov theorem in the autonomous case, that is the function F depends only on u.

Hypothesis of Ovsyannikov theorem. First we state the hypothesis of the autonomous Ovsyannikov theorem.

1. $\{X_\delta\}_{0<\delta\le 1}$ is a scale of decreasing Banach spaces, i.e.,
$$X_\delta\subset X_{\delta'},\ ||\cdot||_{\delta'}\le||\cdot||_\delta,\quad 0<\delta'<\delta\le 1.$$
2. $F:X_\delta\to X_{\delta'}$ is a function such that for any given $u_0\in X_1$ and $R>0$ there exist L and M positive numbers, depending on u_0 and R, such that for any $0<\delta'<\delta\le 1$ and all $u,v\in X_\delta$ with $\|u-u_0\|_\delta<R$ and $\|v-u_0\|_\delta<R$ we have the following "Lipschitz condition"
$$\|F(u)-F(v)\|_{\delta'}\le\frac{L}{\delta-\delta'}\|u-v\|_\delta, \tag{3.9}$$
and
$$||F(u_0)||_\delta\le\frac{M}{1-\delta},\quad 0<\delta<1. \tag{3.10}$$
3. For $0<\delta'<\delta<1$ and $a>0$, if the function $t\longmapsto u(t)$ is holomorphic on $\{t\in\mathbb{C}:|t|<a(1-\delta)\}$ with values in X_δ and $\sup_{|t|<a(1-\delta)}\|u(t)-u_0\|_\delta<R$, then the function $t\longmapsto F(u(t))$ is holomorphic on $\{t\in\mathbb{C}:|t|<a(1-\delta)\}$ with values in $X_{\delta'}$.

Next, we state an autonomous version of Ovsyannikov theorem, which as we mentioned earlier in addition to existence and uniqueness provides an estimate about the analytic lifespan of the solution.

Theorem 3.3 (Baouendi–Goulaouic [2], improved in [3]). *Assume that the scale of Banach spaces X_δ and the function $F(u)$ satisfy the above conditions (1)–(3). For given $u_0 \in X_1$ and $R > 0$ set*

$$T = \frac{R}{16LR + 8M}. \tag{3.11}$$

Then there exists a unique solution $u(t)$ to the Cauchy problem (3.7), which for every $\delta \in (0,1)$ is a holomorphic function in the disc $D(0, T(1-\delta))$ valued in X_δ satisfying

$$\sup_{|t|<T(1-\delta)} \|u(t) - u_0\|_\delta < R, \quad 0 < \delta < 1. \tag{3.12}$$

Remark 2 The novelty in Theorem 3.3 is that it contains estimate (3.11) for the *analytic lifespan* of the solution $u(t)$. A slightly more general version of Theorem 3.3, where $F = F(u,t)$ but no estimate on T, was proved by Baouendi and Goulaouic [2], Ovsyannikov [59], Nirenberg [56], Nishida [57], and Treves [69].

3.2 The Cauchy problem for CH with analytic initial data

By using our autonomous version of Ovsyannikov theorem we are going to present our result about the Cauchy problem for the Camassa-Holm equation. For this we use the spaces $G^{\delta,s}_{0<\delta\le 1}$. It is easily seen that they form a scale of decreasing Banach spaces like the spaces X_δ in the autonomous Ovsyannikov theorem for

$$\begin{aligned} \frac{du}{dt} &= F(u) = (1-\partial_x^2)^{-1}[-3uu_x + 2u_x u_{xx} + uu_{xxx}], \\ u(0) &= u_0. \end{aligned} \tag{3.13}$$

Also, these spaces and F(u) satisfy condition (1) and (3) in the autonomous Ovsyannikov theorem. Furthermore, in order to obtain the analytic lifespan for the solution to the Cauchy problem for the CH we need good estimates.

For this we begin with the properties of the $G^{\delta,s}$ and the estimates needed to prove the three conditions of the autonomous Ovsyannikov theorem. These are listed in the following lemma whose proof is straightforward and is omitted.

Lemma 3.1 *If $0 < \delta' < \delta \le 1$, $s \ge 0$ and $\varphi \in G^{\delta,s}$ on the circle or the line, then*

$$||\partial_x \varphi||_{\delta',s} \le \frac{e^{-1}}{\delta - \delta'} ||\varphi||_{\delta,s} \tag{3.14}$$

$$||\partial_x \varphi||_{\delta,s} \le ||\varphi||_{\delta,s+1} \tag{3.15}$$

$$||(1-\partial_x^2)^{-1}\varphi||_{\delta,s+2} = ||\varphi||_{\delta,s} \tag{3.16}$$

$$||(1-\partial_x^2)^{-1}\varphi||_{\delta,s} \le ||\varphi||_{\delta,s} \tag{3.17}$$

$$||\partial_x(1-\partial_x^2)^{-1}\varphi||_{\delta,s} \le ||\varphi||_{\delta,s}. \tag{3.18}$$

Furthermore, we shall need an algebra property for these spaces, which is the main result in the following lemma.

Lemma 3.2 ([4] Lemma 4). *For $\varphi \in G^{\delta,s}$ on the circle or the line the following properties hold true:*

(1) *If $0 < \delta' < \delta$ and $s \ge 0$, then $||\cdot||^2_{\delta',s} \le ||\cdot||^2_{\delta,s}$; i.e. $G^{\delta,s} \hookrightarrow G^{\delta',s}$.*
(2) *If $0 < s' < s$ and $\delta > 0$, then $||\cdot||^2_{\delta,s'} \le ||\cdot||^2_{\delta,s}$; i.e. $G^{\delta,s} \hookrightarrow G^{\delta,s'}$.*
(3) *For $s > 1/2$ and $\varphi, \psi \in G^{\delta,s}$ we have*

$$||\varphi\psi||_{\delta,s} \le c_s ||\varphi||_{\delta,s} ||\psi||_{\delta,s}, \tag{3.19}$$

where $c_s = \sqrt{2(1+2^{2s})\sum_{k=0}^{\infty} \frac{1}{\langle k \rangle^{2s}}}$ in the periodic case and $c_s = \sqrt{2(1+2^{2s})\int_0^{\infty} \frac{1}{\langle \xi \rangle^{2s}} d\xi}$ in the non-periodic case.

Now we are ready to describe the proof of the following important result that provides the desired analytic lifespan T.

Theorem 3.4 ([3]). *Let $s > -\frac{1}{2}$. If $u_0 \in G^{1,s+2}$ on the circle or the line, then there exists a positive time T, which depends on the initial data u_0 and s, such that for every $\delta \in (0,1)$, the Cauchy problem (3.13) has a unique solution u which is a holomorphic function in the disc $D(0, T(1-\delta))$ valued in $G^{\delta,s+2}$. Furthermore, the analytic lifespan T satisfies the estimate*

$$T \approx \frac{1}{||u_0||_{G^{1,s}}}. \tag{3.20}$$

Proof We need only to prove the condition (2) in the autonomous Ovsyannikov theorem. We start by recalling that the CH equation can be written in the following form

$$\begin{aligned} \frac{du}{dt} = &F(u) \doteq -\partial_x \\ &\left(\frac{1}{2}u^2 + \left(1-\partial_x^2\right)^{-1}\left[u^2 + \frac{1}{2}(\partial_x u)^2\right]\right). \end{aligned} \tag{3.21}$$

Applying Lemma 3.1 and the triangle inequality we get

$$\begin{aligned} &||F(u) - F(v)||_{\delta',s+2} \\ &\le \frac{e^{-1}}{\delta - \delta'} \\ &\quad \left(\frac{1}{2}||u^2 - v^2||_{\delta,s+2} + ||u^2 - v^2||_{\delta,s} \right. \\ &\quad \left. + \frac{1}{2}||(\partial_x u)^2 - (\partial_x v)^2||_{\delta,s}\right). \end{aligned}$$

Also, applying the algebra property (3.19) and inequality (3.15) we get the estimates

$$||u^2 - v^2||_{\delta,s} \le ||u^2 - v^2||_{\delta,s+2} \le c_s||u - v||_{\delta,s+2}||u + v||_{\delta,s+2}, \tag{3.22}$$

$$||(\partial_x u)^2 - (\partial_x v)^2||_{\delta,s} = ||\partial_x(u - v)\partial_x(u + v)||_{\delta,s} \le c_s||u - v||_{\delta,s+2}||u + v||_{\delta,s+2}. \tag{3.23}$$

Finally, bounding $||u + v||_{\delta,s+2}$ as follows

$$||u + v||_{\delta,s+2} \le ||u - u_0||_{\delta,s+2} + ||v - u_0||_{\delta,s+2} + 2||u_0||_{\delta,s+2} \le 2(R + ||u_0||_{1,s+2})$$

and combining the above three inequalities gives the desired estimate (3.9) with

$$L = 4e^{-1}c_s(R + ||u_0||_{1,s+2}), \tag{3.24}$$

where c_s is given in Lemma 3.2

Next we prove (3.10) for CH. Using the properties of our scale of Banach spaces $G^{\delta,s}$ stated in Lemmas 3.1 and 3.2 for $0 < \delta' < \delta \le 1$ we have

$$||\partial_x(u_0^2)||_{\delta',s+2} \le \frac{e^{-1}c_s||u_0||^2_{\delta,s+2}}{\delta - \delta'},$$
$$||\partial_x(1 - \partial_x^2)^{-1}(u_0^2)||_{\delta',s+2} \le \frac{e^{-1}c_s||u_0||^2_{\delta,s+2}}{\delta - \delta'},$$
$$||\partial_x(1 - \partial_x^2)^{-1}(\partial_x u_0)^2||_{\delta',s+2} \le \frac{e^{-1}c_s||u_0||^2_{\delta,s+2}}{\delta - \delta'}.$$

Combining these we get the inequality

$$||F(u_0)||_{\delta',s+2} \le \frac{2e^{-1}c_s||u_0||^2_{\delta,s+2}}{\delta - \delta'},$$

which, by replacing δ' by δ and δ by 1, gives the desired estimate (3.10), with

$$M = 2e^{-1}c_s||u_0||^2_{1,s+2}. \tag{3.25}$$

Now, we are ready to complete the proof of Theorem 3.4. For any u_0 in $G^{1,s+2}$ and $R > 0$, according to (3.24) and (3.25) and thanks to Theorem 3.3 we have

$$T = \frac{R}{16LR + 8M} = \frac{R}{16C(R + ||u_0||_{1,s+2})R + 4C||u_0||^2_{1,s+2}}, \tag{3.26}$$

where $C = 4e^{-1}c_s$ and there exists a unique solution $u(t)$ to the Cauchy problem (3.13), which for every $\delta \in (0, 1)$ is a holomorphic function in $D(0, T(1 - \delta)) \to G^{\delta,s+2}$ and

$$\sup_{|t|<T(1-\delta)} \|u(t) - u_0\|_{\delta,s+2} < R. \tag{3.27}$$

Thus, by letting $R = ||u_0||_{1,s+2}$ we obtain

$$T = \frac{e}{144c_s} \cdot \frac{1}{||u_0||_{1,s+2}}. \tag{3.28}$$

This completes the proof of Theorem 3.4. □

Estimate (3.20) besides being interesting on its own merit, it is also the key ingredient for proving continuity for the solution map. More precisely, for the CH equation we have the following important result.

Theorem 3.5 *If $s > -\frac{1}{2}$, then the data-to-solution map $u(0) \mapsto u(t)$ of the Cauchy problem (3.13) for the CH equation is continuous from $G^{\delta,s+2}$ into the solutions space.*

3.3 Global analytic CH solutions and the evolution of the uniform radius of analyticity.

Here, we consider the Cauchy problem for the Camassa–Holm (CH) equation on the line

$$\begin{cases} u_t = -u\partial_x u - \partial_x(1 - \partial_x^2)^{-1}\left[u^2 + \frac{1}{2}(\partial_x u)^2\right] \\ u(x, 0) = u_0(x), \quad x \in \mathbb{R}, \ t \ge 0, \end{cases} \tag{3.29}$$

and study the problem of analyticity of the smooth solutions for initial data $u_0(x)$ that are analytic on the line and can be extended as holomorphic functions in a strip around the x-axis. Under the condition that the McKean quantity $(1 - \partial_x^2)u_0(x)$ (see [53, 54]) does not change sign we obtain explicit lower bounds on the radius of spatial analyticity $r(t)$ at any time $t \ge 0$. We recall that the Cauchy problem for the CH equation is globally well-posed for initial data in $H^\infty(\mathbb{R}) = \bigcap_{s\ge 0} H^s(\mathbb{R})$ and satisfying the McKean condition (see, e.g., [5, 64]). Furthermore, under the above analyticity assumption on initial data, it has been shown in [5] that the solution to the CH Cauchy problem is globally analytic in $x \in \mathbb{R}$, $t \ge 0$, and a lower bound of double exponential decay was derived for the radius of space analyticity at later times. More precisely, the lower bound for $r(t)$ is of the form $L_3 \exp(-L_1 \exp(L_2 t))$, where L_1, L_2 and L_3 are appropriate positive constants.

More recently we improved the double exponential decay above by replacing it with a single exponential. In order to do this we shall need the following spaces of analytic functions, which were first introduced by Kato and Masuda (see [47]). For each $r > 0$, we define $A(r)$ to be the set of all real-valued functions f that can be extended analytically in the strip $S(r)$ of width $2r$ around the x-axis in the complex plane and also belong in $L^2(S(r'))$ for every $0 < r' < r$. More precisely, we have

$$A(r) \doteq \{f : \mathbb{R} \to \mathbb{R} : f(z) \text{ is analytic in } S(r)\} \cap \{f : f \in L^2(S(r')) \text{ for all } 0 < r' < r\}, \tag{3.30}$$

where $S(r) = \{z \in \mathbb{C} : -\infty < \mathrm{Re}\, z < \infty, -r < \mathrm{Im}\, z < r\}$. Also, we note that $A(r)$ is a Fréchet space with these $L^2(S(r'))$-norms as the generating system of seminorms.

We would like to point out that the topology that we are going to use on $A(r)$ was set by Kato and Masuda in [47]. More precisely, in [47] they used the following set of norms

$$\|f\|^2_{\sigma,s} = \sum_{j=0}^{\infty} \frac{1}{j!^2} e^{2j\sigma} \|\partial_x^j f\|_s^2, \quad 0 < e^{\sigma} < r, \quad s \geq 0, \tag{3.31}$$

where $\|\cdot\|_s$ denotes the standard Sobolev norm. Furthermore, we note that the space $A(r)$ is a Fréchet space with the set of norms $\{\|\cdot\|_{\sigma,s}\}$, $e^{\sigma} < r$, $s \geq 0$. The equivalence of the set of norms (3.31) to the previous ones given in the definition of the spaces $A(r)$ is proved in the following key lemma.

Lemma 3.3 (See Lemma 2.2 in [47]). *If $f \in A(r)$ then $\|f\|_{\sigma,s} < \infty$ for any σ, with $e^{\sigma} < r$, and $s \geq 0$. Conversely, if $f \in H^{\infty}(\mathbb{R})$ satisfies $\|f\|_{\sigma,s} < \infty$ for some $s \geq 0$ and for each $e^{\sigma} < r$ then $f \in A(r)$.*

Now, we are ready to state our improved lower bound for the radius of space analyticity for CH.

Theorem 3.6 ([42]). *Let $u_0 \in G^{\delta_0,\theta}(\mathbb{R})$, $\delta_0 > 0, \theta > \frac{3}{2}$. Also, assume that the McKean quantity $m_0(x) = (1-\partial_x^2)u_0(x)$ does not change sign so that the Cauchy problem (3.29) has a unique global solution $u \in C([0,\infty);H^{\infty}(\mathbb{R}))$. Then, for every time t the solution $u(t,\cdot)$ has an analytic continuation that belongs to the space $A(\delta(t))$ with $\delta(t) > 0$. Furthermore, for any time $T \geq 0$ we have the following lower bound for the radius of space analyticity $r(t)$*

$$r(t) \geq \delta(t) \doteq A^{-1}(1 + C_1 B t)^{-1} \exp\{-C_0 ||u_0||_{H^1(\mathbb{T})} t\}, \quad t \in [0,T], \tag{3.32}$$

where $A \geq 1$, $B \geq \frac{9}{4}||u_0||_{H^s(\mathbb{R})}$, C_0, C_1 are positive constants and T is a given positive number.

For the proof of Theorem 3.6 we refer the reader to [42]. It has been motivated by the work of Kukavica and Vicol on the Euler equations [51] and [52].

Open question. Is the single exponential decay estimate (3.32) for CH optimal?

4 The periodic BBM equation with analytic data

Here we consider the Cauchy problem for the periodic Benjamin-Bona-Mahony (BBM) equation

$$\begin{cases} \partial_t u + \partial_x u + u\partial_x u - \partial_x^2 \partial_t u = 0, \\ u(0,x) = u_0(x), \ t \in \mathbb{R}, \ x \in \mathbb{T}, \end{cases} \tag{4.1}$$

with data in $G^{\delta,s}(\mathbb{T})$ and discuss the analyticity properties of its solution, following our recent work in [43]. The BBM or regularized long-wave equation was derived in [6] and [61] as a model for the unidirectional propagation of long-crested, surface water waves. It is an alternative to the classical Korteweg-de Vries equation (see [11] and [50])

$$\partial_t u + \partial_x u + u\partial_x u + \partial_x^3 u = 0. \tag{4.2}$$

If the initial data belong in a Sobolev space H^s, $s \geq 0$, then it has been shown by Bona and Tzvetkov [8] that the Cauchy problem for the BBM equation is globally well-posed. On the other hand, Panthee [60] has shown that the BBM equation is ill-posed for initial data that belong in H^s, $s < 0$.

The main result to be discussed here is about global solutions to the BBM Cauchy problem with initial data in $G^{\delta,s}$, and the evolution of the uniform radius of spatial analyticity. It reads as follows.

Theorem 4.1 ([43]). *For $u_0 \in G^{\delta_0,s}(\mathbb{T})$ with $\delta_0 > 0$ and $s \geq 0$, the Cauchy problem (4.1) has a global in time solution $u(t)$ such that for any $T > 0$ we have*

$$u \in C^{\omega}\big([-T,T], G^{\delta(T),s}(\mathbb{T})\big) \tag{4.3}$$

with $\delta(T) > 0$. Furthermore, the radius of spatial analyticity $r(T)$ satisfies the lower bound estimate

$$r(T) \geq \delta(T) = \min\{\delta_0, cT^{-1}\}, \ T > 0, \quad \text{where} \quad c = c(u_0, \delta_0, s) > 0. \tag{4.4}$$

We mention that for the non-periodic Cauchy problem of the BBM equation with analytic initial data, Bona and Grujić [7] have proved that the radius of spatial analyticity $r(t)$ satisfies the lower bound estimate $r(t) \geq c_0(r_0^{-1} + t + t^{2/3})^{-1}$, for all $t \geq 0$ and some $c_0 > 0$.

The proof of this result is done in two steps. In the first step we study the BBM by viewing it as an ODE on $G^{\delta,s}(\mathbb{T})$ and applying a simpler version of a method developed for the Cauchy problem of the Camassa-Holm equation with data in an analytic space (see [3]). This way we obtain a local solution whose uniform radius of analyticity persists during its lifespan, which is given explicitly. The second step consist of deriving an approximate conservation law, which is based on the fact the H^1 norm is conserved by BBM solutions. This provides a certain control on the growth of the $G^{\delta,s}(\mathbb{T})$-norm of the solution u(t) at time t which allows us to extend the solution for all times in $G^{\delta(t),s}(\mathbb{T})$ if $\delta(t)$ is chosen as in (4.4). This strategy is motivated by the recent advances in the study of the KdV equation on the line [65] and the circle [44], as well as the quartic generalized KdV equation on the line [67] and the modified Kawahara equation on line [62] (see also [66] where the 1-D Dirac–Klein–Gordon equation is considered). In these works Bourgain type spaces are used to prove the local existence of solution and also to obtain an approximate conservation law in $G^{\delta,s}$ spaces. For example, in the case of the KdV this approximate conservation law is based on the fact that its solutions conserve the L^2-norm and that the KdV bilinear estimates in Bourgain spaces hold for $s > -\frac{3}{4}$ on the line, and for $s > -\frac{1}{2}$ on the circle (see [10, 20, 49]). For the uniform radius of spatial analyticity r(t), these yield the asymptotic lower bound of $r(t) \geq c_\varepsilon t^{-\frac{4}{3}-\varepsilon}$ on the line, and $r(t) \geq ct^{-2}$ on the circle. Finally, it is worth mentioning that in the case of the Camassa–Holm equation (another nonlocal equation) it is proved in [42] that its radius of spatial analyticity $r(t)$ satisfies the asymptotic lower bound of $r(t) \geq ce^{-bt}$ for some positive constants b and c.

4.1 An abstract ODE Theorem and local BBM solutions

We begin by writing the BBM equation in the following nonlocal form

$$\begin{cases} \dfrac{du}{dt} = F(u) \doteq -\left(1-\partial_x^2\right)^{-1}\partial_x\left[\dfrac{1}{2}u^2+u\right] \\ u(0) = u_0(x) \in G^{\delta,s}(\mathbb{T}). \end{cases} \tag{4.5}$$

To prove that BBM in this form is an ODE on the Hilbert space $G^{\delta,s}(\mathbb{T})$ with $\delta > 0$ and $s \geq 0$, we need the following two estimates:

$$||(1-\partial_x^2)^{-1}\partial_x(\varphi\psi)||_{\delta,s} \leq c_s||\varphi||_{\delta,s}||\psi|||_{\delta,s}, \tag{4.6}$$

where $c_s^2 = 2^s\left(1+\frac{\pi^2}{3}\right)$, and

$$||(1-\partial_x^2)^{-1}\partial_x\varphi||_{\delta,s} \leq ||\varphi||_{\delta,s}, \tag{4.7}$$

where $\varphi, \psi \in G^{\delta,s}(\mathbb{T})$ with $\delta > 0$ and $s \geq 0$.

Using these two properties we see that the BBM equation in its nonlocal form (4.5) is an ordinary differential equation (ODE) on the Hilbert space $G^{\delta,s}(\mathbb{T})$, if $s \geq 0$ and $\delta \geq 0$. More precisely, if $u \in G^{\delta,s}(\mathbb{T})$ then $F(u) \doteq -\left(1-\partial_x^2\right)^{-1}\partial_x\left[\frac{1}{2}u^2+u\right] \in G^{\delta,s}(\mathbb{T})$.

The existence and uniqueness of solution for the BBM Cauchy problem (4.5) will follow from solving the following more general Cauchy problem

$$\begin{cases} \dfrac{du}{dt} = F(u) \\ u(0) = u_0 \in X, \end{cases} \tag{4.8}$$

where the space $G^{\delta,s}(\mathbb{T})$ is replaced by a Banach space $(X, ||\cdot||_X)$ and the function F has the following properties.

1. $F : X \longmapsto X$ is a function such that for any given $u_0 \in X$ and $R > 0$ there exist L and M positive numbers, depending on u_0 and R, such that for all $u, v \in X$ with $\|u - u_0\|_X < R$ and $\|v - u_0\|_X < R$ we have
$$\|F(u) - F(v)\|_X \leq L\|u - v\|_X, \tag{4.9}$$
and
$$||F(u_0)||_X \leq M. \tag{4.10}$$
2. For $T > 0$, if the function $t \longmapsto u(t)$ is holomorphic on $\{t \in \mathbb{C} : |t| < T\}$ with values in X and $\sup_{|t|<T}\|u(t) - u_0\|_X < R$, then the function $t \longmapsto F(u(t))$ is holomorphic on $\{t \in \mathbb{C} : |t| < T\}$ with values in X.

For the more general abstract ODE Cauchy problem (4.8) we have the following result.

Theorem 4.2 *Assume that the space X and the function F satisfy the properties (1) and (2) above. For given data $u_0 \in X$ and $R > 0$ set*

$$T \doteq \frac{R}{RL + 2M}. \tag{4.11}$$

Then there exists a unique solution $u(t)$ to the Cauchy problem (4.8) which is a holomorphic function on the disc $D(0, T)$ valued in X satisfying

$$\sup_{|t|<T} ||u(t) - u_0||_X < R. \tag{4.12}$$

Proof Its proof follows the lines of the proof of the autonomous Ovsyannikov theorem, with the appropriate simplifications.

Next, applying the abstract ODE Theorem 4.2 to the Cauchy problem (4.8) for the nonlocal BBM, we obtain the following local well-posedness result.

Theorem 4.3 *Let $s \geq 0, \delta > 0$ and $u_0 \in G^{\delta,s}(\mathbb{T})$. Then there exists a lifespan $T_{\delta,s} = T_{\delta,s}(||u_0||_{\delta,s})$ given by the formula*

$$T_{\delta,s} = \frac{1}{3(1 + c_s||u_0||_{\delta,s})}, \tag{4.13}$$

such that the periodic Cauchy problem for BBM (4.5) has a unique solution u in the space

$$E_{T_{\delta,s}} \doteq \{\mathcal{H}(|t| \leq T_{\delta,s}; G^{\delta,s}(\mathbb{T})) : |u - u_0|_{T_{\delta,s}} \leq \frac{||u_0||_{\delta,s}}{2}\}, \tag{4.14}$$

where $|w|_{T_{\delta,s}} = \sup_{|t|\leq T_{\delta,s}} ||w(t)||_{\delta,s} < \infty$, satisfying the following estimate

$$||u(t)||_{\delta,s} \leq 2||u_0||_{\delta,s}, \quad \text{for} \quad 0 \leq t \leq T_{\delta,s}. \tag{4.15}$$

The constant c_s in the lifespan is as in (4.6), that is $c_s^2 = 2^s\left(1+\frac{\pi^2}{3}\right)$.

This way we obtain a local solution whose uniform radius of spatial analyticity persists during its lifespan, which is given explicitly by (4.13).

4.2 Approximate conservation law

We start by recalling that solutions $u(t)$ to the BBM equation conserve the H^1 norm, that is

$$||u(t)||^2_{H^1(\mathbb{T})} \doteq \int_{\mathbb{T}} [u^2(t) + u_x^2(t)]dx = ||u(0)||^2_{H^1(\mathbb{T})}. \tag{4.16}$$

We now state an approximate conservation law for the BBM equation in $G^{\delta,1}(\mathbb{T})$ space, which for $\delta = 0$ gives the H^1 conservation law (4.16) just presented above.

Theorem 4.4 *Fix $\theta \in (0, 1]$. For $u_0 \in G^{\delta,1}(\mathbb{T})$, $\delta > 0$, let $u \in C^\omega([-T_{\delta,1}, T_{\delta,1}], G^{\delta,1}(\mathbb{T}))$ be the local solution of (4.1) obtained in Theorem 4.3. Then, for $0 < \rho \le T_{\delta,1}$ we have that*

$$\sup_{t\in[0,\rho]} ||u(t)||^2_{\delta,1} \le ||u(0)||^2_{\delta,1} + C_\theta \delta^\theta \rho \sup_{t\in[0,\rho]} ||u(t)||^3_{\delta,1}, \quad \text{where} \quad C_\theta = 4^{\theta+1}\Big(1+\frac{\pi^2}{3}\Big)^{1/2}. \tag{4.17}$$

Applying the size estimate (4.15) for the solution, that is $\sup_{t\in[0,\rho]} ||u(t)||_{\delta,1} \le 2||u(0)||^3_{\delta,1}$, we obtain the following key corollary, since $0 < \rho \le T_{\delta,1} \le 1$.

Corollary 4.1 *Under the assumptions of Theorem 4.4 we have the following almost conservation law*

$$\sup_{t\in[0,\rho]} ||u(t)||^2_{\delta,1} \le ||u(0)||^2_{\delta,1} + L_\theta \delta^\theta ||u(0)||^3_{\delta,1}, \quad \text{where} \quad L_\theta = 2^3 C_\theta. \tag{4.18}$$

Outline of Proof of Theorem 4.4 Defining the function U by $U \doteq e^{\delta|D_x|}u$ or $\widehat{U}(t,\xi) = e^{\delta|\xi|}\widehat{u}(t,\xi)$, where u is the real-valued solution to our BBM Cauchy problem described in the statement of Theorem 4.4, we see that first U is a real-valued function, and second U is a solution to the inhomogeneous BBM equation

$$\partial_t U + \partial_x U + U\partial_x U - \partial_x^2\partial_t U = F, \quad \text{where} \quad F \doteq \frac{1}{2}\partial_x\big[U^2 - e^{\delta|D_x|}(u^2)\big]. \tag{4.19}$$

Since $||U(t)||^2_{H^1(\mathbb{T})} = ||u(t)||^2_{\delta,1}$, using equation (4.19) and integration by parts we find that

$$\frac{d}{dt}||U(t)||^2_{H^1(\mathbb{T})} = 2\int_{\mathbb{T}} UFdx. \tag{4.20}$$

Integrating (4.20) over $[0, t]$ with $0 \le t \le T_{\delta_0,1}$, and using the last equality we obtain

$$||u(t)||^2_{\delta,1} = ||u(0)||^2_{\delta,1} + 2\int_0^t \int_{\mathbb{T}} UFdxdt'. \tag{4.21}$$

Then, estimating the term $\int_{\mathbb{T}} UFdx$ appropriately (see [43]) we get the inequality

$$2\Big|\int_{\mathbb{T}} UFdx\Big| \le 4^{\theta+1}\delta^\theta\Big(1+\frac{\pi^2}{3}\Big)^{1/2}||u||^3_{\delta,1}, \tag{4.22}$$

which combined with (4.21) gives the desired almost approximation law (4.17). ☐

4.3 Outline of the proof of Theorem 4.1 for $s = 1$

The general case $s \ge 0$ and $s \ne 1$ follows from the case $s = 1$ by exploiting the relations in of $G^{\delta,s}$ spaces. Also, by a simple change of variables we can assume $t \ge 0$. So, for given data $u_0 \in G^{\delta_0,1}(\mathbb{T})$ with $\delta_0 > 0$, applying Theorem 4.3 (restricted to $t \ge 0$), it gives a unique solution $u \in C^\omega([0, T_{\delta_0,1}]; G^{\delta_0,1}(\mathbb{T}))$ to the Cauchy problem (4.1) satisfying the size estimate

$$||u(t)||_{\delta_0,1} \le 2||u_0||_{\delta_0,1}, \quad 0 \le t \le T_{\delta_0,1}, \tag{4.23}$$

where $T_{\delta_0,1}$ is the following estimate for the lifespan

$$T_{\delta_0,1} = \frac{1}{3(1+c_1||u_0||_{\delta_0,1})}. \tag{4.24}$$

Next, we define the maximal lifespan by

$$T^* \doteq \sup\{T_{\delta_0,1} \,:\, u \in C^\omega([0, T_{\delta_0,1}]; G^{\delta_0,1}(\mathbb{T})) \text{ solves (4.1) \& satisfies (4.23)}\}, \tag{4.25}$$

and distinguish two possible cases. The first case is $T^* = \infty$, which means that $u \in C^\omega([0,\infty); G^{\delta_0,1}(\mathbb{T}))$ and thus we have **persistence** of the uniform radius of spatial analyticity of $u(t)$ for all time. That is

$$r(T) = \delta_0, \text{ for all } T > 0, \tag{4.26}$$

which proves Theorem 4.1 in this case. The second case is T^* to be finite, which means $u \in C^\omega([0, T^*); G^{\delta_0,s}(\mathbb{T}))$ and

$$\infty > T^* \ge T_{\delta_0,s} = \frac{1}{3(1+c_s||u_0||_{\delta_0,s})} > \frac{1}{3(1+2c_s||u_0||_{\delta_0,1})} = \rho^*. \tag{4.27}$$

Now, taking enough time-steps of length ρ^* we can arrive at any $T \ge T^*$. More precisely, there is a positive integer n (namely the integer part of T/ρ^*) such that

$$n\rho^* \le T < (n+1)\rho^*.$$

Now, if δ satisfies the following size conditions

$$0 < \delta \le \delta_0 \quad \text{and} \quad (n+1)L_\theta \delta^\theta 2^{\frac{3}{2}} ||u(0)||_{\delta_0,1} \le 1, \tag{4.28}$$

then applying induction on $k \in \{1, 2, \dots, n+1\}$, with the first step being the almost conservation law, after $n+1$ steps we arrive at the estimate

$$\sup_{t\in[0,(n+1)\rho^*]} ||u(t)||^2_{\delta,1} \le 2||u(0)||^2_{\delta_0,1}. \tag{4.29}$$

In other words we prove the implication

$$\begin{aligned} &0 < \delta \le \delta_0 \ \textit{and} \ (n+1)L_\theta \delta^\theta 2^{\frac{3}{2}} ||u(0)||_{\delta_0,1} \le 1 \implies \\ &\quad u \in C^\omega([0,(n+1)\rho^*], G^{\delta,1}(\mathbb{T})). \end{aligned} \tag{4.30}$$

Since $n\rho^* \le T < (n+1)\rho^*$ we have that

$$n+1 \le \frac{T}{\rho^*} + 1 < \frac{T}{\rho^*} + \frac{T}{\rho^*} = \frac{2T}{\rho^*}.$$

Therefore, the second δ-size condition (4.28) holds if we chose δ to satisfy the condition

$$\begin{aligned} &\frac{2T}{\rho^*} L_\theta \delta^\theta 2^{\frac{3}{2}} ||u(0)||_{\delta_0,1} \le 1 \iff \delta \le cT^{-\frac{1}{\theta}} \\ &\quad \text{where } c = \Big(\frac{\rho^*}{2L_\theta 2^{3/2}||u(0)||_{\delta_0,1}}\Big)^{\frac{1}{\theta}}. \end{aligned} \tag{4.31}$$

Furthermore, since $[0,T] \subset [0,(n+1)\rho^*]$, from (4.30) and (4.31) we conclude that if $0 < \delta \le \delta_0$ and $0 < \delta \le cT^{-\frac{1}{\theta}}$ then $u \in C^\omega([0,T], G^{\delta,1}(\mathbb{T}))$. Choosing the biggest δ satisfying both conditions gives

$$\begin{aligned} &u \in C^\omega\big([0,T], G^{\delta(T),1}(\mathbb{T})\big) \quad \text{with} \\ &\quad \delta(T) = \min\{\delta_0, cT^{-\frac{1}{\theta}}\}, \quad T > 0. \end{aligned} \tag{4.32}$$

which proves Theorem 4.1 in the case $s = 1$. □

Open question: Is the estimates $r(t) \ge ct^{-1}$ optimal for the periodic BBM equation?

5 The KdV Cauchy problem with analytic initial data

Here, we discuss analyticity properties in the spatial and time variables for solutions to the Cauchy problem of the periodic Korteweg–de Vries equation with analytic initial data.

5.1 Analyticity in spatial variable

We begin by presenting a lower bound estimate for the uniform radius of spatial analyticity following [44].

Theorem 5.1 ([44]). *If $\delta_0 > 0$, $s \ge -1/2$, $\varphi \in G^{\delta_0,s}(\mathbb{T})$ real-valued, then for any $T > 0$ the solution to ivp*

$$\begin{cases} \partial_t u + \partial_x^3 u + u\partial_x u = 0, \\ u(x,0) = \varphi(x), \end{cases} \tag{5.1}$$

satisfies $u \in C\big([-T,T]; G^{\delta(T),s}(\mathbb{T})\big)$ where

$$\begin{aligned} &r(T) \ge \delta(T) = \min\Big\{\delta_0, \frac{c}{T^2}\Big\}, \quad \text{with} \\ &\quad c = c(\varphi, \delta_0, s) > 0. \end{aligned} \tag{5.2}$$

We mention that on $\mathbb{R}$ a decay like t^{-12} was proved by Bona-Grujić-Kalisch [9], and like $t^{-(\frac{4}{3}+\epsilon)}$ it was by Selberg-Silva [65].

The proof of Theorem 5.1 is based on a local analyticity result proved in [41], and an almost L^2-conservation law proved in [44]. We begin with the local well-posedness in $G^{\delta,s}(\mathbb{T})$ spaces.

Theorem 5.2 ([41]). *Given $\delta > 0$ and $s \ge -\frac{1}{2}$, then for any $u_0 \in G^{\delta,s}(\mathbb{T})$, that is*

$$\|u_0\|_{G^{\delta,s}(\mathbb{T})} = \|u_0\|_{\delta,s} = \Big(\sum_{k\in\mathbb{Z}} e^{2\delta|k|}\langle k\rangle^{2s}|\hat{u}_0(k)|^2\Big)^{1/2} < \infty,$$

there exists a time $T_0 = T_0(\|u_0\|_{G^{\delta,s}(\mathbb{T})}) > 0$ and a solution $u \in C\big([-T_0,T_0]; G^{\delta,s}(\mathbb{T})\big)$ of the Cauchy problem (5.1). Moreover,

$$T_0 = \frac{c_0}{\big(1 + \|u_0\|_{G^{\delta,s}(\mathbb{T})}\big)^a}$$

for some constants $a, c_0 > 0$ depending only on s. Also, we have

$$\sup_{|t|\le T_0} ||u(\cdot,t)||_{\delta,s} \le 2C\|u(0)\|_{\delta,s}$$

for some constant $C > 0$ depending only on s.

The proof of local well-posedness is based on bilinear estimates in Bourgain space $X^{s,b}$ on $\mathbb{T}\times\mathbb{R}$, defined by the norm

$$\|u\|_{X^{s,b}} = \left(\sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \langle k\rangle^{2s}\langle \tau - k^3\rangle^{2b}|\hat{u}(k,\tau)|^2\, d\tau\right)^{1/2},$$

where $\langle k\rangle = \sqrt{1+k^2}$ and $\hat{u}(k,\tau)$ is the Fourier transform

$$\hat{u}(k,\tau) = \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{T}} e^{-i(kx+\tau t)}u(x,t)\,dx\,dt \quad (k\in\mathbb{Z}, \tau\in\mathbb{R}).$$

Following [20], we iterate in the norms

$$\|u\|_{Y^s} = \|u\|_{X^{s,1/2}} + \left(\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\left(\int_{\mathbb{R}}|\widehat{u}(k,\tau)|\,d\tau\right)^2\right)^{1/2},$$

where the inclusion of the last term ensures that $Y^s \hookrightarrow C(\mathbb{R};H^s(\mathbb{T}))$. The nonlinearity is then estimated in the norm

$$\|u\|_{Z^s} = \|u\|_{X^{s,-1/2}} + \left(\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\left(\int_{\mathbb{R}}\frac{|\widehat{u}(k,\tau)|}{\langle \tau-k^3\rangle}\,d\tau\right)^2\right)^{1/2}.$$

We will also use the corresponding spaces $X^{\delta,s,b}$, $Y^{\delta,s}$ and $Z^{\delta,s}$, where u is replaced by $U = e^{\delta|D_x|}u$ in the definition of the norm. Thus, $Y^{\delta,s} \hookrightarrow C(\mathbb{R};G^{\delta,s}(\mathbb{T}))$. We denote by $Y_I^{\delta,s}$ the restriction to a time interval I, defined by the norm

$$\|u\|_{Y_I^{\delta,s}} = \inf\left\{\|v\|_{Y^{\delta,s}} \,:\, v\in Y^{\delta,s},\ v=u \text{ on } \mathrm{T}\times\mathrm{Int}(I)\right\},$$

where $\mathrm{Int}(I)$ is the interior of I. The restriction $Z_I^{\delta,s}$ is similarly defined.

The next ingredient for deriving the lower bound (5.2) for the uniform radius of spatial analyticity for KdV is the following almost conservation in $G^{\delta,0}(\mathbb{T})$, which is based on the L^2 concervation law

$$\int_{\mathbb{T}} u^2(t)dx = \int_{\mathbb{T}} u^2(0)dx.$$

Theorem 5.3 (Almost conservation law). *Given $\delta > 0$ and $u_0 \in G^{\delta,0}(\mathbb{T})$, let $u \in C([-\rho,\rho];G^{\delta,0}(\mathbb{T}))$ be the local solution obtained in theorem above (with $s = 0$). Then*

$$\sup_{|t|\le\rho}||u(t)||^2_{\delta,0} \le ||u(0)||^2_{\delta,0} + C\delta^{1/2}||u(0)||^3_{\delta,0} \tag{5.3}$$

for some constant $C > 0$.

Remark 3 Observe that the conservation index $s = 0$ is well above the critical local well-posedness index $s = -1/2$.

We recall that $\int_{\mathbb{R}} u^2(x,t)\,dx$ is conserved for a KdV solution u, since

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}}u^2dx &= \int_{\mathbb{T}}u\partial_t u\,dx = -\int_{\mathbb{T}}u\partial_x^3u\,dx - \int_{\mathbb{T}}u^2\partial_x u\,dx \\ &= \frac{1}{2}\int_{\mathbb{T}}\partial_x(\partial_x u\partial_x u)dx - \frac{1}{3}\int_{\mathbb{T}}\partial_x(u^3)\,dx = 0.\end{aligned} \tag{5.4}$$

Next we define the (real-valued) function

$$U \doteq e^{\delta|D_x|}u, \quad \text{which gives} \quad \|U(t)\|^2_{L^2(\mathbb{T})} = \|u(t)\|^2_{G^{\delta,0}(\mathbb{T})}. \tag{5.5}$$

Since u satisfies KdV, we see that U satisfies the forced KdV

$$\partial_t U + \partial_x^3 U + U\partial_x U = F,$$

where

$$F = \frac{1}{2}\partial_x\big((e^{\delta|D_x|}u)\cdot(e^{\delta|D_x|}u) - e^{\delta|D_x|}(u\cdot u)\big).$$

Doing for U similar L^2-conservation computations we get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}}U^2dx = \int_{\mathbb{T}}UFdx. \tag{5.6}$$

Integrating in time from 0 to $t \in [0,T]$, we get

$$\|u(t)\|^2_{G^{\delta,0}(\mathbb{T})} \le \|u(0)\|^2_{G^{\delta,0}(\mathbb{T})} + 2\left|\int_0^t\int_{\mathbb{T}}UF(x,t')\,dx\,dt'\right|. \tag{5.7}$$

Now applying the Cauchy–Schwarz inequality we can get

$$\left|\int_0^t\int_{\mathbb{T}}UF(x,t')\,dx\,dt'\right| \le C\|U\|_{Y_I^0}\|F\|_{Z_I^0}, \quad I=[0,t]. \tag{5.8}$$

And, using the KdV bilinear estimates, which hold for $s \ge -\frac{1}{2}$, we get

$$\|F\|_{Z_I^0} \le C\delta^{1/2}\|U\|^2_{Y_I^0}. \tag{5.9}$$

Also, using the solution size estimate from the local theorem, we get

$$\|U\|_{Y_I^0} = \|u\|_{Y_I^{\delta,0}} \le C\|u(0)\|_{G^{\delta,0}(\mathbb{T})}. \tag{5.10}$$

Finally, the above relations and letting $T = \rho$, we get the almost conservation law

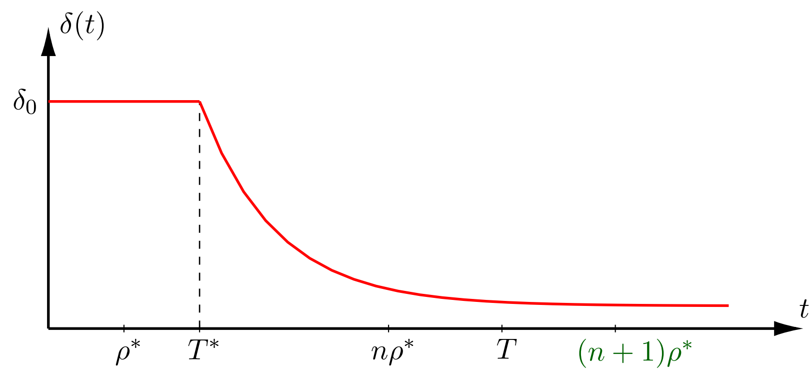
$$\sup_{|t|\le\rho}||u(t)||^2_{\delta,0} \le ||u(0)||^2_{\delta,0} + C\delta^{1/2}||u(0)||^3_{\delta,0}.$$

Outline of main theorem proof for $s = 0$ ($t \ge 0$). Denote by T^* the maximal lifespan for which the solution corresponding to the initial data $u_0 \in G^{\delta_0,0}(\mathbb{T})$ remains in $G^{\delta_0,0}(\mathbb{T})$. If $T^* = \infty$ then $r(t) = \delta_0$ and we are done. Otherwise,

$$\begin{aligned}\infty > T^* \ge T_0 &\doteq \frac{c_0}{\left(1+\|u_0\|_{G^{\delta_0,0}(\mathbb{T})}\right)^a} \\ &> \frac{c_0}{\left(1+2\|u_0\|_{G^{\delta_0,0}(\mathbb{T})}\right)^a} \doteq \rho^*.\end{aligned} \tag{5.11}$$

Now, for any given $T > T^*$ there is $n \in \mathbb{N}$ such that

$$n\rho^* \le T < (n+1)\rho^*. \tag{5.12}$$



Claim: *If for δ the conditions*

$$0 < \delta \le \delta_0 \quad \text{and} \quad (n+1)C\delta^{1/2}2^{3/2}\|u_0\|_{G^{\delta_0,0}(\mathbb{T})} \le 1 \tag{5.13}$$

hold, then for each $k \in \{1, 2, \ldots, n+1\}$ we have that

$$\sup_{t\in[0,k\rho^*]} ||u(t)||^2_{\delta,0} \le ||u(0)||^2_{\delta,0} + kC\delta^{1/2}2^{\frac{3}{2}}||u(0)||^3_{\delta_0,0} \tag{5.14}$$

and

$$\sup_{t\in[0,k\rho^*]} ||u(t)||^2_{\delta,0} \le 2||u(0)||^2_{\delta_0,0}. \tag{5.15}$$

This claim is proved by induction, where the first step is provided by the almost conservation law. Applying estimate (5.15) with $k = n + 1$ we get

$$\sup_{t\in[0,(n+1)\rho^*]} ||u(t)||^2_{\delta,0} \le 2||u(0)||^2_{\delta_0,0}, \tag{5.16}$$

if δ-size conditions (5.13) hold. In other words we have proved the implication

$$0 < \delta \le \delta_0 \quad \textit{and} \quad (n+1)C\delta^{1/2}2^{\frac{3}{2}}||u(0)||_{\delta_0,0} \le 1 \implies u \in C([0,(n+1)\rho^*], G^{\delta,0}(\mathbb{T})). \tag{5.17}$$

Since $n\rho^* \le T < (n+1)\rho^*$ we have that

$$n + 1 \le \frac{T}{\rho^*} + 1 < \frac{T}{\rho^*} + \frac{T}{\rho^*} = \frac{2T}{\rho^*}.$$

Therefore, the second δ-size condition (5.13) holds if we chose δ to satisfy the condition

$$\frac{2T}{\rho^*}C\delta^{1/2}2^{\frac{3}{2}}||u(0)||_{\delta_0,0} \le 1 \iff \delta \le cT^{-2} \quad \text{where } c = \Big(\frac{\rho^*}{2C2^{3/2}||u(0)||_{\delta_0,0}}\Big)^2. \tag{5.18}$$

Furthermore, since $[0,T] \subset [0,(n+1)\rho^*]$, from (5.17) and (5.18) we conclude that if $0 < \delta \le \delta_0$ and $0 < \delta \le cT^{-2}$ then $u \in C([0,T], G^{\delta,0}(\mathbb{T}))$. Choosing the biggest δ satisfying both conditions gives

$$u \in C\big([0,T], G^{\delta(T),0}(\mathbb{T})\big) \quad \text{with} \quad \delta(T) = \min\{\delta_0, cT^{-2}\}, \quad T > 0, \tag{5.19}$$

which completes the proof of the theorem in the case $s = 0$. The general case $s \ge -1/2$ is reduced to the case $s = 0$.

Open Question. Does the KdV radius of analyticity persist for all time?

5.2 Time regularity for KdV and non-analytic solutions

In 1977 Trubowitz [71] proved that a spatially periodic solution of the KdV equation, $\partial_t u = 3u\partial_x u - \frac{1}{2}\partial_{xxx}u$, which is initially real analytic is spatially real analytic for all time. In 1986 Kato and Masuda [47] showed that if the initial state of the KdV type equation $\partial_t u = -\partial_x^3 u - a(u)\partial_x u$, where $x \in \mathbb{R}$, $t \ge 0$, and $a(\lambda)$ is real analytic in $\lambda \in \mathbb{R}$, has a analytic continuation that is analytic and L^2 in a strip containing the real axis, then the solution has the same property for all time, though the width of the strip might decrease with time. Here we will show that analyticity in time variable of the solutions to the KdV equation fails. However, we will show that in time the solution belongs to G^3 for initial data analytic. More precisely, for the Cauchy problem for the KdV equation

$$\begin{cases} \partial_t u = \partial_x^3 u + u\partial_x u, \\ u(0,x) = u_0(x), \quad t \in \mathbb{R}, \quad x \in \mathbb{T} \text{ or } \mathbb{R}, \end{cases} \tag{5.20}$$

following [15], we shall present examples demonstrating the non-analyticity of the solution in time.

Theorem 5.4 ([15]). *The solution to the KdV initial valaue problem (5.20) with initial data analytic may not be analytic in the time variable t. More precisely, in the periodic case, if*

$$\varphi(x) = \frac{-e^{ix}}{2-e^{ix}} = -\sum_{k=1}^{\infty} 2^{-k} e^{ikx}, \tag{5.21}$$

then u is not analytic in t near $t = 0$. While in the non-periodic case, if

$$\varphi(x) = (i-x)^{-2}, \tag{5.22}$$

then u is not analytic in t near $t = 0$. Finally, if we replace $\varphi(x)$ with its real part then we obtain a real-valued solution u which is not analytic in $t = 0$.

Proof The main tool of the proof is the following result, which proof can be done by induction.

Lemma 5.1 *If $u(x, t)$ is a solution to the initial value problem (5.20), then*

$$\partial_t^j u = \partial_x^{3j} + \sum_{|\alpha|+2\ell=3j+2} C_\alpha(\partial_x^{\alpha_1} u)\cdots(\partial_x^{\alpha_\ell} u), \quad \text{with } C_\alpha \geq 0. \tag{5.23}$$

Since for the periodic case we have $\varphi(x) = \frac{-e^{ix}}{2-e^{ix}} = -\sum_{k=1}^{\infty} 2^{-k} e^{ikx}$ then

$$\varphi^{(n)}(x) = -\sum_{k=1}^{\infty} 2^{-k}(ik)^n e^{ikx} \tag{5.24}$$

and therefore

$$\varphi^{(n)}(0) = i^{n+2} A_n \tag{5.25}$$

where

$$A_n = \sum_{k=1}^{\infty} 2^{-k} k^n > 2^{-n} n^n. \tag{5.26}$$

Let $u(x, t)$ be a solution to the initial value problem (5.20) with initial data $\varphi(x) = -\sum_{k=1}^{\infty} 2^{-k} e^{ikx}$. Then by (5.23), we have

$$\begin{aligned} &\partial_t u(0,0) \\ &\quad = \varphi^{3j}(0) + \sum_{|\alpha|+2\ell=3j+2} C_\alpha(\varphi^{\alpha_1}(0))\cdots(\varphi^{\alpha_\ell}(0)) \\ &\quad = \Big(A_{3j} + \sum_{|\alpha|+2\ell=3j+2} C_\alpha A_{\alpha_1}\cdots A_{\alpha_\ell}\Big) i^{3j+2}, \end{aligned} \tag{5.27}$$

and by (5.26), we have that for any j,

$$|\partial_t^j u(0,0)| \geq A_{3j} > 2^{-3j}(3j)^{3j} > (j!)^2. \tag{5.28}$$

Therefore $u(x, t)$ is not analytic in the t variable at the point $(0, 0)$. The proof of the other cases is similiar. □

5.3 G^3 regularity in time for the KdV

For the Airy equation $\partial_t u + \partial_x^3 u = 0$, we see that one time-derivative is equal three space-derivatives. Therefore, if the solution is analytic in x, that is $\partial_x^k u$ grow like $k!$, then the time derivatives $\partial_t^k u$ grow like $(3k)!$. This means that the solution is in Gevrey class 3 in time. Following [34], here we prove that this phenomenon is also true for the KdV equation.

Theorem 5.5 ([34]). *The solution $u(x, t)$ to the periodic KdV initial value problem (5.20) with analytic initial data belongs to G^3 in the time variable t.*

Proof We already know that the solution $u(x, t)$ is analytic in the spatial variable (see [71] and [31]). We shall use the analyticity estimates obtained in [31] to complete the proof. More precisely, there exist $C > 0$ and $\delta > 0$ such that

$$|\partial_x^k u(x,t)| \leq C^{k+1} k!, \ k = 0,1,2,\dots, \ t \in (-\delta, \delta), \quad x \in \mathbb{T}. \tag{5.29}$$

In order to prove Theorem 5.5 it is enough to prove the following

Lemma 5.2 *For $k = 0, 1, \dots$ and $j = 0, 1, 2, \dots$ the following inequality holds true*

$$\left|\partial_t^j \partial_x^k u(x,t)\right| \leq C^{k+j+1}(k+3j)!(C^2 + C/2)^j, \quad \text{for} \quad t \in (-\delta,\delta), \quad x \in \mathbb{T}. \tag{5.30}$$

In fact, taking $k = 0$ we obtain

$$\left|\partial_t^j u(x,t)\right| \leq C^{j+1}(3j)!(C^2 + C/2)^j \leq C_1^{j+1}(j!)^3$$

and therefore we can conclude that u is G^3 in time t variable for all $x \in \mathbb{T}$.

Outline of the proof of Lemma 5.2. We will prove it by using induction on j. For $j = 0$ inequality (5.30) holds for all $k \in \{0, 1, 2, \dots\}$ since it is nothing else but inequality (5.29). For $j = 1$, $k \in \{0, 1, 2, \dots\}$ and by using the KdV equation we obtain

$$\begin{aligned} \partial_t \partial_x^k u &= \partial_x^{k+3} u + \partial_x^k(u\partial_x u) \\ &= \partial_x^{k+3} u + \sum_{p=0}^{k} \binom{k}{p} \partial_x^{k-p} u \partial_x^{p+1} u. \end{aligned} \tag{5.31}$$

First, from (5.29) we obtain that

$$|\partial_x^{k+3}u(x,t)| \le C^{k+3+1}(k+3)! \le C^{k+1+1}(k+3\cdot 1)!C^2,$$
$$t \in (-\delta,\delta),\ \ x \in \mathbb{T}. \tag{5.32}$$

Now we notice that

$$\begin{aligned}
&\left|\sum_{p=0}^{k}\binom{k}{p}\partial_x^{k-p}u\partial_x^{p+1}u\right| \\
&\le \sum_{p=0}^{k}\frac{k!}{p!(k-p)!}C^{k-p+1}(k-p)!C^{p+1+1}(p+1)! \\
&= C^{k+3}k!\sum_{p=0}^{k}(p+1) = C^{k+3}k!(k+1)(k+2)/2 \\
&= C^{k+3}(k+2)!/2 = C^{k+1+1}(k+2)!C/2 \\
&\le C^{k+1+1}(k+3)!C/2,
\end{aligned} \tag{5.33}$$

for $t \in (-\delta,\delta)$, $x \in \mathbb{T}$, where we have used the fact that $\sum_{p=0}^{k}(p+1) = (k+1)(k+2)/2$. It follows from (5.32) and (5.33) that

$$|\partial_t\partial_x^k u(x,t)| \le C^{k+1+1}(k+3.1)!(C^2 + C/2),$$

for $t \in (-\delta,\delta)$, $x \in \mathbb{T}$, which complete the proof in this case.

Now supposing that (5.30) holds for all derivatives in t of order $\le j$ and $k \in \{0,1,2,\dots\}$, and following the lines of what we have done in the first step, i.e., $j=1$, we are able to prove that (5.30) holds for $j+1$ and $k \in \{0,1,2,\dots\}$. The proof is complete. □

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