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RESEARCH EXPOSITION

Jordan Groups and Geometric Properties of Manifolds

Tatiana Bandman[1] · **Yuri G. Zarhin**[2]

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Abstract
The aim of this note is to draw attention to recent results about the so called Jordan property of groups. (The name was motivated by a classical theorem of Jordan about finite subgroups of matrix groups). We explore interrelations between geometric properties of complex projective varieties and compact Kähler manifolds and the Jordan property (or the lack of it) of their automorphism groups of birational and biregular selfmaps, and of bimeromorphic and biholomorphic maps, respectively.

Keywords Automorphism groups of compact complex manifolds · Complex tori · Conic bundles · Jordan properties of groups

Mathematics Subject Classification 32M05 · 32M18 · 14E07 · 32L05 · 32J18 · 32J27 · 14J50 · 57S25

1 Introduction

The aim of this note is to draw attention to the so called *Jordan property* of groups that was recently actively studied. The property was explicitly formulated by Jean-Pierre Serre and Vladimir Popov in this century, and the name goes back to a classical result of Jordan [14] about finite subgroups of complex matrix groups. Though defined for arbitrary groups, in special situations it bears a strong geometric meaning. A more detailed review on this topic may be found in [11].

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✉ Yuri G. Zarhin
zarhin@math.psu.edu

Tatiana Bandman
bandman@math.biu.ac.il

1 Department of Mathematics, Bar-Ilan University, 5290002 Ramat Gan, Israel

2 Department of Mathematics, Pennsylvania State University, University Park, PA 16802, USA

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We will use the standard notation $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{C}$ for the set of positive integers, the ring of integers, the fields of rational and complex numbers, respectively. If q is a prime (or a prime power) then we write $\mathbb{F}_q$ for the (finite) q-element field. In this note we consider the following groups.

- $\mathrm{Bir}(X)$ of all birational self-maps of an irreducible complex algebraic variety X;
- $\mathrm{Bim}(X)$ of all bimeromorphic self-maps of a connected complex manifold X;
- $\mathrm{Diff}(X)$ of all diffeormorphisms of a smooth real manifold X;
- $\mathrm{Aut}_{an}(X)$ and $\mathrm{Aut}(X)$ of all automorphisms of complex or algebraic variety, respectively.

Remark 1 If X is a smooth projective variety over the field of complex numbers then $\mathrm{Bim}(X) = \mathrm{Bir}(X)$. In addition, $\mathrm{Aut}_{an}(X) = \mathrm{Aut}(X)$; we will denote both groups as $\mathrm{Aut}(X)$ when no confusion can arise.

Sometimes these groups are finite; for example, $\mathrm{Bim}(X)$ is finite if X is a compact connected complex manifold of general type (i.e., it has maximal possible Kodaira dimension $\varkappa(X) = \dim X$ [17]). However, in general, the groups $\mathrm{Bim}(X)$ may be infinite and non-algebraic. One of the most interesting and important examples of such groups in birational geometry is the *Cremona group* $\mathrm{Cr}_n = \mathrm{Bir}(\mathbb{P}^n)$ where $\mathbb{P}^n$ is the n-dimensional complex projective space. If $n \geq 2$, then Cr_n is a huge non-abelian non-algebraic group. To understand the structure of such groups one is tempted to consider their less complicated subgroups: finite, abelian or their combinations. This is where the Jordan properties come in.

Definition 1 A group G is called *Jordan* if there is a finite positive integer J such that every finite subgroup B of G contains an abelian subgroup A that is normal in B and such that the index $[B : A] \leq J$. The smallest such J is called the *Jordan constant* of G, denoted by J_G, [35, Question 6.1], [22, Definition 2.1].

The study of Jordan properties was inspired by the following fundamental results of Jordan and Serre (see [14], [36, Theorem 9.9], and [35, Theorem 5.3] respectively).

Theorem 1 (Theorem of Jordan) *The group* $\mathrm{GL}_n = \mathrm{GL}_n(\mathbb{C})$ *is Jordan.*

Theorem 2 (Theorem of Serre) *The Cremona group* $\mathrm{Cr}_2 = \mathrm{Bir}(\mathbb{P}^2)$ *is Jordan,* $J_{\mathrm{Cr}_2} \leq 2^{10}3^4 5^2 7$.

(Later the exact value $J_{\mathrm{Cr}_2} = 7200$ was found by Yasinsky [41].)

Example 1 It follows from Theorem 1 that every linear algebraic group over any field of characteristic zero is Jordan. Moreover, every connected real (or complex) Lie group is Jordan (Popov [24]).

Example 2 It is well known that GL_n contains a subgroup of order $(n+1)!$ that is isomorphic to the full symmetric group $\mathbf{S}_{n+1}$ of permutations on $(n+1)$ letters. Indeed, permutations of the coordinates in $(n+1)$-dimensional vector space $\mathbb{C}^{n+1}$ leave invariant the hyperplane $H = \{\sum_1^{n+1} x_i = 0\} \cong \mathbb{C}^n$. If $n \geq 4$ then $n+1 \geq 5$ and $\mathbf{S}_{n+1}$ is a nonabelian group that does not contain a proper abelian normal subgroup.

(Actually, its only proper normal subgroup is the alternating group $\mathbf{A}_{n+1}$ that is simple nonabelian.) This implies that if $n \geq 4$ then

$$J_{\mathrm{GL}_n} \geq J_{\mathbf{S}_{n+1}} = (n+1)!. \tag{1}$$

The equality holds if $n \geq 71$ or $n = 63, 65, 67, 69$ [12].

Example 3 Finite subgroups of the group $\mathrm{GL}_2 = \mathrm{GL}_2(\mathbb{C})$ were classified in the XIX century [16] (see also [39, Chap. 3, Sect. 6]). In particular, GL_2 contains a subgroup of order 120 that is isomorphic to $\mathrm{SL}(2, \mathbb{F}_5)$. Its largest abelian normal subgroup C consists of two scalars $\{1, -1\}$ (see below) and the corresponding quotient $\mathrm{SL}(2, \mathbb{F}_5)/C$ is isomorphic to the simple nonabelian alternating group $\mathbf{A}_5$.

It follows that $J_{\mathrm{GL}_2(\mathbb{C}))} \geq 60$. Actually, $J_{\mathrm{GL}_2(\mathbb{C}))} = 60$.

Example 4 (Example of a non-Jordan group) Let p be a prime and $\overline{\mathbb{F}}_p$ an algebraic closure of the field $\mathbb{F}_p$. Then $\mathrm{SL}(2, \overline{\mathbb{F}}_p)$ is **not** Jordan.

Indeed, if m is a positive integer and $q = p^m \geq 4$, then $\mathrm{SL}(2, \mathbb{F}_q) \subset \mathrm{SL}(2, \overline{\mathbb{F}}_p)$.

Recall that $\mathrm{SL}(2, \mathbb{F}_q)$ is a finite noncommutative group of order $(q^2 - 1)q$ such that its only proper normal subgroup $C \subsetneq \mathrm{SL}(2, \mathbb{F}_q)$ consists of one or two scalars.

Thus the values of indices

$$[\mathrm{SL}(2, \mathbb{F}_q) : C] = (q^2 - 1)q/2 \text{ or } (q^2 - 1)q$$

are unbounded when m tends to infinity. Hence $\mathrm{SL}(2, \overline{\mathbb{F}}_p)$ is *not* Jordan.

In his paper [22] Popov asked whether for any algebraic variety X the groups $\mathrm{Aut}(X)$ and $\mathrm{Bir}(X)$ are Jordan. This question stimulated an intensive and fruitful activity, see Sect. 2 below.

The following "Jordan properties" of groups are also very useful.

Definition 2 1. A group G is called *bounded* if the orders of its finite subgroups are bounded by a universal constant that depends only on G [22, Definition 2.9].
2. A Jordan group G is called *strongly Jordan* [7, 26] if there is a positive integer m such that every finite subgroup of G is generated by at most m elements.
3. A group G is called *very Jordan* [9] if there exist a commutative normal subgroup G_0 of G and a bounded group F that sit in a short exact sequence

$$1 \to G_0 \to G \to F \to 1. \tag{2}$$

Example 5 (Examples of bounded groups) The matrix group $\mathrm{GL}(n, \mathbb{Q})$ and its subgroup $\mathrm{GL}(n, \mathbb{Z})$ are bounded.

This is a celebrated result of Minkowski (1887), see [36, Sect. 9.1]. Actually, Minkowski gave an explicit upper bound $M(n)$ for the orders of finite subgroups of $\mathrm{GL}(n, \mathbb{Q})$ (ibid, see also [34]).

Example 6 The multiplicative group $\mathbb{C}^*$ of the field $\mathbb{C}$ is commutative, (hence, Jordan) but not bounded. The same is true for the group of translations of any complex torus of positive dimension.

Remark 2 1. Every finite group is bounded, Jordan, and very Jordan.
2. Every commutative group is Jordan and very Jordan.
3. Every finitely generated commutative group is bounded. Indeed, such a group is isomorphic to a finite direct sum with every summand isomorphic either to $\mathbb{Z}$ or to $\mathbb{Z}/n\mathbb{Z}$ where n is a positive integer.
4. A subgroup of a Jordan group is Jordan. A subgroup of a very Jordan group is very Jordan.
5. "Bounded" implies "very Jordan", "very Jordan" implies "Jordan".
6. "Bounded" implies "strongly Jordan." On the other hand, "very Jordan" does not imply "strongly Jordan." For example, a direct sum of infinitely many copies of $\mathbb{Z}/2\mathbb{Z}$ is commutative but has finite subgroups with any given minimal number of generators.

2 Jordan properties of groups Aut(*X*), Bir(*X*), Bim(*X*), and Diff(*X*)

In this section we sketch certain facts, methods and tools related to the study of the Jordan properties of groups arising from complex geometry.

Example 7 Let X be a smooth irreducible projective curve (Riemann surface) of genus g. Then $\mathrm{Aut}(X) = \mathrm{Bir}(X) = \mathrm{Bim}(X)$. We have:

- If $g > 1$ then $\mathrm{Aut}(X)$ is finite, hence bounded and Jordan.
- If $g = 0$ then $\mathrm{Aut}(X) = \mathrm{PGL}(2, \mathbb{C})$ is Jordan (by the Jordan Theorem), strongly Jordan, but not bounded and not very Jordan.
- If $g = 1$, i.e., X is an elliptic curve, then it is a commutative algebraic group that acts on itself by translations. Moreover, $X \subset \mathrm{Aut}(X)$ is a normal commutative subgroup of finite index, namely $[\mathrm{Aut}(X) : X] \leq 6$. It follows that $\mathrm{Aut}(X)$ is very Jordan, strongly Jordan, but not bounded.

Example 8 Winkelmann [40] and Popov [23] proved the existence of a connected non-compact Riemann surface M such that $\mathrm{Aut}(M)$ contains an isomorphic copy of every finitely presented (in particular, every finite) group G. In particular, $\mathrm{Diff}(M)$ is not Jordan.

Example 9 The automorphism group $\mathrm{Aut}(A)$ of an abelian variety A is strongly Jordan and very Jordan. Moreover, if d is a positive integer then there are universal constants $J(d)$ and $R(d)$ that depend only on d and such that if A is a d-dimensional abelian variety then every finite subgroup of $\mathrm{Aut}(A)$ may be generated by $r \leq R(d)$ elements and $J_A \leq J(d)$.

Proof Let L_A be a lattice in $\mathbb{C}^d$ such that $A = \mathbb{C}^d/L_A$. Thus A is isomorphic as a group to $(\mathbb{R}/\mathbb{Z})^{2d}$, hence every finite subgroup has at most $2d$ generators.

Let $T_A \subset \mathrm{Aut}(A)$ be the (sub)group of translations

$$t_a : A \to A, \ \to x + a, \ (a \in A).$$

Then T_A is isomorphic to A as a group. There is an exact sequence:

$$0 \to T_A \to \mathrm{Aut}(A) \to \mathrm{Aut}(L_A) \cong \mathrm{GL}(2d, \mathbb{Z}).$$

Since T_A is very abelian and the group $\mathrm{GL}(2d, \mathbb{Z})$ is bounded, $\mathrm{Aut}(A)$ is very Jordan and the corresponding constants are bounded by universal constants that depend only on d. □

As of today (June 2024), there are no examples of complex *algebraic varieties* (compact or non-compact) with non-Jordan $\mathrm{Aut}(X)$. If X is a compact complex connected manifold, then $\mathrm{Aut}(X)$ carries the natural structure of a (not necessarily connected) complex Lie group [5]. The identity component $\mathrm{Aut}_0(X)$ of $\mathrm{Aut}(X)$ is Jordan for every compact complex space X [24, Theorems 5 and 7].

The group $\mathrm{Aut}(X)/\,\mathrm{Aut}_0(X)$ of connected components of $\mathrm{Aut}(X)$ is bounded if X is Kähler [9, Proposition 1.4].

It is known that the group $\mathrm{Aut}(X)$ is Jordan if

- X is projective (Meng and Zheng [18]);
- X is a compact complex Kähler manifold (Kim [15]);
- X is a compact complex space in the Fujiki Class $\mathscr{C}$ (Meng et al. [19]; see also [29] for Moishezon threefolds).

Moreover, $\mathrm{Aut}(X)$ is very Jordan if the Kodaira dimension $\varkappa(X)$ of X is non-negative, or if X is a $\mathbb{P}^1$-bundle over a certain non-uniruled complex manifold [9–11].

Remark 3 Recall that the Kodaira dimension $\varkappa(X)$ is a numerical invariant of a variety X that can take on values $-\infty, 0, 1, 2, \ldots, \dim X$. As was already mentioned, if $\varkappa(X) = \dim\ X$, then X is called *a variety of general type*. Roughly speaking, it is rigid. For example, the group $\mathrm{Aut}(X)$ is finite, and the set of regular maps from any projective variety Y onto X is finite as well. It cannot be covered by a family of rational curves. At the other side of the spectrum ($\varkappa(X) = -\infty$) are, in particular, uniruled varieties. A compact complex variety X is *uniruled* if there exist a compact complex variety Y, a proper complex closed subspace $Z \subset Y$, and a meromorphic dominant map $f : Y \times \mathbb{P}^1 \to X$ such that $\dim(f(y \times \mathbb{P}^1)) = 1$ for any $y \in Y \backslash Z$. If $\dim X \leq 3$ then $\varkappa(X) = -\infty$ implies that X is uniruled. Any projective space is uniruled.

The structure of the groups $\mathrm{Bir}(X)$ and $\mathrm{Bim}(X)$ of birational and bimeromorphic selfmaps, respectively, is more complicated. It appears that uniruled varieties play a special role with respect to Jordan properties.

There are examples of

- a projective variety X_{pr} with non-Jordan group $\mathrm{Bir}(X_{pr})$, namely

$$X_{pr} := E \times \mathbb{P}^1$$

where E is any elliptic curve [42];

- a non-algebraic connected compact complex manifold X_c with non-Jordan group $\mathrm{Bim}(X_c)$:

$$X_c := T \times \mathbb{P}^1,$$

where T is any non-algebraic complex torus of positive algebraic dimension [43];
- a smooth compact real manifold M with non-Jordan group $\mathrm{Diff}(M)$ with M being the direct product of 2-dimensional real torus by 2-dimensional sphere (Csikós et al. [13]). Note that $\mathbb{P}^1$ is a 2-dimensional sphere as a real manifold.

All these examples are essentially the same. Let us note their main features: all those objects are

- uniruled (covered by rational curves);
- direct products with a torus T;
- a torus T carries no rational curves and the group T is an algebraic, commutative, not bounded group.

It seems that the Jordan property (or rather its absence) of the groups $\mathrm{Bir}(X)$, or $\mathrm{Bim}(X)$ for a complex manifold (or projective varietiy) X correlate with such geometric features as being uniruled over a non-uniruled positive dimensional base or being a direct product.

Let us illustrate it in the case of surfaces by the following assertion.

Theorem 3 [22] *If X is an irreducible projective surface then* $\mathrm{Bir}(X)$ *is Jordan unless X is birational to a product $E \times \mathbb{P}^1$ of an elliptic curve E and $\mathbb{P}^1$.*

Let us sketch the ideas involved in the proof. They are basic for this theory and, in a more sophisticated form, are widely used.

We will restrict ourselves to the smooth situation. Recall that a smooth surface X has a *minimal model* X_m (that is smooth and contains no (-1) curves, see, e.g., [38]).
Case 1. $\varkappa(X) \geq 0$. Then $\mathrm{Bir}(X) = \mathrm{Bir}(X_m) = \mathrm{Aut}(X_m)$.

Every automorphism $f \in \mathrm{Aut}(X_m)$ induces the automorphism $\psi(f)$ of the Néron-Severi group $\mathrm{NS}(X_m)$ (the group of connected components of $\mathrm{Pic}(X)$.) Let $G_i := \ker(\psi)$. This is a complex Lie group that may be included into the exact sequence:

$$0 \longrightarrow G_i \xrightarrow{i} \mathrm{Aut}(X_m) \xrightarrow{\psi} \mathrm{Aut}(\mathrm{NS}(X)). \tag{3}$$

It is known that

- G_i has finitely many connected components;
- the identity component G_i^0 of G_i is a connected algebraic group;
- Being a connected algebraic group, G_i^0 is Jordan;
- The Néron-Severi group $\mathrm{NS}(X)$ is a finitely generated abelian group; in particular, its torsion subgroup F is finite and the quotient $NS(X))/F$ is isomorphic to the free abelian group $\mathbb{Z}^\rho$ of finite (positive) rank ρ where ρ is the Picard number of X. This implies that the kernel of the natural homomorphism

$$\mathrm{Aut}(\mathrm{NS}(X)) \to \mathrm{Aut}(NS(X)/F) \cong \mathrm{GL}(\rho, \mathbb{Z})$$

is finite. By the theorem of Minkowski, $\mathrm{GL}(\rho, \mathbb{Z})$ is bounded. This implies that $\mathrm{Aut}(\mathrm{NS}(X))$ is bounded as well.

Now Eq. (3) implies that $\mathrm{Bir}(X) = \mathrm{Aut}(X_m)$ is *Jordan.*

Case 2. $\varkappa(X) = -\infty$

As was already mentioned, the case of $\mathrm{Cr}_2(\mathbb{C}) = \mathrm{Bir}(\mathbb{P}^2)$ is due to Serre (see Theorem 2 above).

If the surface is birational to a direct product $X_m := B \times \mathbb{P}^1$ of a curve B of genus $g \geq 1$ and the projective line then every birational automorphism $f \in \mathrm{Bir}(X_m) \cong \mathrm{Bir}(X)$ is fiberwise. It means that it can be included into the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \pi\downarrow & & \pi\downarrow \\ B & \xrightarrow{\tau(f)} & B \end{array}. \tag{4}$$

Here $\pi : X \to B$ is the natural projection and $\tau(f) \in \mathrm{Aut}(B)$.

The subgroup $G_0 = \{f \in \mathrm{Aut}(X_m) | \tau(f) = \mathrm{id}\} \subset \mathrm{PSL}(2, K)$, where $K = \mathbb{C}(B)$ is the field of rational functions on B, is Jordan.

Once more we have an exact sequence

$$0 \longrightarrow G_0 \xrightarrow{i} \mathrm{Aut}(X_m) \xrightarrow{\tau} G_B \tag{5}$$

where $G_B = \psi(\mathrm{Aut}(X_m)) \subset \mathrm{Aut}(B)$ is finite if genus $g > 1$.

Thus if the genus $g(B) > 1$ then Eq. (5) implies that $\mathrm{Bir}(X_m) \cong \mathrm{Bir}(X)$ is Jordan. □

The special case: X is birational to $E \times \mathbb{P}^1$ where E is an elliptic curve, is left.

Theorem 4 [42] *If X is birational to $E \times \mathbb{P}^1$ then* $\mathrm{Bir}(X)$ *is not Jordan.*

The proof of this Theorem is done in two steps. First, for every $N \in \mathbb{N}$ a certain group $\mathfrak{G}_N$ is constructed and its Jordan number is shown to be N. Then for every $N \in \mathbb{N}$ a surface S_N is built such that

- S_N is birational to $E \times \mathbb{P}^1$;
- $\mathrm{Aut}(S_N)$ contains a group $G_N \cong \mathfrak{G}_N$.

It follows that $\mathrm{Bir}(E \times \mathbb{P}^1)$ contains a subgroup G_N with $J_{G_N} = N$ for every $N \in \mathbb{N}$ thus is not Jordan. Let us give some details.

Step 1: Analogues of the Heisenberg groups that were used by Mumford [21]. Let

- $\mathbf{K}$ be a finite commutative group of order $N > 1$;
- $\mu_N \subset \mathbb{C}^*$ be the multiplicative group of Nth roots of unity;
- $\hat{\mathbf{K}} = \mathrm{Hom}(\mathbf{K}, \mu_N)$—the dual of $\mathbf{K}$.

The Mumford *theta group* $\mathfrak{G}_{\mathbf{K}}$ for $\mathbf{K}$ is the group of matrices of the type

$$\begin{pmatrix} 1 & \alpha & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix}$$

where $\alpha \in \hat{\mathbf{K}}$, $\gamma \in \mathbb{C}^*$, and $\beta \in \mathbf{K}$. The product $\alpha(\beta) \in \mathbb{C}^*$ of $\alpha \in \hat{\mathbf{K}}$ and $\beta \in \mathbf{K}$ is used in order to define a certain natural non-degenerate alternating bilinear form $e_{\mathbf{K}}$ on $\mathbf{H}_{\mathbf{K}} = \mathbf{K} \times \hat{\mathbf{K}}$ with values in $\mathbb{C}^*$ [42, p. 302]. This group may be included into a short exact sequence

$$1 \to \mathbb{C}^* \to \mathfrak{G}_{\mathbf{K}} \to \mathbf{H}_{\mathbf{K}} \to 1$$

where the image of $\mathbb{C}^*$ is the center of $\mathfrak{G}_{\mathbf{K}}$. These groups are Jordan and

$$J_{\mathfrak{G}_{\mathbf{K}}} = \sqrt{\#(\mathbf{H}_{\mathbf{K}})} = N = \#(\mathbf{K}).$$

In particular, let us put $\mathfrak{G}_N := \mathfrak{G}_{\mathbb{Z}/N\mathbb{Z}}$, i.e., $K = \mathbb{Z}/N\mathbb{Z}$. Then $J_{\mathfrak{G}_N} = N$.

Step 2: Constructing surfaces S_N.

Fix a point $P \in E$ and denote by $[P]$ the corresponding divisor on E. Choose an integer $N > 1$ and consider the divisor $N[P]$ on E. Let $L_{N[P]}$ be the holomorphic line bundle on E that corresponds to $N[P]$. Let $\mathscr{L}_N$ be the total space of the line bundle $L_{N[P]}$. Let $S_N = \overline{\mathscr{L}_N}$ be its *projective closure/compactification*, i.e., $S_N = \mathscr{L}_N \cup \mathscr{T}_\infty$, where $\mathscr{T}_\infty$ is the "infinite" section of $L_{N[P]}$. Actually, $\overline{\mathscr{L}_N}$ is the $\mathbb{P}^1$-bundle over E that is the projectivization of the rank two vector bundle $L_N \oplus \mathbf{1}_E$, where $\mathbf{1}_E = E \times \mathbb{C}$ is the trivial line bundle over E. Thus, S_N is a ruled surface birational to $E \times \mathbb{P}^1$.

Let $G(N)$ be the subgroup of all those $f \in \mathrm{Aut}(S_N)$ that may be included into the following commutative diagram:

$$\begin{array}{ccc} \overline{L_N} & \xrightarrow{f} & \overline{L_N} \\ {\scriptstyle p}\downarrow & & {\scriptstyle p}\downarrow \\ E & \xrightarrow{T_Q} & E \end{array}$$

Here $p : S_N \to E$ is the natural projection, $E(N)$ stands for the subgroup of points in E of order dividing N, point $Q \in E[N]$ is a point of order dividing N, and $T_Q : E \to E$ is the translation map $e \to e + Q$. Moreover, f induces $\mathbb{C}$-linear isomorphisms between the fibers of p over e and $e + Q$.

On $E \times \mathbb{P}^1$ elements of the group $G(N)$ induce birational maps and form a subgroup $G_N \subset \mathrm{Bir}(E \times \mathbb{P}^1)$ that may be described as follows.

$G(N) = \{(Q, f), Q \in E(N), f \in \mathbb{C}(E)^*$ such that $(f) = N[P + Q] - N[P]\}$ is acting as

$$(y, t) \in E \times \mathbb{P}^1 \longrightarrow (Q, f)(y, t) = (Q + y, f(y)t).$$

Here (f) is the divisor of a rational function f.

By a result of Mumford [21, Sect. 1, Corollary of Theorem 1] that the group G_N is isomorphic to $\mathfrak{G}_N$; hence $J_{G_N} = N$. Thus, $J_{\mathrm{Bir}(E\times\mathbb{P}^1)} \geq J_{G_N} = N$ for all N, i.e., $\mathrm{Bir}(E \times \mathbb{P}^1)$ is *not* Jordan.

Based on the proof of the non-Jordanness of $\mathrm{Bir}(E \times \mathbb{P}^1)$ [42], Csikós et al. [13] constructed a counterexample to

Conjecture of Ghys (1997, see [23], Conjecture 3) If M is a connected compact smooth real manifold then $\mathrm{Diff}(M)$ is Jordan.

Let us describe their counterexample. From the *real* point of view, $\mathbb{P}^1$ is the two-dimensional sphere $\mathbb{S}^2$, E is the two-dimensonal real torus $\mathbb{T}^2$, and S_N is an oriented $\mathbb{S}^2$-bundle over $\mathbb{T}^2$.

As a smooth manifold, S_N is diffeomorphic to the product $\mathbb{T}^2 \times \mathbb{S}^2$ if and only if N is *even*. Therefore for each even N we have

$$G_N \hookrightarrow \mathrm{Diff}\left(\mathbb{T}^2 \times \mathbb{S}^2\right).$$

Since the set of J_{G_N} for positive even integers N is unbounded, the group $\mathrm{Diff}(\mathbb{T}^2 \times \mathbb{S}^2)$ is *not* Jordan.

Remark 4 If X is a complex compact surface with non-negative Kodaira dimension then $\mathrm{Bir}(X)$ is even bounded unless it is one of the following [30, Theorem 1.1]:

- a complex torus (in particular an abelian surface);
- a bielliptic surface;
- S_{K1}—a surface of Kodaira dimension 1;
- S_K—a Kodaira surface (it is not a Kähler surface). See [30, Theorem 1.1].

Moreover [28], if X is a projective threefold, then $\mathrm{Bir}(X)$ is not Jordan if and only if X is birational to a direct product $E \times \mathbb{P}^2$ or $S \times \mathbb{P}^1$, where a surface S is one of the surfaces listed above in this Remark.

For complex projective varieties Yu. Prokhorov and C. Shramov, and C. Birkar proved the following

Theorem 5 *Let X be a projective irreducible variety of dimension n. Then the following hold.*

(i) *The group* $\mathrm{Bir}(X)$ *is bounded provided that X is non-uniruled and has irregularity* $q(X) = 0$ *[26, Theorem 1.8].*
(ii) *The group* $\mathrm{Bir}(X)$ *is Jordan provided that X is non-uniruled [26, Theorem 1.8].*
(iii) *The group* $\mathrm{Bir}(X)$ *is Jordan provided that X has irregularity* $q(X) = 0$ *[26, Theorem 1.8], [3].*

Here $q(X) = \dim_{\mathbb{C}} H^1(X, \mathcal{O}_X)$ is the irregularity of X. In particular, the Cremona group Cr_n of any rank n is Jordan [3, 27].

The group $\mathrm{Diff}(M)$ of all diffeomorphisms of a smooth manifold M also appeared to be Jordan for certain classes of manifolds.

Namely, Zimmerman [44] proved that if M is compact and $\dim(M) \leq 3$ then $\mathrm{Diff}(M)$ is Jordan. The Jordan property of $\mathrm{Diff}(M)$ was studied by I. Mundet i Riera. In particular, he proved [20] that $\mathrm{Diff}(M)$ is Jordan if M is one of the following:

1. open acyclic manifolds,
2. compact manifolds (possibly with boundary) with nonzero Euler characteristic,

3. homology spheres.

So, in high dimensions the situation is very similar: the group $\mathrm{Bim}(X)$ or $\mathrm{Bir}(X)$ is mostly Jordan, and the worst case from the Jordan properties point of view is the following: a uniruled variety X with $q(X) > 0$ (or fibered over a non-uniruled base) that has many sections (such as a direct product). A typical example of such a variety X is a $\mathbb{P}^1$-bundle over a complex torus T of positive dimension.

The need of "many sections" may be demonstrated by the case of projective non-trivial conic bundles.

Definition 3 A regular surjective map $f : X \to Y$ of smooth irreducible projective complex varieties is a *conic bundle* over Y if the generic fiber $\mathcal{X} := \mathcal{X}_f$ is an absolutely irreducible curve over $k(Y)$ with genus 0 (see [32, 33].)

Recall that the *generic fiber* of f is an irreducible smooth projective curve $\mathcal{X}_f$ over the field $K := \mathbb{C}(Y)$ such that its field of rational functions $K(\mathcal{X}_f)$ coincides with $\mathbb{C}(X)$. Notice that K-points in $\mathcal{X}_f$ correspond to a rational sections of the conic bundle $f : X \to Y$. If such a K-point exists, then $\mathcal{X}_f$ is isomorphic over K to the projective line $\mathbb{P}^1_K$ and X is birational to $Y \times \mathbb{P}^1$ (over $\mathbb{C}$).

Remark 5 There are different definitions of a notion of conic bundle. The classical one is *three-dimensional quadric bundle* over $\mathbb{P}^2$ (see [1, Definition 1.1], [2]). Yu. Prokhorov in [25, Definition 3.1] defines a conic bundle as a a proper flat morphism of nonsingular varieties $\pi : X \to Y$ such that it is of relative dimension 1 and the anticanonical divisor $-K_X$ is relatively ample.

Theorem 6 [7] *Let X be a conic bundle over a non-uniruled smooth irreducible projective variety Y with* $\dim(Y) \geq 2$. *If X is not birational to $Y \times \mathbb{P}^1$ then* $\mathrm{Bir}(X)$ *is Jordan.*

Let us sketch the proof.

If $f : X \to Y$ is a conic bundle and Y is non-uniruled, then every $\phi \in \mathrm{Bir}(X)$ is fiberwise [see (4)].

It follows that there is an exact sequence of groups:

$$0 \to \mathrm{Bir}_{\mathbb{C}(Y)}(\mathcal{X}_f) \to \mathrm{Bir}(X) \to \mathrm{Bir}(Y); \tag{6}$$

Since Y is non-uniruled, the group $\mathrm{Bir}(Y)$ is Jordan, thanks to Theorem 5. Moreover, it is strongly Jordan (see [7, Corollary 3.8 and its proof]). Let us compute $\mathrm{Bir}_K(\mathcal{X}_f)$ (recall that $K = \mathbb{C}(Y)$). We have

1. $\mathrm{Bir}(\mathcal{X}_f) = \mathrm{Aut}(\mathcal{X}_f)$, since $\dim(\mathcal{X}_f) = 1$.
2. Since X is *not* birational to $Y \times \mathbb{P}^1$, the genus 0 curve $\mathcal{X}_f$ has **no** K-points and therefore there exists a ternary quadratic form

$$q(T) = a_1 T_1^2 + a_2 T_2^2 + a_3 T_3^2$$

over K such that

- all a_i are nonzero elements of K;
- $q(T) = 0$ if and only if $T = (0, 0, 0)$) (this means that q is *anisotropic*);
- $\mathcal{X}_f$ is biregular over K to the plane projective quadric

$$\mathbf{X}_q := \{(T_1 : T_2 : T_3) \mid q(T) = 0\} \subset \mathbb{P}^2_K.$$

3. K is a field of characteristic zero that contains all roots of unity.

Now we can use the following fact that was proven in [7]).

Theorem 7 [7] *Suppose that K is a field of characteristic zero that contains all roots of unity, $d \geq 3$ an odd integer, V a d-dimensional K-vector space and let $q : V \to K$ be a quadratic form such that $q(v) \neq 0$ for all nonzero $v \in V$. Let us consider the projective quadric $X_q \subset \mathbb{P}(V)$ defined by the equation $q = 0$, which is a smooth projective irreducible $(d-2)$-dimensional variety over K. Let $\mathrm{Aut}(X_q)$ be the group of biregular automorphisms of X_q. Let G be a finite subgroup in $\mathrm{Aut}(X_q)$. Then G is commutative, all its non-identity elements have order 2 and the order of G divides 2^{d-1}.*

(See [37] where a variant of Theorem 7 was later proven for anisotropic reductive K-groups.)

Thus if G is a nontrivial finite subgroup of $\mathrm{Aut}(\mathcal{X}_f)$ then either $G \cong \mathbb{Z}/2\mathbb{Z}$ or $G \cong (\mathbb{Z}/2\mathbb{Z})^2$.

Applying Eq. (6), we get that $\mathrm{Bir}(X)$ is Jordan.

We summarize now what we know about the Jordan properties when X is a $\mathbb{P}^1$-bundle over a complex torus T of positive dimension n. First, let us recall basic facts about complex tori [4].

For a complex torus T there exists its *algebraic model* T_0 such that:

- T_0 is an abelian variety;
- there is a holomorphic surjective homomorphism $p : T \to T_0$ with connected kernel that is *universal* in a sense that every homomorphism from T to any abelian variety factors uniquely through p;
- the field $\mathbb{C}(T)$ of meromorphic functions on T coincides with $p^*(\mathbb{C}(T_0))$, i.e., every meromorphic function on T is the lift of a rational function on T_0;
- by definition, the algebraic dimension $a(T)$ is $\dim_{\mathbb{C}} T_0$.

Now we are ready to state our

Summary.

1. We may consider T as a real manifold T_r. It follows from the counterexample to the Ghys Conjecture that
   If $\dim_{\mathbb{R}}(T_r) \geq 2$ and $X = \mathbb{S}^2 \times T_r$ then $\mathrm{Diff}(X)$ is not Jordan.
2. Since T is a complex torus, it is a connected compact Kähler manifold.

   2.1 Suppose that $a(T) = \dim(T) = n$. This means that T is algebraic, i.e., is an abelian variety. If $X = \mathbb{P}^1 \times T$ then $\mathrm{Bir}(X)$ is not Jordan (see Theorem 4). If X is not birational to $\mathbb{P}^1 \times T$ then $\mathrm{Bir}(X)$ is Jordan (see Theorem 6).

   2.2 Suppose that $0 < a(T) < n$. Then T is a non-algebraic torus and $n > 1$. (In dimension 1 all complex tori are algebraic—they are the famous elliptic curves.) If $X = \mathbb{P}^1 \times T$ (or has at least three sections) then $\mathrm{Bim}(X)$ is not Jordan [43].

2.3 Suppose that $a(T) = 0$. Then $n \geq 2$ and T is non-algebraic. This is a "very general" case: in a "versal" family [4] of all complex tori of a given dimension $n \geq 2$ the subset of tori with algebraic dimension zero is dense. (See [9] for explicit examples of such tori in all dimensions $n \geq 2$.) If $a(T) = 0$ then any $\mathbb{P}^1$-bundle X over T that is not biholomorphic to the direct product $\mathbb{P}^1 \times T$ has at most two sections and $\mathrm{Bim}(X) = \mathrm{Aut}(X)$ is Jordan [9].

3 Some open problems

Let us mention some open problems. Fix a positive integer n.

Varieties with non-Jordan group $\mathrm{Bir}(X)$. Let $\mathscr{V}_n$ and $\mathscr{X}_n$ be the class of connected complex projective varieties V (respectively, complex compact manifolds X) of dimension n such that the group $\mathrm{Bir}(V)$ (respectively, $\mathrm{Bim}(X)$ is not Jordan. For $n \leq 3$ these classes are well described (see [22, 26, 28–31, 42]). It is known that $A \times \mathbb{P}^n \in \mathscr{V}_{n+k}$ if A is an abelian variety of positive dimension k, and $T \times \mathbb{P}^n \in \mathscr{X}_{n+k}$ if T is a complex torus of dimension k and positive algebraic dimension.

Question 1 Assume that V is a non-uniruled smooth projective variety and $Y = V \times \mathbb{P}^n$. Is $\mathrm{Bir}(Y)$ non-Jordan? More generally, how to describe $\mathscr{V}_n$ and $\mathscr{X}_n$?

Quasiprojective varieties Assume that W is a smooth quasiprojective variety that is an open subset of a smooth projective variety X. Then

$$\mathrm{Aut}(W) \subset \mathrm{Bir}(X).$$

If $\mathrm{Bir}(X)$ is not Jordan, then, *a priori*, the same may be true for $\mathrm{Aut}(W)$. However, to the best of our knowledge there is no example of a complex algebraic variety W with non-Jordan $\mathrm{Aut}(W)$. It is known that $\mathrm{Aut}(W)$ is Jordan if either

- $\dim W \leq 3$ and W is *not* birational to $E \times \mathbb{P}^2$, where E is an elliptic curve [6] or
- W is quasiprojective and birational to a product $A \times \mathbb{P}^1$, where A is a smooth irreducible positive-dimensional projective variety that contains no rational curves. (See [8].)

Question 2 Does there exist a complex algebraic variety W with non-Jordan group $\mathrm{Aut}(W)$?

Line bundles over tori of positive algebraic dimension The statement of Summary 2.2 remains true if the direct product $X = \mathbb{P}^1 \times T$ is replaced by the "natural compactification" X_L of the total space of a holomorphic line bundle $L = p^*(L_0)$ on X where L_0 is any holomorphic line bundle on the algebraic model T_0 of T and $p : T \to T_0$ the universal homomorphism. Here by natural compactification X_L we mean the projectivization of the total space of the rank 2 holomorphic vector bundle $L \oplus \mathbf{1}_T$ where $\mathbf{1}_T = T \times \mathbb{C}$ is the trivial holomorphic line bundle. (Summary 2.2 still remains true even if just the Chern class of L coincides with the Chern class of $p^*(L_0)$ for some holomorphic line bundle on T_0.) See [43].

Question 3 Does Summary 2.2 remains true for X_L for an arbitrary holomorphic line bundle L on T?

Poor manifolds. The statement of Summary 2.3 remains true if the torus T is replaced by any *poor* manifold [9].

Definition 4 We say that a compact connected complex manifold Y of positive dimension is *poor* if it enjoys the following properties.

- Y does not contain closed analytic subspaces of codimension 1 (hence, a fortiori, the algebraic dimension $a(Y)$ of Y is 0);
- Y does not contain rational curves.

Any complex torus T with $\dim(T) \geq 2$ and $a(T) = 0$ is poor.
There are examples of poor $K3$ surfaces.

Question 4 Find a classification of poor manifolds.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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