



# Hénon Maps: A List of Open Problems

Julia Xénelkis de Hénon<sup>1</sup>

Received: 13 December 2023 / Revised: 18 May 2024 / Accepted: 7 June 2024  
© Institute for Mathematical Sciences (IMS), Stony Brook University, NY 2024

## Abstract

We propose a set of questions on the dynamics of Hénon maps from the real, complex, algebraic and arithmetic points of view.

## Contents

1	Introduction (C. Favre, T. Firsova, L. Palmisano, J. Raissy, and G. Vigny (Eds.))	.....
2	Real Hénon Maps (P. Berger)	.....
2.1	Strange Attractors	.....
2.2	Non-statistical Behavior	.....
2.3	Conservative Dynamics	.....
3	Dissipative Real Hénon Maps (S. Crovisier and E. Pujals)	.....
3.1	Mild Dissipation	.....
3.2	Maps with Zero Entropy	.....
3.3	Set of Periodic Points	.....
4	Symbolic Dynamics for Real Hénon and Lozi Maps (S. Štimac)	.....
4.1	Example	.....
5	Hénon Maps Tangent to the Identity (X. Buff)	.....
6	Quasi-Hyperbolicity and Uniform Hyperbolicity (E. Bedford)	.....
6.1	Complex Hénon Maps	.....
6.2	Quasi-Expanding Hénon Maps	.....
6.3	Surface Automorphisms	.....
6.4	Real Maps	.....
7	Parameter Loci for the Hénon Family (Y. Ishii)	.....
7.1	Connectedness Locus	.....
7.2	Horseshoe Locus	.....
7.3	Isentropes	.....
8	Topology and Rigidity of Hénon Maps (R. Dujardin)	.....
9	Statistical Properties of Complex Hénon Maps (F. Bianchi and T.-C. Dinh)	.....
9.1	Thermodynamics for Hénon Maps	.....
9.2	Statistical Properties of Equilibrium States and Spectral Gap for the Transfer Operators	.....
9.3	Higher Dimension and Other Generalizations	.....
10	Towards Higher Dimensions and Complex Differential Geometry (C. Favre)	.....
10.1	Hénon Maps and the Group of Polynomial Automorphisms of $\mathbb{C}^2$	.....
10.2	Hénon Maps and Compact Complex Manifolds	.....
11	Hénon Maps Over Number Fields (P. Ingram)	.....

✉ Julia Xénelkis de Hénon  
gabriel.vigny@u-picardie.fr

<sup>1</sup> LAMFA: Laboratoire Amienois de mathématique fondamentale et appliquée Amiens, Amiens, France

## 1 Introduction (C. Favre, T. Firsova, L. Palmisano, J. Raissy, and G. Vigny (Eds.))

A workshop ‘Dynamics of Hénon maps: Real, Complex and Beyond’ took place at BIRS, Banff in April 2023. The purpose of this meeting was to bring together international experts working on various aspects of Hénon maps. Recall that these maps are two-dimensional discrete dynamical systems which are ubiquitous in low dimensional dynamics, and among the most studied examples exhibiting chaotic behavior. Quadratic Hénon maps

$$H_{a,c}(x, y) := (ay + x^2 + c, ax)$$

are arguably the simplest examples. Here  $a$  and  $c$  are fixed parameters and  $x, y$  are affine coordinates.<sup>1</sup> These maps can be analyzed over the real numbers using techniques from smooth dynamical systems, or over the complex numbers and then complex analysis and geometry play crucial roles. They are amenable to generalizations, by replacing  $x^2 + c$  by higher degree polynomials, or even transcendental maps, and we may consider finite composition of such maps. We may also consider them with coefficients in number fields, and look at them from the perspective of arithmetic dynamics. Many of the most recent breakthroughs were actually made by combining several techniques coming from these different fields. It was delightful to attend series of talks blending so many different ideas. Many interesting questions were raised during the conference, a fact which encouraged us to collect them in a single text.

Hénon introduced his family of maps in the real domain as a simplified model of the Poincaré section of the first return map of the Lorenz flow [87]. In 1976, Hénon made numerical experiments for the map  $H_{\sqrt{0.3}, -1.4}$ <sup>2</sup> and observed that an initial point of the plane either approaches a set of points known since then as the Hénon strange attractor, or diverges to infinity under iterations. The Hénon attractor has a fractal nature: it is smooth in the unstable direction and has a Cantor-like structure in the transversal direction. This led Hénon to conjecture the existence of an ergodic measure which restricts to the Lebesgue measure in the smooth direction (a.k.a. an SRB measure). In 1981, Jakobson [96] proved the existence of a set of positive Lebesgue measure of parameters  $c$  for which  $x^2 + c$  displays an SRB measure. In the 90’s, Benedicks and Carleson [24] reworked Jakobson’s theorem and further generalized it to describe the dynamics of Hénon maps. They proved  $H_{a,c}$  display a strange attractor for  $a$  small and for a set of parameters  $c$  of positive Lebesgue measure. Benedicks–Carleson’s breakthrough has been further developed by Mora-Viana [120], Wang-Young [150], and Takahasi [142].

In 1996, during his inaugural lecture at Collège de France, Yoccoz proposed an alternative approach to prove the Hénon conjecture, with Sinai’s positive entropy con-

<sup>1</sup> The exact definition of (quadratic) Hénon maps may differ from section to section in these notes, as one might want to conjugate them by affine transformations to exploit various aspects of the original equations.

<sup>2</sup> Hénon actually considered the map  $h(x, y) = (1 - 1.4 \cdot x^2 + y, 0.3 \cdot x)$  which is affinely conjugate to it.

jecture lying in the horizon. To this end, he introduced a combinatorial and topological approach, based on the notion of strong regularity, that he used to give yet another proof of Jakobson's theorem [154]. The second step of Yoccoz' program was completed more recently by Berger in [25] who generalized this notion of strong regularity, leading him in particular to an alternative proof of Benedicks–Carleson theorem.

The theory of Hénon maps in the complex domain started with the seminal work of Friedland and Milnor [81], who used Jung's theorem to show that every polynomial automorphism of the complex affine plane is affinely conjugate to either an affine map, or a map preserving the pencil  $x = \text{cst}$ , or to a finite compositions of generalized Hénon maps (the latter class being usually called complex Hénon maps nowadays). In the early 90's Hubbard and his collaborators developed a topological approach to the study of Hénon maps giving description of the Fatou sets [91, 93]. Hubbard pointed out that Hénon maps appeared as natural generalizations of quadratic polynomials (by taking  $a \rightarrow 0$  in  $H_{a,c}$ ), which he used with Oberste-Vorth to give topological description of Julia sets [92].

An important breakthrough was the introduction of pluripotential techniques to construct invariant currents by Fornæss and Sibony [79] and by Bedford and Smillie [17]. The latter authors, partly with Lyubich, further developed in a series of influential papers (e.g., [15, 16, 18]) a thorough study of the ergodic properties of Hénon mappings and of their stability properties linked to hyperbolicity. These works were pursued and generalized to other invertible maps by many others including Diller, Cantat, Dujardin, Dinh and Sibony [42, 55, 58, 62, 68, 69]. Very recently, Bianchi and Dinh [31] made significant progress in the study of the fine statistical properties of the maximal entropy measure.

It is intriguing to see in retrospect how these seemingly simple maps have produced such an elaborate and successful theory. Note however that the results obtained so far are most complete in the case of dissipative maps (that is  $|a| < 1$ ), while the understanding of the conservative case (that is  $|a| = 1$ ) remains less developed.

In the past decade, the algebraic and arithmetic aspects of dynamical systems defined by rational maps have also been developed extensively. We refer to the survey [23] in which one can find a large set of open problems in this emerging field. For Hénon maps defined over a number field or over a function field, Silverman [141], and later Kawaguchi [100], constructed a suitable height function that lead to interesting analogs of the Northcott property (see also [94]). Hénon maps have also been the testing ground of some important conjectures in arithmetic dynamics like the Kawaguchi–Silverman conjecture on arithmetic degrees [101], or the dynamical Manin–Mumford problem which was partially solved by Dujardin and Favre using both height and Pesin theories [73]. Deep connections exist between the arithmetic of these systems and pluripotential theoretic techniques: it is for instance possible to retrieve the equidistribution of repelling periodic orbits using a theorem on the equidistribution of points of small height by Yuan [156].

The study of Hénon maps is still very active, as shown by the recent breakthroughs in the study of wandering domains. On the one hand, Ou [126] has proved the absence of wandering domains for strongly dissipative doubly infinitely period-doubling renormalizable real Hénon maps. On the other hand, Berger and Biebler [27] exhibited wandering domains for complex Hénon maps in 2023 by mixing deep techniques

coming from both real and complex dynamics. We also witness exciting new developments extending the already rich theory of Hénon maps to more general systems such as transcendental diffeomorphisms of the complex plane [4–6], or higher dimensional invertible rational maps, [54, 60, 63, 64, 85], where questions arising from complex dynamics led to profound developments in complex geometry such as PB currents, density currents and superpotential theory. Several developments have been made also in the context of higher dimensional unfoldings of homoclinic tangencies with a rank one saddle point, [128, 148, 151]. We hope that gathering these questions and open problems at one place will reinforce the community and attract new generations of researchers to work on these beautiful and rich objects.

## 2 Real Hénon Maps (P. Berger)

We propose a set of questions on the dynamics of Hénon maps in the real domain, or more generally on entire diffeomorphisms of  $\mathbb{R}^2$ .

### 2.1 Strange Attractors

Attractors play an important role in the study of dynamical systems since the 60's (Lorenz attractor, Hénon strange attractor, etc.). This notion is quite flexible and can cover many different situations in which a substantial set of points (either in the topological or measurable sense) is converging to some invariant compact subset. We refer to Milnor [114] for a discussion of various possible definitions of attractors. For instance, a measure-theoretical attractor is an invariant compact subset which attracts a set of Lebesgue positive measure and which is minimal with this property.

In the case of unimodal interval dynamics, measure-theoretical attractors can be classified into four types: cyclic, solenoidal, interval and wild (see, e.g., [36]). The latter two classes are arguably the most interesting. Jakobson [96] proved the abundance of quadratic maps displaying a stochastic interval of attractors (induced by an SRB measure). On the other hand, Bruin–Keller–Nowicki–van Strien [39] showed the existence of a polynomial unimodal map displaying a wild attractor: an invariant Cantor set attracting Lebesgue almost every point and included in a transitive interval.

Van Strien [146, Question 1.9] asked whether a suitable analog of wild attractors could exist for Hénon maps (of some degree). More precisely, one can ask:

**Question 1** Can we find a Hénon map which admits a wild attractor, i.e., a Cantor set which attracts a set of Lebesgue positive measure and which is strictly included in a transitive set ?

Avila and Lyubich have announced a positive answer to this problem in the quadratic Hénon family.

Returning to stochastic attractors, observe that the existence of a parameter  $c \in \mathbb{R}$  for which the quadratic map  $x \mapsto x^2 + c$  displays an absolutely continuous measure is easy to ensure. It suffices to pick a parameter  $c$  such that the post-critical orbit is finite but not periodic. In fact, much more is known. Lyubich ruled out the existence

of wild attractors for real quadratic maps [108]. He also showed in [109] the following “regular or stochastic Dichotomy”: for Lebesgue almost every quadratic map, almost all orbits either converge to an attracting cycle, or they are equidistributed with respect to an absolutely continuous invariant measure with positive entropy.

In dimension 2, the existence of a positive measure set of parameters of Hénon maps displaying an attractor supporting an invariant SRB measure<sup>3</sup> is a fundamental result, whose proof still remains difficult and lengthy [24, 25, 142, 150].

**Question 2** Is there a quick proof for the existence of SRB for some parameters of the Hénon maps?

A positive answer to this question might help finding new examples of stochastic attractors in the Hénon maps.

## 2.2 Non-statistical Behavior

Let  $f$  be any smooth dynamical system. We say that a point  $x$  has a non-statistical behavior (or simply is non-statistical) if its sequence of empirical measures  $e_n(x) := \frac{1}{n} \sum_{k=0}^{n-1} \delta_{x_k}$  is not converging. We say that  $f$  is non-statistical if there is a positive Lebesgue measure set of points with non convergent empirical measures. Ruelle [135] asked whether non-statistical dynamics could exist persistently.<sup>4</sup>

In polynomial dynamics, two phenomena give rise to non-statistical dynamics. The first one was discovered by Hofbauer and Keller:

**Theorem 1** [88] *There exist uncountably many  $c \in \mathbb{R}$  such that the quadratic polynomial  $P_c(x) := x^2 + c$  has non-statistical dynamics. More precisely, Lebesgue almost every non-escaping point  $x$  has non-statistical behavior.*

In [143], Talebi gave a counterpart of this result for rational functions on the Riemann sphere. In these two results, the set of non-statistical points is of full measure, but of empty interior.

The second occurrence of non-statistical dynamics is related to the notion of wandering stable component that we now introduce.

**Definition 1** A stable domain of  $f$  is a connected open subset  $U$  such that

$$\lim_{n \rightarrow \infty} d(f^n(x), f^n(y)) = 0$$

for all  $x, y \in U$ . A stable component is a maximal stable domain. A stable component is wandering if it is not preperiodic.

Colli and Vargas [48] gave the first example of a smooth dynamical system having a wandering stable component formed by points with non-statistical behavior. In [102], Kiriki and Soma constructed a locally dense set of such dynamics in the  $C^r$ -category with  $r < \infty$ . This also occurs for polynomial maps:

<sup>3</sup> i.e., a measure whose conditional measures along unstable curves are absolutely continuous.

<sup>4</sup> Ruelle used another terminology and talked about historical behavior.

**Theorem 2** [27] *There is a locally dense set of real sextic polynomials  $P(x) = x^6 + a_4x^4 + \dots + a_0$ , for which the map  $(x, y) \mapsto (P(x) - by, x)$  displays a wandering stable component containing only points with non-statistical behavior.*

The proof of this theorem actually implies the existence of a wandering Fatou component at the same parameters for its complex counterpart. This is in sharp contrast with the one-dimensional situation for which no wandering Fatou components exist by Sullivan's theorem.

Conversely, we can ask whether a counterpart of Hofbauer-Keller phenomenon appears within the Hénon family. We can formulate this question in more precise terms.

**Question 3** Do there exist  $\varepsilon > 0$  and a locally dense set<sup>5</sup>  $E$  of quadratic Hénon maps for which every point starting in some set of Lebesgue measure at least  $\varepsilon$  has non-statistical behavior?

By [143, Theorem 1.14], this would imply the existence of a generic set of the closure of  $E$  of non-statistical dynamics.

### 2.3 Conservative Dynamics

Let  $f: S \rightarrow S$  be any homeomorphism of a closed surface  $S$ . An annular rotation domain for  $f$  is by definition an  $f$ -invariant open annulus that does not contain any periodic points. Such domains play an important role in conservative dynamics. The next result can be deduced from the works of [80, 103, 104, 153] (Le Calvez, private communication).

**Theorem 3** *Let  $S$  be a closed surface endowed with a symplectic form. Let  $f: S \rightarrow S$  be any symplectomorphism of class  $C^r$  with  $r \geq 1$  that contains at least one periodic point and satisfying the following conditions.*

1. *Any eigenvalue of any periodic point does not belong to  $\{e^{2\pi ip/q} : p/q \in \mathbb{Q}\}$ .*
2. *For every hyperbolic periodic points  $P, Q \in \text{Per}(f)$ ,  $W^s(P)$  is transverse to  $W^u(Q)$ .*
3. *Every elliptic point  $P \in \text{Per}(f)$  is surrounded by arbitrarily close KAM circles.*
4. *There are no annular rotation domain.*

*Then  $\bigcup_{P \in \text{Per}(f)} W^s(P)$  is dense in  $S$ .*

It is natural to ask whether the third condition is superfluous. The recent result [125] suggests that this may be the case.

**Question 4** Is there a real, conservative, polynomial Hénon map with an annular rotation domain? Is there an open set of such real, conservative, polynomial Hénon maps?

An annular rotation domain is said to be trivial when the whole dynamics is conjugate to a rotation.

<sup>5</sup> i.e., whose closure has non-empty interior.

**Question 5** Is there an entire<sup>6</sup> symplectomorphism of  $\mathbb{R}^2$  with a non trivial annular rotation domain?

Recently, an entire map of the cylinder  $\mathbb{R} \times \mathbb{R}/\mathbb{Z}$  having a bounded rotation domain on which the dynamics is not conjugate to a rotation has been exhibited in [26], thereby disproving a conjecture by Birkhoff [32]. The construction in [26] also gives an example of a symplectic entire automorphism of  $\mathbb{C} \times \mathbb{C}/\mathbb{Z}$  without periodic point and with a non-empty set of (Lyapunov) unstable points.<sup>7</sup> Hence Question 5 may shed light on the following intriguing problem.

**Problem 1** Construct an entire symplectic automorphism of  $\mathbb{C}^2$  without periodic point and with non-empty set of (Lyapunov) unstable points with bounded orbit.

### 3 Dissipative Real Hénon Maps (S. Crovisier and E. Pujals)

#### 3.1 Mild Dissipation

Quadratic real Hénon maps

$$f_{c,b}(x, y) := (x^2 + c - by, x),$$

with Jacobian  $b$  close to zero share some properties of the quadratic family on the interval: some results are obtained by perturbative methods (for instance [24, 53]) and are known when  $|b|$  is extremely tiny. In [50] another approach has been introduced which allows to reduce the dynamics to a one-dimensional system.

**Definition 2** The map  $f_{c,b}$  is *mildly dissipative* if it is dissipative (i.e.,  $|b| < 1$ ) and if for any ergodic invariant measure  $\mu$  which is not supported on a sink, and for  $\mu$ -almost every point  $x$ , both components of the stable curve  $W^s(x) \setminus \{x\}$  are unbounded.

Under this assumption, and in restriction to any open topological disc  $D \subset \mathbb{R}^2$  that is compactly mapped inside itself, the dynamics is semi-conjugated to a (non-trivial) continuous map of a real tree. Other strong consequences can be derived (e.g., a closing lemma, or a description of zero entropy dynamics, see below). Using Wiman's theorem (in the same spirit like in [75, 110]), one can prove that Hénon maps are mildly dissipative once  $|b| < 1/4$ . In this way we obtain dynamical informations for *all* Hénon maps having their Jacobian in  $(-1/4, 1/4)$  and not only for those satisfying  $|b| \ll 1$ . One expects that this property extends beyond the bound obtained through Wiman's theorem.

**Question 6** Which real Hénon maps  $f_{c,b}$  are mildly dissipative? Is this property satisfied by all Hénon maps with  $|b| < 1$ ?

In some cases [50] proves that the mild dissipation is an open property, but we don't know if this holds in general.

<sup>6</sup> i.e., an analytic map which extends to a holomorphic map of  $\mathbb{C}^2$ .

<sup>7</sup> An orbit  $(x_n)_{n \geq 0}$  is Lyapunov stable if for any  $y_0$  close enough to  $x_0$ , then  $y_n$  stays close to  $x_n$  for all  $n \geq 0$ .

### 3.2 Maps with Zero Entropy

It is well-known that the quadratic maps  $f_c(x) := x^2 + c$  have their topological entropy equal to zero exactly when  $c$  belongs to some interval  $[c_0, +\infty)$ . At the critical parameter  $c_0$ , the dynamics exhibits an odometer, which is the limit set of an infinite sequence of successive renormalizations of period 2. This result persists inside any line  $c \mapsto f_{c,b}$ , provided  $|b|$  is smaller than some number  $\varepsilon > 0$  small, as it has been shown in [53]. Let us consider the locus in the parameter space where the topological entropy vanishes:

$$\mathcal{E}_0 := \{(b, c) \in \mathbb{R}^2, h_{\text{top}}(f_{c,b}) = 0\}.$$

Inside the strip  $(-\varepsilon, \varepsilon) \times \mathbb{R}$  this set is bounded by an analytic arc  $\{(b, c_0(b)), |b| < \varepsilon\}$ . Moreover any map  $f_{c,b}$  with  $c > c_0(b)$  can be renormalized at most finitely many times; and for an open and dense subset of these parameters, the dynamics is Morse Smale (i.e., is supported by finitely many hyperbolic periodic orbits). When  $c = c_0(b)$ , the sequence of renormalizations converges towards a particular unimodal map of the interval.

When  $|b|$  is larger but smaller than  $1/4$  (so that it is mildly dissipative), [51] describes the dynamics of  $f_{c,b} \in \mathcal{E}_0$ . In particular, all maps  $f_{c,b} \in \partial\mathcal{E}_0$  are infinitely renormalizable (with renormalization periods eventually equal to 2), solving a conjecture by Tresser (which is still open when we don't assume the mild dissipation). One may wonder if the converse holds.

**Question 7** Let us consider any infinitely renormalizable mildly dissipative map  $f_{c,b} \in \mathcal{E}_0$ . Is it the limit of maps with positive entropy? Is it the limit of Morse-Smale maps?

It is also natural to try to describe the boundary  $\partial\mathcal{E}_0$ : is it a (piecewise smooth) arc? One would like to implement the strategy developed in [53] for  $b$  close to 0:

**Question 8** Let us consider any infinitely renormalizable mildly dissipative map in  $\mathcal{E}_0$ . Does the sequence of renormalizations converge?

The recent preprint [49] is related to the above two questions, the authors introduce a class of infinitely renormalizable, unicritical diffeomorphisms of the disk (with a non-degenerate “critical point”) and show that under renormalization, those maps eventually become Hénon-like, and then converge super-exponentially fast to the space of one-dimensional unimodal maps.

We also don't know if different combinatorics of the renormalizations may occur.

**Question 9** Does there exist a mildly dissipative map  $f_{c,b} \in \mathcal{E}_0$  which admits odd periods?

### 3.3 Set of Periodic Points

One would like to describe the dynamics through its periodic orbits. To any periodic point  $p$ , one associates two Lyapunov exponents  $\lambda^-(p) \leq \lambda^+(p)$ . When they do not vanish and have different sign, we say that  $p$  is a saddle.



As mentioned previously, for mildly dissipative real Hénon maps the set of periodic points is dense in the union of the supports of the invariant probability measures [50]. The same property holds for any complex Hénon maps [71].

**Question 10** For any real Hénon map, does the closure of the set of periodic orbits support all the invariant probability measures?

The next step is to describe how periodic saddles are organized. We say that two saddles  $p, q$  are homoclinically related if there exists  $k \geq 0$  such that the invariant curves  $W^u(p), W^s(f^k(q))$  (and  $W^s(p), W^u(f^k(q))$  as well) intersect transversally. This defines an equivalence relation which decomposes the set of periodic saddles into its homoclinic classes. There may exist infinitely many periodic saddles which are not homoclinically related, but we conjecture that their hyperbolicity should drop.

**Question 11** For any map  $f_{c,b}$  and any infinite set of periodic saddles  $(p_n)$  which are pairwise not homoclinically related, do we have  $\min(|\lambda^-(p_n)|, |\lambda^+(p_n)|) \xrightarrow{n \rightarrow \infty} 0$ ?

This questions goes beyond Hénon maps, but [51] implies a positive answer in the particular case of mildly dissipative  $f_{c,b} \in \mathcal{E}_0$ .

### 4 Symbolic Dynamics for Real Hénon and Lozi Maps (S. Štímac)

Kneading theory is a combinatorial tool to understand the dynamics of a piecewise monotone map from the interval to itself and was developed by Milnor and Thurston [115]. Applications extend from the topological classification to the computation of the entropy, to the counting of periodic orbits, and the construction of measures of maximal entropy. We propose several problems connected to the extension of this theory to real Hénon maps and the Lozi maps  $\tilde{H}_{a,b}, L_{a,b} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ ,

$$\tilde{H}_{a,b}(x, y) = (1 + y - ax^2, bx), \quad L_{a,b}(x, y) = (1 + y - a|x|, bx),$$

respectively.<sup>8</sup> The Lozi maps are piecewise affine map that display the same fold and bend behavior as the Hénon maps, but are usually easier to analyze technically [76, 117].

In [118], the authors developed a kneading theory for the Lozi maps  $L_{a,b}$  with  $(a, b) \in \mathcal{M}$ , where  $\mathcal{M} = \{(a, b) \in \mathbb{R}^2 : b > 0, a\sqrt{2} - b > 2, 2a + b < 4\}$  is a set of parameters for which Misiurewicz proved the existence of a strange attractor (for details see [118] and [119]). A kneading sequence  $\bar{k}$  is defined as the itinerary of a turning point  $T$ , where turning points are points of transversal intersections of the  $x$ -axis and the unstable manifold  $W^u$  of the fixed point  $X$  of the attractor. Any kneading sequence is a bi-infinite sequence of  $+$  and  $-$ .

The kneading set  $\mathcal{K} = \{\bar{k}^n : n \in \mathbb{Z}\}$  is the set of all kneading sequences  $\bar{k}^n, n \in \mathbb{Z}$ , and every kneading sequence  $\bar{k} = \bar{k}^n$ , for some  $n \in \mathbb{Z}$ , has the following form:

$$\bar{k} = \infty_{+w} \pm \cdot \vec{k}_0,$$

<sup>8</sup> Observe that  $\tilde{H}_{a,b}$  is affinely conjugated to  $H_{\sqrt{b}, -a}$  from the introduction.

where  $\infty+ = \dots + + +$ ,  $w = w_0 \dots w_m$ , for some  $m \in \mathbb{N}_0$ ,  $\vec{k}_0 = k_0 k_1 k_2 \dots$ ,  $w_0 = -$ ,  $k_0 = +$ ,  $w_i, k_j \in \{-, +\}$  for  $i = 1, \dots, m$  and  $j \in \mathbb{N}$ , and the little black square  $\cdot$  indicates where the 0th coordinate is located. Here for  $\pm$  one can substitute any of  $+$  and  $-$ .

In [118], the authors prove that  $\mathfrak{K}$  characterizes all itineraries of all points of the attractor of  $L_{a,b}$ . The proof is given in two steps. We say that an itinerary is  $W^u$ -admissible if it is realized by a point on the unstable manifold  $W^u$ . We first have:

**Theorem 4** *A sequence  $\infty + \vec{p}_n$ , where  $\vec{p}_n = p_n p_{n+1} \dots$  such that  $p_n = -$  for some  $n \in \mathbb{Z}$ , is  $W^u$ -admissible if and only if for every kneading sequence  $\infty + w \pm \cdot \vec{k}_0$ , such that  $w = p_n p_{n+1} \dots p_{n+m}$  for some  $m \in \mathbb{N}_0$ , we have  $\sigma^{m+2}(\vec{p}_n) \leq \vec{k}_0$ , where  $\leq$  is the parity-lexicographical ordering.*

Next, we equip the symbolic space with its natural product topology. Using topological arguments, one may prove:

**Theorem 5** *A sequence  $\vec{p} = \dots p_{-2} p_{-1} \cdot p_0 p_1 \dots$  is admissible if and only if for every positive integer  $n$  there is a  $W^u$ -admissible sequence  $\vec{q} = \dots q_{-2} q_{-1} \cdot q_0 q_1 \dots$  such that  $p_{-n} \dots p_n = q_{-n} \dots q_n$ .*

**Problem 2** *Describe the set of kneading sequences  $\mathfrak{K}$ .*

In [95], Ishii developed formulas that can be used to obtain a relation between parameters  $a, b$ , a turning point  $T = (x_T, 0)$  of the Lozi map  $L_{a,b}$ , and its itinerary  $\vec{k}$  (that is a kneading sequence of  $L_{a,b}$ ). This relation is  $p(a, b, \vec{k}) = x_T = q(a, b, \vec{k})$ , where  $p = p(a, b, \vec{k})$  is given in formula [94, (4.2)] and  $q = q(a, b, \vec{k})$  is given in formula [94, (4.3)]. Therefore, every kneading sequence  $\vec{k}$  gives an equation

$$p(a, b, \vec{k}) = q(a, b, \vec{k}). \tag{1}$$

Numerical experiments show that if one has two kneading sequences,  $\vec{k}^0$  of the rightmost turning point  $T_0$  and  $\vec{k}^{-1}$  of the leftmost turning point  $T_{-1}$ , and if these two turning points lie in the stable manifolds of some periodic points with small periods, then it is possible to calculate  $a$  and  $b$  from the corresponding two equations, implying that these two kneading sequences govern all other kneading sequences.

### 4.1 Example

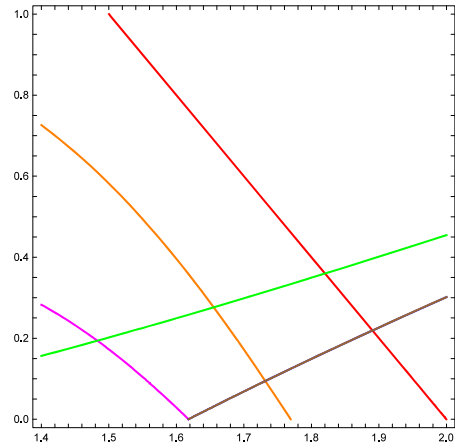
Let  $\vec{k}^0 = \infty + \pm \cdot + - - + \infty$  and  $\vec{k}^{-1} = \infty + - \pm \cdot (+ -) \infty$ . The equation  $p(a, b, \vec{k}^0) = q(a, b, \vec{k}^0)$  reads

$$a^4 - 6a^2 - 4a + 4b^2 + a^2b + (a^3 + 2a - ab)\sqrt{4b + a^2} = 0, \tag{2}$$

and the equation  $p(a, b, \vec{k}^{-1}) = q(a, b, \vec{k}^{-1})$  reads

$$\frac{4(-a^2 - 2b^2 + 2b + a\sqrt{a^2 - 4b})}{a - 2b - \sqrt{a^2 - 4b}} - (2 + a - \sqrt{a^2 + 4b})(3a - \sqrt{a^2 + 4b}) = 0. \tag{3}$$

**Fig. 1** Graph of (2) is in orange and of (3) in green. Graph of (1) for  $\bar{k} = \infty + \pm \cdot (+ - -)^\infty$  is in magenta and for  $\bar{k} = \infty + - \pm \cdot (+ - -)^\infty$  in brown. Graph of the line  $2a + b = 4$ , that is a boundary line of the Misiurewicz set, is in red



Using the “NSolve” command of Wolfram Mathematica produces a unique solution to this system of equations in the region  $a \in [1, 2], b \in [0, 1]$ . This solution is approximately  $a = 1.655319602968851744592, b = 0.2765071079677260998121$ , see Fig. 1.

**Question 12** Is it true that any two distinct kneading sequences determine a unique pair of parameters  $(a, b)$ , and in that way govern all the other kneading sequences of  $\mathfrak{K}$ ?

Very recently, in [38], the authors developed a kneading theory for the Hénon maps  $H_{a,b}$  within a set of parameters  $\mathcal{WY}$  for which Wang and Young proved the existence of a strange attractor. This set has positive measure and consist of maps which are strongly dissipative. We refer to [38, 150] for details.

**Problem 3** Describe the set of kneading sequences  $\mathfrak{K}$  of the Hénon map  $H_{a,b}$ , with  $(a, b) \in \mathcal{WY}$ .

**Question 13** Is it true that any two distinct kneading sequences of the Hénon map  $H_{a,b}$ , with  $(a, b) \in \mathcal{WY}$ , determine a unique pair of parameters  $(a, b)$ , and in that way govern all the other kneading sequences in  $\mathfrak{K}$  of  $H_{a,b}$ ?

### 5 Hénon Maps Tangent to the Identity (X. Buff)

We propose to investigate the local dynamics of some specific Hénon maps. Consider the quadratic complex Hénon map  $H_2: \mathbb{C}^2 \rightarrow \mathbb{C}^2$  defined by

$$H_2(x, y) = (y, x + y^2).$$

The origin is a fixed point and  $H_2^{\circ 2}$  is tangent to the identity at the origin.

Note that  $H_2$  restricts to an orientation reversing diffeomorphism  $H_2: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ . The dynamics in  $\mathbb{R}^2$  is well understood. There is an analytic map  $\phi_2: \mathbb{R} \rightarrow \mathbb{R}^2$  such that

$$H_2 \circ \phi_2(t) = \phi_2(t + 1) \text{ and } \phi_2(t) \sim \left( \frac{-2}{t}, \frac{-2}{t} \right) \text{ as } t \rightarrow +\infty.$$

The curve  $\phi_2(\mathbb{R})$  is invariant by  $H_2$  and within  $\phi_2(\mathbb{R})$ , every orbit converges to the origin in  $\mathbb{R}^2$ . Outside the origin and  $\phi_2(\mathbb{R})$ , every orbit diverges to infinity (see Fig. 2).

**Question 14** Can we describe the dynamics of  $H_2$  near the origin in  $\mathbb{C}^2$  ?

Before specifying this question, let us consider the Hénon map  $H_3: \mathbb{C}^2 \rightarrow \mathbb{C}^2$  defined by

$$H_3(x, y) = (y, x + y^3).$$

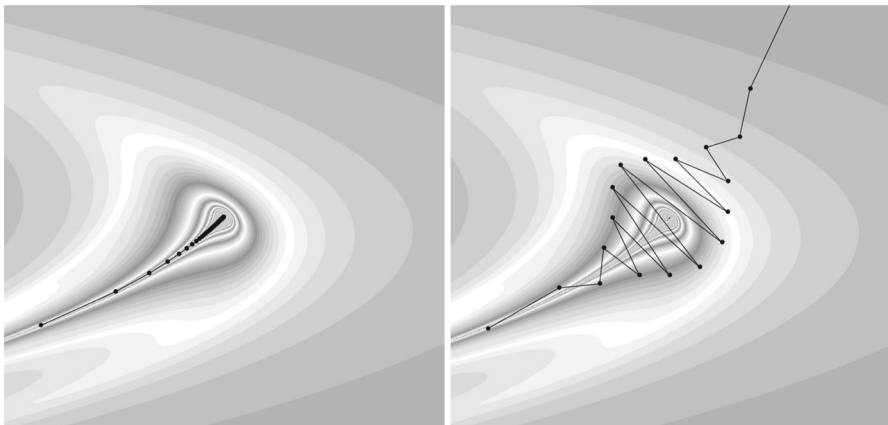
This Hénon map also preserves  $\mathbb{R}^2$  and the dynamics in  $\mathbb{R}^2$  is also completely understood. There is an analytic map  $\phi_3: \mathbb{R} \rightarrow \mathbb{R}^2$  such that

$$H_3 \circ \phi_3(t) = -\phi_3(t + 1) \text{ and } \phi_3(t) \sim \left( \frac{1}{\sqrt{t}}, \frac{-1}{\sqrt{t}} \right) \text{ as } t \rightarrow +\infty.$$

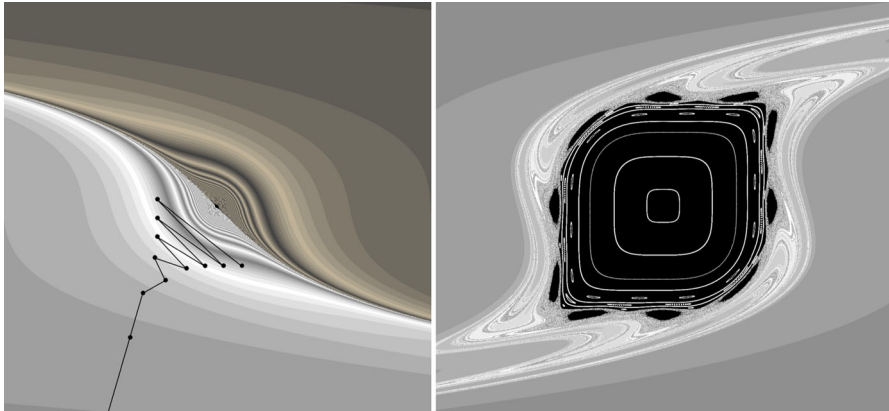
The curves  $\phi_3(\mathbb{R})$  and  $-\phi_3(\mathbb{R})$  are exchanged by  $H_3$ . Within those curves, every orbit converges to the origin. Outside those curves and the origin, every orbit diverges to infinity (see Fig. 3 left).

Set  $\omega = e^{i\frac{\pi}{8}}$  so that  $\omega^9 = -\omega$  and consider the real planes  $\Pi_1 \subset \mathbb{C}^2$  and  $\Pi_2 \subset \mathbb{C}^2$  defined by

$$\Pi_1 = \left\{ (\omega x, \omega^3 y) : x \in \mathbb{R}, y \in \mathbb{R} \right\} \text{ and } \Pi_2 = \left\{ (\omega^3 x, \omega y) : x \in \mathbb{R}, y \in \mathbb{R} \right\}.$$



**Fig. 2** Left: an orbit converging to the origin. Right: an orbit diverging to infinity



**Fig. 3** Left: The points are colored according to whether  $x + y$  tends to  $+\infty$  (dark grey) or to  $-\infty$  (light grey). Right: The dynamics of  $H_3^{\circ 2} : \Pi_1 \rightarrow \Pi_1$  exhibits KAM phenomena

Observe that  $H_3$  exchanges the planes  $\Pi_1$  and  $\Pi_2$ :

$$H_3(\omega x, \omega^3 y) = (\omega^3 y, \omega(x - y^3)) \text{ and } H_3(\omega^3 x, \omega y) = (\omega y, \omega^3(x + y^3)).$$

The dynamics of  $H_3^{\circ 2} : \Pi_1 \rightarrow \Pi_1$  is much more complex than that of  $H_3 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  (see Fig. 3 right).

The second iterate of  $H_3$  is tangent to the identity at the origin. More precisely

$$H_3^{\circ 2}(x, y) = (x, y) + (y^3, x^3) + \mathcal{O}(\|x, y\|^4).$$

It follows that near the origin, the orbits of  $H_3^{\circ 2}$  shadow the orbits of the vector field

$$\vec{v}_3 = y^3 \partial_x + x^3 \partial_y.$$

The vector field is a Hamiltonian vector field. It is tangent to the level curves of the function

$$\Phi_3 = x^4 - y^4.$$

Note that

$$\Phi_3 \left( \begin{matrix} \omega x \\ \omega^3 y \end{matrix} \right) = i(x^4 + y^4)$$

so that the intersection of the level curves of  $\Phi_3$  with the real plane  $\Pi_1$  are topological circles. Those topological circles are invariant by the flow of the vector field  $\vec{v}_3$ . It follows from the theory of Kolmogorov–Arnold–Moser that in any neighborhood of the origin, there is a set of positive Lebesgue measure of topological circles which are

invariant by  $H_3$  and on which  $H_3$  is analytically conjugate to a rotation  $\mathbb{R}/\mathbb{Z} \ni t \mapsto t + \theta \in \mathbb{R}/\mathbb{Z}$  with bounded type rotation number  $\theta \in (\mathbb{R} \setminus \mathbb{Q})/\mathbb{Z}$ . Those invariant circles are separated by small saddle cycles and small elliptic cycles. The analytic conjugacies extend to complex neighborhoods of  $\mathbb{R}/\mathbb{Z}$  in  $\mathbb{C}/\mathbb{Z}$ . This proves that  $H_3$  has lots of Herman rings.

Coming back to our initial problem, observe that the second iterate of  $H_2$  is also tangent to the identity at the origin with

$$H_2^{\circ 2}(x, y) = (x, y) + (y^2, x^2) + \mathcal{O}(\|x, y\|^3).$$

It follows that near the origin, the orbits of  $H_2^{\circ 2}$  shadow the orbits of the vector field

$$\vec{v}_2 = y^2 \partial_x + x^2 \partial_y.$$

The vector field is also a Hamiltonian vector field. The vector field  $\vec{v}_2$  is tangent to the level curves of the function

$$\Phi_2 = x^3 - y^3.$$

We can no longer apply the theory of Kolmogorov-Arnold-Moser since there is no invariant real-plane on which the level curves of  $\Phi_2$  are topological circles. However, we may wonder whether the complex dynamics of  $H_2$  exhibits KAM phenomena.

We say that  $H_2$  has small cycles if for any neighborhood  $U$  of the origin  $\mathbf{0}$  in  $\mathbb{C}^2$ , there exists a cycle of  $H_2$  which is entirely contained in  $U \setminus \{\mathbf{0}\}$ .

**Question 15** Does  $H_2$  have small cycles?

**Question 16** Does  $H_2$  have both small saddle cycles and small elliptic cycles?

We say that  $H_2$  has a Herman ring with rotation number  $\theta \in (\mathbb{R} \setminus \mathbb{Q})/\mathbb{Z}$  if there exists an annulus  $V = \{z \in \mathbb{C}/\mathbb{Z} : \text{Im}(z) < h\}$  with  $h > 0$ , a holomorphic map  $\phi: V \rightarrow \mathbb{C}^2$ , and an integer  $n \geq 2$  such that

$$\forall z \in V, \quad H_2^{\circ n} \circ \phi(z) = \phi(z + \theta).$$

**Question 17** Does  $H_2$  have a Herman ring?

If the answer is yes, we may consider the set  $\Theta \subset (\mathbb{R} - \mathbb{Q})/\mathbb{Z}$  of rotation numbers  $\theta$  such that  $H_2$  has a Herman ring with rotation number  $\theta$ .

**Question 18** Does  $\Theta$  have positive Lebesgue measure? More precisely, is 0 a Lebesgue density point of  $\Theta$ ?

We believe that the answers to the previous questions are all affirmative. Regarding the following question, we do not have an opinion.

**Question 19** Assume  $H_2$  has a Herman ring with bounded type rotation number  $\theta$ . Is it possible to find parameters  $a \in \mathbb{D} \setminus \{0\}$  arbitrarily close to 1 such that the dissipative Hénon map  $H : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  defined by

$$H(x, y) = (ay, x + y^2).$$

has a Herman ring with rotation number  $\theta$ ?

## 6 Quasi-Hyperbolicity and Uniform Hyperbolicity (E. Bedford)

### 6.1 Complex Hénon Maps

Any Hénon map  $H_{a,P}(x, y) := (ay + P(x), x)$  where  $a \in \mathbb{C}^*$  and  $P \in \mathbb{C}[x]$  is a polynomial of degree  $d \geq 2$  induces a polynomial automorphism of the affine plane  $H_{a,P} : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ .

For a general polynomial automorphism  $f : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ , write  $(f(x, y)) = (P(x, y), Q(x, y))$  and define its degree  $\deg(f) := \max\{\deg(P), \deg(Q)\}$ . In a celebrated article [81], Friedland and Milnor have proved the following remarkable result (see Sect. 10.1 below for more details). If the sequence of degrees  $\deg(f^n)$  is unbounded, then  $f$  is actually conjugated to a composition of Hénon maps  $H_{a_1, P_1} \circ \dots \circ H_{a_k, P_k}$ , and  $\deg(f) = \deg(P_1) \dots \deg(P_k)$ .

Any such composition will be called a generalized Hénon map.

### 6.2 Quasi-Expanding Hénon Maps

Suppose  $f$  is a generalized Hénon map of degree  $d \geq 2$ , and let  $S$  denote its set of (periodic) saddle points. It is known to be infinite, and its distribution represents the unique measure of maximal entropy, see [16].

Given a saddle point  $p$ , we denote by  $E_p^u$  the set

$$E_p^u = \{v \in T_p(\mathbb{C}^2) : \|Df_p^n v\| \rightarrow 0 \text{ as } n \rightarrow -\infty\}.$$

Let  $W_{\text{loc}}^u(p)$  denote its local unstable manifold, and let  $W^u(p)$  be its global unstable manifold.

We now introduce the following three conditions measuring the expansion of  $f$ .

**Condition 1** For each  $p \in S$ , there is a metric on  $E_p^u$  which is expanded by  $Df_p$  with a uniform bound independent on  $p$ . More specifically, this means that there exists  $\kappa > 1$  such that for each  $p$ , there is a metric  $\|\cdot\|_p$  on  $E_p^u$  so that for each  $v \in E_p^u$ , we have

$$\|Df_p v\|_{f(p)} \geq \kappa \|v\|_p.$$

If  $f$  is a Hénon map, then following Hubbard [90], we may define a Green function

$$G^+(x, y) = \lim_{n \rightarrow \infty} \frac{1}{d^n} \log \max\{1, |f^n(x, y)|\}.$$

We let  $W_r^u(p)$  denote the connected component of  $B(p, r) \cap W^u(p)$  containing  $p$ , where  $W^u(p)$  is the unstable manifold at  $p$  and  $B(p, r)$  the euclidean ball in  $\mathbb{C}^2$ .

**Condition 2** The unstable manifolds  $W^u(p)$  satisfy the proper, locally bounded area condition: there exist  $\varepsilon > 0$  and  $A < \infty$  such that for each  $\delta > 0$  there is an  $\eta > 0$  such that for each saddle point  $p$  we have:  $W_\varepsilon^u(p)$  is closed in  $B(p, \varepsilon)$ ,  $\text{Area}(W_\varepsilon^u(p)) \leq A$ , and  $\sup_{W_\delta^u(p)} G^+ \geq \eta$  (see [20, Corollary 3.5]).

Recall that each unstable manifold is uniformized by an entire map  $\xi_p : \mathbb{C} \rightarrow W^u(p) \subset \mathbb{C}^2$  with  $\xi_p(0) = p$ . Using the Green function, we may normalize it by putting  $\hat{\xi}_p(\zeta) := \xi_p(\alpha\zeta)$  so that

$$\max_{|\zeta| \leq 1} G^+(\hat{\xi}_p(\zeta)) = 1. \tag{4}$$

The last condition we want to mention is:

**Condition 3** The normalized maps  $\{\hat{\xi}_p : p \in \mathcal{S}\}$  form a normal family of entire manifolds.

A consequence of Condition 3 is that for all  $x \in \bar{\mathcal{S}}$  we define a family of unstable manifolds by setting  $W^u(x) := \xi_x(\mathbb{C})$ .

For generalized Hénon maps, Conditions 1, 2, and 3 are equivalent (see [20]), and in case one/all of them hold, we say that  $f$  is *quasi-expanding*. A map  $f$  is said to be *quasi-hyperbolic* if both  $f$  and  $f^{-1}$  are quasi-expanding.

Furthermore, a theorem from [12] asserts a quasi-hyperbolic map is uniformly hyperbolic if and only if there is no tangency between  $W^u(x_1)$  and  $W^s(x_2)$  for any  $x_1, x_2$  in the closure of  $\mathcal{S}$ .

Recall the following standard definitions (see [17]). Let  $K^\pm$  denote the set of points with bounded forward orbits for  $f^\pm$ . We denote  $J^\pm := \partial K^\pm$ . We also define  $J := J^+ \cap J^-$  and  $J^*$  to be the closure of  $\mathcal{S}$  (hence  $J^* \subset J$ ).

**Question 20** If  $f$  is quasi-hyperbolic, then is  $\text{int}(K^+)$  the union of a finite number of basins of sink orbits?

**Question 21** If  $f$  is quasi-hyperbolic, is  $J = J^*$ ?

**Question 22** If  $f$  is quasi-hyperbolic, then is there no wandering Fatou component?

**Question 23** If  $f$  is quasi-hyperbolic, do the unstable slices satisfy a John-type condition (as in [19])?

**Question 24** If  $f$  is quasi-hyperbolic and dissipative, and if  $J$  is connected, do the external rays land at  $J$ ? Is  $J$  a finite quotient of the real solenoid?



Let us discuss the properties involved in this group of questions. In dimension one, quasi-expansion is related to the Misiurewicz condition. In dimension two, quasi-hyperbolic dynamics is expected to be close to hyperbolic. The answers are “yes” to these questions in the uniformly hyperbolic case; the first three follow from [17], and the last two follow from [19]. The properties in Questions 20 and 23 do not hold for general Hénon maps. In connection with Question 22, Berger and Biebler [27] have found wandering domains for some Hénon maps of degree 6 (which are not known to be quasi-hyperbolic). The question whether  $J = J^*$  is well-known. In addition to the hyperbolic case [17], it has been shown to hold in a few other cases [72, 86, 111], but the case of a generalized Hénon map remains elusive.

### 6.3 Surface Automorphisms

Let  $X$  be any compact complex Kähler surface, and let  $f : X \rightarrow X$  be any holomorphic automorphism having positive topological entropy  $\log \lambda$ ,  $\lambda > 1$ . By Cantat [42],  $X$  is isomorphic to either the blow-up of  $\mathbb{P}^2$  at at least 10 points, or a K3 surface, or an Enriques surface, or an Abelian surface.

In this context, Condition 1 is still meaningful, but we do not have a Green function  $G^+$ . However, since the dynamical degree of  $f$  is  $\lambda > 1$ , there exists an expanded positive closed  $(1, 1)$  current  $T^+$  with  $f^*T^+ = \lambda T^+$  (see [42]). In this case, we can replace the normalization (4) with a condition involving the mass of a slice of the current  $T^+$ :

$$\text{Mass}(T^+|_{\xi_p(|\zeta|<1)}) = 1 \tag{5}$$

Thus we may formulate a Condition 2' to be the same as Condition 2 with the normalization (4) replaced by (5), and we may formulate Condition 3' similarly.

**Question 25** Are Conditions 2' and 3' equivalent to Condition 1? And do Questions 20, 21, 22 above hold for quasi-hyperbolic surface automorphisms?

### 6.4 Real Maps

Let us now suppose that  $f$  is a real surface automorphism. In other words, we suppose  $X$  to be projective and defined by real polynomial equations, and  $f$  to be also defined over the real numbers. We may thus consider the restriction map  $f_{\mathbb{R}}$  to the set of real points  $X_{\mathbb{R}}$ . Observe that  $h_{top}(f_{\mathbb{R}}) \leq \log \lambda$ . It was proved in [20] that for generalized Hénon maps  $h_{top}(f_{\mathbb{R}}) = \log \lambda$  implies  $f$  to be quasi-hyperbolic.

**Question 26** If  $f$  is a real surface automorphism such that the entropy of  $f_{\mathbb{R}}$  is the same as the entropy of  $f$ , does it follow that  $f$  is quasi-hyperbolic?

We refer to [57] for a discussion of real surface automorphisms satisfying this condition on the entropy.

## 7 Parameter Loci for the Hénon Family (Y. Ishii)

### 7.1 Connectedness Locus

Consider the complex Hénon family:

$$f_{c,b}(x, y) := (x^2 + c - by, x),$$

where  $(c, b) \in \mathbb{C}^2$  is a parameter.<sup>9</sup> Let  $J_{c,b}$  be the Julia set of  $f_{c,b}$ : by definition this is the intersection between the boundaries of the sets of points having bounded forward (resp. backward) orbits. By extension, we let  $J_{c,0}$  be the Julia set of  $p(z) = z^2 + c$ . The *connectedness locus* of  $f_{c,b}$  is defined as

$$\mathcal{M} = \{(c, b) \in \mathbb{C}^2 : J_{c,b} \text{ is connected}\}.$$

**Conjecture 1**  $\mathcal{M}$  is disconnected.

It has been shown that  $\mathcal{M} \cap \mathbb{R}^2$  is disconnected [3], which partially supports the conjecture above.

### 7.2 Horseshoe Locus

We say that  $f_{c,b}$  is a *complex hyperbolic horseshoe* if  $J_{c,b}$  is a hyperbolic set for  $f_{c,b}$  and the restriction  $f_{c,b} : J_{c,b} \rightarrow J_{c,b}$  is topologically conjugate to the full 2-shift. The *complex hyperbolic horseshoe locus* is defined as

$$\mathcal{H}_{\mathbb{C}} = \{(c, b) \in \mathbb{C}^2 : f_{c,b} \text{ is a complex hyperbolic horseshoe}\}.$$

One can see that  $\mathcal{H}_{\mathbb{C}}$  is not simply connected since the monodromy representation:

$$\rho : \pi_1(\mathcal{H}_{\mathbb{C}}) \longrightarrow \text{Aut}(\{0, 1\}^{\mathbb{Z}})$$

of the fundamental group of  $\mathcal{H}_{\mathbb{C}}$  (with the base-point at  $(c, b) = (-4, 0)$ ) to the group of shift-commuting automorphisms of  $\{0, 1\}^{\mathbb{Z}}$  is non-trivial (see, e.g., [1, 21]).

**Question 27** Is the locus  $\mathcal{H}_{\mathbb{C}}$  connected?

For  $(c, b) \in \mathbb{R}^2$ , we can consider the restriction of  $f_{c,b}$  to  $\mathbb{R}^2$  and we can analogously define the *real hyperbolic horseshoe locus*  $\mathcal{H}_{\mathbb{R}} \subset \mathbb{R}^2$ . One of the main result of [2] states that  $\mathcal{H}_{\mathbb{R}}$  is connected and simply connected (see also [22]).

### 7.3 Isentropes

Take again  $(c, b) \in \mathbb{R}^2$  and consider the restriction  $f_{c,b}|_{\mathbb{R}^2} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ . Let  $h_{\text{top}}(f_{c,b}|_{\mathbb{R}^2})$  be the topological entropy of the real Hénon map  $f_{c,b}|_{\mathbb{R}^2}$ . For every

<sup>9</sup> We include the case  $b = 0$  to simplify the presentation.

$0 \leq \alpha \leq \log 2$ , the *isentrope* is defined as

$$\mathcal{E}_\alpha = \{(c, b) \in \mathbb{R}^2 : h_{\text{top}}(f_{c,b}|_{\mathbb{R}^2}) = \alpha\}.$$

In a topological term, monotonicity of the topological entropy of the real Hénon map  $f_{c,b}|_{\mathbb{R}^2}$  can be formulated as

**Question 28** (van Strien [147]) Is the isentrope  $\mathcal{E}_\alpha$  connected for any  $0 \leq \alpha \leq \log 2$  ?

Milnor and Tresser [116] showed it is true for cubic polynomials. The main result of [2] implies that the locus  $\mathcal{E}_{\log 2}$  is connected and simply connected (see also [22]).

Several articles attempt at giving lower bounds for topological entropy of real Hénon maps, e.g., [122, 123]. Among others, the paper [122] rigorously showed that  $h_{\text{top}}(f_{c,b}|_{\mathbb{R}^2}) > 0.46469$  for the classical Hénon’s parameter, and this bound is believed to be close to optimal. For upper bounds, the paper [16] has shown that  $h_{\text{top}}(f_{c,b}|_{\mathbb{R}^2}) < \log 2$  if and only if the Julia set of  $f_{c,b}$  (as a complex dynamical system) is not contained in  $\mathbb{R}^2$ . However, there is no algorithm which provides rigorous (non-trivial) upper bounds.

We thus propose the following problem.

**Question 29** Construct an algorithm to compute a rigorous upper bound for the topological entropy of a real Hénon map  $f_{c,b}|_{\mathbb{R}^2}$ .

Probably the only existing formula for (non-trivial) upper bound is given in [155]. However, according to Yomdin himself, the bound in the current form is far from sharp and would not give non-trivial ones.

## 8 Topology and Rigidity of Hénon Maps (R. Dujardin)

For polynomials and rational maps in dimension 1, there is a well-known list of exceptional examples whose Julia sets and dynamical properties are unexpectedly regular: Chebyshev polynomials, monomial mappings and Lattès examples. They can be characterized in many different ways, see e.g., [46, 157].

For generalized Hénon maps (as defined in Sect. 6.1) it is expected that no such exceptional example exists, but not so many actual results in this direction are known:

- Brunella proved in [40] that a generalized Hénon map cannot preserve an algebraic foliation of  $\mathbb{C}^2$ , i.e., a singular algebraic foliation by holomorphic curves. Here by preserving we mean that  $f$  maps leaves into leaves.
- Bedford and Kim proved in [13, 14] that neither  $J^+$  nor  $J^-$  (see Sect. 6.2 for a definition) can be a smooth  $C^1$  submanifold, nor a semi-analytic set.

Here we propose a few rigidity questions related to these results.

The first question is about a quantitative reinforcement of the Bedford–Kim theorem. Recall from the introduction, the definition of the standard quadratic Hénon map  $H_{a,c}(x, y) := (ay + x^2 + c, ax)$ , and the definition of  $K^+$  and  $J^+$  from Sect. 6.2.

For  $(a, c)$  close to  $(0, 0)$ ,  $H_{a,c}$  is a small perturbation of the monomial map  $(x, 0) \mapsto (x^2, 0)$ , whose Julia set is smooth, and in this case  $J_{a,c}^+$  is a topological 3-manifold, see [92, §7], and [144, §9].

**Question 30** Give an asymptotic expansion of the Hausdorff dimension of  $J_{a,c}^+$  as  $(a, c)$  tends to  $(0, 0)$ . In particular is there a uniform lower bound of  $\dim(J_{a,c}^+)$  of the form  $\dim(J_{a,c}^+) \geq 3 + h(a)$  with  $h(a) > 0$  in the neighborhood of  $c = 0$ ?

Note that  $(a, c) \mapsto \dim(J_{a,c}^+)$  is real analytic in the domain where  $H_{a,c}$  is hyperbolic (this was proved for one-dimensional maps by Ruelle in [133], and by Wolf [152] for polynomial automorphisms). It is not clear whether the dimension of the Julia set remains real-analytic when  $H_{a,c}$  degenerates to a unimodal map, for instance in a full neighborhood of  $(a, c) = (0, 0)$ .

Can we make Brunella's theorem local? More precisely:

**Question 31** Is it possible to find a generalized complex Hénon map  $f$ , an open set  $U \subset \mathbb{C}^2$  intersecting  $J^+$  and a holomorphic foliation of  $U$  such that  $J^+ \cap U$  is a union of leaves of this foliation?

We conjecture that the answer to this question is “no”. The answer is presumably easier if we assume that  $U \cap J^* \neq \emptyset$ . It is also possible that if  $f$  is dissipative (i.e.,  $|\det(Df)| < 1$ ), the assumption that  $J^-$  is foliated is stronger than the assumption that  $J^+$  is foliated (see [11, § 2]).

This would imply in particular that a generalized Hénon map cannot preserve a (transcendental) holomorphic foliation  $\mathcal{F}$  of  $\mathbb{C}^2$ . Indeed in such a case, consider the leaf  $L$  through a saddle periodic point  $p$ :  $L$  must be mapped into itself by  $f^n$ , hence coincide with the stable  $W^s(p)$  or unstable  $W^u(p)$  manifolds (see again §6.2 for a discussion of these objects); changing  $f$  to  $f^{-1}$  if necessary, we may assume that  $L = W^s(p)$ , and since  $W^s(p)$  is dense in  $J^+$  it follows that  $J^+$  is a union of leaves of  $\mathcal{F}$ .

**Remark 1** Note that the basin of attraction of the super-attracting point at infinity  $\Omega(f) := \{(x, y), |f^n(x, y)| \rightarrow \infty\}$  carries a natural (transcendental) holomorphic foliation which is  $f$ -invariant defined by the holomorphic 1-form  $\partial G^+$ , see [91, §7]. However this foliation does not extend to  $\mathbb{C}^2$  (otherwise it would extend holomorphically to  $\mathbb{P}^2$  which is absurd, see [70, §3]).

Related results include:

- the classification of holomorphic Anosov diffeomorphisms on surfaces by Ghys [82], in which a basic step is to prove that stable and unstable laminations are actually holomorphic foliations;
- the classification of birational maps preserving algebraic foliations by Cantat and Favre [43];
- the work of Pinto, Rand and others on the smooth rigidity of hyperbolic diffeomorphisms on surfaces (see e.g., [130]).

If the stable lamination is holomorphic, then by holonomy the unstable slices are holomorphically equivalent. We can now forget the foliation and ask about holomorphic equivalence of stable/unstable slices.

**Question 32** Under which circumstances is it possible that there exist saddle points  $p$  and  $q$ , and relative open subsets  $U \subset W^u(p)$  and  $V \subset W^u(q)$  such that  $U \cap K^+$  is biholomorphic to  $V \cap K^+$ ?

One obvious possibility is that  $p$  and  $q$  belong to the same cycle, and that the biholomorphism is induced by the action of  $f$ . We suspect that this is the only possibility.

A variant of this problem is when  $p$  and  $q$  are associated to different mappings.

**Question 33** Let  $f_1, f_2$  be two generalized Hénon maps. Under which circumstances is it possible that some local unstable slice of  $f_1$  (i.e., a set of the form  $K^+ \cap U$ , where  $U$  is a relatively open subset of an unstable manifold) is biholomorphically equivalent to an unstable slice of  $f_2$ ?

We expect that this can happen only if  $f_1$  and  $f_2$  are related by some algebraic correspondence. This question was raised in [73, Remark 4.4] for  $f_2 = f_1^{-1}$ , and a complete understanding would imply the main conjecture of [73]. The analogous question of existence of local biholomorphisms between Julia sets for 1-dimensional rational maps was addressed in [74, 98, 107].

Since a local unstable slice of a generalized Hénon maps contains essentially a complete information about unstable multipliers, the previous question is reminiscent of the classical “spectral rigidity” problem:

**Question 34** To which extent is a generalized Hénon map determined by the list of its unstable multipliers (resp. by the list of moduli of its unstable multipliers)?

We refer to [97] for a proof that the list of moduli of all multipliers determine a finite set of conjugacy classes of rational map of the Riemann sphere.

## 9 Statistical Properties of Complex Hénon Maps (F. Bianchi and T.-C. Dinh)

We denote in this section by  $f$  a complex Hénon map and by  $\mu$  its unique measure of maximal entropy [16, 17, 140]. We are interested in the statistical properties of  $\mu$  and of other natural invariant measures associated to such systems.

### 9.1 Thermodynamics for Hénon Maps

Consider a continuous function  $\phi: \mathbb{C}^2 \rightarrow \mathbb{R}$ , that will be called a *weight*. Following [132] one can define the *pressure*  $P(\phi)$  as

$$P(\phi) := \sup (h_\nu + \langle \nu, \phi \rangle),$$

where the supremum is taken over all invariant probability measures  $\nu$  for  $f$  and  $h_\nu$  denotes the measure-theoretic entropy of  $\nu$ . A measure  $\nu_0$  maximising the above supremum is called an *equilibrium state* associated to  $\phi$  and is necessarily ergodic when it is unique. The equilibrium state associated to  $\phi \equiv 0$  is the measure of maximal entropy  $\mu$ . We refer to [131] for an account on the properties of equilibrium states in one-dimensional complex dynamics and to [7, 45, 47] and references therein for the case of real Hénon maps and diffeomorphisms of compact manifolds satisfying some hyperbolicity assumptions.

**Problem 4** Prove the existence and the uniqueness of the equilibrium state  $\mu_\phi$  associated to any sufficiently regular weight  $\phi$  (for instance, every Hölder continuous  $\phi$ , and possibly with some bound on  $\max \phi - \min \phi$ ).

Recall that saddle points are equidistributed with respect to the measure of maximal entropy [15]. Namely, we have

$$\frac{1}{d^n} \sum_{x \in SP_n} \delta_x \rightarrow \mu, \tag{6}$$

where  $d$  is the algebraic degree of  $f$  (or, equivalently,  $\log d$  is the topological entropy of  $f$ , and the measure-theoretic entropy of  $\mu$ ) and  $SP_n$  is the set of the saddle  $n$ -periodic points of  $f$ .

**Question 35** Suppose  $\phi$  is sufficiently regular, so that the equilibrium state  $\mu_\phi$  exists and is unique. Is it true that

$$\frac{1}{e^{nP(\phi)}} \sum_{x \in SP_n} e^{S_n(\phi)} \delta_x \rightarrow \mu_\phi \quad ? \tag{7}$$

A version of the previous question has been established in [29] in the (expanding) setting of endomorphisms of  $\mathbb{P}^k_{\mathbb{C}}$ , and in particular for polynomials maps on  $\mathbb{C}$ .

Of a somehow different flavour, we recall that an explicit speed of convergence in (6) is unknown. We believe it is a very natural and challenging question to quantify such convergence when testing against sufficiently regular functions.

**Question 36** Is the convergence (6) exponentially fast against Hölder continuous observables? Is that also the case for (7)?

## 9.2 Statistical Properties of Equilibrium States and Spectral Gap for the Transfer Operators

Suppose the existence and the uniqueness of an equilibrium state  $\mu_\phi$  have been established. The (deterministic) problem of describing all orbits in the support of  $\mu_\phi$  is essentially impossible as this support should be contained in the set of points with chaotic behaviour in both forward and backward time. It is natural to adopt a probabilistic (or statistical) approach to this problem, to consider an *observable*  $g : \mathbb{C}^2 \rightarrow \mathbb{R}$ , and to view the sequence  $\{g \circ f^j\}_{j \in \mathbb{N}}$  as a sequence of random variables on the probability space  $(\mathbb{C}^2, \mu_\phi)$ . Since  $\mu_\phi$  is invariant, these random variables have the same distribution. They are however not independent, since they arise from a deterministic setting. The first goal is thus to show that the correlations  $\langle \mu_\phi, g \circ f^{j_1} \cdot g \circ f^{j_2} \rangle - \langle \mu_\phi, g \rangle^2$  go to zero in a quantifiable way, as  $|j_2 - j_1| \rightarrow \infty$ , see for instance [149, Problem 2]. When this happens and the convergence is fast enough, the sequence  $\{g \circ f^j\}_{j \in \mathbb{N}}$  is then expected to satisfy a list of properties which are typical of independent identically distributed (i.i.d.) random variables.

As a first step, since  $\mu_\phi$  is ergodic, Birkhoff theorem asserts that

$$\frac{1}{n}S_n(g)(x) := \frac{1}{n}(g(x) + g \circ f(x) + \dots + g \circ f^{n-1}(x)) \rightarrow \langle \mu_\phi, g \rangle := \int_{\mathbb{C}^2} g \, d\mu_\phi \tag{8}$$

for  $\mu_\phi$ -almost every  $x$  and every  $g \in L^1(\mu_\phi)$ . This can be seen as a version of the law of large numbers in this setting. The next step is to show the Central Limit Theorem (CLT) for sufficiently regular observables. As in the case of i.i.d. random variables, this CLT gives the rate of the above convergence (8).

**Problem 5** Show that  $\mu_\phi$  satisfies the CLT for Hölder continuous observables. Namely, prove that, for any Hölder continuous observable  $g$ , there exists  $\sigma \geq 0$  such that for any interval  $I \subset \mathbb{R}$ , we have

$$\begin{aligned} &\lim_{n \rightarrow \infty} \mu_\phi \left( \left\{ \frac{S_n(g) - n\langle \mu_\phi, g \rangle}{\sqrt{n}} \in I \right\} \right) \\ &= \begin{cases} 1 & \text{when } I \text{ is of the form } I = (-\delta, \delta) \text{ if } \sigma^2 = 0, \\ \frac{1}{\sqrt{2\pi\sigma^2}} \int_I e^{-t^2/(2\sigma^2)} \, dt & \text{if } \sigma^2 > 0. \end{cases} \end{aligned}$$

In the case of the maximal entropy measure  $\mu$ , the CLT was established in [31]. A natural question is also to characterize the observables for which  $\sigma = 0$ .

Sequences of (almost) independent random variables are also expected to satisfy large deviations properties. Recall that a coboundary  $g$  is an observable of the form  $\varphi \circ f - \varphi$ .

**Problem 6** Show that  $\mu_\phi$  satisfies the Large Deviation Principle (LDP) for Hölder continuous observables. Namely, prove that, for any Hölder continuous observable  $g$  with  $\langle \mu_\phi, g \rangle = 0$  and which is not a coboundary, there exists a non-negative, strictly convex function  $c$  which is defined on a neighborhood of  $0 \in \mathbb{R}$ , vanishes only at  $0$ , and such that, for all  $\epsilon > 0$  sufficiently small,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mu_\phi \left( \left\{ x \in X : \frac{S_n(g)(x)}{n} > \epsilon \right\} \right) = -c(\epsilon).$$

Note that this question is still open even for the measure of maximal entropy.

A possible unified approach to the above statistical properties would be to find Banach spaces (containing all Hölder continuous functions) where a suitable Ruelle-Perron-Frobenius (transfer) operator associated to  $f$  would turn out to be a strict contraction on the complement of an invariant line, see for instance [7, 83, 132]. In the case of endomorphisms of  $\mathbb{P}^k_{\mathbb{C}}$  in any dimension (in particular for any 1-dimensional complex polynomial), this is the main result of [28] (see also [113, 134]).

**Question 37** Do there exist norms for functions on the Julia set which are bounded on Hölder continuous functions, contract (on the complement of an invariant line) under

the action of  $f_*$  (or, more generally, of  $f_*(e^{\phi-P(\phi)} \cdot)$ ), and such that the contraction is stable by perturbation of  $\phi$ ?

In the case of hyperbolic maps, such a good Banach space has been introduced by Blank–Keller–Liverani [35, 106]. The norm is obtained by combining a regularity condition on the unstable manifolds together with a dual condition on the stable manifolds. Note that this was the starting point of a long story (see, e.g., [8, 84]). As Hénon maps are only non-uniformly hyperbolic (so that stable and unstable manifolds do not behave nicely in general) the above result does not apply here.

A positive answer to Question 37 would also give a unified proof for many statistical properties of independent interest (including the Large Deviations as in Problem 6), without the need of an *ad hoc* proof for each of them. For instance, the Local Central Limit Theorem (LCLT) and the Almost Sure Invariance Principle (ASIP) are both satisfied by sequences of i.i.d., and provide stronger results than the CLT, see, e.g., [83, 129] for definitions and criteria.

**Problem 7** Let  $\mu_\phi$  as in Problem 4. Show that all Hölder continuous observables which are not coboundaries satisfy the LCLT and the ASIP with respect to  $\mu_\phi$ .

Recall that the ASIP implies the Almost Sure Central Limit Theorem and the Law of the Iterated Logarithm.

### 9.3 Higher Dimension and Other Generalizations

Until now, we restricted our attention to Hénon maps, i.e., polynomial automorphisms of  $\mathbb{C}^2$ . On the other hand, Problems 6 and 7, and Question 37 make perfect sense for the equilibrium measures of general rational maps once this measure has been successfully defined. We review below some partial results that have been obtained in more general (invertible) settings than Hénon maps.

A polynomial automorphism of  $\mathbb{C}^k$  is said to be regular if the indeterminacy sets of the extensions to  $\mathbb{P}_{\mathbb{C}}^k$  of  $f$  and  $f^{-1}$  are non-empty and disjoint (observe that every Hénon map in dimension 2 satisfies this assumption, as these two sets are two distinct points). The construction of the measure of maximal entropy is given in [140], and the equidistribution of saddle points with respect to this measure is proved in [64] (see [31, 58] for the exponential mixing and the CLT in this case). More generally, one can also consider birational meromorphic maps of  $\mathbb{P}_{\mathbb{C}}^k$ , see [10, 54, 56, 69] for the construction of the measure of maximal entropy and its properties.

Given integers  $1 \leq p < k$  and open bounded convex domains  $M \Subset \mathbb{C}^p$  and  $N \Subset \mathbb{C}^{k-p}$ , a *horizontal-like map* is a proper holomorphic map from a vertical subset to a horizontal subset of  $M \times N$  which geometrically expands in  $p$  directions and contracts in  $k - p$  directions, see [62] for the precise definition. In this setting, the unique measure of maximal entropy has been constructed and studied in [59, 62, 68]. In the invertible case, the CLT for this measure can be deduced from [31].

One can also consider automorphisms of compact Kähler manifolds, see for instance [42, 61, 69] for the construction of the measure of maximal entropy and its properties. This setting shares a number of features with that of Hénon maps (in dimension 2) and regular automorphisms (in any dimensions). On the other hand, the compactness of



the manifold makes it more difficult to apply pluripotential techniques as in the case of Hénon maps. For instance, the proof of the CLT for the measure of maximal entropy, given in [30], requires the use of the theory of *superpotentials* on such manifolds [63].

## 10 Towards Higher Dimensions and Complex Differential Geometry (C. Favre)

### 10.1 Hénon Maps and the Group of Polynomial Automorphisms of $\mathbb{C}^2$

Let  $\text{Aut}[\mathbb{C}^2]$  be the group of polynomial automorphisms of  $\mathbb{C}^2$ . Recall the definition of degrees of a polynomial automorphism of the affine plane from Sect. 6.1. Jung [99] proved that the group  $\text{Aut}[\mathbb{A}_{\mathbb{C}}^2]$  is generated by affine transformations and triangular maps of the form  $E_P(x, y) := (x, y + P(y))$ . And the more precise version of Friedland-Milnor’s main theorem ([81]) states that either  $f \in \text{Aut}[\mathbb{A}_{\mathbb{C}}^2]$  is conjugated to a generalized Hénon maps  $H_{a_1, P_1} \circ \dots \circ H_{a_k, P_k}$  and  $\deg(f^n) \asymp (d_1 \dots d_k)^n$  for all  $n$ ; or  $\deg(f^n)$  remains bounded and  $f$  is conjugated to an affine map or to a triangular map.

Let us state the following general problem in vague terms.

**Problem 8** Describe the growth type of the sequence  $\{\deg(f^n)\}$  for any polynomial automorphism  $f$  of  $\mathbb{C}^d$ ,  $d \geq 3$ .

Very few results are known. Recall that Russakovski and Shiffman [137] observed that

$$\deg(f^{n+m}) \leq \deg(f^n) \deg(f^m)$$

for all  $n, m \geq 0$  so that the following limit  $\lambda(f) := \lim_n \deg(f^n)^{1/n}$  exists. We refer to it as the dynamical degree of  $f$ .

The case of cubic automorphisms on  $\mathbb{C}^3$ , and the case of automorphisms obtained as a composition of an affine transformation and a triangular one were considered by Blanc and Van Santen [33, 34]. Their computations lead them to formulate the following intriguing problem. A weak Perron number is an algebraic integer  $\lambda \geq 1$  such that all its Galois conjugates satisfy  $|\mu| \leq \lambda$ .

**Question 38** Is the dynamical degree of any polynomial automorphism of  $\mathbb{C}^d$  equal to a weak Perron number of degree  $\leq d - 1$ ?

It has been proven in [52], that  $\lambda(f)$  is an algebraic number of degree  $\leq 6$  for any polynomial automorphism  $f$  of  $\mathbb{C}^3$ .

The case  $\lambda(f) = 1$  is also particularly interesting.

**Question 39** Suppose  $f$  is a polynomial automorphism of  $\mathbb{C}^d$  satisfying  $\lambda(f) = 1$ . Is it true that  $\deg(f^n) \asymp n^k$  for some  $k \in \mathbb{N}$ ? Moreover, if  $k \geq 1$ , does  $f$  preserve a rational fibration?

Urech proved that  $\deg(f^n)$  tends to infinity whenever it is unbounded, see [145]. His result was made stronger by Cantat and Xie in [44]: there exists a universal function  $\sigma : \mathbb{N} \rightarrow \mathbb{N}$  such that  $\limsup \sigma = \infty$  and  $\deg(f^n) \geq \sigma(n)$ . They raised the following weaker form of the previous problem.

**Question 40** Suppose that  $\lambda(f) = 1$ , and  $\deg(f^n)$  is unbounded for some polynomial automorphism  $f$  of  $\mathbb{C}^d$ . Does there exist  $C > 0$  such that  $\deg(f^n) \geq Cn$ ?

Let  $\text{Tame}(3)$  be the subgroup of polynomial automorphisms of  $\mathbb{C}^3$  which is generated by affine and triangular transformations. A theorem of Shestakov and Umirbaev [138] states that  $\text{Tame}(3)$  is a strict subgroup of the full group of polynomial automorphisms of  $\mathbb{C}^3$  (as opposed to the 2-dimensional situation). A decisive progress on the structure of  $\text{Tame}(3)$  was recently made by Lamy and Przytycky [105], who constructed a CAT(0)-complex  $\mathcal{C}$  over which  $\text{Tame}(3)$  acts by isometries.

**Question 41** Is it possible to characterize those  $f \in \text{Tame}(3)$  for which  $\lambda(f) = 1$  in terms of their action on  $\mathcal{C}$ ?

### 10.2 Hénon Maps and Compact Complex Manifolds

Consider any generalized Hénon map  $f = H_{a_1, P_1} \circ \dots \circ H_{a_k, P_k}$  as in the previous section. Recall that  $f$  extends to the projective plane  $\mathbb{P}^2_{\mathbb{C}}$  as a birational map contracting the line at infinity to the super-attracting fixed point  $p = [1 : 0 : 0]$ . The topology of the basin of attraction of this point  $\Omega(f) := \{q \in \mathbb{C}^2, f^n(q) \rightarrow p\}$  has been explored by Hubbard and Oberste-Vorth [91]. They also observed that the map  $f$  acts properly discontinuously on  $\Omega(f)$  so that the space of orbits  $S(f) := \Omega(f)/\langle f \rangle$  is naturally a complex surface. One can then construct a compact complex surface  $\check{S}(f)$  having an isolated normal singularity at a point  $0 \in \check{S}(f)$  such that  $\check{S}(f) \setminus \{0\}$  is biholomorphic to  $S(f)$ . The minimal resolution of  $\check{S}(f)$  is a compact complex surface  $\bar{S}(f)$  that is non-Kähler, contains no smooth rational curve of self-intersection  $-1$ , and satisfies  $b_1(\bar{S}(f)) = 1$ . In Kodaira’s classification of surfaces [9], it belongs to the class VII<sub>0</sub> which is arguably the most mysterious class of compact complex surfaces. Dloussky and Oeljeklaus [66] have investigated when these surfaces carry global holomorphic vector fields.

**Question 42** Let  $f$  be any generalized Hénon map. Describe the set of all generalized Hénon maps  $g$  such that  $\bar{S}(f)$  is biholomorphic to  $\bar{S}(g)$ .

It follows from [77, Proposition 2.1] that under the preceding assumptions,  $f$  and  $g$  have the same degree and the same jacobian. Some partial results have been obtained by R. Pal for maps of the form  $H_{a, P}$  [127] generalizing former works on quadratic Hénon maps by Bonnot-Radu-Tanase [37].

The surface  $\bar{S}(f)$  carries a unique (singular) holomorphic foliation by curves, whose pull-back to  $\mathbb{C}^2 \setminus K_f^+$  is the holomorphic foliation  $\mathcal{F}^+$  given by  $\partial G_f^+ = 0$  (see §8 above).

**Question 43** Let  $f$  and  $g$  be two generalized Hénon maps. Suppose  $h: \mathbb{C}^2 \setminus (K_f^+ \cup K_f^-) \rightarrow \mathbb{C}^2 \setminus (K_g^+ \cup K_g^-)$  is a homeomorphism mapping the leaves of  $\mathcal{F}_f^+$  and  $\mathcal{F}_f^-$  to the leaves of  $\mathcal{F}_g^+$  and  $\mathcal{F}_g^-$ , respectively.

Is it true that  $h$  is either holomorphic or anti-holomorphic and extends as a polynomial automorphism of  $\mathbb{C}^2$ ?

This problem has been investigated by Lyubich–Robertson [112] and then by Firsova [78]. They proved that any conjugacies between  $\mathbb{C}^2 \setminus (K^+ \cap K^-)$  of quadratic Hénon maps that are small perturbation of one-dimensional hyperbolic polynomials is affine or anti-affine if it preserves both foliations.

Returning to surfaces of the form  $\bar{S}(f)$ , observe that they carry only finitely many rational curves that are all contracted to the singular point  $0 \in \check{S}(f)$ . One can also prove that it carries a unique holomorphic foliation which is induced by the Levi flats of the Green function  $G^+$  on  $\Omega(f)$ , see Remark 1.

An interesting feature of the complex surface  $\bar{S}(f)$  is that it admits a family of charts  $(U_i, \phi_i)$  where  $U_i$  is an open cover of  $\bar{S}(f)$ , and  $\phi_i: U_i \rightarrow \mathbb{C}^2$  is an open immersion such that  $\phi_{ij}$  is the restriction to an open domain of a birational self-map of  $\mathbb{P}_{\mathbb{C}}^2$ .

A complex manifold which admits a holomorphic atlas whose transition maps are restriction of birational maps of  $\mathbb{P}_{\mathbb{C}}^d$  is said to carry a birational structure.

The following problem is extracted from [65].

**Question 44** Does any non-Kähler compact complex surface admit a birational structure?

This question is extremely challenging, and reduces to the case of VII<sub>0</sub> surfaces. One can ask whether any deformation  $S$  of a surface  $\bar{S}(f)$  associated to a polynomial automorphism  $f$  as above admits a birational structure. This is true when the surface satisfies  $b_2(S) \leq 3$ , see [65].

Analogs of the construction of  $\bar{S}(f)$  have been explored by Oeljeklaus and Renaud in [124] for some quadratic polynomial automorphisms of  $\mathbb{C}^3$ , and further expanded by Ruggiero [136, Chapter 4]. A polynomial automorphism  $f \in \text{Aut}[\mathbb{C}^3]$  is said to be regular if the indeterminacy locus  $I(f)$  of its extension to  $\mathbb{P}_{\mathbb{C}}^3$  is disjoint from  $I(f^{-1})$ . This notion was introduced by Sibony in [139]. Let  $\Omega(f)$  be the basin of attraction of  $I(f^{-1})$ : this is an open  $f$ -invariant set over which  $f$  acts properly discontinuously. As above, denote by  $S(f)$  the quotient space  $\Omega(f)/\langle f \rangle$ .

**Problem 9** Let  $f \in \text{Aut}[\mathbb{C}^d]$  be any regular polynomial automorphism.

1. Prove that one can find a compact complex manifold  $\bar{S}(f)$  and an open immersion  $S(f) \subset \bar{S}(f)$  such that the complement  $\bar{S}(f) \setminus S(f)$  is a divisor.
2. Prove that  $\bar{S}(f)$  is unique up to bimeromorphism.
3. Describe complex objects on  $\bar{S}(f)$  (analytic subvarieties, vector fields, holomorphic foliations, positive closed currents,...). Compute its deformation space.

It is unclear how to extend this construction to a larger class of polynomial automorphisms of  $\mathbb{C}^3$ . However when  $\lambda(f)^2 > \lambda(f^{-1})$  an invariant valuation on the ring

of polynomial functions in three variables<sup>10</sup> is known to exist by [52], which suggests the following question.

**Question 45** Let  $f \in \text{Aut}[\mathbb{C}^3]$  be any polynomial automorphism such that  $\lambda(f)^2 > \lambda(f^{-1})$ . Prove the existence of a projective compactification  $X$  of  $\mathbb{C}^3$  such that the induced birational map  $f: X \dashrightarrow X$  admits a super-attracting fixed point  $p$  on the divisor at infinity.

Once such a compactification has been found, one can consider the basin of attraction  $\Omega$  of the point  $p$  and try to construct a compactification of the space of  $f$ -orbits in  $\Omega$  as above.

### 11 Hénon Maps Over Number Fields (P. Ingram)

Consider a sufficiently large field  $k$ , for example a number field. In general, one should expect to be able to construct Hénon maps of the form

$$f(x, y) = (y, F(y) - \delta x)$$

with cycles of length  $\deg(F) + 2$ . Write  $F(y) = a_0 + \dots + a_d y^d$ , and let  $y_0, \dots, y_{d+1}$  be variables ranging over  $k$ . Then, insisting that  $f$  sends

$$(y_0, y_1) \rightarrow (y_1, y_2) \rightarrow \dots \rightarrow (y_d, y_{d+1}) \rightarrow (y_{d+1}, y_0) \rightarrow (y_0, y_1) \tag{9}$$

is the same as insisting that

$$a_0 + a_1 y_n + \dots + a_d y_n^d - \delta y_{n-1} = y_{n+1},$$

for all  $n \pmod{d+2}$ . One checks that the determinant of the associated Vandermonde-like matrix

$$\begin{pmatrix} 1 & y_0 & \dots & y_0^d & -y_{d+1} \\ 1 & y_1 & \dots & y_1^d & -y_0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & y_{d+1} & \dots & y_{d+1}^d & -y_d \end{pmatrix}$$

is not identically zero (e.g., substituting  $y_{d+1} = y_d$  into this determinant gives  $\pm(y_d - y_{d-1}) \prod_{0 \leq i < j \leq d} (y_j - y_i)$ , which is itself not identically zero), and so this matrix is invertible over some affine-open subset of  $k^{d+1}$ . Here, one can find coefficients  $a_0, \dots, a_d$ , and  $\delta$  of  $f$  which enact (9).

Recently, Hyde and Doyle [67] exhibited single-variable polynomials over number fields with more preperiodic points than this sort of naive interpolation construction

<sup>10</sup> One way to interpret geometrically such an object is to say that it picks an irreducible subvariety in any algebraic compactification of  $\mathbb{C}^3$  in a compatible way.

gives. One might ask if similar phenomena could be exploited for generalized Hénon maps.

**Question 46** Over a number field  $K$ , is it possible to construct infinite families of generalized Hénon maps of algebraic degree  $d$ , and  $K$ -rational cycles of length at least  $d + 3$ ? Can one construct maps with  $N_d$  periodic points, where  $N_d - d \rightarrow \infty$ , or even  $N_d/d \rightarrow \infty$  as  $d \rightarrow \infty$ ?

Next, it is natural to ask about bounds in the other direction. In analogy to the Uniform Boundedness Conjecture of Morton and Silverman [121], it is natural to posit the following, in which  $\text{Per}(f)$  is the set of periodic points of  $f$  over the algebraic closure of  $K$ .

**Conjecture 2** Let  $K$  be a number field, let  $B \geq 1$ , and let  $d \geq 2$ . Then as  $f$  varies over generalized Hénon maps of degree  $d$  over  $K$ , the quantity

$$\#\{P \in \text{Per}(f) : [K(P) : K] \leq B\}$$

is bounded just in terms of  $[K : \mathbb{Q}]$ ,  $d$ , and  $B$ .

We have already seen why this bound must depend on  $d$ , and adjoining periodic points of  $f$  to the base field shows why the bound must depend on  $[K : \mathbb{Q}]$ . As a starting point for further exploration, we mention two more readily falsifiable conjectures.

**Conjecture 3** (See [94]) Over  $\mathbb{Q}$ ,  $(x, y) \mapsto (y, y^2 + c + x)$  has no point of period  $N$ , other than  $N \in \{1, 2, 3, 4, 6, 8\}$ .

**Conjecture 4** For all but finitely many  $\delta \in \mathbb{Q}$ , the  $\mathbb{Q}$ -rational periodic points of any  $f(x, y) = (y, y^2 + c - \delta x)$  with  $c \in \mathbb{Q}$  have period dividing 2.

It should be noted that some infinite families of examples show that both of these conjectures, if true, would be sharp.

It seems reasonable to posit something even stronger than Conjecture 2. Write  $\log^+ x = \log \max\{1, x\}$  for  $x \in \mathbb{R}^+$ , and for an absolute value  $|\cdot|_v$ , set

$$\|x_1, \dots, x_m\|_v = \max\{|x_1|_v, \dots, |x_m|_v\}.$$

Recall that a number field  $K$  comes equipped with a standard set  $M_K$  of absolute values, and we define the naive Weil height of  $P \in \mathbb{A}^N(K)$  to be

$$h(P) = \sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log^+ \|P\|_v.$$

Kawaguchi [100] constructed a canonical height  $\hat{h}_f$  associated to a generalized Hénon map, which differs by a controllable amount from the naive height, and interacts favorably with the dynamics of  $f$ , satisfying for example

$$\hat{h}_f \circ f + \hat{h}_f \circ f^{-1} = \left(d + \frac{1}{d}\right) \hat{h}_f,$$

and  $\hat{h}_f(P) = 0$  if and only if  $P$  is periodic. In light of partial results in this direction [94], it seems reasonable to conjecture the following strengthening of Conjecture 2.

**Conjecture 5** *Let  $K$  be a number field, let  $B \geq 1$ , and let  $d \geq 2$ . Then, there exist an  $\epsilon > 0$  and constant  $C$  (depending on these data) such that, as  $f$  varies over generalized Hénon maps of degree  $d$  over  $K$ , the quantity*

$$\#\{P \in (\overline{K})^2 : [K(P) : K] \leq B \text{ and } \hat{h}_f(P) < \epsilon h(f) + C\}$$

*is bounded uniformly, where  $h(f)$  is the height of the tuple of coefficients of  $f$ .*

Finally, let  $f$  be a generalized Hénon map defined over a number field  $K$  with good reduction away from some finite set  $S$  of primes (so, the coefficients are  $S$ -integers, and  $a_d$  and  $\delta$  are  $S$ -units), and let  $P_0 \in (\mathcal{O}_{K,S})^2$  be some non-periodic point. Set  $P_{n+1} = f(P_n)$ , for  $n \geq 0$ , and

$$\mathfrak{a}_n = \text{gcd}(x(P_n) - x(P_0), y(P_n) - y(P_0)) \subseteq \mathcal{O}_{K,S}.$$

That is,  $\mathfrak{a}_n$  is the largest ideal such that  $f^n(P) \equiv P$  modulo  $\mathfrak{a}_n$ . Then  $\mathfrak{a}_n$  is a divisibility sequence, i.e.,  $m \mid n \Rightarrow \mathfrak{a}_m \mid \mathfrak{a}_n$ , which we will call a *Hénon divisibility sequence*. In the case  $\mathcal{O}_{K,S} = \mathbb{Z}$ , we may identify the ideals with their unique positive generators, and think of this as a sequence of positive integers.

**Example 1**  $f(x, y) = (y, y^2 + x - 2)$ ,  $P = (2, 3)$ .

$\mathfrak{a}_n : 1, 1, 8, 3, 1, 8, 1, 3, 8, 1, 5, 48, 11, 1, 8, 51, 1, 8, 1, 3, 8, 5, 7, 288, 13, 11, 8, 3, 1, \dots$

In analogy to other divisibility sequences, we probably expect this sequence to grow slowly. By comparing to the height of  $P_n$ , one easily obtains an upper bound of size  $C^{d^n}$ , for some  $C$ , on the norm of each of the terms in the gcd, but the gcd itself should usually be much smaller.

**Theorem 6** (Bugeaud, Corvaja, and Zannier [41]) *If  $a, b \geq 2$  are multiplicatively independent integers and  $\epsilon > 0$ , then*

$$\text{gcd}(a^n - 1, b^n - 1) \ll_{\epsilon} e^{\epsilon n}.$$

**Theorem 7** (Huang [89]) *If  $f(x), g(x) \in \mathbb{Z}[x]$  of degree  $d \geq 2$ , then “generically” and under Vojta’s Conjecture*

$$\text{gcd}(f^n(a) - \alpha, g^n(b) - \beta) \ll_{\epsilon} e^{\epsilon d^n}.$$

On the other hand, it is certainly true that every prime  $\mathfrak{p} \subseteq \mathcal{O}_{K,S}$  divides some term in the sequence, since the image of  $P$  in the residue field  $\mathcal{O}_{K,S}/\mathfrak{p}$  must be periodic of period at most  $\text{Norm}(\mathfrak{p})^2$ .

**Question 47** Under what circumstance is it true that

$$\log \text{Norm}(\mathbf{a}_n) = o(d^n)$$

as  $n \rightarrow \infty$ ? Under what circumstance does there exist an ideal  $\mathfrak{b} \subset \mathcal{O}_{K,S}$  such that  $\mathfrak{a}_n | \mathfrak{b}$  for infinitely many  $n$ ? Note that, since  $\mathfrak{a}_1 | \mathfrak{a}_n$  for all  $n$ ,  $\mathfrak{b} = \mathfrak{a}_1$  is a reasonable candidate.

**Acknowledgements** The idea to collect this list of problem arose after holding a problem session at a BIRS conference which was lively chaired by M. Abate. We thank all participants of this session and the other contributors to this list to have generously shared their ideas and problems on Hénon maps. We are also grateful to Zin Arai, André de Carvalho, Sébastien Gouëzel, Stéphane Lamy, Misha Lyubich, Joseph Silverman, Liz Vivas, and the referee for their careful reading and their comments on previous versions of this manuscript.

**Data availability** Not applicable.

## References

1. Arai, Z.: On loops in the hyperbolic locus of the complex Hénon map and their monodromies. *Phys. D* **334**, 133–140 (2016)
2. Arai, Z., Ishii, Y.: On parameter loci of the Hénon family. *Commun. Math. Phys.* **361**(2), 343–414 (2018)
3. Arai, Z., Ishii, Y.: In preparation (2023)
4. Arosio, L., Benini, A.M., Fornæss, J.E., Peters, H.: Dynamics of transcendental Hénon maps. *Math. Ann.* **373**(1–2), 853–894 (2019)
5. Arosio, L., Benini, A.M., Fornæss, J.E., Peters, H.: Dynamics of transcendental Hénon maps III: Infinite entropy. *J. Mod. Dyn.* **17**, 465–479 (2021)
6. Arosio, L., Benini, A.M., Fornæss, J.E., Peters, H.: Dynamics of transcendental Hénon maps-II. *Math. Ann.* **385**(3–4), 975–999 (2023)
7. Baladi, V.: Positive Transfer Operators and Decay of Correlations. *Advanced Series in Nonlinear Dynamics*, vol. 16. World Scientific Publishing Co. Inc., River Edge (2000)
8. Baladi, V., Tsujii, M.: Anisotropic Hölder and Sobolev spaces for hyperbolic diffeomorphisms. *Ann. Inst. Fourier (Grenoble)* **57**(1), 127–154 (2007)
9. Barth, W.P., Hulek, K., Peters, C.A.M., Van de Ven, A.: *Compact complex surfaces*, volume 4 of *Ergeb. Math. Grenzgeb.*, 3. Folge, 2nd enlarged edn. Springer, Berlin (2004)
10. Bedford, E., Diller, J.: Energy and invariant measures for birational surface maps. *Duke Math. J.* **128**(2), 331–368 (2005)
11. Bedford, E., Dujardin, R.: Topological and geometric hyperbolicity criteria for polynomial automorphisms of  $\mathbb{C}^2$ . *Ergodic Theory Dyn. Syst.* **42**(7), 2151–2171 (2022)
12. Bedford, E., Guerini, L., Smillie, J.: Hyperbolicity and quasi-hyperbolicity in polynomial diffeomorphisms of  $\mathbb{C}^2$ . *Pure Appl. Math. Q.* **18**(1), 5–32 (2022)
13. Bedford, E., Kim, K.: No smooth Julia sets for polynomial diffeomorphisms of  $\mathbb{C}^2$  with positive entropy. *J. Geom. Anal.* **27**(4), 3085–3098 (2017)
14. Bedford, E., Kim, K.: Julia sets for polynomial diffeomorphisms of  $\mathbb{C}^2$  are not semianalytic. *Doc. Math.* **24**, 163–173 (2019)
15. Bedford, E., Lyubich, M., Smillie, J.: Distribution of periodic points of polynomial diffeomorphisms of  $\mathbb{C}^2$ . *Invent. Math.* **114**(2), 277–288 (1993)
16. Bedford, E., Lyubich, M., Smillie, J.: Polynomial diffeomorphisms of  $\mathbb{C}^2$ . IV: the measure of maximal entropy and laminar currents. *Invent. Math.* **112**(1), 77–125 (1993)
17. Bedford, E., Smillie, J.: Polynomial diffeomorphisms of  $\mathbb{C}^2$ : currents, equilibrium measure and hyperbolicity. *Invent. Math.* **103**(1), 69–99 (1991)
18. Bedford, E., Smillie, J.: Polynomial diffeomorphisms of  $\mathbb{C}^2$ . III. Ergodicity, exponents and entropy of the equilibrium measure. *Math. Ann.* **294**(3), 395–420 (1992)

19. Bedford, E., Smillie, J.: Polynomial diffeomorphisms of  $\mathbb{C}^2$ . VII: hyperbolicity and external rays. *Ann. Sci. Éc. Norm. Supér.* (4) **32**(4), 455–497 (1999)
20. Bedford, E., Smillie, J.: Polynomial diffeomorphisms of  $\mathbb{C}^2$ . VIII: quasi-expansion. *Am. J. Math.* **124**(2), 221–271 (2002)
21. Bedford, E., Smillie, J.: The Hénon family: the complex horseshoe locus and real parameter space. In: *Complex dynamics. Twenty-five years after the appearance of the Mandelbrot set. Proceedings of an AMS-IMS-SIAM joint summer research conference, Snowbird, UT, USA, June 13–17, 2004*, pp. 21–36. American Mathematical Society (AMS), Providence (2006)
22. Bedford, E., Smillie, J.: Real polynomial diffeomorphisms with maximal entropy. II: small Jacobian. *Ergodic Theory Dyn. Syst.* **26**(5), 1259–1283 (2006)
23. Benedetto, R., Ingram, P., Jones, R., Manes, M., Silverman, J.H., Tucker, T.J.: Current trends and open problems in arithmetic dynamics. *Bull. Am. Math. Soc. (N.S.)* **56**(4), 611–685 (2019)
24. Benedicks, M., Carleson, L.: The dynamics of the Hénon map. *Ann. Math. (2)* **133**(1), 73–169 (1991)
25. Berger, P.: Abundance of non-uniformly hyperbolic Hénon-like endomorphisms. *Astérisque* **410**, 53–177 (2019)
26. Berger, P.: Analytic pseudo-rotations. (Preprint) [arxiv: 2210.03438](https://arxiv.org/abs/2210.03438) (2022)
27. Berger, P., Biebler, S.: Emergence of wandering stable components. *J. Am. Math. Soc.* **36**(2), 397–482 (2023)
28. Bianchi, F., Dinh, T.-C.: Equilibrium states of endomorphisms of  $\mathbb{P}^k$ : spectral stability and limit theorems. (Preprint) [arXiv:2204.02856](https://arxiv.org/abs/2204.02856) (2022) (**To appear in Geometric and Functional Analysis**)
29. Bianchi, F., Dinh, T.-C.: Equilibrium states of endomorphisms of  $\mathbb{P}^k$  I: existence and properties. *J. Math. Pures Appl.* **9**(172), 164–201 (2023)
30. Bianchi, F., Dinh, T.-C.: Exponential mixing of all orders and CLT for automorphisms of compact Kähler manifolds. (Preprint) [arXiv:2304.13335](https://arxiv.org/abs/2304.13335) (2023)
31. Bianchi, F., Dinh, T.-C.: Every complex Hénon map is exponentially mixing of all orders and satisfies the CLT. *Forum Math. Sigma* **12**, e4 (2024)
32. Birkhoff, G.D.: Some unsolved problems of theoretical dynamics. *Science* **94**, 598–600 (1941)
33. Blanc, J., van Santen, I.: Automorphisms of the affine 3-space of degree 3. *Indiana Univ. Math. J.* **71**(2), 857–912 (2022)
34. Blanc, J., van Santen, I.: Dynamical degrees of affine-triangular automorphisms of affine spaces. *Ergodic Theory Dyn. Syst.* **42**(12), 3551–3592 (2022)
35. Blank, M., Keller, G., Liverani, C.: Ruelle–Perron–Frobenius spectrum for Anosov maps. *Nonlinearity* **15**(6), 1905–1973 (2002)
36. Blokh, A.M., Lyubich, M.: Measurable dynamics of  $S$ -unimodal maps of the interval. *Ann. Sci. École Norm. Sup. (4)* **24**(5), 545–573 (1991)
37. Bonnot, S., Radu, R., Tanase, R.: Hénon mappings with biholomorphic escaping sets. *Complex Anal. Synerg.* **3**(1), 3 (2017)
38. Boroński, J.P., Štímac, S.: The pruning front conjecture, wandering domains, and a classification of Hénon maps in the presence of strange attractors. (Preprint) [arXiv:2302.12568](https://arxiv.org/abs/2302.12568) (2023)
39. Bruin, H., Keller, G., Nowicki, T., van Strien, S.: Wild Cantor attractors exist. *Ann. Math. (2)* **143**(1), 97–130 (1996)
40. Brunella, M.: Minimal models of foliated algebraic surfaces. *Bull. Soc. Math. Fr.* **127**(2), 289–305 (1999)
41. Bugeaud, Y., Corvaja, P., Zannier, U.: An upper bound for the g.c.d of  $a^n - 1$  and  $b^n - 1$ . *Math. Z.* **243**(1), 79–84 (2003)
42. Cantat, S.: Dynamique des automorphismes des surfaces  $K3$ . *Acta Math.* **187**(1), 1–57 (2001)
43. Cantat, S., Favre, C.: Birational symmetries of foliated surfaces. *J. Reine Angew. Math.* **561**, 199–235 (2005)
44. Cantat, S., Xie, J.: On degrees of birational mappings. *Math. Res. Lett.* **27**(2), 319–337 (2020)
45. Carrasco, P.D., Rodríguez-Hertz, F.: Equilibrium states for center isometries. *J. Inst. Math. Jussieu First View* **23**, 1–61 (2023)
46. Casale, G.: Enveloppe galoisienne d’une application rationnelle de  $\mathbb{P}^1$ . *Publ. Mat.* **50**(1), 191–202 (2006)
47. Climenhaga, V., Pesin, Y., Zelerowicz, A.: Equilibrium measures for some partially hyperbolic systems. *J. Mod. Dyn.* **16**, 155–205 (2020)
48. Colli, E., Vargas, E.: Non-trivial wandering domains and homoclinic bifurcations. *Ergodic Theory Dyn. Syst.* **21**(6), 1657–1681 (2001)



49. Crovisier, S., Lyubich, M., Pujals, E., Yang, J.: Renormalization of Unicritical Diffeomorphisms of the Disk (2014). [arXiv:2401.13559](https://arxiv.org/abs/2401.13559)
50. Crovisier, S., Pujals, E.: Strongly dissipative surface diffeomorphisms. *Comment. Math. Helv.* **93**(2), 377–400 (2018)
51. Crovisier, S., Pujals, E., Tresser, C.: Mildly dissipative diffeomorphisms of the disk with zero entropy. (Preprint) [arXiv:2005.14278v3](https://arxiv.org/abs/2005.14278v3) 2023 (**To appear in Acta Mathematica**)
52. Dang, N.-B., Favre, C.: Spectral interpretations of dynamical degrees and applications. *Ann. Math.* (2) **194**(1), 299–359 (2021)
53. De Carvalho, A., Lyubich, M., Martens, M.: Renormalization in the Hénon family. I. Universality but non-rigidity. *J. Stat. Phys.* **121**(5–6), 611–669 (2005)
54. De Thélin, H., Vigny, G.: Entropy of meromorphic maps and dynamics of birational maps. *Mém. Soc. Math. Fr. (N.S.)* **122**, vi+98 (2010)
55. Diller, J.: Dynamics of birational maps of  $\mathbb{P}^2$ . *Indiana Univ. Math. J.* **45**(3), 721–772 (1996)
56. Diller, J., Favre, C.: Dynamics of bimeromorphic maps of surfaces. *Am. J. Math.* **123**(6), 1135–1169 (2001)
57. Diller, J., Kim, K.: Entropy of real rational surface automorphisms. *Exp. Math.* **30**(2), 172–190 (2021)
58. Dinh, T.-C.: Decay of correlations for Hénon maps. *Acta Math.* **195**, 253–264 (2005)
59. Dinh, T.-C., Nguyễn, V.-A., Sibony, N.: Dynamics of horizontal-like maps in higher dimension. *Adv. Math.* **219**(5), 1689–1721 (2008)
60. Dinh, T.-C., Sibony, N.: Dynamics of regular birational maps in  $\mathbb{P}^k$ . *J. Funct. Anal.* **222**(1), 202–216 (2005)
61. Dinh, T.-C., Sibony, N.: Green currents for holomorphic automorphisms of compact Kähler manifolds. *J. Am. Math. Soc.* **18**(2), 291–312 (2005)
62. Dinh, T.-C., Sibony, N.: Geometry of currents, intersection theory and dynamics of horizontal-like maps. *Ann. Inst. Fourier (Grenoble)* **56**(2), 423–457 (2006)
63. Dinh, T.-C., Sibony, N.: Super-potentials for currents on compact Kähler manifolds and dynamics of automorphisms. *J. Algebraic Geom.* **19**(3), 473–529 (2010)
64. Dinh, T.-C., Sibony, N.: Equidistribution of saddle periodic points for Hénon-type automorphisms of  $\mathbb{C}^k$ . *Math. Ann.* **366**(3–4), 1207–1251 (2016)
65. Dloussky, G.: Special birational structures on non-Kählerian complex surfaces. *J. Math. Pures Appl.* (2) **106**(1), 76–122 (2016)
66. Dloussky, G., Oeljeklaus, K.: Surfaces of class VII<sub>0</sub> and Hénon automorphisms. *C. R. Acad. Sci. Paris Sér. I Math.* **328**(7), 609–612 (1999)
67. Doyle, J.R., Hyde, T.: Polynomials with many rational preperiodic points. (Preprint) [arXiv:2201.11707](https://arxiv.org/abs/2201.11707) (2022)
68. Dujardin, R.: Hénon-like mappings in  $\mathbb{C}^2$ . *Am. J. Math.* **126**(2), 439–472 (2004)
69. Dujardin, R.: Laminar currents and birational dynamics. *Duke Math. J.* **131**(2), 219–247 (2006)
70. Dujardin, R.: Some remarks on the connectivity of Julia sets for 2-dimensional diffeomorphisms. In: Devaney, R.L., Keen, L. (eds.) *Complex Dynamics*, volume 396 of *Contemp. Math.*, pp. 63–84. American Mathematical Society, Providence (2006)
71. Dujardin, R.: A closing lemma for polynomial automorphisms of  $\mathbb{C}^2$ . Number 415. *Some aspects of the theory of dynamical systems: a tribute to Jean-Christophe Yoccoz*, vol. I, pp. 35–43. (2020)
72. Dujardin, R.: Saddle hyperbolicity implies hyperbolicity for polynomial automorphisms of  $\mathbb{C}^2$ . *Math. Res. Lett.* **27**(3), 693–709 (2020)
73. Dujardin, R., Favre, C.: The dynamical Manin–Mumford problem for plane polynomial automorphisms. *J. Eur. Math. Soc. (JEMS)* **19**(11), 3421–3465 (2017)
74. Dujardin, R., Favre, C., Gauthier, T.: When do two rational functions have locally biholomorphic Julia sets? *Trans. Am. Math. Soc.* **376**(3), 1601–1624 (2023)
75. Dujardin, R., Lyubich, M.: Stability and bifurcations for dissipative polynomial automorphisms of  $\mathbb{C}^2$ . *Invent. Math.* **200**(2), 439–511 (2015)
76. Elhadj, Z.: *Lozi Mappings. Theory and Applications*, With a Foreword by Julien Clinton Sprott and an Introduction by René Lozi. CRC Press, Boca Raton (2014)
77. Favre, C.: Classification of 2-dimensional contracting rigid germs and Kato surfaces. *I. J. Math. Pures Appl.* (2) **79**(5), 475–514 (2000)
78. Firsova, T.: The critical locus for complex Hénon maps. *Indiana Univ. Math. J.* **61**(4), 1603–1641 (2012)

79. Fornæss, J.E., Sibony, N.: Complex Hénon mappings in  $\mathbb{C}^2$  and Fatou-Bieberbach domains. *Duke Math. J.* **65**(2), 345–380 (1992)
80. Franks, J., Le Calvez, P.: Regions of instability for non-twist maps. *Ergodic Theory Dyn. Syst.* **23**(1), 111–141 (2003)
81. Friedland, S., Milnor, J.: Dynamical properties of plane polynomial automorphisms. *Ergodic Theory Dyn. Syst.* **9**(1), 67–99 (1989)
82. Ghys, É.: Holomorphic Anosov systems. *Invent. Math.* **119**(3), 585–614 (1995)
83. Gouëzel, S.: Limit theorems in dynamical systems using the spectral method. In: *Hyperbolic Dynamics, Fluctuations and Large Deviations*, volume 89 of *Proc. Sympos. Pure Math.* pp. 161–193. American Mathematical Society, Providence (2015)
84. Gouëzel, S., Liverani, C.: Banach spaces adapted to Anosov systems. *Ergodic Theory Dyn. Syst.* **26**(1), 189–217 (2006)
85. Guedj, V., Sibony, N.: Dynamics of polynomial automorphisms of  $\mathbb{C}^k$ . *Ark. Mat.* **40**(2), 207–243 (2002)
86. Guerini, L., Peters, H.: Julia sets of complex Hénon maps. *Int. J. Math.* **29**(7), 1850047 (2018)
87. Hénon, M.: A two-dimensional mapping with a strange attractor. *Commun. Math. Phys.* **50**(1), 69–77 (1976)
88. Hofbauer, F., Keller, G.: Quadratic maps without asymptotic measure. *Commun. Math. Phys.* **127**(2), 319–337 (1990)
89. Huang, K.: Generalized greatest common divisors for orbits under rational functions. *Monatsh. Math.* **191**(1), 103–123 (2020)
90. Hubbard, J.H.: The Hénon mapping in the complex domain. *Notes Rep. Math. Sci. Engrg.*, vol. 2, pp. 101–111. Academic Press, Inc., Orlando, FL (1986)
91. Hubbard, J.H., Oberste-Vorth, R.W.: Hénon mappings in the complex domain. I: The global topology of dynamical space. *Publ. Math. Inst. Hautes Étud. Sci.* **79**, 5–46 (1994)
92. Hubbard, J.H., Oberste-Vorth, R.W.: Hénon mappings in the complex domain. II. Projective and inductive limits of polynomials. In: *Real and complex dynamical systems (Hillerød, 1993)*, volume 464 of *NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci.*, pp. 89–132. Kluwer Acad. Publ., Dordrecht (1995)
93. Hubbard, J.H., Papadopol, P., Veselov, V.: A compactification of Hénon mappings in  $\mathbb{C}^2$  as dynamical systems. *Acta Math.* **184**(2), 203–270 (2000)
94. Ingram, P.: Canonical heights for Hénon maps. *Proc. Lond. Math. Soc.* (3) **108**(3), 780–808 (2014)
95. Ishii, Y.: Towards a kneading theory for Lozi mappings. I: a solution of the pruning front conjecture and the first tangency problem. *Nonlinearity* **10**(3), 731–747 (1997)
96. Jakobson, M.V.: Absolutely continuous invariant measures for one-parameter families of one-dimensional maps. *Commun. Math. Phys.* **81**(1), 39–88 (1981)
97. Ji, Z., Xie, J.: Homoclinic orbits, multiplier spectrum and rigidity theorems in complex dynamics. *Forum Math.* **11**, 37 (2023)
98. Ji, Z., Xie, J.: Local rigidity of Julia sets. (Preprint) [arxiv:2302.02562](https://arxiv.org/abs/2302.02562) (2023)
99. Jung, H.: Beispiele für die Anwendung von Anaglyphen-Raumbildern in der Geophysik. *Z. Geophys.* **17**(291–308), 1942 (1942)
100. Kawaguchi, S.: Canonical height functions for affine plane automorphisms. *Math. Ann.* **335**(2), 285–310 (2006)
101. Kawaguchi, S., Silverman, J.H.: On the dynamical and arithmetic degrees of rational self-maps of algebraic varieties. *J. Reine Angew. Math.* **713**, 21–48 (2016)
102. Kiriki, S., Soma, T.: Takens’ last problem and existence of non-trivial wandering domains. *Adv. Math.* **306**, 524–588 (2017)
103. Koropecski, A.: Aperiodic invariant continua for surface homeomorphisms. *Math. Z.* **266**(1), 229–236 (2010)
104. Koropecski, A., Le Calvez, P., Nassiri, M.: Prime ends rotation numbers and periodic points. *Duke Math. J.* **164**(3), 403–472 (2015)
105. Lamy, S., Przytycki, P.: Almost a building for the tame automorphism group. *Ann. Henri Lebesgue* **4**, 605–651 (2021)
106. Liverani, C.: Decay of correlations. *Ann. of Math.* (2) **142**(2), 239–301 (1995)
107. Luo, Y.: On the inhomogeneity of the Mandelbrot set. *Int. Math. Res. Not. IMRN* **8**, 6051–6076 (2021)

108. Lyubich, M.: Combinatorics, geometry and attractors of quasi-quadratic maps. *Ann. Math. (2)* **140**(2), 347–404 (1994)
109. Lyubich, M.: Almost every real quadratic map is either regular or stochastic. *Ann. Math. (2)* **156**(1), 1–78 (2002)
110. Lyubich, M., Peters, H.: Classification of invariant Fatou components for dissipative Hénon maps. *Geom. Funct. Anal.* **24**(3), 887–915 (2014)
111. Lyubich, M., Peters, H.: Structure of partially hyperbolic Hénon maps. *J. Eur. Math. Soc. (JEMS)* **23**(9), 3075–3128 (2021)
112. Lyubich, M., Robertson, J.W.: The critical locus and rigidity of foliations of complex henon maps. (preprint) [arXiv:2101.12148](https://arxiv.org/abs/2101.12148) (2021)
113. Makarov, N., Smirnov, S.: On “thermodynamics” of rational maps. I. Negative spectrum. *Commun. Math. Phys.* **211**(3), 705–743 (2000)
114. Milnor, J.: On the concept of attractor. *Commun. Math. Phys.* **99**(2), 177–195 (1985)
115. Milnor, J., Thurston, W.: On iterated maps of the interval. In: *Dynamical systems* (College Park, MD, 1986–87), Volume 1342 of *Lecture Notes in Math.*, pp. 465–563. Springer, Berlin (1988)
116. Milnor, J., Tresser, C.: On entropy and monotonicity for real cubic maps (with an appendix by Adrien Douady and Pierrette Sentenac). *Commun. Math. Phys.* **209**(1), 123–178 (2000)
117. Misiurewicz, M.: Strange attractors for the Lozi mappings. In: *Nonlinear dynamics* (Internat. Conf., New York, 1979), Volume 357 of *Ann. New York Acad. Sci.*, pp. 348–358. New York Acad. Sci., New York (1980)
118. Misiurewicz, M., Štímac, S.: Symbolic dynamics for Lozi maps. *Nonlinearity* **29**(10), 3031–3046 (2016)
119. Misiurewicz, M., Štímac, S.: Lozi-like maps. *Discrete Contin. Dyn. Syst.* **38**(6), 2965–2985 (2018)
120. Mora, L., Viana, M.: Abundance of strange attractors. *Acta Math.* **171**(1), 1–71 (1993)
121. Morton, P., Silverman, J.H.: Rational periodic points of rational functions. *Int. Math. Res. Not.* **1994**(2), 97–109 (1994)
122. Newhouse, S., Berz, M., Grote, S., Makino, K.: On the estimation of topological entropy on surfaces. In: *Geometric and probabilistic structures in dynamics. Workshop on dynamical systems and related topics in honor of Michael Brin on the occasion of his 60th birthday*, College Park, MD, USA, March 15–18, 2008, pp. 243–270. American Mathematical Society (AMS), Providence (2008)
123. Newhouse, S., Pignataro, T.: On the estimation of topological entropy. *J. Stat. Phys.* **72**(5–6), 1331–1351 (1993)
124. Oeljeklaus, K., Renaud, J.: Compact complex threefolds of class  $L$  associated to polynomial automorphisms of  $\mathbb{C}^3$ . *Publ. Mat. Barc.* **50**(2), 401–411 (2006)
125. Oliveira, F., Contreras, G.: No elliptic points from fixed prime ends. (Preprint) [arXiv:2205.14768](https://arxiv.org/abs/2205.14768) (2023)
126. Ou, D.-S.: Nonexistence of wandering domains for strongly dissipative infinitely renormalizable Hénon maps at the boundary of chaos. *Invent. Math.* **219**(1), 219–280 (2020)
127. Pal, R.: Relation between Hénon maps with biholomorphic escaping sets. *Math. Ann.* **388**(4), 4355–4382 (2024)
128. Palis, J., Viana, M.: High dimension diffeomorphisms displaying infinitely many periodic attractors. *Ann. Math. (2)* **140**(1), 207–250 (1994)
129. Philipp, W., Stout, W.: Almost sure invariance principles for partial sums of weakly dependent random variables. *Mem. Am. Math. Soc.* **2**(161), iv+140 (1975)
130. Pinto, A.A., Rand, D.A., Ferreira, F.: *Fine Structures of Hyperbolic Diffeomorphisms*. Springer Monographs in Mathematics. Springer, Berlin (2009)
131. Przytycki, F., Urbański, M.: *Conformal Fractals: Ergodic Theory Methods*. London Mathematical Society Lecture Note Series, vol. 371. Cambridge University Press, Cambridge (2010)
132. Ruelle, D.: Thermodynamic formalism, volume 5 of *Encyclopedia of Mathematics and its Applications*. Addison-Wesley Publishing Co., Reading, MA, 1978. The mathematical structures of classical equilibrium statistical mechanics, With a foreword by Giovanni Gallavotti and Gian-Carlo Rota
133. Ruelle, D.: Repellers for real analytic maps. *Ergodic Theory Dyn. Syst.* **2**(1), 99–107 (1982)
134. Ruelle, D.: Spectral properties of a class of operators associated with conformal maps in two dimensions. *Commun. Math. Phys.* **144**(3), 537–556 (1992)
135. Ruelle, D.: Historical behaviour in smooth dynamical systems. In: *Global analysis of dynamical systems*, pp. 63–66. *Inst. Phys., Bristol* (2001)

136. Ruggiero, M.: The valuative tree, rigid germs and kato varieties. *Tesi di Perfezionamento in Matematica* (2011)
137. Russakovskii, A., Shiffman, B.: Value distribution for sequences of rational mappings and complex dynamics. *Indiana Univ. Math. J.* **46**(3), 897–932 (1997)
138. Shestakov, I.P., Umirbaev, U.U.: The tame and the wild automorphisms of polynomial rings in three variables. *J. Am. Math. Soc.* **17**(1), 197–227 (2004)
139. Sibony, N.: Dynamics of rational maps on  $\mathbb{P}^k$ . In: *Dynamique et géométrie complexes*. Société Mathématique de France, pp. 97–185. American Mathematical Society, Providence (1999)
140. Sibony, N.: *Dynamique des applications rationnelles de  $\mathbb{P}^k$* . In: *Dynamique et géométrie complexes* (Lyon, 1997), volume 8 of *Panor. Synthèses*, pages ix–x, xi–xii, 97–185. Soc. Math. France, Paris (1999)
141. Silverman, J.H.: Geometric and arithmetic properties of the Hénon map. *Math. Z.* **215**(2), 237–250 (1994)
142. Takahasi, H.: Abundance of non-uniform hyperbolicity in bifurcations of surface endomorphisms. *Tokyo J. Math.* **34**(1), 53–113 (2011)
143. Talebi, A.: Non-statistical rational maps. *Math. Z.* **302**(1), 589–608 (2022)
144. Tanase, R.: Complex Hénon maps and discrete groups. *Adv. Math.* **295**, 53–89 (2016)
145. Urech, C.: Remarks on the degree growth of birational transformations. *Math. Res. Lett.* **25**(1), 291–308 (2018)
146. van Strien, S.: One-dimensional dynamics in the new millennium. *Discrete Contin. Dyn. Syst.* **27**(2), 557–588 (2010)
147. van Strien, S.: Milnor’s conjecture on monotonicity of topological entropy: results and questions. In: *Frontiers in complex dynamics*. In: *Celebration of John Milnor’s 80th birthday*. Based on a conference, Banff, Canada, February 2011, pp. 323–337. Princeton University Press, Princeton (2014)
148. Viana, M.: Strange attractors in higher dimensions. *Bol. Soc. Bras. Mat. (N.S.)* **24**(1), 13–62 (1993)
149. Viana, M.: Stochastic dynamics of deterministic systems. *Colóquios Brasileiros de Matemática*. IMPA (1997)
150. Wang, Q., Young, L.-S.: Strange attractors with one direction of instability. *Commun. Math. Phys.* **218**(1), 1–97 (2001)
151. Wang, O., Young, L.-S.: Toward a theory of rank one attractors. *Ann. Math. (2)* **167**(2), 349–480 (2008)
152. Wolf, C.: Dimension of Julia sets of polynomial automorphisms of  $\mathbb{C}^2$ . *Mich. Math. J.* **47**(3), 585–600 (2000)
153. Xia, Z.: Area-preserving surface diffeomorphisms. *Commun. Math. Phys.* **263**(3), 723–735 (2006)
154. Yoccoz, J.-C.: A proof of Jakobson’s theorem. *Astérisque* **410**, 15–52 (2019)
155. Yomdin, Y.: Local complexity growth for iterations of real analytic mappings and semicontinuity moduli of the entropy. *Ergodic Theory Dyn. Syst.* **11**(3), 583–602 (1991)
156. Yuan, X.: Big line bundles over arithmetic varieties. *Invent. Math.* **173**(3), 603–649 (2008)
157. Zdunik, A.: Parabolic orbifolds and the dimension of the maximal measure for rational maps. *Invent. Math.* **99**(3), 627–649 (1990)

**Publisher’s Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.